Progress In Electromagnetics Research, *Volume 12, 107–132, 1996*

USE OF FRACTIONAL INTEGRATION TO PROPOSE SOME "FRACTIONAL" SOLUTIONS FOR THE SCALAR HELMHOLTZ EQUATION

N. Engheta

1. Introduction

The idea of generalization of differential and integral operators with integer orders to those with non-integer (e.g., fractional) orders is a subject in mathematics [1] which dates back to the late part of seventeenth century. For many years this area of mathematics, which is known as *fractional calculus* , has been the subject of study for many researchers in pure and applied mathematics [1–3]. This powerful mathematical tool has been applied to treatment of selected problems in various fields of science. The bibliography prepared by Ross and reprinted in the monograph by Oldham and Spanier [1, pp. 3–15] is an excellent list of historical survey in this area.

The scalar Helmholtz equation is one of the most basic and important equations encountered in mathematical treatment of various phenomena in many areas of physical sciences and engineering [4,5].

This partial differential equation (operator), along with its Green's function, has long been studied in great detail in mathematical physics, and its applications in treating various problems have been extensively addressed [see, e.g., 4–7]. For the one-, two-, and three-dimensional cases, the scalar and dyadic Green's functions of Helmholtz operator have been used to describe wave propagation and scattering in classical electrodynamics [see, e.g., 4]. In a conventional Cartesian coordinate system (x, y, z), it is required [4, Eq. 7.2.5] that the scalar Green's function of the Helmholtz equation satisfies the following equation

$$\nabla^2 G(\vec{r},\, \vec{r}_0;\, k) + k^2 G(\vec{r},\, \vec{r}_0;\, k) = -\delta(\vec{r} - \vec{r}_0) \tag{1.1}$$

where $\delta(\vec{r} - \vec{r}_0) \equiv \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)$ is the three-dimensional Dirac delta function, $\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z$ and $\vec{r}_0 = x_0\vec{a}_x + y_0\vec{a}_y + z_0\vec{a}_z$ are position vectors for the observation and source points, respectively, $\vec{a}_x$, $\vec{a}_y$, and $\vec{a}_z$ are the unit vectors in the coordinate system, ∇^2 is the Laplacian operator which in the three-dimensional Cartesian coordinate system is expressed as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, k is a scalar constant for a homogeneous isotropic space, and $G(\vec{r},\, \vec{r}_0;\, k)$ is the scalar Green's function satisfying the above equation subject to certain boundary conditions. For the unbounded space (i.e., with boundary condition of vanishing field at infinity), and for the scalar constant k independent of position, it can be shown that $G(\vec{r},\, \vec{r}_0;\, k) = G(|\vec{r} - \vec{r}_0|;\, k)$. In this case (and for the rest of this paper), without loss of generality, one can take $\vec{r}_0 = 0$, and thus for the canonical cases of one-, two-, and three-dimensional delta function sources located at $\vec{r}_0 = 0$ we write the following well-known scalar Green's functions in unbounded isotropic media with Sommerfeld's radiation condition [7, p. 188-190, Section 28, Eq. 2] satisfied. [1] . For the one-dimensional Dirac delta function $\delta_1(\vec{r}) \equiv \delta(x)$ (i.e., a uniform plate source located at the $y - z$ plane), [2] the scalar Green's function for Eq. (1.1) is known

[1] Here the outgoing wave is assumed to satisfy the Sommerfeld radiation condition . More explicitly, if Eq. (1.1) is considered as the time-harmonic wave equation for a phenomenon with exp($-i\omega$ t) time dependence, the outgoing wave will satisfy $\underset{|\vec{r}| \to \infty}{\text{limit}} \left(ikG - \frac{\partial G}{\partial |\vec{r}|} \right) \to 0$ [7].

[2] It must be noted that our embedding physical space is taken to be always a three-dimensional space. When we refer to one-, two- or three-dimensional delta functions, we mean the corresponding delta

[4, Eq. 7.2.19] to be

$$G_1(\vec{r}, \vec{r}_0; k) = G_1(|x|; k) = i\frac{e^{ik|x|}}{2k} \tag{1.2}$$

The subscript for δ and G denotes the "dimensionality" of the Dirac delta function source. In the two-dimensional case, where the two-dimensional Dirac delta function source $\delta_2(\vec{r}) \equiv \delta(x)\,\delta(y)$ (i.e., a uniform line source along the z-axis) is considered, one gets the following Green's function [4, Eq. 7.2.18] for Eq. (1.1)

$$G_2(\vec{r}, \vec{r}_0; k) = G_2(\sqrt{x^2+y^2}; k) = \frac{i}{4}H_0^{(1)}\,(k\sqrt{x^2+y^2}) \tag{1.3}$$

where $H_0^{(1)}(\cdot)$ is the zeroth-order Hankel function of the first kind [8, Ch. 9]. Finally for the three-dimensional delta function $\delta_3(\vec{r}) \equiv \delta(x)\,\delta(y)\,\delta(z)$ at the origin, the scalar Green's function [4, Eq. 7.2.17] for Eq. (1.1) is

$$G_3(\vec{r}, \vec{r}_0; k) = G_3(\sqrt{x^2+y^2+z^2}; k) = \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}} \tag{1.4}$$

In these well-known Green's functions, the boundary is assumed to be at infinity and the outgoing wave is shown. (The time harmonic excitation with $\exp(-i\omega\,\mathrm{t})$ is assumed here.) The Green's functions of the Helmholtz equation with higher-dimensional Dirac delta functions ($n \geq 4$ where n being an integer order dimension) have been analyzed and reported in the literature [5, pp. 772–775, pp. 803–804; 7, pp. 232–234].

In light of the above Green's functions for the integer-dimensional delta functions, one can raise the following interesting questions: Does there exist a source distribution which can be considered *effectively* as the "intermediate" step between integer-dimensional Dirac delta functions? If such a source is given as the source term in the inhomogeneous Helmholtz equation, what would the particular solution of this equation look like? Would that solution be regarded as the "intermediate" or "fractional" solution between the well-known Green's

---

function is a plate, a line, or a point source, respectively. In Section V, the embedding space is an n-dimensional space.

functions for Eq. (1.1) for integer-dimensional Dirac delta function sources? To clarify these questions and to bring some physical insights into the motivation behind these questions, let us consider the variation of the magnitude of above-noted conventional Green's functions (given in Eqs. (1.2) through (1.4)) in the far zone. In one-dimensional case, for the far-zone region where $k|x| >> 1$, it can be observed that $|G_1(|x|;k)|_{k|x|>>1} \rightarrow \left(\frac{1}{2k}\right)$ which will be independent of x. For the two-dimensional case, we get

$$\left|G_2(\sqrt{x^2+y^2};k)\right|_{k\sqrt{x^2+y^2}>>1} \rightarrow \frac{1}{4}\sqrt{\frac{2}{\pi k\sqrt{x^2+y^2}}} \propto (x^2+y^2)^{-1/4} \equiv \rho^{-1/2}$$

which shows that the magnitude of G_2 drops as $\rho^{-1/2}$ where $\rho \equiv \sqrt{x^2+y^2}$ denotes the distance of the observation point from the line source. And for the three-dimensional case, one gets

$$\left|G_3(\sqrt{x^2+y^2+z^2};k)\right|_{k\sqrt{x^2+y^2+z^2}>>1} \rightarrow \frac{1}{4\pi\sqrt{x^2+y^2+z^2}} \propto R^{-1}$$

