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SCALARIZATION APPROACH IN THE THEORY OF EM WAVES IN A PLANE-LAYERED BIANISOTROPIC MEDIUM: GENERAL FORMULATIONS

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1. Introduction

Plane-layered (bi)anisotropic structures may find potential applications in a vast variety of areas including novel antenna radomes and absorbing coatings design, integrated circuit technology, optoelectronics, etc. This creates a growing need for the tractable models and methods to analyze EM phenomena in a layered (bi)anisotropic medium.

In view of a severe algebraic complexity of Maxwell's equations for such a medium, the success of theoretical investigation is critically dependent on analytical techniques. A fruitful coordinate-free approach for bianisotropic media has been developed in [1–3], extending some of the original methods for anisotropic media from [4]. This analytical approach operates with the vectorial field amplitudes and the dyadic constitutive parameters without invoking the preferred coordinate system. It is applicable to plane-wave propagation problems involving homogeneous media or discrete-layered structures. An alternative strategy stemming from the works [5,6] is to eliminate the field components parallel to the axis of stratification and to operate with a system of four coupled differential equations with respect to the transverse field components. Within the framework of this formulation, an ingenuous analytical approach has been presented for solving radiation and propagation problems in anisotropic [7,8] and bianisotropic [9] media. It is based on the representation of a solution to the aforementioned system of differential equations in homogeneous regions in the matrix exponential form and the subsequent use of the Caley-Hamilton theorem. This approach is applicable to homogeneous or piecewise-homogeneous layered structures. The case of a continuous layered slab can be treated via the just cited methods by approximately dividing the inhomogeneous region into thin homogeneous layers. However, such a technique requires the numerical solution of a characteristic equation in the complex domain for each artificial layer and the numerical treatment of the exponentially growing factors depicting non-uniform plane waves. In view of these complexities a direct numerical solution of the respective differential system may appear advantageous. The efficient finite-difference numerical schemes to deal with the matrix differential equations for anisotropic slabs satisfied by the transmission or the reflection matrices have been proposed in [10] and [11], respectively.

In spite of the progress that has been already achieved, there exists an opportunity to further simplify the analytical and numerical treatment of the EM problems in layered bianisotropic media by invoking the scalarization approach [12–16]. In this context, a systematic analytical method has been developed [13,14,16] that synthesizes the ideas of the coordinate-free approach and the elimination technique and comprises a variety of radiation and propagation problems in anisotropic layered media. The basic advantages of this method are as follows:

a) the number of the unknown quantities—scalar potentials—is reduced to two rather than four in the past,
b) the respective field representation through scalar potentials, as well as the resulting problem formulation for these potentials is

independent of the choice of a Cartesian coordinate system where the constitutive parameters are specified,

c) there is no need to employ the matrix quantities and formulations. The quantities of physical interest are explicitly related to the solution to a respective problem for scalar potentials. The latter includes standard Sturm-Liouville differential operators, with the coefficients being expressed invariantly and in a highly symmetrical manner through the dyadic constitutive parameters,

d) a unified consideration of media with the piecewise-constant and the smooth variation of the EM properties is provided.

A sacrifice for these advantages was a huge amount of algebraic manipulations performed at the intermediate stages of developing the ultimate formulations [16].

In this paper, the scalarization approach of [13,14,16] is extended to include the basic excitation, plane-wave diffraction and normal mode propagation problems for an arbitrarily bianisotropic plane-layered medium bounded by a surface with an arbitrary anisotropic non-local impedance. As well, the paper can be thought of as a natural generalization of the work [17] which has been considering the spectral-domain dyadic Green's functions in layered chiral media. Namely, we deal with a medium where the constitutive relations for a time-harmonic $(\sim e^{-i\omega t})$ EM field are stated as

$$\vec{D} = \hat{\varepsilon} \circ \vec{E} + \hat{\xi} \circ \vec{H}, \quad \vec{B} = \hat{\mu} \circ \vec{H} - \hat{\zeta} \circ \vec{E} \tag{1}$$

The complex dyadic quantities $\hat{\varepsilon}, \hat{\mu}, \hat{\xi},$ and $\hat{\zeta}$ have arbitrary geometrical structure and vary in any prescribed way in the z direction but do not depend on $\vec{r} = (x, y, 0)$. The variation with z may be smooth or piecewise-constant (discrete-layered structure) or any combination of these.

