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BI-ISOTROPIC LAYERED MIXTURES

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1. Introduction

Electromagnetostatics in bianisotropic media was probably first studied in 1971 [1], and for this kind of media, the electric and magnetic static fields do not appear independently. The bi-isotropie medium is a special case of such a general medium. And the reciprocal chiral medium is a special case of the bi-isotropic medium. Recently, there have been an increasing interest in the bi-isotropic medium [2–6].

In this paper, the theory of polarizabilities of a chiral sphere and a layered dielectric ellipsoid introduced in [7, 8] will be extended to cover a layered bi-isotropic ellipsoid. The Maxwell-Garnett formula and the effective medium theory of heterogeneous dielectric media will be generalized to derive the four parameters of layered bi-isotropic mixtures. The successive steps are not very novel, but it was considered worthwhile to spell them out for reference purposes.

2. Polarizabilities of Bi-iostropic Layered Ellipsoids

2.1 The Quasi-Static Fields in a Bi-isotropic Medium.

The static problem can be formulated in terms of scalar potentials, because the curl of the electric and magnetic fields vanishes in a bi-isotropic medium and the electric and magnetic fields can be expressed as [6]

$$\overline{E} = -\nabla\phi \tag{1}$$

$$\overline{H} = -\nabla\phi^m \tag{2}$$

$$\overline{D} = \epsilon_r\epsilon_0\overline{E} + \xi_r\sqrt{\mu_0\epsilon_0}\,\overline{H}\epsilon_0 = -\left[\epsilon_r\epsilon_0\nabla\phi + \xi_r\sqrt{\mu_0\epsilon_0}\,\nabla\phi^m\right] \tag{3}$$

$$\overline{B} = \sqrt{\mu_0\epsilon_0}\,\zeta_r\overline{E} + \mu_r\mu_0\overline{H} = -\left[\zeta_r\sqrt{\mu_0\epsilon_0}\,\nabla\phi + \mu_r\mu_0\nabla\phi^m\right] \tag{4}$$

The four parameters ϵ_r, μ_r, ξ_r and ζ_r, are assumed to be constants in a rectangular coordinate system and no attempt to interpret the medium physically will be made in this paper. Because there are no sources within the bi-isotropic medium,from $\nabla\cdot\overline{D} = 0$ and $\nabla\cdot\overline{B} = 0$ we have $\nabla\cdot\overline{E} = 0$ and $\nabla\cdot\overline{H} = 0$, hence both potential ϕ and ϕ^m satisfy the Laplace equation [6]

$$\nabla^2\phi = 0 \tag{5}$$

$$\nabla^2\phi^m = 0 \tag{6}$$

In Section 2.3, we shall show that this formulation is more suitable for using the boundary conditions than that of [7].

2.2 General Solution of the Laplace Equation.

Consider a confocal ellipsoid consisting of N layers of different medium parameters, lying in a background medium of parameters $(\epsilon_0, \mu_0, \xi_0, \zeta_0)$ according to the geometry shown in Fig. 1. The surface layer of the scatterer has parameters $(\epsilon_{r1}, \mu_{r1}, \xi_{r1}, \zeta_{r1})$, the next outermost ellipsoidal shell has parameters $(\epsilon_{r2}, \mu_{r2}, \xi_{r2}, \zeta_{r2})$, the next is $(\epsilon_{r3}, \mu_{r3}, \xi_{r3}, \zeta_{r3})$, etc. Finally, the core is of parameters $(\epsilon_{rN}, \mu_{rN}, \xi_{rN}, \zeta_{rN})$. The incident static electric and magnetic fields are assumed along the x axis of the ellipsoid without loss of generality [8]. The N ellipsoid boundaries are assumed to be confocal,i.e.,

$$a_i^2 - a_j^2 = b_i^2 - b_j^2 = c_i^2 - c_j^2 \tag{7}$$

for all pairs i, j. Where a_i, b_i, c_i are the semiaxes of the ith ellipsoid boundary. This means that the ellipsoidal boundaries between the layers are the constant-coordinate surface $\xi = \xi_i$ in the ellipsoidal coordinate system (for ellipsoidal coordinates and the general solution of Laplace equation in it see [8] and references therein), and the general solution of Laplace equation are only dependent on one coordinate ξ. The ellipsoidal coordinates (ξ, η, ζ) are defined by the three real roots of the following cubic equation of u.

