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AN ALTERNATIVE APPROACH TO THE HERTZ VECTOR

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1. Introduction

In this paper, we shall develop a theory of the Hertz vector in electromagnetism which is based on the standard theory, but departs from it by introducing two constraints. Consequently, the interpretation of the Hertz vector is now quite different. It is no longer linked by the wave equation to the charge and current sources, but rather it appears as a quantity which relates to the familiar scalar and vector potentials ϕ and $\mathbf{A}$ in essentially the same way that the fields $\mathbf{E}$ and $\mathbf{B}$ relate to the charge and current densities ρ and $\mathbf{j}$ (Maxwell's equations). We first give a brief summary of the standard theory and then in the next section introduce the modified approach.

Throughout, it is assumed that there are no polarisable or magnetisable media present, so $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$.

The Hertz vector was important in the early development of the classical theory of electromagnetic fields from a radiating source. Since then, it has received surprisingly little attention in fundamental electromagnetism, is unknown to many physicists, and is not even mentioned in several textbooks (notable exceptions being [1–2]). A thorough and authoritative account has, however, been given in Ref. [3] and with tensor formulation in Ref. [4]. A number of more recent articles have been written to revive interest, including Refs. [5–7].

The most important points which emerge are as follows. The (electric) Hertz vector $\mathbf{Z}_e$ (sometimes denoted $\mathbf{\Pi}_e$) is defined such that the scalar and vector potentials are given respectively by

$$\phi = -\mathrm{div}\ \mathbf{Z}_\mathrm{e}, \quad \mathbf{A} = \frac{1}{c^2}\ \frac{\partial \mathbf{Z}_\mathrm{e}}{\partial t} \tag{1a,b}$$

which automatically ensure that the Lorentz gauge condition

$$\mathrm{div}\ \mathbf{A} = -\frac{1}{c^2}\ \frac{\partial \phi}{\partial t} \tag{2}$$

is satisfied. $\mathbf{Z}_e$ is then at the basis of a powerful method for calculating radiation fields due to electric dipole and multipole distributions. If there also exist sources of magnetic dipole and multipole radiation, it is profitable to add a term curl $\mathbf{Z}_m$ to the RHS of equation (1b) [3]. $\mathbf{Z}_m$ is called the magnetic Hertz vector. Since div curl $\mathbf{Z}_m = 0$, equation (2) is still satisfied.

The equations of propagation of the Hertz vector are then

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{Z}_e = \frac{\mathbf{p}}{\varepsilon_0}, \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{Z}_m = \mu_0 \mathbf{m} \tag{3a,b}$$

where $\mathbf{p}$ and $\mathbf{m}$, known as the stream potentials, are related to the electric and magnetic dipole moments of the source. It is convenient to regard $\mathbf{Z}_e$ and $\mathbf{Z}_m$ as having their origins in $\mathbf{p}$ and $\mathbf{m}$ respectively. Thus for a given source (e.g. an oscillating electric dipole), one can deduce $\mathbf{Z}_e$ and/or $\mathbf{Z}_m$, and hence the fields $\mathbf{E}$ and $\mathbf{B}$ over space.

2. Alternative Approach

We shall here take a rather different approach to the interpretation of the Hertz vector. The power of solving problems involving radiating sources will be sacrificed, but the role of the modified Hertz vector in fundamental electromagnetism will be made manifest.

Following at first the previous section, we define a complex Hertz vector $\mathbf{Y}$ which satisfies the Lorentz gauge condition (2),

$$\phi = \text{div } \mathbf{Y}, \quad \mathbf{A} = -\frac{1}{c^2}\frac{\partial \mathbf{Y}}{\partial t} + \text{curl (any vector field)} \tag{4a,b}$$

and choose that the arbitrary field in the latter equation be $-(i/c)\mathbf{Y}$.