The medium fills the halfspace $0 < z < +\infty$ and is bounded at $z = 0$ by an impedance surface characterized by the dyadic impedance $\hat{L}$ (see Figure 1) . The quantity $\hat{L}$ generally is an operator which acts on the vectorial functions of $\vec{r}$ which are tangential to the boundary $z = 0$. This is characteristic of a penetrable boundary (see [18,19] and Appendix B) or to a statistically rough interface [20,21]. For a well conducting surface the operator $\hat{L}$ reduces to multiplication by a scalar factor representing the impedance of an underlying medium [22, 23]. In case of perfect conductivity, $\hat{L} = 0$. The boundary $z = 0$

is homogeneous with respect to its physical properties. Formally this means that on applying the operator $\hat{L}$ to a spatial harmonic

$$\vec{B}(\vec{r}) = \vec{A}(\vec{\chi}) \exp(i\vec{\chi} \circ \vec{r}) \tag{2}$$

with the vector amplitude $\vec{A} = (A_x, A_y, 0)$ and arbitrary complex wave vector $\vec{\chi} = (\chi_x, \chi_y, 0)$ one gets the same spatial harmonic multiplied by a dyadic factor $\hat{L}(\vec{\chi})$:

$$\hat{L} \circ \vec{B}(\vec{r}) \equiv \exp(i\vec{\chi} \circ \vec{r})\, \hat{L}(\vec{\chi}) \circ \vec{A}(\vec{\chi}) \tag{3}$$

The dyad $\hat{L}(\vec{\chi})$ obeys the relations $\vec{z}_0 \circ \hat{L}(\vec{\chi}) = \hat{L}(\vec{\chi}) \circ \vec{z}_0 \equiv 0$ where $\vec{z}_0$ is the unit vector along the z axis, and is otherwise an arbitrary complex dyadic function of $\vec{\chi}$ variable. It represents the dyadic impedance of the boundary $z = 0$ in the spectral domain.

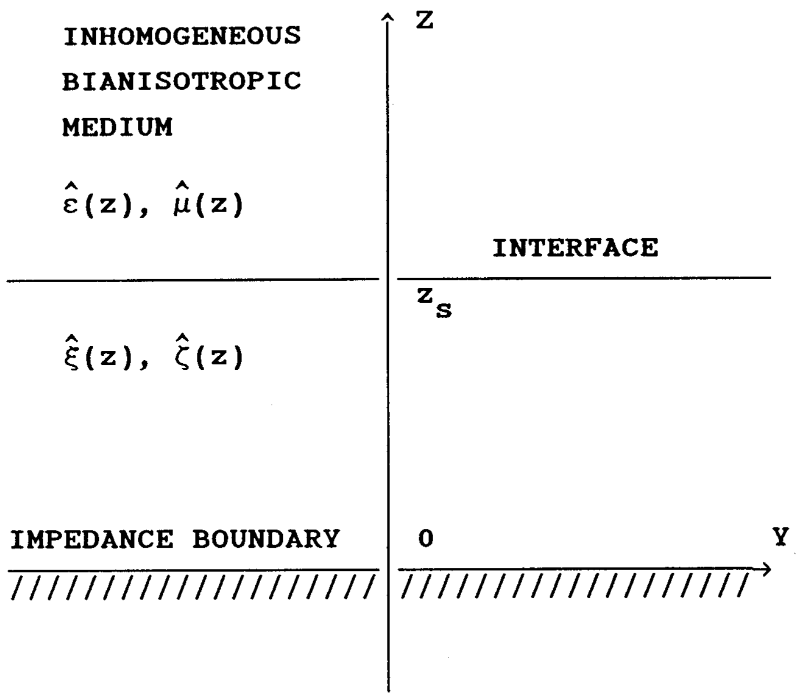

Figure 1. Two-dimensional sketch of the problem.