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1\,, \qquad a > b > c \tag{8}$$

The coordinates ξ, η, ζ are the root that lies in the range $-c^2 < \xi < \infty, -b < \eta < -c^2, -a^2 < \zeta < -b^2$ respectively. Constant- ξ surfaces are ellipsoids all confocal to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{9}$$

Therefore, $\xi_1 = 0$ is the equation of the surface of the outermost ellipsoid and

$$\xi = \xi_k = c_k^2 - c_1^2 = b_k^2 - b_1^2 = a_k^2 - a_1^2 \tag{10}$$

is that of the surface of the kth ellipsoidal boundary, where $a_1 = a, b_1 = b,$ and $c_1 = c$. The incident x-directed static electric and magnetic fields of amplitudes E_L and H_L respectively, polarizated along the a-axis of the layered ellipsoid, have the potentials ϕ_0, ϕ_0^m :

$$\phi_0(\overline{r}) = -E_L x = -E_L \sqrt{\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{(b^2-a^2)(c^2-a^2)}} \tag{11}$$

$$\phi_0^m(\overline{r}) = -H_L x = -H_L \sqrt{\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{(b^2-a^2)(c^2-a^2)}} \tag{12}$$

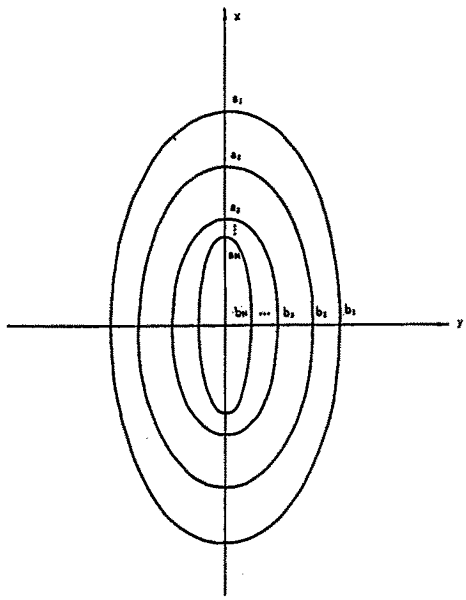

Figure 1 Multilayer ellipsoid of the problems.

Then the potentials in the k th region can be expressed as [8]

$$\phi_0(\overline{r}) = -E_L x \left[C_k - \frac{D_k}{2} \int_\xi^\infty \frac{ds}{(s+a_1^2)R_1(s)} \right] \tag{13}$$

$$\phi_0^m(\overline{r}) = -E_L x \left[C_k^m - \frac{D_k^m}{2} \int_\xi^\infty \frac{ds}{(s+a_1^2)R_1(s)} \right] \tag{14}$$

$$R_1(s) = \sqrt{(s+a_1^2)(s+b_1^2)(s+c_1^2)} \tag{15}$$

where $a_1, b_1, c_1,$ are the semiaxes of the outermost ellipsoid seperating parameters $(\epsilon_{r1}, \mu_{r1}, \xi_{r1}, \zeta_{r1})$ and $(\epsilon_{r0}, \mu_{r0}, \xi_{r0}, \zeta_{r0})$. The boundary separating medium parameters $(\epsilon_{rk}, \mu_{rk}, \xi_{rk}, \zeta_{rk})$ and $(\epsilon_{r(k+1)}, \mu_{r(k+1)}, \xi_{r(k+1)}, \zeta_{r(k+1)})$ is the ellipsoid with semiaxes a_{k+1}, b_{k+1}, and c_{k+1} where the coordinate ξ has the value ξ_{k+1}. The author intend to use E_L in (14) for the purpose of easy formulation.