Putting now $\mathbf{Y} = \mathbf{Y}_e + ic\mathbf{Y}_m$, where $\mathbf{Y}_e$ and $\mathbf{Y}_m$ are both real, we have

$$\begin{aligned} &\phi = \text{div } (\mathbf{Y}_\mathrm{e} + ic\mathbf{Y}_\mathrm{m}) \\ &\mathbf{A} = -\frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{Y}_e + ic\mathbf{Y}_m) - \frac{i}{c}\text{curl } (\mathbf{Y}_\mathrm{e} + ic\mathbf{Y}_\mathrm{m}) \end{aligned} \tag{5a,b}$$

the real and imaginary parts of which give

$$\text{div}\mathbf{Y}_\mathrm{e} = \phi \tag{6a}$$

$$\text{div}\mathbf{Y}_\mathrm{m} = 0 \tag{6b}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{Y}_e}{\partial t} + \text{curl } \mathbf{Y}_\mathrm{m} = \mathbf{A} \tag{6c}$$

$$\text{curl } \mathbf{Y}_\mathrm{e} + \frac{\partial \mathbf{Y}_\mathrm{m}}{\partial \mathrm{t}} = 0 \tag{6d}$$

These bear a striking similarity to Maxwell's equations; indeed, the latter are produced by the substitutions $\mathbf{Y}_e \to \mathbf{E}$, $\mathbf{Y}_m \to \mathbf{B}$, $\phi \to \frac{\rho}{\varepsilon_0}$ and $\mathbf{A} \to \mu_0\mathbf{j}$. To achieve this, it has been necessary to part company with tradition by changing the signs of $\mathbf{Y}_e$ and $\mathbf{Y}_m$ (compare equations (4) with (1)). More significantly, two extra constraints (6b,d) have been introduced which do not appear in the standard theory.

From (6) and the identity (curl curl = grad div - ∇^2), the electric field is given by

$$\begin{aligned} \mathbf{E} &= -\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial \mathrm{t}} \\ &= -\text{grad div } \mathbf{Y}_\mathrm{e} + \frac{1}{c^2}\frac{\partial^2 \mathbf{Y}_\mathrm{e}}{\partial t^2} - \frac{\partial}{\partial t}\text{curl } \mathbf{Y}_\mathrm{m} \\ &= \frac{1}{c^2}\frac{\partial^2 \mathbf{Y}_e}{\partial t^2} - \text{grad div } \mathbf{Y}_\mathrm{e} + \text{curl curl } \mathbf{Y}_\mathrm{e} \end{aligned} \tag{7}$$

so

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{Y}_e = \mathbf{E} \tag{8}$$

Also, the magnetic flux density is

$$\begin{aligned}\mathbf{B} &= \text{curl } \mathbf{A} \qquad (9)\\ &= -\frac{1}{c^2}\frac{\partial}{\partial t}\text{curl } \mathbf{Y}_\text{e} + \text{curl curl } \mathbf{Y}_\text{m} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{Y}_\text{m} + \text{curl curl } \mathbf{Y}_\text{m}\end{aligned}$$

so

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{Y}_m = \mathbf{B} \tag{10}$$

Eqs. (8) and (10) are the equations of propagation for $\mathbf{Y}_e$ and $\mathbf{Y}_m$. They are quite different from Eqs. (3), and may be interpreted that the fields $\mathbf{Y}_e$ and $\mathbf{Y}_m$ have as their sources the **E** and **B** fields, whereas in the traditional approach, they originate in the dipole moments of the charge and current distributions.

The general solutions to (8) and (10) are found by appealing to the well known result that the equation of propagation for the vector potential

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} = \mu_0\mathbf{j}$$

has a solution at time t

$$\mathbf{A} = \frac{\mu_0}{\mathbf{4\pi}}\int\frac{[\mathbf{j}]}{r}d\tau$$

where $[\mathbf{j}]$ is the current density at time $t - r/c$.

It follows that

$$\mathbf{Y}_e = \frac{\mathbf{1}}{\mathbf{4\pi}}\int\frac{[\mathbf{E}]}{r}d\tau, \quad \mathbf{Y}_m = \frac{\mathbf{1}}{\mathbf{4\pi}}\int\frac{[\mathbf{B}]}{r}d\tau \tag{11a, b}$$

3. The Equations of Electromagnetism

The equations of electromagnetism are elegantly formulated using a non-commutative but associative algebra, isomorphic with the quaternion algebra, in which scalar and vector quantities may be added to form a single entity. For further details, the reader is referred to an earlier paper [8]. For convenience, we re-state in what follows some of the more important results we shall use.