The basic scalarized formulation for a general EM problem in such a medium is presented in Section 2. Next, in Section 3 the usefulness of the scalarization method is demonstrated by the solution to the spectral domain Green's functions in a general bianisotropic medium and the asymptotic evaluation of the spatial Green's functions in case where a point source and an observation point are located within the bianisotropic slab and in the adjacent space, respectively. It is also shown that the spectral dyadic Green's functions are essentially the generalized functions, or distributions. Their regular and delta function constituents are explicitly found, and their singular behavior at a source plane is revealed. In Section 4, assuming planar excitation in the interface an excitation problem in the spectral domain is solved, and the closed-form asymptotic expressions for the spatial dyadic Green's functions accounting for the interface sources are obtained. The fundamental importance of the results of Sections 3 and 4 in solving radiation and scattering problems for microwave integrated circuits is obvious. Finally, in Section 5 the concept of the equivalent impedance of a penetrable boundary is employed to produce an elegant coordinate-free solution to a plane-wave diffraction problem and a natural mode propagation problem involving a bianisotropic slab.

2. Scalarization of Field Quantities in the Spectral Domain

The main purpose of this section is to obtain an analytical representation for the vector field amplitudes through scalar potentials referring to a generally bianisotropic plane-layered medium with an impedance boundary and to derive the formulation for a general EM problem in terms of these potentials.

The starting problem for the field intensity vectors $\vec{E}(\vec{R})$, $\vec{H}(\vec{R})$, $\vec{R} = (x, y, z)$ in the halfspace $z > 0$ consists of Maxwell's equations

$$\begin{aligned} \nabla \times \vec{H} + ik_0(\hat{\varepsilon} \circ \vec{E} + \hat{\xi} \circ \vec{H}) &= (4\pi/c)\vec{J} \\ \nabla \times \vec{E} - ik_0(\hat{\mu} \circ \vec{H} - \hat{\zeta} \circ \vec{E}) &= -(4\pi/c)\vec{M} \end{aligned} \tag{4}$$

within the continuous parts of a medium, customary continuity conditions for the tangential EM field $\vec{E}_\tau, \vec{H}_\tau$ at every interface $z =$ const, (const > 0), free of the impressed sources, boundary conditions at the interface $z = z_s, (z_s > 0)$, supporting the impressed sources

$$\{\vec{z}_0 \times \vec{H}_\tau\} = (4\pi/c)\vec{J}_s, \quad \{\vec{z}_0 \times \vec{E}_\tau\} = -(4\pi/)\vec{M}_s \tag{5}$$

and the impedance boundary condition at $z = +0$

$$\vec{E}_\tau - \hat{L} \circ \vec{z}_0 \times \vec{H}_\tau = 0 \tag{6}$$

Here $k_0 = \omega/c$, c is a speed of light in free space, $\vec{J} \equiv \vec{J}(\vec{R})$, $\vec{M} \equiv \vec{M}(\vec{R})$ are the volume densities and $\vec{J}_s \equiv \vec{J}_s(\vec{r})$, $\vec{M}_s \equiv \vec{M}_s(\vec{r})$ the surface densities of the impressed electric and magnetic currents, $\{f(z)\} \equiv f(z+0) - f(z-0)$ for any function of variable z. We note that in Maxwell's equations (4), the constitutive relations (1) have been utilized. Equations (4) and (5) preserve their form under the following transformations:

$$\begin{aligned}
&\vec{E} \to \vec{H}\ , \quad \vec{H} \to -\vec{E} \\
&\vec{J} \to \vec{M}\ , \quad \vec{J}_s \to \vec{M}_s, \quad \vec{M} \to -\vec{J}, \quad \vec{M}_s \to -\vec{J}_s \\
&\hat{\varepsilon} \leftrightarrow \hat{\mu}, \quad \hat{\xi} \leftrightarrow \hat{\zeta}
\end{aligned} \tag{7}$$

This symmetry property will be traced in subsequent formulations.