2.3 The Boundary Condition

Connections between the amplitudes in adjacent regions is obtained through four interface conditions

$$\phi_k = \phi_{k+1} \tag{16}$$

$$\phi_k^m = \phi_{k+1}^m \tag{17}$$

$$\epsilon_{rk}\widehat{n}\cdot\nabla\phi_k + \xi_{rk}\widehat{n}\cdot\phi_k^m = \epsilon_{r(k+1)}\widehat{n}\cdot\nabla\phi_{(k+1)} + \xi_{r(k+1)}\widehat{n}\cdot\nabla\phi_{k+1}^m \tag{18}$$

$$\zeta_{rk}\widehat{n}\cdot\nabla\phi_k + \mu_{rk}\widehat{n}\cdot\phi_k^m = \zeta_{r(k+1)}\widehat{n}\cdot\nabla\phi_{(k+1)} + \mu_{r(k+1)}\widehat{n}\cdot\nabla\phi_{k+1}^m \tag{19}$$

Substituting (13), (14) into (16)–(19), we have [8]

$$C_k - D_k M_k = C_{k+1} - D_{k+1} M_k \tag{20}$$

$$C_k^m - D_k^m M_k = C_{k+1}^m - D_{k+1}^m M_k \tag{21}$$

$$\begin{aligned}&\epsilon_{rk}\left[C_k + D_k M_k^1\right] + \xi_{rk}\left[C_k^m + D_k^m M_k^1\right]\\&= \epsilon_{r(k+1)}\ \left[C_{k+1} + D_{k+1}M_k^1\right] + \xi_{r(k+1)}\ \left[C_{k+1}^m + D_{k+1}^m M_k^1\right]\end{aligned} \tag{22}$$

$$\begin{aligned}&\zeta_{rk}\left[C_k + D_k M_k^1\right] + \mu_{rk}\left[C_k^m + D_k^m M_k^1\right]\\&= \zeta_{r(k+1)}\ \left[C_{k+1} + D_{k+1}M_k^1\right] + \mu_{r(k+1)}\ \left[D_{k+1}^m + D_{k+1}^m M_k^1\right]\end{aligned} \tag{23}$$

where [8]

$$M_k = \frac{N_k^x}{a_k b_k c_k} = \frac{1}{2}\int_0^\infty \frac{ds^1}{(s^1 + a_k^2)\sqrt{(s^1 + a_k^2)(s^1 + b_k^2)(s^1 + c_k^2)}} \tag{24}$$

$$M_k^1 = \frac{1}{R_1(\xi_{k+1})} - M_k \tag{25}$$

From the boundary conditions (20)–(23), the field amplitudes in the k th region can be calculated from the amplitudes in the $(k+1)$ th region. In a matrix form

$$\begin{bmatrix} C_k \\ C_k^m \\ D_k \\ D_k^m \end{bmatrix} = \begin{bmatrix} \overline{C}_k \\ \overline{D}_k \end{bmatrix} = \overline{\overline{B}}_{k,k+1} \begin{bmatrix} C_{k+1} \\ C_{k+1}^m \\ D_{k+1} \\ D_{k+1}^m \end{bmatrix} = \overline{\overline{B}}_{k,k+1} \cdot \begin{bmatrix} \overline{C}_{k+1} \\ \overline{D}_{k+1} \end{bmatrix} \tag{26}$$

$$\overline{\overline{B}}_{k,k+1} = \frac{1}{\Delta_k} \begin{bmatrix} b_{k11} & b_{k12} & b_{k13} & b_{k14} \\ b_{k21} & b_{k22} & b_{k23} & b_{k24} \\ b_{k31} & b_{k32} & b_{k33} & b_{k34} \\ b_{k41} & b_{k42} & b_{k43} & b_{k44} \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} C_{k+1} \\ C_{k+1}^m \\ D_{k+1} \\ D_{k+1}^m \end{bmatrix} = \overline{\overline{F}}_{k+1,k} \begin{bmatrix} C_k \\ C_k^m \\ D_k \\ D_k^m \end{bmatrix} \tag{28}$$

$$\overline{\overline{F}}_{k+1,k} = \frac{1}{\Delta_k^1} \begin{bmatrix} f_{k11} & f_{k12} & f_{k13} & f_{k14} \\ f_{k21} & f_{k22} & f_{k23} & f_{k24} \\ f_{k31} & f_{k32} & f_{k33} & f_{k34} \\ f_{k41} & f_{k42} & f_{k43} & f_{k44} \end{bmatrix} \tag{29}$$

where $\overline{\overline{B}}_{k,k+1}, \overline{\overline{F}}_{k+1,k}$ are the backward (outward) and the forward (inward) propagation matrices introduced in [8] respectively. All the elements of $\overline{\overline{B}}_{k,k+1}$, and $\overline{\overline{F}}_{k+1,k}$ are given in Appendix A.