The general product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined to be $\mathbf{ab} = \mathbf{a}.\mathbf{b} + i\mathbf{a} \times \mathbf{b}$, hence $\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1$, $\mathbf{ij} = i\mathbf{k}$ etc. From these, the result of $\nabla(= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})$ operating on the sum of a scalar α and a vector $\mathbf{a}$ is

$$\nabla(\alpha + \mathbf{a}) = \text{grad } \alpha + \text{div } \mathbf{a} + i \text{ curl } \mathbf{a} \tag{12}$$

From (12),

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t} \mp \nabla\right)&\left(\mp \mathbf{Y}_e + ic\mathbf{Y}_m\right) \\ &= \mp \frac{1}{c}\frac{\partial \mathbf{Y}_e}{\partial t} + i\frac{\partial \mathbf{Y}_m}{\partial t} + \text{div } \mathbf{Y}_\text{e} + i \text{ curl } \mathbf{Y}_\text{e} \\ &\quad \mp ic \text{ div } \mathbf{Y}_\text{m} \pm c \text{ curl } \mathbf{Y}_\text{m}\end{aligned}$$

which from (6), reduces to

$$\left(\frac{1}{c}\frac{\partial}{\partial t} \mp \nabla\right)\left(\mp \mathbf{Y}_e + ic\mathbf{Y}_m\right) = \phi \pm c\mathbf{A} \tag{13a}$$

It is interesting to compare this result with the following, which are readily proved. From (2), (7), (9) and (12), the potentials ϕ and $\mathbf{A}$ are related to the fields $\mathbf{E}$ and $\mathbf{B}$ by

$$\left(\frac{1}{c}\frac{\partial}{\partial t} \mp \nabla\right)\left(\phi \pm c\mathbf{A}\right) = \mp \mathbf{E} + ic\mathbf{B} \tag{13b}$$

Furthermore,

$$\left(\frac{1}{c}\frac{\partial}{\partial t} \mp \nabla\right)\left(\mp \mathbf{E} + ic\mathbf{B}\right) = \frac{1}{\varepsilon_0}\left(\rho \pm \frac{\mathbf{j}}{c}\right) \tag{13c}$$

the real and imaginary, scalar and vector parts of which are Maxwell's equations. To these may be appended

$$\mathrm{S}\left\{\left(\frac{1}{c}\frac{\partial}{\partial t}\pm\nabla\right)\ \left(\rho\pm\frac{\mathbf{j}}{c}\right)\right\}=0 \tag{13d}$$

where S denotes the scalar part. This is the equation of continuity

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\ \mathbf{j}=0$$

Equations (13a-d) form a natural progression, with the vector $\mathbf{Y}_e+ic\mathbf{Y}_m$ as a natural description of the electromagnetic field, having a time/space dimension which is one higher than the 4-vector $\phi+c\mathbf{A}$, and having properties which are similar to those of $\mathbf{E}+ic\mathbf{B}$.

4. Lorentz Transformation (LT)

The analogy between $\mathbf{Y}_e+ic\mathbf{Y}_m$ and $\mathbf{E}+ic\mathbf{B}$ allows us to write down the LTs for the components of $\mathbf{Y}_e$ and $\mathbf{Y}_m$ immediately, by comparison with the familiar LTs for $\mathbf{E}$ and $\mathbf{B}$,

$$\begin{array}{ll} Y'_{ex}=Y_{ex} & Y'_{mx}=Y_{mx} \\ Y'_{ey}=\gamma\Big(Y_{ey}-vY_{mz}\Big) & Y'_{my}=\gamma\Big(Y_{my}+\frac{v}{c^2}Y_{ez}\Big) \\ Y'_{ez}=\gamma\Big(Y_{ez}+vY_{my}\Big) & Y'_{mz}=\gamma\Big(Y_{mz}-\frac{v}{c^2}Y_{ey}\Big) \end{array}$$

where the primed frame is moving with a velocity v in the x direction relative to the unprimed frame, and $\gamma=(1-v^2/c^2)^{-\frac{1}{2}}$.