where $R \equiv \sqrt{x^2+y^2+z^2}$ is the distance of the observation point to the point source at the origin. If an observer sits along the x axis in the far zone (i.e., $kx >> 1$) and measures the variation of the magnitude of the Green's function along the x axis, (s)he would notice that for one-dimensional delta function source, the magnitude of Green's function does not change with x, whereas for the two-dimensional source, this magnitude drops as $|x|^{-1/2}$. Now we can recast the above questions in a different way: Would it be possible to have a scalar solution for Eq. (1.1) (but with a different source term) whose magnitude, in some particular direction, say x, drops not as fast as $|x|^{-1/2}$, but also not independent of x? In other words, could it drop as $|x|^{(1-f)/2}$ where f is a real number between 1 and 2? [2] What kind of source distribution should one have in order to obtain such a solution for Eq. (1.1) with such a property? It must be noted that the idea of slowly varying localized wave carrying energy in space-time has been the subject of interest for many researchers and the solutions such as focus wave modes, localized wave transmission, electromagnetic missiles, and the likes have been suggested [9–12]. In this paper, however, we use a totally different approach based on fractional calculus, to find solutions of Helmholtz equations with the distributed source describable as fractional integral of higher-dimensional Dirac delta functions. Using the concept of fractional differintegration [3] we present a way to find such solutions for Eq. (1.1) with their corresponding "intermediate" source distribution. We call these solutions "fractional" solutions, since the behavior of their magnitudes in the far zone in a certain direction follow power-law dependence with a rate (power exponent) being a continuous fractional number (e.g., in the above example, power exponent between $-1/2$ and zero.) We present our analysis in detail for the case that is intermediate between one- and two-dimensional delta function source, and we will then address the generalization of these solutions for any fractional f where $n-1 < f < n$, i.e., solution of the scalar Helmholtz equation with a source term that is effectively an intermediate case between $(n-1)$ and n-dimensional Dirac delta function sources.

2. Fractional Differintegral Relation Between One- and Two-Dimensional Scalar Green's Functions of Helmholtz Operator

Let us consider the two-dimensional Green's function for Eq. (1.1) described in Eq. (1.3). It is well known [see, e.g., 15, Section 5–5] that this Green's function can be written for $x \geq 0$ as the following integral transform

$$G_2(\sqrt{x^2+y^2};\, k) = \frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{\exp(i\sqrt{k^2-\beta^2}x + i\beta y)}{\sqrt{k^2-\beta^2}} d\beta \tag{2.1}$$

where the contour of integration in the complex β-plane runs from $-\infty$ to $+\infty$ along the real β axis with infinitesimally small indentations below and above branch points $+k$ and $-k$, respectively. The branch cuts at $\beta = k$ and $\beta = -k$ are drawn in the upper half and the lower half of the complex β plane, respectively. The expression given in Eq. (2.1) was obtained through expansion of the

[3] The term "differintegral", which was coined by Oldham and Spanier in their book [1] to denote "derivatives or integrals to arbitrary order", is used in the present work.

two-dimensional Dirac delta function along the y axis as $\delta_2(\vec{r}) \equiv \delta(x)\,\delta(y) = \dfrac{\delta(x)}{2\pi}\displaystyle\int_{-\infty}^{+\infty} \exp(i\beta y)\,d\beta$, and for each term $\delta(x)\,\exp(i\beta y)$ as a source for Eq. (1.1), the solution for the region $x \geq 0$ is written as $\frac{i\exp(i\sqrt{k^2-\beta^2}x+i\beta y)}{2\sqrt{k^2-\beta^2}}$. For the region $x \leq 0$, $\frac{i\exp(-i\sqrt{k^2-\beta^2}x+i\beta y)}{2\sqrt{k^2-\beta^2}}$ should be used instead. For the rest of this paper, we assume $x \geq 0$.

One of the commonly used definitions for fractional integral is that known as the Riemann-Liouville integrals [1, p. 49, Eq. 3.2.3]. It is based on generalization of Cauchy's repeated integration formula which is for the n-fold integration of a given function $f(x)$

$$\begin{aligned} {}_aD_x^{-n} f(x) &\equiv \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \ldots \int_a^{x_1} f(x_0)dx_0 \\ &= \frac{1}{(n-1)!}\int_a^x (x-u)^{n-1} f(u)du \end{aligned} \tag{2.2}$$

where ${}_aD_x^{-n}$ denotes the nth-order integration (with respect to x) with the lower limit of the integrals being a [1, p. 38, Eq. 2.7.2]. Now if one replaces $-n$ with ν which is a non-integer negative number and bears in mind that $(n-1)! = \Gamma(n)$, one will obtain the Riemann-Liouville definition of fractional integrations. That is

$${}_aD_x^{\nu} f(x) \equiv \frac{1}{\Gamma(-\nu)}\int_a^x (x-u)^{-\nu-1} f(u)\,du \qquad \text{for } \nu < 0 \tag{2.3}$$

where $\Gamma(\cdot)$ is the Gamma function. This is one of the main (and commonly used) definitions of fractional integration of a function of one variable. For other definitions, the reader is referred to [1, Ch. 3].[4] When we compared Eq. (2.1) with the Riemann-Liouville definition of fractional integration (Eq. 2.3), we asked ourselves the following question: Would it be possible to write the expression given for two-dimensional Green's function in Eq. (2.1) evaluated at $y = 0$,[5] in terms of the fractional integration of some other function? To answer

---

[4] It is worth mentioning that in [1] symbols like $\frac{d^\nu f(x)}{d(x-a)^\nu}$ are used instead of ${}_aD_x^\nu f(x)$ which is used by H. T. Davis in [3]. In this paper, we use the latter.

[5] Needless to say, the problem is independent of z coordinate since the two-dimensional Green's function is being considered.

this question, we consider the observation points along the x axis, i.e., points with $y = 0$.[5] Then, instead of deforming the contour into a steepest descent contour as is usually done, we expand the integral of Eq. (2.1) in terms of four integrals along four sections of the real β axis.

$$\begin{aligned} G_2(x, y = 0; k) =& \frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{\exp(i\sqrt{k^2 - \beta^2}x)}{\sqrt{k^2 - \beta^2}} d\beta \\ =& \int_{-\infty}^{-k} + \int_{-k}^{0} + \int_{0}^{k} + \int_{k}^{\infty} \end{aligned} \tag{2.4}$$

The integration along the indentations at the branch points $\beta = \pm k$ can be shown to be nil when the radius of indentation approaches zero. Since the integrand in Eq. (2.4) is an even function of β, the four integrals can be written as the sum of two integrals over the two segments of the positive real β axis. That is

$$\begin{aligned} G_2(x, y = 0; k) =& \frac{i}{2\pi} \int_{0}^{k} \frac{\exp(i\sqrt{k^2 - \beta^2}x)}{\sqrt{k^2 - \beta^2}} d\beta \\ &+ \frac{1}{2\pi} \int_{k}^{\infty} \frac{\exp(-\sqrt{\beta^2 - k^2}x)}{\sqrt{\beta^2 - k^2}} d\beta \end{aligned} \tag{2.5}$$

Using change of variables of $u \equiv \sqrt{k^2 - \beta^2}$ for the first integral in Eq. (2.5) and $q \equiv \sqrt{\beta^2 - k^2}$ for the second integral of Eq. (2.5), we obtain

$$G_2(x,\, y = 0;\, k) = \frac{i}{2\pi} \int_{0}^{k} \frac{\exp(iux)}{\sqrt{k^2 - u^2}} du + \frac{1}{2\pi} \int_{0}^{\infty} \frac{\exp(-qx)}{\sqrt{q^2 + k^2}}\, dq \tag{2.6}$$

These two integrals can be evaluated exactly in the closed form [13, pages 427 and 322] as

$$\text{First Integral } I_1 = \left(\frac{1}{4}\right) [iJ_0(kx) - \mathbf{H}_0(kx)] \tag{2.7a}$$

$$\text{Second Integral } I_2 = \left(\frac{1}{4}\right) [\mathbf{H}_0(kx) - N_0(kx)] \tag{2.7b}$$

where $J_0(\cdot)$ and $N_0(\cdot)$ are the Bessel and Neumann functions of zeroth order, and $\mathbf{H}_0(\cdot)$ is the Struve function of zeroth order [8, p. 496] (We use the symbol $N_0(\cdot)$ instead of $Y_0(\cdot)$). The sum of