The propagation matrices can be used to calculate the field amplitudes in the core region as functions of those outside the scatterer and vice versa

$$\begin{bmatrix} \overline{C}_0 \\ \overline{D}_0 \end{bmatrix} = \overline{\overline{B}}_{0,1} \cdot \overline{\overline{B}}_{1,2} \cdots \overline{\overline{B}}_{N-1,N} \begin{bmatrix} \overline{C}_N \\ \overline{D}_N \end{bmatrix}$$

$$= \overline{\overline{B}}_{0,N} \begin{bmatrix} \overline{C}_N \\ \overline{D}_N \end{bmatrix} = \begin{bmatrix} \overline{\overline{b}}_{11} & \overline{\overline{b}}_{12} \\ \overline{\overline{b}}_{21} & \overline{\overline{b}}_{22} \end{bmatrix} \begin{bmatrix} \overline{C}_N \\ \overline{D}_N \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} \overline{C}_N \\ \overline{D}_N \end{bmatrix} = \overline{\overline{F}}_{N,N-1} \cdot \overline{\overline{F}}_{N-1,N-2} \cdots \overline{\overline{F}}_{1,0} \begin{bmatrix} \overline{C}_0 \\ \overline{D}_0 \end{bmatrix}$$

$$= \overline{\overline{F}}_{N,0} \begin{bmatrix} \overline{C}_1 \\ \overline{D}_0 \end{bmatrix} = \begin{bmatrix} \overline{\overline{f}}_{11} & \overline{\overline{f}}_{12} \\ \overline{\overline{f}}_{21} & \overline{\overline{f}}_{22} \end{bmatrix} \begin{bmatrix} \overline{C}_0 \\ \overline{D}_0 \end{bmatrix} \tag{31}$$

where $\overline{\overline{B}}_{0,N}$ and $\overline{\overline{F}}_{N,0}$ are the total backward and forward propagation matrices respectively, $\overline{\overline{b}}_{ij}, \overline{\overline{f}}_{ij} (i, j = 1, 2)$ are all 2×2 matrices.

In the region outside the ellipsoid the incoming electric and magnetic fields are of amplitudes E_L and H_L respectively and hence (see Eqs. (13), (14))

$$\overline{C}_0 = \begin{pmatrix} 1 \\ H_L/E_L \end{pmatrix} \qquad \overline{D}_N = 0 \tag{32}$$

because there are no outgoing fields in the bi-isotropic medium core region. Therefore, the scattering–field coefficients $\overline{D}_0$ and the coefficients of homogeneous field in the core region $\overline{C}_N$ can be solved

$$\overline{D}_0 = \overline{\overline{b}}_{12} \cdot \overline{\overline{b}}_{11}^{-1} \begin{pmatrix} 1 \\ H_L/E_L \end{pmatrix} \tag{33}$$

$$\overline{C}_N = \overline{\overline{b}}_{11}^{-1} \begin{pmatrix} 1 \\ H_L/E_L \end{pmatrix} \tag{34}$$

where $\overline{\overline{B}}_{0,N}$ and $\overline{\overline{F}}_{N,0}$ are the total backward and forward propagation matrices respectively.

The boundary conditions of the perfectly conducting core at $\xi = \xi_N$ yield

$$C_N = D_N \frac{N_N^x}{a_N b_N c_N} = C_{11} D_N \tag{35}$$

$$C_N^m = \frac{a_N b_N c_N}{R_1(\xi_N)} = C_{22} D_N^m \tag{36}$$

In other words,

$$\overline{C}_N = \overline{\overline{C}} \cdot \overline{D}_N, \qquad \overline{\overline{C}} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \tag{37}$$