It is easily verified that these may be condensed into a single equation

$$\mp\mathbf{Y}'_e+ic\mathbf{Y}'_m=e^{\pm\frac{1}{2}\mathbf{i}\theta}\Big(\mp\mathbf{Y}_e+ic\mathbf{Y}_m\Big)e^{\mp\frac{1}{2}\mathbf{i}\theta}$$

where $\cosh\ \theta=\gamma$, $\sinh\theta=\beta\gamma$, $\beta=v/c$ and $e^{\pm\frac{1}{2}\mathbf{i}\theta}=\cosh\frac{1}{2}\theta\pm\mathbf{i}\sinh\frac{1}{2}\theta$.

Similar contractions, which we have shown previously [8; also 9] are

$$\phi' \pm c\mathbf{A}' = e^{\mp\frac{1}{2}\mathbf{i}\theta}\Big(\phi \pm c\mathbf{A}\Big)e^{\mp\frac{1}{2}\mathbf{i}\theta}$$

$$\mp\mathbf{E}' + ic\mathbf{B}' = e^{\pm\frac{1}{2}\mathbf{i}\theta}\Big(\mp\mathbf{E} + ic\mathbf{B}\Big)e^{\mp\frac{1}{2}\mathbf{i}\theta}$$

$$\rho' \pm \frac{\mathbf{j}'}{c} = e^{\mp\frac{1}{2}\mathbf{i}\theta}\Big(\rho \pm \frac{\mathbf{j}}{c}\Big)e^{\mp\frac{1}{2}\mathbf{i}\theta}$$

5. Examples

We now deduce the Hertz vector $\mathbf{Y}$ for some specific cases:

(a) stationary point charge q at the origin of coordinates

Clearly here $\mathbf{Y}_e$ and $\mathbf{Y}_m$ should have no time dependence. Therefore, from equations (6), div $\mathbf{Y}_e = q/(4\pi\varepsilon_0 r)$, where r is the distance from the origin, curl $\mathbf{Y}_e = 0$ and $\mathbf{Y}_m = 0$. In spherical polar coordinates, $\mathbf{Y}_e$ should depend only on r, hence

$$\mathrm{div}\mathbf{Y}_\mathrm{e} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2\,Y_\mathrm{er}\Big) = \frac{q}{4\pi\varepsilon_0 r}$$

giving

$$\mathbf{Y}_e = \left(\frac{q}{8\pi\varepsilon_0} + \frac{C}{r^2}\right)\hat{\mathbf{r}}$$

where C is a constant of integration, and $\hat{\mathbf{r}}$ is the unit vector in the r direction. It is interesting to note that (if $C = 0$), $\mathbf{Y}_e$ has the same magnitude at all points in space.

(b) uniform magnetic field

A uniform magnetic field $\mathbf{B}$ in the z direction may be represented by $\phi = 0$, $A_x = -\frac{1}{2}By$, $A_y = \frac{1}{2}Bx$, $A_z = 0$, since then curl $\mathbf{A} = \mathbf{B}$. Again, $\mathbf{Y}_e$ and $\mathbf{Y}_m$ should have no time dependence, so equations (6) give

$$\mathbf{Y}_e = 0,\ \mathrm{div}\ \mathbf{Y}_\mathrm{m} = 0,\ \mathrm{curl}\ \mathbf{Y}_\mathrm{m} = \mathbf{A}$$

which are satisfied by

$$\mathbf{Y}_m = -\frac{1}{4}B\Big(x^2 + y^2\Big)\mathbf{k}$$

where $\mathbf{k}$ is the unit vector in the z direction.