Let us assume that the impressed sources represent a spatial harmonic with a wave vector $\vec{\chi}$:

$$\begin{aligned}
\vec{J}(\vec{R}) &= \vec{J}(\vec{\chi}, z)\exp(i\vec{\chi} \circ \vec{r}), \quad \vec{M}(\vec{R}) = \vec{M}(\vec{\chi}, z)\exp(i\vec{\chi} \circ \vec{r}) \\
\vec{J}_s(\vec{r}) &= \vec{J}_s(\vec{\chi})\exp(i\vec{\chi} \circ \vec{r}), \quad \vec{M}_s(\vec{r}) = \vec{M}_s(\vec{\chi})\exp(i\vec{\chi} \circ \vec{r})
\end{aligned} \tag{8}$$

The quantities $\vec{J}(\vec{\chi}, z)$, $\vec{M}(\vec{\chi}, z)$ and $\vec{J}_s(\vec{\chi})$, $\vec{M}_s(\vec{\chi})$ stand for vector source amplitudes, $\vec{\chi}$ may be complex if not stated otherwise. The exceptional place of the particular case (8) among the innumerable variety of the impressed sources is explained by the fact that any of them can be represented as proper superposition of spatial harmonics like (8) (e.g. via the inverse double Fourier transform in the $\vec{\chi}$ domain).

We seek for a solution to the above described problem in the form

$$\vec{E}(\vec{R}) = \vec{E}(\vec{\chi}, z)\exp(i\vec{\chi} \circ \vec{r}), \quad \vec{H}(\vec{R}) = \vec{H}(\vec{\chi}, z)\exp(i\vec{\chi} \circ \vec{r}) \tag{9}$$

where $\vec{E}(\vec{\chi}, z)$, $\vec{H}(\vec{\chi}, z)$ denote the vector field amplitudes. A question arises how to express these vectorial quantities through auxiliary scalar functions. An efficient method which leads to a minimal number of these functions is presented below.

It is an easy matter to verify that the vectors $\vec{\chi}$ and $\vec{z}_0 \times \vec{\chi}$ are linearly independent if $\vec{\chi} \circ \vec{\chi} \neq 0$, i.e., if $\chi_x \neq i\chi_y$. Assuming that this demand is satisfied introduce a basis set of vectors [13,16,24]

$$\vec{a}_1 = \vec{n}, \qquad \vec{a}_t = \vec{z}_0 \times \vec{n}, \qquad \vec{a}_z = \vec{z}_0 \tag{10}$$

where $\vec{n} = \vec{\chi}/\chi$ is a unit vector: $\vec{n} \circ \vec{n} = 1$, $\chi = (\vec{\chi} \circ \vec{\chi})^{1/2}$, and the branch of a square root is chosen, for definiteness, to satisfy the condition $0 \leq \arg \chi < \pi$. We apply the definition (10) also to the case $\chi_x = \chi_y = 0$. This time $\vec{n}$ is supposed to be an arbitrarily fixed unit vector lying in the plane $z = 0$. The case where $\chi_x = i\chi_y \neq 0$ needs separate treatment which is omitted. By construction, the unit vectors (10) obey the orthogonality relations

$$\vec{a}_\sigma \circ \vec{a}_\tau = 0, \qquad (\sigma \neq \tau) \tag{11}$$

where $\sigma, \tau = l, t, z$. Making scalar products of the Maxwell's equations with the unit vectors $\vec{a}_z$ and $\vec{a}_t$, one arrives after some tedious algebra at the direct formulae expressing $\vec{a}_{z,1} \circ \vec{E}(\vec{\chi}, z)$ and $\vec{a}_{z,1} \circ \vec{H}(\vec{\chi}, z)$ in terms of scalar potentials

$$\mathcal{E}(\vec{\chi}, z) \equiv \vec{a}_t \circ \vec{E}(\vec{\chi}, z), \quad \mathcal{H}(\vec{\chi}, z) \equiv \vec{a}_t \circ \vec{H}(\vec{\chi}, z) \tag{12}$$

and their first derivatives with respect to z. Subsequent reconstruction of the vectors $\vec{E}(\vec{\chi}, z), \vec{H}(\vec{\chi}, z)$ via their projections on the basis set (10) brings the result we are seeking:

$$\begin{aligned} \vec{E}(\vec{\chi}, z) &= \vec{v}_e \, \mathcal{E}(\vec{\chi}, z) - \vec{w}_e \, \mathcal{H}(\vec{\chi}, z) \\ &\qquad + (4i\pi/ck_0)\Big[\hat{\alpha}_{ee} \circ \vec{J}(\vec{\chi}, z) - \hat{\alpha}_{em} \circ \vec{M}(\vec{\chi}, z)\Big] \\ \vec{H}(\vec{\chi}, z) &= \vec{v}_m \, \mathcal{H}(\vec{\chi}, z) + \vec{w}_m \, \mathcal{E}(\vec{\chi}, z) \\ &\qquad + (4i\pi/ck_0)\Big[\hat{\alpha}_{mm} \circ \vec{M}(\vec{\chi}, z) + \hat{\alpha}_{me} \circ \vec{J}(\vec{\chi}, z)\Big] \end{aligned} \tag{13}$$