Thus the scattering–field coefficients $\overline{D}_0$ and $\overline{D}_N$ can be written as

$$\overline{D}_0 = [\overline{\overline{b}}_{21} \cdot \overline{\overline{C}} + \overline{\overline{b}}_{22}] \cdot [\overline{\overline{b}}_{11} \cdot \overline{\overline{C}} + \overline{\overline{b}}_{12}]^{-1} \cdot \overline{C}_0 \tag{38a}$$

$$\overline{D}_N = [\overline{\overline{b}}_{11} \cdot \overline{\overline{C}} + \overline{\overline{b}}_{12}]^{-1} \cdot \overline{C}_0 \tag{38b}$$

$$\overline{C}_0 = \begin{pmatrix} 1 \\ H_L/E_L \end{pmatrix} \tag{38c}$$

or

$$\overline{D}_0 = [\overline{\overline{C}} \cdot \overline{\overline{f}}_{22} - \overline{\overline{f}}_{12}]^{-1} \cdot [\overline{\overline{f}}_{11} - \overline{\overline{C}} \cdot \overline{\overline{f}}_{21}] \cdot \overline{C}_0 \tag{39a}$$

$$\overline{D}_N = \overline{\overline{f}}_{21} \cdot \overline{C}_0 + \overline{\overline{f}}_{22} \cdot \overline{D}_0 \tag{39b}$$

We would like to emphasize that Eqs. (35)–(39) are for the case with perfectly conducting core, while Eqs. (32)–(34) are for the case with a bi-isotropic medium core.

2.4 Polarizability Dyadics

As shown by Sihvola and Lindell, the equivalent dipole moments p_e, p_m can be expressed in terms of D_0, D_0^m as follows:

$$p_e = \frac{4\pi}{3} \epsilon_0 D_0 E_L \tag{40a}$$

$$p_m = \frac{4\pi}{3}\mu_0 D_0^m E_L \tag{40b}$$

In a matrix notation

$$\begin{aligned}\begin{bmatrix} p_e \\ p_m \end{bmatrix} &= \frac{4\pi}{3}\begin{bmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{bmatrix}\begin{bmatrix} D_0 E_L \\ D_0^m E_L \end{bmatrix} \\ &= \frac{4\pi}{3}\begin{bmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{bmatrix}\begin{bmatrix} D_0 E_L \\ D_0^m E_L \end{bmatrix}\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}\cdot\begin{bmatrix} E_L \\ H_L \end{bmatrix} \\ &= \begin{bmatrix} \alpha_{ee} & \alpha_{em} \\ \alpha_{me} & \alpha_{mm} \end{bmatrix}\cdot\begin{bmatrix} E_L \\ H_L \end{bmatrix}\end{aligned} \tag{41}$$

where we have unified the notation in (33) and (34) as well as (38) and (39). Notice that in the above, $\overline{\overline{\alpha}}$ means $\overline{\overline{\alpha^x}}$, and the above analysis can easily adapted to $\overline{\overline{\alpha^y}}$ and $\overline{\overline{\alpha^z}}$. If we designate $\overline{\overline{\alpha}}_{rs} = \sum_{i=1}^3 \alpha_{rs}^i \widehat{x}_i\widehat{x}_i$ $(r, s = e, m)$, where $\widehat{x}_i$ is the unit vector along the i th orthogonal semiaxes of the ellipsoid, we can generalize (41) as

$$\begin{bmatrix} \overline{p}_e \\ \overline{p}_m \end{bmatrix} = \begin{bmatrix} \overline{\overline{\alpha}}_{ee} & \overline{\overline{\alpha}}_{em} \\ \overline{\overline{\alpha}}_{me} & \overline{\overline{\alpha}}_{mm} \end{bmatrix}\begin{bmatrix} \overline{E}_L \\ \overline{H}_L \end{bmatrix} \tag{42}$$

3. Macroscopic Parameters of Ordered Layered-Ellipsoid Mixtures

The purpose of this section is to derive the mixing formula of a bi-isotropic mixture containing multilayer ellipsoids. Let the background medium be of parameters $(\epsilon_0, \mu_0, \xi_0, \zeta_0)$ as before, and let there be n ellipsoids inclusions per unit volume. Consider first the case that all the ellipsoids are aligned equally in the mixture.