(c) plane electromagnetic wave

Consider now a plane sinusoidal electromagnetic wave travelling in the positive z direction and polarised with its electric field in the x direction. In the usual notation,

$$\mathbf{E} = E_0 \mathbf{i} e^{j(\omega t - kz)}, \quad \mathbf{B} = (E_0/c)\mathbf{j} e^{j(\omega t - kz)}$$

For simplicity, we take the scalar potential ϕ to be zero, so there are no free charges present. The vector potential is then

$$\mathbf{A} = j(E_0/\omega)\mathbf{i} e^{j(\omega t - kz)}$$

which satisfies (2), (7) and (9).

Since $\mathbf{E}$ is everywhere in the x direction, then from (11a), so is $\mathbf{Y}_e$; similarly, $\mathbf{Y}_m$ is in the y direction. The only non-vanishing components are Y_{ex} and Y_{my} , for which (6a-d) give

$$\frac{\partial Y_{ex}}{\partial x} = \frac{\partial Y_{my}}{\partial y} = \frac{\partial Y_{ex}}{\partial y} = \frac{\partial Y_{my}}{\partial x} = 0$$

$$\frac{-1}{c^2}\frac{\partial Y_{ex}}{\partial t} - \frac{\partial Y_{my}}{\partial z} = \frac{jE_0}{\omega} e^{j(\omega t - kz)}$$

$$\frac{\partial Y_{ex}}{\partial z} + \frac{\partial Y_{my}}{\partial t} = 0$$

Hence, Y_{ex} and Y_{my} depend only on t and z . Clearly the time dependence should be $e^{j\omega t}$, but the equations cannot be satisfied by a spatial dependence e^{-jkz} . A full analysis gives a solution

$$Y_{ex} = \left\{ \frac{E_0}{2k^2}\left(-\frac{1}{2} - jkz\right) + \frac{\alpha}{2} \right\} e^{j(\omega t - kz)} + \beta e^{j(\omega t + kz)}$$

$$cY_{my} = \left\{ \frac{E_0}{2k^2}\left(\frac{1}{2} - jkz\right) + \frac{\alpha}{2} \right\} e^{j(\omega t - kz)} - \beta e^{j(\omega t + kz)}$$

where α and β are arbitrary constants. Putting $\alpha = \beta = 0$ gives

$$\mathbf{Y}_e = \mathbf{i}\frac{E_0}{2k^2}\left(-\frac{1}{2} - jkz\right) e^{j(\omega t - kz)}$$

$$c\mathbf{Y}_m = \mathbf{j}\frac{E_0}{2k^2}\left(\frac{1}{2} - jkz\right)e^{j(\omega t - kz)}$$

It is interesting to note that the amplitudes vary linearly with z.

(d) oscillating electric dipole

We now analyse an electric dipole with moment $p_0 \cos\omega t$ oscillating in the z direction with angular frequency ω. The equations for the potentials are [10]

$$\phi = \frac{kp_0 \cos\theta}{4\pi\varepsilon_0 r}\left(\frac{1}{kr}\cos\omega t' - \sin\omega t'\right)$$

$$\mathbf{A} = \frac{\omega p_0 \sin\omega t'}{4\pi\varepsilon_0 c^2 r}\left(-\cos\theta\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

where $\omega t' = \omega t - kr$, $k = 2\pi/\lambda$ and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ are the usual unit vectors in spherical polar coordinates.

A full analysis, too detailed to give here, shows that the solutions to (6) are

$$\mathbf{Y}_e = \frac{kp_0 \sin\theta}{8\pi\varepsilon_0}\left(\frac{1}{kr}\cos\omega t' - \sin\omega t'\right)\hat{\boldsymbol{\theta}}$$

$$\mathbf{Y}_m = -\frac{kp_0}{8\pi\varepsilon_0 c}\sin\theta\sin\omega t'\,\hat{\boldsymbol{\phi}}$$

At large r, $\mathbf{Y}_e \pm ic\mathbf{Y}_m \rightarrow -\frac{kp_0}{8\pi\varepsilon_0}\sin\theta\sin\omega t'\left(\hat{\boldsymbol{\theta}} \pm i\,\hat{\boldsymbol{\phi}}\right)$

The results for $\mathbf{Y}_e$ and $\mathbf{Y}_m$ are quite different from those of the traditional approach [1], in which $\mathbf{Y}_m = 0$, and $\mathbf{Y}_e$ falls off as $1/r$. These latter expressions would not be valid in our analysis, since (6d) would not be satisfied.