These formulae constitute a desired representation of vector field amplitudes through two scalar potentials $\mathcal{E}, \mathcal{H}$. Here $\vec{w}_\beta, \vec{v}_\beta, (\beta = e, m)$, are vector differential operators acting on z and depending on $\chi, \vec{n}$:

$$\begin{aligned}\vec{w}_e &= \vec{n}t_l + \vec{z}_0\, t_z + k_0{}^{-1}\Big[\chi(\vec{z}_0 p_{ll} - \vec{n}p_{lz}) + i(\vec{n}p_{zz} - \vec{z}_0\, p_{zl})\partial_z\Big] \\ \vec{v}_e &= \vec{z}_0 \times \vec{n} + \vec{n}v_l + \vec{z}_0 v_z + k_0{}^{-1}\Big[\chi(\vec{n}r_{lz} - \vec{z}_0 r_{ll}) + i(\vec{z}_0 r_{zl} - \vec{n}r_{zz})\partial_z\Big]\end{aligned} \tag{14}$$

$$\begin{aligned}&\vec{w}_e \to \vec{w}_m\ , \qquad (t_\sigma \to u_\sigma\ , \quad p_{\sigma\tau} \to q_{\sigma\tau}) \\ &\vec{v}_e \to \vec{v}_m\ , \qquad (v_\sigma \to w_\sigma\ , \quad r_{\sigma\tau} \to s_{\sigma\tau})\end{aligned} \tag{15}$$

$\partial_z = \partial/\partial z, \quad \hat{\alpha}_{ee}, \ldots, \hat{\alpha}_{mm}$ are the dyadic functions of $\vec{n}, z$:

$$\begin{aligned}\hat{\alpha}_{ee}(\vec{n}, z) &= \vec{n}\vec{z}_0\, p_{lz} + \vec{z}_0\vec{n}\, p_{zl} - \vec{n}\vec{n}\, p_{zz} - \vec{z}_0\vec{z}_0\, p_{ll} \\ \hat{\alpha}_{em}(\vec{n}, z) &= \vec{n}\vec{z}_0 r_{lz} + \vec{z}_0\vec{n}\, r_{zl} - \vec{n}\vec{n}\, r_{zz} - \vec{z}_0\vec{z}_0 r_{ll}\end{aligned} \tag{16}$$

$$\hat{\alpha}_{ee} \to \hat{\alpha}_{mm}, \quad (p_{\sigma\tau} \to q_{\sigma\tau}); \qquad \hat{\alpha}_{em} \to \hat{\alpha}_{me}, \quad (r_{\sigma\tau} \to s_{\sigma\tau}) \tag{17}$$

Scalar functions $p_{\sigma\tau}, q_{\sigma\tau}, r_{\sigma\tau}, s_{\sigma\tau}, t_\sigma, u_\sigma, v_\sigma$ and w_σ of the variables $\vec{n}, z$ are given in Appendix A.

Continuing the previous line, take a scalar product of the Maxwell's equation (4) and the remaining vector $\vec{a}_t$. Substitution of the formulae already obtained for $\vec{a}_{z,1} \circ \vec{E}(\vec{\chi}, z), \vec{a}_{z,1} \circ \vec{H}(\vec{\chi}, z)$ into the relations arising yields the following result:

$$\begin{aligned}D_{ss}\mathcal{H}(\vec{\chi}, z) + D_{sp}\,\mathcal{E}(\vec{\chi}, z) &= (4\pi/c)\, q_s(\vec{\chi}, z) \\ -D_{ps}\mathcal{H}(\vec{\chi}, z) + D_{pp}\,\mathcal{E}(\vec{\chi}, z) &= (4\pi/c)\, q_p(\vec{\chi}, z)\end{aligned} \tag{18}$$