Define the effective parameters of the mixture $\overline{\overline{\epsilon}}_{\text{eff}}, \overline{\overline{\mu}}_{\text{eff}}, \overline{\overline{\xi}}_{\text{eff}}$, and $\overline{\overline{\zeta}}_{\text{eff}}$ by the coefficients in the macroscopic constitutive relations between the average flux densities and the average fields $\overline{E}_0, \overline{H}_0$

$$\begin{bmatrix} \langle\overline{D}\rangle \\ \langle\overline{B}\rangle \end{bmatrix} = \begin{bmatrix} \overline{\overline{\epsilon}}_{\text{eff}} & \overline{\overline{\xi}}_{\text{eff}} \\ \overline{\overline{\zeta}}_{\text{eff}} & \overline{\overline{\mu}}_{\text{eff}} \end{bmatrix}\cdot\begin{bmatrix} \overline{E}_0 \\ \overline{H}_0 \end{bmatrix} \tag{43}$$

The flux densities are calculated from the electric and magnetic polarizations $\langle\overline{P}_e\rangle, \langle\overline{P}_m\rangle$ due to the dipole moments of the scatterers in the mixture:

$$\begin{bmatrix}\langle\overline{D}\rangle\\ \langle\overline{B}\rangle\end{bmatrix} = \begin{bmatrix}\epsilon_0 & \xi_0\\ \zeta_0 & \mu_0\end{bmatrix}\begin{bmatrix}\overline{E}_0\\ \overline{H}_0\end{bmatrix} + \begin{bmatrix}\langle\overline{P}_e\rangle\\ \langle\overline{P}_m\rangle\end{bmatrix} \tag{44}$$

The average polarization is the dipole moment density:

$$\begin{bmatrix}\langle\overline{P}_e\rangle\\ \langle\overline{P}_m\rangle\end{bmatrix} = n\begin{bmatrix}\overline{p}_e\\ \overline{p}_m\end{bmatrix} = n\begin{bmatrix}\overline{\overline{\alpha}}_{ee} & \overline{\overline{\alpha}}_{em}\\ \overline{\overline{\alpha}}_{me} & \overline{\overline{\alpha}}_{mm}\end{bmatrix}\begin{bmatrix}\overline{E}_L\\ \overline{H}_L\end{bmatrix} \tag{45}$$

It is observed that the exciting fields $\overline{E}_L$ and $\overline{H}_L$ are not the same as the average fields $\overline{E}_0$ and $\overline{H}_0$ but rather than the Lorentian fields [9] larger than the incident fields that include contributions from the surrounding polarization, whose effect comes through the depolarization dyadic [9], (see ref. [3] Eq. (12))

$$\begin{bmatrix}\overline{E}_L\\ \overline{H}_L\end{bmatrix} = \begin{bmatrix}\overline{E}_0\\ \overline{H}_0\end{bmatrix} + \frac{\delta}{v_0}\overline{\overline{L}}\cdot\begin{bmatrix}\mu_0 & -\xi_0\\ -\zeta_0 & \epsilon_0\end{bmatrix}\begin{bmatrix}\langle\overline{p}_e\rangle\\ \langle\overline{p}_m\rangle\end{bmatrix} \tag{46a}$$

$$v_0 = \frac{4\pi abc}{3}$$

$$\delta = \left\{\left[i(\xi_0-\zeta_0)+\sqrt{(4(\epsilon_0\mu_0-\xi_0\zeta_0)-(\zeta_0-\xi_0)^2)}\right]\right.$$
$$\left.\cdot\left[i(\zeta_0-\xi_0)+\sqrt{4(\epsilon_0\mu_0-\xi_0\zeta_0)-(\xi_0-\zeta_0)^2}\right]\right\}^{-1}\times 4 \tag{46b}$$

where the depolarization dyadic $\overline{\overline{L}}$ is given by

$$\overline{\overline{L}} = L_1\widehat{x}_1\widehat{x}_1 + L_2\widehat{x}_2\widehat{x}_2 + L_3\widehat{x}_3\widehat{x}_3 \tag{46c}$$

$$L_1 = \frac{1}{2}abc\int_0^\infty ds\,(s+a^2)^{-\frac{3}{2}}(s+b^2)^{-\frac{1}{2}}(s+c^2)^{-\frac{1}{2}} \tag{46d}$$

$$L_2 = \frac{1}{2}abc \int_0^\infty ds\,(s+a^2)^{-\frac{1}{2}}(s+b^2)^{-\frac{3}{2}}(s+c^2)^{-\frac{1}{2}} \tag{46e}$$

$$L_3 = 1 - L_1 - L_2 \tag{46f}$$

The average polarization can be solved from (45) and (46)