I_1 and I_2 becomes $(i/4)H_0^{(1)}(kx)$ which is the zeroth-order Hankel function of the first kind, and agrees with Eq. (1.3). For large argument $kx >> 1$, the leading asymptotic term of the Struve function $\mathbf{H}_0(kx)$ becomes $N_0(kx)$ [8, p. 497, Eq. 12.1.30], and thus the second integral I_2 is negligible. Therefore, the first integral I_1 becomes $(i/4)[J_0(kx) + iN_0(kx)] \equiv (i/4)H_0^{(1)}(kx)$. Therefore, in the far zone one can write

$$\underset{kx>>1}{G_2(x,\, y = 0;\, k)} \approx I_1 = \frac{i}{2\pi}\int_0^k \frac{\exp(iux)}{\sqrt{k^2 - u^2}}\, du \tag{2.8}$$

With one more change of variable $\mathrm{w} \equiv u^2$ in the above equation, we obtain

$$\underset{kx>>1}{G_2(x,\, y = 0;\, k)} \approx \frac{1}{2\pi}\int_0^{k^2} \frac{1}{\sqrt{k^2 - \mathrm{w}}}\frac{i\exp(i\sqrt{\mathrm{w}}x)}{2\sqrt{\mathrm{w}}}\, d\mathrm{w} \tag{2.9}$$

Comparing Eq. (2.9) with the Riemann-Liouville fractional integral shown in Eq. (2.3), and using $h \equiv k^2$, one can conclude that

$$\begin{aligned}\underset{kx>>1}{G_2(x,\, y = 0;\, k)} &\approx \frac{1}{2\sqrt{\pi}}\, {}_0D_h^{-1/2}\left[\frac{i\exp(i\sqrt{h}x)}{2\sqrt{h}}\right] \\ &\equiv \frac{1}{2\sqrt{\pi}}\frac{d^{-1/2}}{d(k^2)^{-1/2}}\left[\frac{i\exp(ikx)}{2k}\right]\end{aligned} \tag{2.10}$$

where h, which is a dummy variable, has been substituted by k^2. [6] In other words, for $kx >> 1$ along the x-axis, the two-dimensional Green's function can be expressed in terms of the 0.5th-order integral or semi-integral (with respect to $h \equiv k^2$ as the variable, and the lower limit for the fractional integration definition as $a = 0$) of the one-dimensional Green's function. Therefore, fractional integration can be used to link the Green's functions (in the far zone along the axis of symmetry) of the Helmholtz operator for the one- and two-dimensional delta function sources.

[6] What this means is that Eq. (2.9) can be explicitly written as $\underset{kx>>1}{G_2(x,y=0;k)} \approx \frac{1}{2\pi}\int_0^{k^2} \frac{1}{\sqrt{k^2-k'^2}}\frac{i\exp(ik'x)}{2k'} d(k'^2)$ which leads to Eq. (2.10).

3. How About νth-Order Fractional Integral of One-Dimensional Green's Function?

In the previous section, we showed that the two-dimensional scalar Green's function for the Helmholtz operator for far points along the x axis can be expressed in terms of certain semi-integral of the one-dimensional Green's function (Eq. (2.10)). More explicitly, if the order of integration on the one-dimensional Green's function is $1/2$, we get some features of two-dimensional Green's function. However, if the order of integration is zero, i.e., no operation is done, we are obviously left with one-dimensional Green's function. What would we get if we extend the fractional integration of one-dimensional Green's function to the fractional order ν between zero and $-1/2$? Would that provide us with some aspects of a solution of Helmholtz equation with a source term which is *effectively* an "intermediate" source between one- and two-dimensional delta functions? In order to answer these equations, let us define ν (fractional order of integration) as a number between 0 and $-1/2$ and f as a number between one and two. That provides $\nu \equiv (1-f)/2$. In this case f is a descriptor of intermediary of the source which is regarded as an intermediate source between one- and two-dimensional Dirac delta functions. Therefore let us write the following expression for $\Psi_f(x, y=0; k)$ shown below when $kx >> 1$ where subscript "f" denotes "fractional" aspect of this solution.[7]

---

[7] Here and for the rest of this paper, we assume that quantities such as x, y, u, q, β, etc. are physically dimensionless. Therefore, applying fractional integration with respect to β on terms such as delta-function sources does not introduce an inconsistency with the physical dimensions. If, however, we had assigned physical dimensions to these quantities, we would have had to compensate the effect of fractional integration by multiplying the results by a constant with an appropriate physical dimension. For example, if the two-dimensional source had been a physical current source denoted as $I_0\delta(x)\delta(y)$ with unit of Ampre/m^2, when we had to apply fractional-order integration (of order α) with respect to physical length y on this source, the resulting source should be multiplied by a constant term of physical dimension $l^{-\alpha}$ (where l has the dimension of length) to ensure that the resulting current source has still the dimension of Ampre/m^2.

$$\Psi_f(x, y=0;\, k) \underset{kx>>1}{} \propto \frac{d^{(1-f)/2}}{d(k^2)^{(1-f)/2}} \left[\frac{i\exp(ikx)}{2k}\right]$$

$$\propto \int_0^k \frac{\exp(iux)}{(k^2-u^2)^{(3-f)/2}} du$$

$$\text{for } 1 < f < 2 \tag{3.1}$$

Working backward to find corresponding first and second integrals analogous to those given in Eqs. (2.6) and (2.7), we can write

$$\text{First Integral for fractional case f, } I_1 \propto \int_0^k \frac{\exp(iux)}{(k^2-u^2)^{(3-f)/2}} du$$

$$\text{for } 1 < f < 2 \tag{3.2a}$$

$$\text{Second Integral for fractional case f, } I_2 \propto \int_0^\infty \frac{\exp(-qx)}{(k^2+q^2)^{(3-f)/2}} dq$$

$$\text{for } 1 < f < 2 \tag{3.2b}$$

From the above expressions for I_1 and I_2, finally we propose the following expression for the "fractional" solution of Helmholtz equation for all points $(x,\, y)$ in the two-dimensional Cartesian space,

$$\Psi_f(x,\, y;\, k) \equiv \frac{i}{4\pi} \int_{-\infty}^{+\infty} \frac{\exp(i\sqrt{k^2-\beta^2}x + i\beta y)}{\sqrt{k^2-\beta^2}(i\beta)^{2-f}} d\beta \quad \text{for } 1 < f < 2 \tag{3.3}$$

where the integration is over the contour defined for integral in Eq. (2.1) with additional indentation around the branch point $\beta = 0$.[8]

---

[8] It must be mentioned that for two limits of f, i.e., $f = 1$ and $f = 2$ care must be taken in proper evaluation of Eqs. (3.2) and (3.3). For $f = 2$, one gets the previous expressions for G_2 and its I_1 and I_2. However for $f = 1$, the point $\beta = 0$ in Eq. (3.3) is no longer a branch point but instead a simple pole. As such, the contribution of contour indentations around this pole would not be nil and should be taken into account.