(**e**) TE_{0n} propagation in a waveguide

The final example for consideration concerns wave propagation in a hollow rectangular waveguide with perfectly conducting walls. The guide axis is taken as the z direction and the wall planes as $x = 0$ and a, $y = 0$ and b.

Consider a transverse electric (TE) wave propagating in the positive z direction with its electric field in the x direction. The fields for the TE_{0n} mode are given in Ref. 10.

$$E_x = C \sin \frac{n\pi y}{b} e^{j(\omega t - k_g z)}$$
$$B_y = \frac{C k_g}{\omega} \sin \frac{n\pi y}{b} e^{j(\omega t - k_g z)}$$
$$B_z = \frac{C n\pi}{j\omega b} \cos \frac{n\pi y}{b} e^{j(\omega t - k_g z)}$$

$$E_y = E_z = B_x = 0$$

where C is the maximum amplitude of the E field. The wavenumber k_g in the guide is given by $k^2 - k_g^2 = n^2\pi^2/b^2$, where k is the wavenumber in free space.

If the scalar potential ϕ is arbitrarily equated to zero, the vector potential is given by

$$A_x = \frac{jC}{\omega} \sin \frac{n\pi y}{b} e^{j(\omega t - k_g z)}$$

$$A_y = A_z = 0$$

which satisfy equations (7) and (9).

To obtain $\mathbf{Y}_e$ and $\mathbf{Y}_m$, we see from Eqs. (11) that since $\mathbf{E}$ is in the x direction, then so is $\mathbf{Y}_e$; similarly $\mathbf{Y}_m$ has only y and z components. This, together with equations (6a-d) gives

$$Y_{ey} = Y_{ez} = Y_{mx} = 0$$

$$\frac{\partial Y_{ex}}{\partial x} = \frac{\partial Y_{my}}{\partial x} = \frac{\partial Y_{mz}}{\partial x} = 0$$

$$\frac{\partial Y_{my}}{\partial y} + \frac{\partial Y_{mz}}{\partial z} = 0$$

$$\frac{\partial Y_{ex}}{\partial z} + \frac{\partial Y_{my}}{\partial t} = 0$$

$$-\frac{\partial Y_{ex}}{\partial y} + \frac{\partial Y_{mz}}{\partial t} = 0$$

$$-\frac{1}{c^2}\frac{\partial Y_{ex}}{\partial t} + \frac{\partial Y_{mz}}{\partial y} - \frac{\partial Y_{my}}{\partial z} = \frac{jC}{\omega} \sin \frac{n\pi y}{b} e^{j(\omega t - k_g z)}$$

The solution of these equations is straightforward but rather tedious, and we merely quote one of the possible solutions

$$Y_{ex} = \frac{Cb}{2\pi n} y \cos \frac{n\pi y}{b} e^{j(\omega t - k_g z)}$$

$$Y_{my} = \frac{Cbk_g}{2\pi n\omega} y \cos \frac{n\pi y}{b} e^{j(\omega t - k_g z)}$$

$$Y_{mz} = -\frac{jCb}{2\pi n\omega}\left(\cos \frac{n\pi y}{b} - \frac{n\pi}{b} y \sin \frac{n\pi y}{b}\right) e^{j(\omega t - k_g z)}$$

Various other solutions exist, including one involving functions like $\sin(n\pi y/b) z \exp j\,(\omega t - k_g z)$, but they will not be considered.