$$\begin{aligned}\begin{bmatrix} \langle P_e \rangle \\ \langle P_m \rangle \end{bmatrix} &= n v_0 \begin{bmatrix} \overline{\overline{\alpha}}_{ee} & \overline{\overline{\alpha}}_{em} \\ \overline{\overline{\alpha}}_{me} & \overline{\overline{\alpha}}_{mm} \end{bmatrix} \cdot \left[v_0 \overline{\overline{I}} - \delta \overline{\overline{L}} \begin{bmatrix} \mu_0 & -\xi_0 \\ -\zeta_0 & \epsilon_0 \end{bmatrix} \right]^{-1} \begin{bmatrix} \overline{E}_L \\ \overline{H}_L \end{bmatrix} \\ &= f \begin{bmatrix} \overline{\overline{\gamma}}_{ee} & \overline{\overline{\gamma}}_{em} \\ \overline{\overline{\gamma}}_{me} & \overline{\overline{\gamma}}_{mm} \end{bmatrix}\end{aligned} \tag{47}$$

where $f = nv_0$ is the fractional volume of the bi-isotropic inclusion phase in the mixture. Substituting (47) into (44), we have

$$\overline{\overline{\epsilon}}_{\text{eff}} = \epsilon_0 \overline{\overline{I}} + f \overline{\overline{\gamma}}_{ee} \tag{48a}$$

$$\overline{\overline{\mu}}_{\text{eff}} = \mu_0 \overline{\overline{I}} + f \overline{\overline{\gamma}}_{mm} \tag{48b}$$

$$\overline{\overline{\xi}}_{\text{eff}} = \xi_0 \overline{\overline{I}} + f \overline{\overline{\gamma}}_{em} \tag{48c}$$

$$\overline{\overline{\zeta}}_{\text{eff}} = \zeta_0 \overline{\overline{I}} + f \overline{\overline{\gamma}}_{me} \tag{48d}$$

where

$$\overline{\overline{C}}_{\text{eff}} = \sum_{i=1}^{3} C_{\text{eff}}^i \widehat{x}_i \widehat{x}_i \qquad C = \epsilon, \mu, \xi, \zeta \tag{48e}$$

The equation (48) is the Maxwell-Garnett formula of the bi-isotropic mixture consisting of ordered layered ellipsoids.

In the absence of a strong external aligning field, the layered ellipsoids are randomly distributed. Then the mixture formula for this configuration are

$$C_{\text{eff}} = \frac{1}{3} \sum_{i=1}^{3} C_{\text{eff}}^i \qquad C = \epsilon, \mu, \xi, \zeta \tag{49}$$

Now we turn to the derivation of mixture formula using the effective medium approximation (EMA). Based on EMA (see Ref. [10] and references therein), the effective medium parameters are assumed to be $\epsilon_g, \mu_g, \xi_g, \zeta_g$ and the original mixture is divided into two mixtures.* One (called A) is the original layered ellipsiods with the fractional volume f located in the background medium $\epsilon_g, \mu_g, \xi_g, \zeta_g$. The other (called B) is the original background medium $(\epsilon_g, \mu_g, \xi_g, \zeta_g)$ with the fractional volume $(1 - f)$ and the same ellipsoid geometry as the original outmost ellipsoid located in the background medium $(\epsilon_g, \mu_g, \xi_g, \zeta_g)$. EMA formulates the problem by letting the additional polarizations in the effective medium $(\epsilon_g, \mu_g, \xi_g, \zeta_g)$ to be zero.

$$\begin{bmatrix} \langle \overline{P}_e^A \rangle \\ \langle \overline{P}_m^A \rangle \end{bmatrix} + \begin{bmatrix} \langle \overline{P}_e^B \rangle \\ \langle \overline{P}_m^A \rangle \end{bmatrix} = \begin{bmatrix} \overline{O} \\ \overline{O} \end{bmatrix} \tag{50}$$