The contour is sketched in Fig. 1A. This indentation is taken to be from below the branch point $\beta = 0$. In order for $\Psi_f(x, y; k)$ to satisfy the Helmholtz equation, the source term in the right-hand side of Eq. (1.1) is no longer a Dirac delta function, and must now be modified. It can be shown that the appropriate source for such a case can be written as

$$S_f(x,\, y) = \frac{\delta(x)}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(i\beta y)}{(i\beta)^{2-f}} d\beta \qquad \text{for } 1 < f < 2 \tag{3.4}$$

where the integration is carried out over a contour along the real β axis (in the complex β plane) with the indentation around $\beta = 0$ at described above. The integral in Eq. (3.4) can be evaluated [see, e.g., 14, p. 33, Ch. 3, Eq. 14], and thus this source term $S_f(x, y)$ can be explicitly given as

$$S_f(x,\, y) = \begin{cases} 0 & \text{for } y < 0 \\ \dfrac{\delta(x) y^{1-f}}{\Gamma(2-f)} & \text{for } y > 0 \end{cases} \qquad \text{for } 1 < f < 2 \tag{3.5}$$

However, it is interesting to note that the source distribution derived in Eq. (3.5) can be expressed in terms of fractional integral of two-dimensional Dirac delta function $\delta(x)\,\delta(y)$. From the Riemann-Liouville fractional integral shown in Eq. (2.3), it can be seen that

$$S_f(x,\, y) = {}_{-\infty}D_y^{f-2}[\delta(x)\,\delta(y)] = \begin{cases} 0 & \text{for } y < 0 \\ \dfrac{\delta(x) y^{1-f}}{\Gamma(2-f)} & \text{for } y > 0 \end{cases} \qquad \text{for } 1 < f < 2 \tag{3.6}$$

(Here the lower limit for the Riemann-Liouville fractional integration is taken to be $a = -\infty$. However for the fractional integration of Dirac delta function at the origin, any lower limit $a < 0$ would provide the same results.) [9] It can be easily shown that for the limits of $f = 1$

[9] Needless to say, the expression (3.3), which is the solution of Helmholtz equation with the source term given in Eq. (3.5), can also be obtained as the convolution of the two-dimensional Green's function Eq. (1.3) with $S_f(x, y)$ in (3.5). Here, however, we have followed the approach presented here to highlight the important features of fractional integration.

and $f = 2$, one gets the expected source distributions, i.e., $\delta(x)\,\delta(y)$ for $f = 2$ (two-dimensional delta function that is a line source), and $\delta(x)\,U(y)$ with $U(y)$ as the unit step function in y for $f = 1$. We now go back to the questions we raised in the Introduction about possibility of a solution whose magnitude could drop as $|x|^{(1-f)/2}$ in the far zone along the x direction, and we examine the behavior of $\Psi_f(x, y;\, k)$ along the x axis for far-zone points, i.e., $kx >> 1$ and $y = 0$. For $y = 0$, we can evaluate $\Psi_f(x, y;\, k)$ in Eq. (3.3) in closed form. Using the table of integrals [13, pages 322, 427], after some mathematical steps the expression is thus given as

$$\Psi_f(x, y = 0; k) = \frac{-i\Gamma((f-1)/2)\cos(f\pi/2)}{4\sqrt{\pi}} \left(\frac{x}{2k}\right)^{(2-f)/2} H^{(1)}_{(f-2)/2}(kx)$$

$$\text{for } x > 0 \quad \text{and} \quad \text{for } 1 < f < 2 \tag{3.7}$$

where $H^{(1)}_{(f-2)/2}$ is the Hankel function of order $(f-2)/2$ and of first kind [8, p. 358]. For $kx >> 1$, the magnitude of Eq. (3.7) can be written as

$$\left|\Psi_f(x,\, y = 0;\, k)\right|_{kx>>1} = \frac{\Gamma((f-1)/2)|\cos(f\pi/2)|}{4\pi 2^{(1-f)/2} k^{(3-f)/2}} |x|^{(1-f)/2}$$

$$\propto |x|^{(1-f)/2} \qquad \text{for } 1 < f < 2 \tag{3.8}$$

The limiting cases of $f = 1$ and $f = 2$ for Eqs. (3.7) and (3.8) can be shown to provide the expected results for the one- and two-dimensional Green's function for far-zone points along the x axis. For $f = 2$, Eq. (3.7) will give the value $\frac{i}{4}H_0^{(1)}(kx)$ which is $G_2(x,\, y = 0;\, k)$ along the x axis, and in the far zone along this axis Eq. (3.8) will provide $|x|^{-1/2}$ as the dependence on x. For $f = 1$, using the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ for $0 < \mathrm{Re}\, z < 1$ [8, p. 256, Eq. 6.1. 17], it can be shown that Eq. (3.7) will become $\frac{i}{4k}\exp(ikx)$ which is half of the value of $G_1(x;\, k)$ shown in Eq. (1.2). The factor $1/2$ is resulted from the fact that in the limit of $f = 1$ the source term $S_f(x, y)$ in Eq. (3.6) represents a unit step function in y, and due to such source distribution in the $y - z$ plane, the value of the solutions of the Helmholtz equation along the x axis is the result of the "symmetric" or "even" part of this source distribution, namely, the "even" part of the unit step function along the y axis. This even part of the source is a uniform source along the y axis (in the $y - z$ plane) with a magnitude

$1/2$, and hence the factor $1/2$ for the limit of $G_f(x, y = 0; k)$ when $f \to 1$. The remaining part of the unit step function is a function with the value of $-1/2$ for $y < 0$, and $+1/2$ for $y > 0$. Due to the asymmetric or "odd" nature of this remaining part of source distribution, there is no contribution from it to the observation points along the x axis. To remove the problem of nonsymmetric distribution of the source obtained in Eq. (3.6), let us consider only the even portion of this source (even with respect to the y variable), namely

$$\begin{aligned} {}_eS_f(x, y) &= \frac{[S_f(x, y) + S_f(x, -y)]}{2} \\ &= (1/2)[{}_{-\infty}D_y^{f-2}\delta(x)\,\delta(y) + {}_{-\infty}D_{-y}^{f-2}\delta(x)\,\delta(y)] \\ &= \frac{\delta(x)|y|^{1-f}}{2\Gamma(2-f)} \qquad \text{for } 1 < f < 2 \end{aligned} \tag{3.9}$$

For this symmetric source distribution, $\Psi_f(x\,, y\; =\; 0; k)$ given in Eq. (3.7) remains unchanged, since the x axis is the axis of symmetry for ${}_eS_f(x, y)$.

From Eqs. (3.7) and (3.8), it can be seen that it is possible to introduce a scalar solution for the Helmholtz equation whose magnitude drops as $|x|^{(1-f)/2}$ along the x axis in the far zone, and the corresponding source should be of the form given in Eq. (3.6). For the symmetric distributed source given in Eq. (3.9), the solution for the Helmholtz equation can be written as

$$\begin{aligned} {}_e\Psi_f(x, y; k) \equiv \frac{i}{8\pi}\int_{-\infty}^{+\infty} \frac{\exp(i\sqrt{k^2-\beta^2}x + i\beta y)}{\sqrt{k^2-\beta^2}(i\beta)^{2-f}}d\beta \\ + \frac{i}{8\pi}\int_{-\infty}^{\infty} \frac{\exp(i\sqrt{k^2-\beta^2}x - i\beta y)}{\sqrt{k^2-\beta^2}(i\beta)^{2-f}}d\beta \\ \text{for } x > 0, \quad \text{and} \quad 1 < f < 2 \end{aligned} \tag{3.10}$$

where the pre-subscript "e" in ${}_e\Psi_f(x, y; k)$ indicates the fact that this solution is even symmetric with respect to the $x-z$ plane. It must be noted that in both integrals in Eq. (3.10) the original contours go along the real β axis from $-\infty$ to $+\infty$ with indentation at $\beta = +k$ and $\beta = -k$ as mentioned for Eq. (3.3). However the indentation at $\beta = 0$ for two integrals are different. For the first integral in the right-hand side of Eq. (3.10), the indentation is similar to that of Eq. (3.3) (i.e., the indentation is from below the branch point $\beta = 0$), whereas for the second integral the indentation at $\beta = 0$ is from above the point $\beta = 0$.

When the far-zone observation point is not on the symmetry plane of $x - z$, certain asymptotic evaluation of the integral should be used in order to find the asymptotic expression for ${}_e\Psi_f(x, y; k)$. Let us consider the first integral in the right-hand side of Eq. (3.10). Aside from a factor of $1/2$, this integral is the same as that of Eq. (3.3). When the asymptotic form of this integral is evaluated, the corresponding expression for the second integral can be obtained using a symmetry argument. The contour of integration for this integral is sketched in Fig. 1A.