6. Gauge Transformation

It is well known [11] that ϕ and $\mathbf{A}$ are arbitrary to the extent that the $\mathbf{E}$ and $\mathbf{B}$ fields are unaltered under a gauge transformation $\phi \to \phi - \partial\Lambda/\partial t, \mathbf{A} \to \mathbf{A} + \mathrm{grad}\Lambda$, where Λ is any scalar field. The Lorentz gauge (2) is also unaffected, provided that

$$\frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = \nabla^2 \Lambda$$

Λ is customarily taken to be real, but we allow the freedom that it be complex, so $\Lambda = \Lambda_e + ic\Lambda_m$, where Λ_e and Λ_m are real.

Equations (5) may then be extended to read

$$\phi = \mathrm{div}\ (\mathbf{Y}_e + ic\mathbf{Y}_m) - \frac{\partial}{\partial t}(\Lambda_e + ic\Lambda_m)$$

$$\mathbf{A} = -\frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{Y}_e + ic\mathbf{Y}_m) - \frac{i}{c}\ \mathrm{curl}\ (\mathbf{Y}_e + ic\mathbf{Y}_m) + \mathrm{grad}\ (\Lambda_e + ic\Lambda_m)$$

so that equations (6) now become

$$\mathrm{div}\ \mathbf{Y}_e - \frac{\partial \Lambda_e}{\partial t} = \phi$$

$$\mathrm{div}\ \mathbf{Y}_m - \frac{\partial \Lambda_m}{\partial t} = 0$$

$$\frac{-1}{c^2}\frac{\partial \mathbf{Y}_e}{\partial t} + \text{curl } \mathbf{Y}_m + \text{grad } \Lambda_e = \mathbf{A}$$

$$\text{curl } \mathbf{Y}_e + \frac{\partial \mathbf{Y}_m}{\partial t} - c^2 \text{grad } \Lambda_m = 0$$

With these modified equations, (7), (8), (9), (10) and (13b-d) are of course unaltered, but it is a simple matter to show that (13a) is now

$$\left(\frac{1}{c}\frac{\partial}{\partial t} \mp \nabla\right)\left(-c\Lambda_e \mp \mathbf{Y}_e \pm ic^2\Lambda_m + ic\mathbf{Y}_m\right) = \phi \pm c\mathbf{A}$$

Evidently $c\Lambda_e$ and $c\Lambda_m$ may be regarded as natural partners to $\mathbf{Y}_e$ and $\mathbf{Y}_m$ respectively.

7. Quantum Mechanical Applications

The dimensions of Λ_e, Λ_m, $\mathbf{Y}_e$ and $\mathbf{Y}_m$ are simple combinations of fundamental constants, being respectively $[h/e]$, $[h/ce]$, $[ch/e]$ and $[h/e]$. This fact suggests that there may be applications in quantum theory (and conceivably quantum electrodynamics).

It is well known that the role of the gauge transformation function Λ_e is to introduce an undetectable phase factor into the wave function. An interesting problem is therefore to write the Dirac equation for an electron in a field using Λ and $\mathbf{Y}$, and to enquire whether Λ_e, Λ_m, $\mathbf{Y}_e$ and $\mathbf{Y}_m$ can somehow be incorporated into exponential factors multiplying the field-free wave function. A lengthy analysis reveals that this can be done if $\mathbf{Y}_e$ and $\mathbf{Y}_m$ are constant in time and space, giving a fairly simple version of the Dirac equation. However, this version appears to be incorrect in the general case.

8. Conclusion

To summarise, it has been shown that this alternative formulation leads to Hertz vectors $\mathbf{Y}_e$ and $\mathbf{Y}_m$ which have quite different properties from those of $\mathbf{Z}_e$ and $\mathbf{Z}_m$ which appear in the usual theory. The role of $\mathbf{Y}_e$ and $\mathbf{Y}_m$ is essentially one of a description of the electromagnetic field, from which ϕ, **A, E** and **B** can be derived. $\mathbf{Y}_e$ and

$\mathbf{Y}_m$ are also vector counterparts of the scalar gauge transformation functions Λ_e and Λ_m. The space/time dimensions of Λ and $\mathbf{Y}$, being one step higher than those of ϕ and $\mathbf{A}$, are closely related to those of h/e, indicating that further investigations may reveal applications in quantum mechanics.

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