This yields

$$f \begin{bmatrix} \alpha_{ee}^A & \alpha_{em}^A \\ \alpha_{me}^A & \alpha_{mm}^A \end{bmatrix} + (1-f) \begin{bmatrix} \alpha_{ee}^B & \alpha_{em}^B \\ \alpha_{me}^B & \alpha_{mm}^B \end{bmatrix} = \overline{\overline{O}} \tag{51}$$

i.e.,

$$f\alpha_{rs}^A + (1-f)\alpha_{rs}^B = 0. \qquad r, s = e \text{ or } m \tag{52}$$

where α_{rs}^A and α_{rs}^B have been thoroughly discussed in Section 2 of this paper. From the above four scalar equations we can solve the four unknowns ϵ_g, μ_g, ξ_g, and ζ_g, which have been appeared in α_{rs}^A and α_{rs}^B. Notice that $\alpha_{rs}^A, \alpha_{rs}^B$ can be replaced by α_{rs}^{xA} and α_{rs}^{xB} or α_{rs}^{yA} and α_{rs}^{yB} and α_{rs}^{zA} and α_{rs}^{zB}. So the above procedure is easily adapted to determine $(\overline{\overline{\epsilon}}_g, \overline{\overline{\mu}}_g, \overline{\overline{\xi}}_g, \overline{\overline{\zeta}}_g)$ in the ordered layered-ellipsoid case.

4. Conclusion

In conclusion, the low-frequency electromagnetic scattering of an electrically small layered bi-isotropic ellipsoid immersed in a host bi-isotropic medium was obtained. The polarization dyadic is computed by a recursive algorithm. The Maxwell-Garnett formula is derived for the layered-ellipsoid bi-isotropic mixture. And the effective medium approximation is also used to analyze this mixture.

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* After submitting this paper, a EMA treatment of the bi-isotropic mixtures published [11]. Numerical calculations of [11] show that the results of EMA are significantly different from those of Maxwell–Garnett formula.

Appendix: Elements of $\overline{\overline{B}}_{k,k+1}$ and $\overline{\overline{F}}_{k+1,k}$

The elements $b_{kij}(i,j=1,2,3,4)$ are as follows:

$$\begin{aligned} b_{k11} &= b_{k31}M_k + 1, & b_{k21} &= b_{k41}M_k + 1 \\ b_{k12} &= b_{k32}M_k, & b_{k22} &= b_{k42}M_k \\ b_{k13} &= b_{k33}M_k - M_k, & b_{k23} &= b_{k43}M_k - M_k \\ b_{k14} &= b_{k34}M_k, & b_{k24} &= b_{k44}M_k \end{aligned} \tag{A1}$$

where

$$b_{k3i} = A_{ki}\mu_{rk} - B_{ki}\xi_{rk} \qquad i = 1,2,3,4, \tag{A2}$$

$$b_{k4i} = B_{ki}\epsilon_{rk} - A_{ki}\zeta_{rk} \qquad i = 1,2,3,4, \tag{A3}$$

$$\begin{aligned} A_{k1} &= \epsilon_{r(k+1)} - \epsilon_{rk} & B_{k1} &= \zeta_{r(k+1)} - \zeta_{rk} \\ A_{k2} &= \xi_{r(k+1)} - \xi_{rk} & B_{k2} &= \mu_{r(k+1)} - \mu_{rk} \\ A_{k3} &= \epsilon_{r(k+1)}M_k^1 + \epsilon_{rk}M_k & B_{k3} &= \zeta_{r(k+1)}M_k^1 + \zeta_{rk}M_k \\ A_{k4} &= \xi_{r(k+1)}M_k^1 + \xi_{rk}M_k & B_{k4} &= \mu_{r(k+1)}M_k^1 + \mu_{rk}M_k \end{aligned} \tag{A4}$$

$$\Delta_k = (M_k + M_k^1)(\epsilon_{rk}\mu_{rk} - \xi_{rk}\zeta_{rk}) \tag{A5}$$

$$\Delta_k^1 = (M_k + M_k^1)(\epsilon_{rk}\mu_{rk} - \xi_{r(k+1)}\zeta_{r(k+1)}) \tag{A6}$$

The elements $f_{kij}(i,j=1,2,3,4)$ can be obtained by exchanging the $(\epsilon_{rk},\mu_{rk},\xi_{rk},\zeta_{rk})$ for $(\epsilon_{r(k+1)},\mu_{r(k+1)},\xi_{r(k+1)},\zeta_{r(k+1)})$ in (A1)–(A4) and replacing b_{kij} by f_{kij}.

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