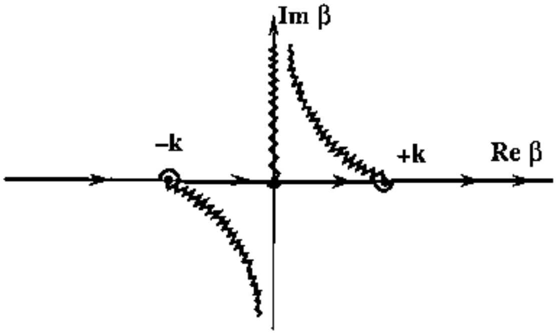

Figure 1A. The original contour of integration for Eq. (3.3), Eq. (2.1), and for the first integral of Eq. (3.10).

The far-zone observation point is at (x, y) with $x > 0$ and $y \neq 0$. As we said earlier, here we take $x > 0$. However, once the results for $x > 0$ are obtained, the corresponding results for $x < 0$ can be easily written using the symmetry argument. In a cylindrical coordinate system attached to our Cartesian coordinate system, x and y can be written as $x = \rho \cos\theta$ and $y = \rho \sin\theta$. Since $x > 0$, the azimuthal angle θ varies between $-\pi/2$ and $+\pi/2$. Using a change of variable $\beta = k \sin\alpha$, the first integral of Eq. (3.10) can be rewritten as

$$\text{Integral}\,1 = \frac{i}{8\pi} \int_C \frac{\exp[ik\rho\cos(\alpha - \theta)]}{(ik\sin\alpha)^{2-f}} d\alpha \tag{3.11}$$

where contour C in the complex α-plane is shown in Fig. 1B.

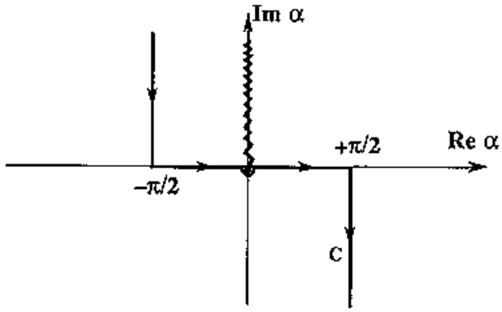

Figure 1B. Contour of integration C for Eq. (3.11).

We also note that the branch point is now at $\alpha_b = 0$ and the corresponding branch cut is along the positive imaginary α axis. For $k\rho >> 1$, this integral can be evaluated using the standard saddle-point technique (see, e.g., 15, pp. 594–599). However, care must be taken in dealing with the branch cut along the imaging α axis. The saddle point is obtained from

$$\frac{d\cos(\alpha-\theta)}{d\alpha} = -\sin(\alpha-\theta) = 0 \qquad \rightarrow \qquad \alpha_s = \theta \tag{3.12}$$

The original contour C should now be modified into the steepest descent path passing through the saddle point at $\alpha_s = \theta$, and should be oriented at $-\pi/4$ and $3\pi/4$ with respect to the real α axis. If the observation point has $y < 0$, then $-\pi/2 < \theta < 0$. Therefore, the steepest descent path does not cross the branch cut along the positive imaginary α axis. The deformed path of integration is shown in Fig. 2A.

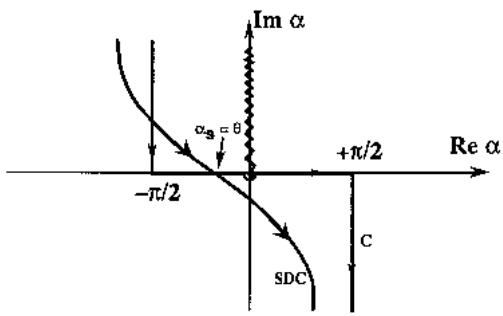

Figure 2A. Contour of integration C and the steepest descent contour (SDC) for Eq. (3.11) when $-\pi/2 < \theta < 0$.

In this case, if θ is not too close to branch point $\alpha_b = 0$, the branch point has essentially no effect, and the approximate value of Integral 1 is obtained from the saddle-point contribution. That is

$$\underset{k\rho>>1}{\text{Integral 1}} \cong \frac{i}{8\pi}(ik\sin\theta)^{f-2}\sqrt{\frac{2\pi}{k\rho}}\,e^{ik\rho-i\pi/4}$$
$$\text{for } -\pi/2 < \theta < 0 \quad \text{and} \quad 1 < f < 2 \tag{3.13}$$

If the observation point has $y > 0$, which requires $0 < \theta < \pi/2$, then the saddle point $\alpha_s = \theta$ is on the right side of branch point $\alpha_b = 0$. In this case, the steepest descent contour should be deformed to go around the branch cut along the positive imaginary α axis. This is shown in Fig. 2B.

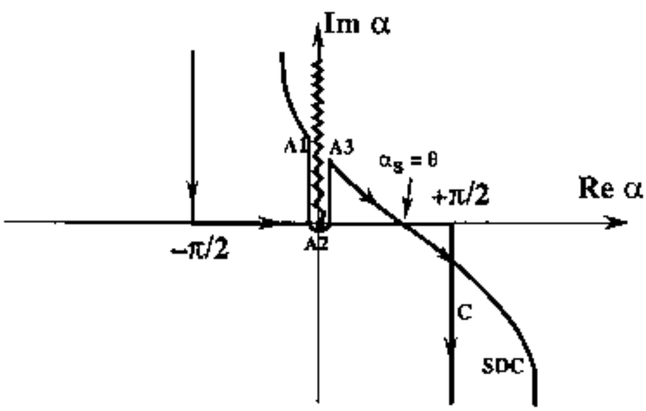

Figure 2B. Contour of integration C and the steepest descent contour (SDC) for Eq. (3.11) when $0 < \theta < \pi/2$.

In this case, two parts contribute to the value of Integral 1: the saddle-point contribution and the branch-cut integration. The contribution from the integration near the saddle point on the steepest descent path has the form similar to that shown in Eq. (3.13). The branch-cut integration, however, has to be evaluated. To do so, we follow the well-known mathematical technique used to evaluate the branch cut integration such as those involved in the problem of radiation of an antenna near the interface of two media [see, e.g., 15, pp. 464–472, Section 15–8]. This branch-cut integration can be rewritten as

$$\begin{aligned}
&\frac{i}{8\pi}\int_{Branch\ Cut}\frac{\exp[ik\rho\cos(\alpha-\theta)]}{(ik\sin\alpha)^{2-f}}d\alpha \\
&\quad=\frac{i}{8\pi}\int_{A1-A2-A3}\frac{\exp[ik\rho\cos(\alpha-\theta)]}{(ik\sin\alpha)^{2-f}}d\alpha \\
&\quad=\frac{i}{8\pi}\int_{A1-A2}\frac{\exp[ik\rho\cos(\alpha-\theta)]}{(ik\sin\alpha)^{2-f}}d\alpha \\
&\qquad-\frac{i}{8\pi}\int_{A3-A2}\frac{\exp[ik\rho\cos(\alpha-\theta)]}{(ik\sin\alpha)^{2-f}}d\alpha
\end{aligned}\tag{3.14}$$

In each of the integrals on the right-hand side of Eq. (3.14), the integration is along the imaginary α axis. However, for the A1–A2 path, the variable $\alpha = |\alpha|\exp(-i3\pi/2)$ and for the A3–A2 path $\alpha = |\alpha|\exp(i\pi/2)$. We now expand the functions $\sin(\alpha)$ and $\cos(\alpha-\theta)$ about $\alpha_b = 0$. We then have

$$\begin{aligned}
&\sin(\alpha)\cong\alpha-\frac{1}{3!}\alpha^3+\ldots \\
&\cos(\alpha-\theta)\cong\cos(\theta)+\alpha\sin(\theta)-\ldots
\end{aligned}\tag{3.15}$$

Substituting these expansions into the two integrals in the right-hand side of Eq. (3.14), we get

$$\begin{aligned}
&\frac{i}{8\pi}\int_{A1-A2}\frac{\exp[ik\rho\cos(\alpha-\theta)]}{(ik\sin\alpha)^{2-f}}d\alpha-\frac{i}{8\pi}\int_{A3-A2}\frac{\exp[ik\rho\cos(\alpha-\theta)]}{(ik\sin\alpha)^{2-f}}d\alpha \\
&\quad\cong\frac{i}{8\pi}\int_{A1-A2}\frac{\exp\ \{ik\rho[\cos(\theta)+\alpha\sin(\theta)]\}}{\left[ik\left(\alpha-\frac{1}{3!}\alpha^3\right)\right]^{2-f}}d\alpha-\frac{i}{8\pi}\int_{A3-A2}\ldots d\alpha
\end{aligned}\tag{3.16}$$

Since the contribution to these integrals comes primarily from the vicinity of $\alpha_b = 0$, we only keep two terms of expansion for $\cos(\alpha-\theta)$ and the first term of expansion for $\sin(\alpha)$, and we let the path of integration from A1-A2 extend to the entire positive imaginary α axis, i.e., $i\infty$ to 0. Thus, we get

$$\begin{aligned}
&\frac{i}{8\pi}\int_{Branch\,Cut}\frac{\exp[ik\rho\cos(\alpha-\theta)]}{(ik\sin\alpha)^{2-f}}d\alpha \\
&\qquad\cong\frac{i\exp ik\rho\cos\theta}{8\pi}[1-e^{i2f\pi}]\int_{i\infty}^{0}\frac{\exp ik\rho\alpha\sin(\theta)}{(ik\alpha)^{2-f}}d\alpha
\end{aligned}\tag{3.17}$$

With the change of variable $i\alpha = -s$, we can rewrite the last integral as

$$\int_{i\infty}^{0} \frac{e^{ik\rho\alpha\sin(\theta)}}{(ik\alpha)^{2-f}} d\alpha = -\frac{i^{2f-3}}{k^{2-f}} \int_{0}^{\infty} s^{f-2} e^{-k\rho s \sin\theta} ds$$
$$= -\frac{ie^{-i\pi f}}{k^{2-f}} \frac{\Gamma(f-1)}{(k\rho\sin\theta)^{f-1}} \tag{3.18}$$

Substituting Eq. (3.18) into Eq. (3.17), after some mathematical steps we obtain

$$\frac{i}{8\pi} \int_{Branch\,Cut} \frac{\exp[ik\rho\cos(\alpha-\theta)]}{(ik\sin\alpha)^{2-f}} d\alpha$$
$$\cong \frac{i}{4k^{2-f}\Gamma(2-f)} \frac{\exp ik\rho\cos\theta}{(k\rho\sin\theta)^{f-1}} \tag{3.19}$$

Therefore, depending on the location of the observation point, the value of Integral 1 in Eq. (3.10) is given for $1 < f < 2$ as

$$\underset{k\rho>>1}{\text{Integral}\,1} \cong \begin{cases} \frac{i}{8\pi}(ik\sin\theta)^{f-2}\sqrt{\frac{2\pi}{k\rho}}e^{ik\rho-i\pi/4}\,, & \text{for } -\frac{\pi}{2} < \theta < 0 \\ \frac{i}{8\pi}(ik\sin\theta)^{f-2}\sqrt{\frac{2\pi}{k\rho}}e^{ik\rho-i\pi/4} & \\ +\frac{i}{4k^{2-f}\Gamma(2-f)} \frac{e^{ikx}}{(ky)^{f-1}}\,, & \text{for } 0 < \theta < \frac{\pi}{2} \end{cases} \tag{3.20}$$

provided that θ is not too small. When $\theta = 0$, we are on the x-z plane, and we have the exact solution of the integral given in Eq. (3.7).[10] In fact, the expression given in Eq. (3.20) is half of the far-zone value of the "fractional" solution of Helmholtz equation shown in Eq. (3.3) for the source given in Eq. (3.5). (The reason for the factor of $1/2$ is that the portion of the source given in Eq. (3.9) for $y > 0$ is half of the source given in Eq. (3.5)). Now the value of Integral 2 of Eq. (3.10) for the far-zone observation points can be obtained similarly

[10] If θ is very close to $\alpha_b = 0$, the result of the saddle-point integration given in Eq. (3.13) and branch-cut integration shown in Eq. (3.19) will no longer be valid. Instead, a uniform asymptotic evaluation of Integral 1 is needed.

using symmetry consideration by replacing θ by $-\theta$, and y by $-y$. Combining the values for Integral 1 and 2 of the right-hand side of Eq. (3.10), we have the following expression for the value of ${}_e\Psi_f(x,\, y;\, k)$ for the far-zone observers when θ is not too small:

$$\begin{aligned} {}_e\Psi_f(x,\, y;\, k) \underset{k\rho>>1}{\cong} & \frac{-i}{4\pi} \cos\left(\frac{f\pi}{2}\right) (k \sin|\theta|)^{f-2} \sqrt{\frac{2\pi}{k\rho}}\, e^{ik\rho - i\pi/4} \\ & + \frac{i}{4k^{2-f}\Gamma(2-f)} \frac{e^{ik|x|}}{(k|y|)^{f-1}} \\ & \qquad \text{for } 1 < f < 2 \end{aligned} \tag{3.21}$$

It is worth noting to see that the above far-zone solution to the Helmholtz equation with the symmetric source given in Eq. (3.9) has two parts: a cylindrical wave which drops as $\rho^{-1/2}$ in the far zone, and a nonuniform plane wave which propagates in the x direction but its amplitude drops with y as $|y|^{1-f}$ for $1 < f < 2$. We can show that when f approaches one of the limits of 1 or 2, the above solution becomes the asymptotic form for the one and two-dimensional cases. More specifically, when $f = 2$, the symmetric source given in Eq. (3.9) represents a two-dimensional Dirac delta function at $x = y = 0$. In this case, the second term in Eq. (3.21) representing a "plane wave" portion of the solution disappears due to the $\Gamma(2-f)$ in its denominator. The first term becomes ${}_e\Psi_{f=2}(x,\, y;\, k) \underset{k\rho>>1}{\cong} \frac{i}{4}\sqrt{\frac{2}{\pi k\rho}} e^{ik\rho - i\pi/4}$ which is the asymptotic form for $\frac{i}{4}H_0^{(1)}(kx)$, as expected. For $f = 1$, the symmetric source in Eq. (3.9) get the form $(1/2)\delta(x)$, which is one-dimensional source, e.g., a source plate at the $y - z$ plane. In this case, the cylindrical term is nil, and the plane-wave part of Eq. (3.21) becomes $(i/4k)\exp(ik|x|)$ which is half of the value of $G_1(x;\, k)$ when the source is $\delta(x)$.

4. Physical Remarks

On a more intuitive basis, we can describe the above analysis as follows: we first consider a two-dimensional delta function (line) source. For this source, the two-dimensional Green's function is a propagating (wave) solution with azimuthal symmetry, and drops as $\rho^{-1/2}$ in the far zone. Now we apply fractional integration (on dimensionless variable y and $-y$, i.e., on both sides with lower limits needed in fractional integral definition as $-\infty$ and $+\infty$, respectively) on the two-dimensional Dirac delta function. The order of such integration is $f-2$ (where f varies between 1 and 2), and this f *effectively* indicates the level of intermediary between one and two dimensions. The result is a symmetric source in the $y-z$ plane which is given in Eq. (3.9). It seems as though fractional integral operation has "smeared" the original Dirac delta function in the $y-z$ plane. Depending on the value of f, the "smearing" can be extended. For this source the solution of Helmholtz equation is given in Eq. (3.10). On the symmetric plane $x-z$, this solution looks like expression given in Eq. (3.7). The magnitude of this solution drops as $|x|^{(1-f)/2}$ in the far zone along the symmetric (i.e., $x-z$) plane. The rate of drop of this magnitude along the x axis depends on the fractional order of integration, or "smearing" of the original source. When f approaches unity, the source becomes $\delta(x)/2$ which is a uniform one-dimensional source located in the $y-z$ plane. For this source, the one-dimensional Green's function is obtained with a constant magnitude in the far zone. When $f=2$, the source becomes a two-dimensional Dirac delta function,[11] and the magnitude of the function decays as $|x|^{-1/2}$ in the far-zone along the x axis. So *effectively*, the two- and one-dimensional Green's functions of Eq. (1.1) have been smoothly "connected" by varying the order of fractional integration of two-dimensional delta functions. Since ${}_e\Psi_f(x, y; k)$ provides a solution whose magnitude along the x axis in the far zone drops as $|x|^{(1-f)/2}$ and since this solution is the "intermediate" solution between the Green's functions of the Helmholtz equations for the one- and two-dimensional Dirac delta function sources, we name it "fractional" solution of Helmholtz equation. For observation points outside the $x-z$ plane (and not too close to this plane), we have also shown that the solution has both cylindrical- and plane-wave portions, and in the limit of $f=1$ and $f=2$, one of these parts disappears and the other represents the far-zone solution.

It is worth mentioning that with appropriate mathematical care one can extend the conditions imposed on f to larger values of f, e.g., $2<f<3$. In that case, what does it mean if we choose f greater than

[11] Care must be taken in determining the value of the source given in Eq. (3.9) when $f=2$. For this value of f, no fractional integration is done on $\delta(x)\delta(y)$. In other words, Eq. (3.9) approaches $\delta(x)\delta(y)$ in the limit when $f\to 2$.

2 ? As an example, let us consider the case where $f = 3$. In this case, the source $S_f(x, y)$ in Eq. (3.5) becomes $\delta(x)\delta'(y)$ where $\delta'(y)$, the first derivative of $\delta(y)$ [4, p. 837], is a doublet function. This source can be thought of as two line sources parallel with the z axis with opposite polarities, located in the $y - z$ plane, and separated along the y axis. The separation and the magnitude of these two line sources are related such that $\delta(x)\delta'(y) \equiv \delta(x) \underset{\Delta \to 0}{\text{limit}} \left\{ \frac{I_0}{\Delta} \left[\delta\left(y + \frac{\Delta}{2}\right) - \delta\left(y - \frac{\Delta}{2}\right) \right] \right\}$. Due to the asymmetry of this source, the solution of Helmholtz equation, in Eq. (3.3), with such a source distribution will be null along the $x - z$ plane, and this is confirmed from Eq. (3.7) for $f = 3$. Although for the case of $f = 3$ the coefficient $\cos(f\pi/2)$ in the solution given in Eq. (3.7) is zero and thus $\Psi_f(x, y = 0; k)$ is zero, it must be noted that its x -dependence in the far zone along the x axis has the form $|x|^{-1}$ (albeit with multiplicative coefficient of zero). Therefore, the values of f in general relates to the rate of drop of this function along the x axis. For observation points outside the $x - z$ plane, one should consider the asymptotic value of Eq. (3.3) which is proportional to the expression given in Eq. (3.20) for Integral 1 . In this case, the plane-wave portion is zero, and the cylindrical part has the form $\sim \sin(\theta)\rho^{-1/2}\exp(ik\rho)$ which is proportional to the asymptotic form of $\sin(\theta)H_1^{(1)}(k\rho)$, the radiation from a doublet source. One of the interesting points about the solution Eq. (3.7) (or Eq. (3.3)) and its corresponding source distribution $S_f(x, y)$ in Eq. (3.5) is that it provides description for all values of f , including non-integer values. Therefore, one is not restricted to using the standard canonical source distributions, namely, uniform (or step function), delta function, doublet functions, etc. Instead, one may also consider, using fractional integration, source distributions such as that given in Eq. (3.6), which are "intermediate" distributions between any one of these canonical ones, and obtain appropriate solutions for Helmholtz equations as given in Eq. (3.3).

5. Generalization to Higher Dimensions

So far we have discussed scalar solutions of Helmholtz equations for sources which lie between the cases of one and two-dimensional Dirac delta functions, i.e. sources given in Eq. (3.6). The above analysis can be straightforwardly generalized to the sources that are intermediate cases between n- and $(n - 1)$-dimensional Dirac delta functions. Without presenting the detailed mathematical steps, in this section we provide mathematical generalization for such intermediate source distribution and its solution to the Helmholtz equation for $n-1 < f < n$ (where n is a positive integer).

In an n-dimensional space with Cartesian coordinates $(x_1, x_2, ..., x_{n-1}, x_n)$, the well-known n-dimensional Dirac delta function [5, pp. 772-775] can be written as

$$S_n(x_1, x_2, \ldots x_{n-1}, x_n) = \delta(x_1)\delta(x_2)\ldots\delta(x_{n-1})\delta(x_n) \tag{5.1}$$

The $(n-1)$-dimensional delta function may be written as

$$S_{n-1}(x_1, x_2, \ldots x_{n-1}) = \delta(x_1)\delta(x_2)\ldots\delta(x_{n-1}) \tag{5.2}$$

Generalizing Eq. (3.6) to the higher dimensional cases where $n-1 < f < n$, we get the following expression for $S_f(x_1, x_2, \ldots, x_{n-1}, x_n)$ when $n-1 < f < n$

$$\begin{aligned} S_f(x_1, x_2, ..., x_{n-1}, x_n) &= {}_{-\infty}D^{f-n}_{x_n}[S_n(x_1, x_2, ..., x_{n-1}, x_n)] \\ &= \begin{cases} 0 & \text{for } x_n < 0 \\ \dfrac{\delta(x_1)\delta(x_2)\ldots\delta(x_{n-1})x_n^{n-f-1}}{\Gamma(n-f)} & \text{for } x_n > 0 \end{cases} \\ &\qquad \text{for } n-1 < f < n \end{aligned} \tag{5.3}$$

Analogous expression for the even part of the source $S_f(x_1, x_2, ..., x_{n-1}, x_n)$ is [12]

$$\begin{aligned} {}_eS_f(x_1, x_2, \ldots, x_{n-1}, x_n) =&(1/2)\,[S_f(x_1, x_2, \ldots, x_{n-1}, x_n) \\ &+S_f(x_1, x_2, \ldots, x_{n-1}, -x_n)] \\ &\qquad \text{for } n-1 < f < n \end{aligned} \tag{5.4}$$

This symmetric source approaches $S_n(x_1, x_2, ..., x_{n-1}, x_n)$ for $f = n$ and becomes $(1/2)\,S_{n-1}(x_1, x_2, ..., x_{n-1})$ when $f = n-1$. Therefore,

[12] As pointed out by an anonymous reviewer of this paper, another generalization can be made when all n delta functions involved in Eq. (5.1) are replaced by their fractional integrations of various orders. Here, however, for the sake of mathematical simplicity and physical clarity we prefer to show one generalization at a time per each step.

intuitively speaking the source introduced in Eq. (5.4) is *effectively* an “intermediate” source between the $(n-1)$ and n-dimensional delta functions.

As for the relationship between the $(n-1)$- and n-dimensional scalar Green’s functions for the Helmholtz equation,[13] following a similar mathematical steps discussed in Section II, we can show that

$$\underset{kR_{n-1}>>1}{G_n(x_1, x_2, ...x_{n-1}, x_n = 0; k)} \approx \frac{1}{2\sqrt{\pi}} {}_0D_h^{-1/2}[G_{n-1}(x_1, x_2...x_{n-1}; \sqrt{h})]$$
$$\equiv \frac{1}{2\sqrt{\pi}} \frac{d^{-1/2}}{d(k^2)^{-1/2}}[G_{n-1}(x_1, x_2...x_{n-1}; k)] \tag{5.5}$$

where $G_n(\cdots)$ and $G_{n-1}(\cdots)$ are symbols for the n- and $(n-1)$-dimensional scalar Green’s functions of the Helmholtz equation, and R_{n-1} is the distance of the observation point in the $(n-1)$ dimensions from the origin, i.e., $R_{n-1} \equiv (x_1^2 + x_2^2 + \cdots + x_{n-1}^2)^{1/2}$. For the solution of Helmholtz equation with the source given in Eq. (5.3), it can be shown that

$$\underset{kR_{n-1}>>1}{\Psi_f(x_1, x_2... x_{n-1}, x_n = 0; k)} \propto \frac{d^{(n-f-1)/2}}{d(k^2)^{(n-f-1)/2}}[G_{n-1}(x_1, x_2... x_{n-1}; k)]$$
$$\text{for } n-1 < f < n \tag{5.6}$$

which, in the complete form, results in the following final form analogous to Eq. (3.3)

$$\Psi_f(x_1, x_2, \cdots, x_{n-1}, x_n; k)$$
$$\equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{G_{n-1}(x_1, x_2, \cdots, x_{n-1}; \sqrt{k^2-\beta^2}) \exp(i\beta x_n)}{(i\beta)^{n-f}} d\beta$$
$$\text{for } n-1 < f < n \tag{5.7}$$

where the integration is carried out over the real β axis in the complex β plane with appropriate indentations for the branch points and poles.

---

[13] For the n-dimensional space, the Helmholtz equation would have the same form as in Eq. (1.1) provided that ∇^2 is written in the n-dimensional form, namely as $\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ and the right-hand side source is also n-dimensional Dirac delta function.

For the symmetric source given in Eq. (5.4), the following solution for the Helmholtz equation can be given

$$
\begin{aligned}
&{}_e\Psi_f(x_1, x_2, \cdots, x_{n-1}, x_n; k) \\
&\quad \equiv \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{G_{n-1}(x_1, x_2, \cdots, x_{n-1}; \sqrt{k^2-\beta^2})\exp(i\beta x_n)}{(i\beta)^{n-f}} d\beta \\
&\quad + \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{G_{n-1}(x_1, x_2, \cdots, x_{n-1}; \sqrt{k^2-\beta^2})\exp(-i\beta x_n)}{(i\beta)^{n-f}} d\beta \\
&\qquad \text{for } n-1 < f < n \qquad (5.8)
\end{aligned}
$$

Care must be exercised for indentation of contours at branch points for the two integrals in Eq. (5.7). The appropriate indentations of the contour around the branch point $\beta = 0$ are not the same for the two integrals in the right-hand side of Eq. (5.8), and these indentations are similar to those given for Eq. (3.10). This solution, which will approach the $(n-1)$ and n-dimensional scalar Green's functions for $f = n-1$ and n respectively, is for the "intermediate" source given in Eq. (5.4). By generalization, the magnitude of this solution in the far-zone points along the symmetry hypersurface, i.e., for points with $x_n = 0$ and $kR_{n-1} >> 1$, drops as

$$
\begin{aligned}
\left|{}_e\Psi_f(x_1, x_2, \cdots, x_{n-1}, x_n = 0; k)\right|_{kR_{n-1}>>1} &\propto \left(\sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}\right)^{\frac{1-f}{2}} \\
&\equiv R_{n-1}{}^{\frac{1-f}{2}} \\
&\text{for } n-1 < f < n \qquad (5.9)
\end{aligned}
$$

Therefore, starting from the Green's function for the Helmholtz operator for integer-dimensional delta function source we can "smoothly collapse" to the $(n-1)$ dimensional case using the sources described as $(n-f)$ th-order integral of n-dimensional Dirac delta functions, where f is any real non-integer number between $(n-1)$ and n. One can then continue this process from $(n-1)$ to $(n-2)$, and from $(n-2)$ to $(n-3)$, and so on until it reaches the Green's function for one-dimensional delta function source.

6. Summary

In this paper, inspired by, and using the concept of fractional calculus we develop solutions for inhomogeneous Helmholtz equation, which are effectively "intermediate" solutions between the scalar Green's functions for Helmholtz operators with integer-dimensional Dirac delta function sources. These solutions, which we refer to as "fractional" solutions, are due to the source distributions describable as fractional integrals of integer-dimensional Dirac delta functions. We have shown, that in the far zone along a certain direction, these solutions can be written as fractional integration (with respect to variable k^2 and with the lower limit of zero) of the scalar Green's function of the lower integer dimension. We have also shown that for the far-zone points along a certain direction, the magnitude of these functions can drop as a function of distance with a rate which is again "intermediate" rate between the two corresponding rates of drop for Green's functions of the neighboring integer dimensional cases. For the case of source distribution intermediate between one- and two-dimensional Dirac delta functions, we have shown that the far-zone field consists of a cylindrical and a plane wave portion. The analysis reported here can have potential applications in study of the far-zone radiation from an extended aperture source with distributed aperture fields similar to the "intermediate" sources discussed here.

Acknowledgments

Portions of these results have been presented in a lecture given by the author at the ecole d'été Internationale sur "*Geometrie Fractale et Hyperbolique*, *Derivation Fractionnaire et Fractale*, *Applications dans les Science de l'Ingenieur et en Economie*," Bordeaux, France, July 3-8, 1994.

References

1. Oldham, K. B., and J. Spanier, *The Fractional Calculus,* Academic Press, New York, 1974. For historical survey prepared by B. Ross and reprinted in Ref [1], the reader is referred to pages 3–15 of this Reference.
2. McBride, A. C., and G. F. Roach (Eds.), *Fractional Calculus, Research Notes in Mathematics,* Pitman Advanced Publishing Program, Boston, 1985.
3. Davis, H. T., *The Theory of Linear Operators,* The Principia Press, Bloomington, Indiana, 1936.
4. Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics,* Mc Graw-Hill, New York, 2 Vols., 64–75, and 276–292, 1953.
5. Hassani, S., *Foundations of Mathematical Physics,* Allyn & Bacon, Boston, 1991.
6. Mathews, J., and R. L. Walker, *Mathematical Methods of Physics,* Benjamin/Cummings, Menlo Park, 1970.
7. Sommerfeld, A., *Partial Differential Equations in Physics,* Vol. VI of Lectures on Theoretical Physics, Academic Press, New York, 1949.
8. Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions,* Dover, New York, 1970.
9. Brittingham, J. N., "Focus Waves Modes in Homogeneous Maxwell's Equations: Transverse Electric Modes," *J. Appl. Phys.,* Vol. 54, No. 3, 1179–1189, 1983.
10. Sezginer, A., "A General Formulation of Focus Wave Modes," *J. Appl. Phys.,* Vol. 57, No. 3, 678–683, 1985.
11. Ziolkowski, R. W., "Exact Solutions of the Wave Equation with Complex Source Locations," *J. Math. Phys.,* vol. 26, No. 4, 861–863, 1985.
12. Wu, T. T., "Electromagnetic Missiles," *J. Appl. Phys.,* Vol. 57, No. 1, 2370–2373, 1985.
13. Gradshteyn, I. S., and I. M. Ryzhik, *Table of Integrals, Series, and Products,* Academic Press, New York 1965.
14. Lighthill, M. J., *Introduction to Fourier Analysis and Generalized Functions,* Cambridge University Press, London, 1958.
15. Ishimaru, A., *Electromagnetic Wave Propagation, Radiation, and Scattering,* Prentice-Hall, Englewood, New Jersey, 1991.