This is a fourth-order system of two coupled ordinary differential equations obeyed by scalar potentials over the interval $0 < z < +\infty$. Here $q_\nu(\vec{\chi}, z)$ depicts a source terms, $D_{\lambda\nu}$ is a Sturm-Liouville operator acting on z, $(\lambda, \nu = p, s)$. Their explicit expressions are:

$$\begin{aligned}q_s(\vec{\chi}, z) = \Big[&\vec{z}_0(\partial_z p_{lz} + i\chi p_{ll} - ik_0 b_z) \\ &- \vec{n}(\partial_z p_{zz} + i\chi p_{zl} + ik_0 b_l)\Big] \circ \vec{J}(\vec{\chi}, z) \\ &+ \Big[\vec{n}_0(\partial_z r_{zz} + i\chi r_{zl} + ik_0 d_l) \\ &- \vec{z}_0(\partial_z r_{lz} + i\chi r_{ll} - ik_0 d_z) - ik_0 \vec{z}_0 \times \vec{n}\Big] \circ \vec{M}(\vec{\chi}, z)\end{aligned} \tag{19}$$

$$q_s \to q_p, \quad (b_\sigma \to c_\sigma,\ d_\sigma \to f_\sigma,\, p_{\sigma\tau} \to q_{\sigma\tau},\\ r_{\sigma\tau} \to s_{\sigma\tau},\ \vec{J} \to -\vec{M},\ \vec{M} \to \vec{J}) \tag{20}$$

$$\begin{aligned} D_{ss} &= \partial_z p_{zz}\partial_z + i\chi(\partial_z p_{lz} + p_{zl}\partial_z) + ik_0(b_l\partial_z - \partial_z t_l) \\ &\quad + k_0^2\delta_\mu - \chi^2 p_{ll} + \chi k_0(b_z - t_z) \\ D_{sp} &= \partial_z r_{zz}\partial_z + i\chi(\partial_z r_{lz} + r_{zl}\partial_z) + ik_0(d_l\partial_z + \partial_z v_l) \\ &\quad + k_0^2\delta_\zeta - \chi^2 r_{ll} + \chi k_0(d_z + v_z) \end{aligned} \tag{21}$$

$$\begin{aligned} D_{ss} &\to D_{pp}, \quad (b_\sigma \to c_\sigma,\ t_\sigma \to u_\sigma,\ p_{\sigma\tau} \to q_{\sigma\tau},\ \delta_\mu \to \delta_\varepsilon) \\ D_{sp} &\to D_{ps}, \quad (d_\sigma \to f_\sigma,\ v_\sigma \to w_\sigma,\ r_{\sigma\tau} \to s_{\sigma\tau},\ \delta_\zeta \to \delta_\xi) \end{aligned} \tag{22}$$

Scalar functions $b_\sigma, c_\sigma\, d_\sigma$ and f_σ of the variables $\vec{n}, z$ are specified in Appendix A.

On assuming that $\vec{J}(\vec{\chi}, z_s \pm 0) = 0$, $\vec{M}(\vec{\chi}, z_s \pm 0) = 0$ by substituting expressions (13) in (5) we obtain at $z = z_s$

$$\begin{aligned} &\Big\{\mathcal{E}(\vec{\chi}, z)\Big\} = (4\pi/c)\vec{n} \circ \vec{M}_s(\vec{\chi}), \qquad \Big\{\mathcal{H}(\vec{\chi}, z)\Big\} = -(4\pi/c)\vec{n} \circ \vec{J}_s(\vec{\chi}) \\ &\Big\{\Big[v_l + k_0{}^{-1}(\chi r_{lz} - ir_{zz}\partial_z)\Big]\,\mathcal{E}(\vec{\chi}, z) \\ &\qquad - \Big[t_l + k_0{}^{-1}(ip_{zz}\partial_z - \chi p_{lz})\Big]\,\mathcal{H}(\vec{\chi}, z)\Big\} = -(4\pi/c)\vec{z}_0 \times \vec{n} \circ \vec{M}_s(\vec{\chi}) \\ &\Big\{\Big[w_l + k_0{}^{-1}(\chi s_{lz} - is_{zz}\partial_z)\Big]\,\mathcal{H}(\vec{\chi}, z) \\ &\qquad + \Big[u_l + k_0{}^{-1}(iq_{zz}\partial_z - \chi q_{lz})\Big]\,\mathcal{E}(\vec{\chi}, z)\Big\} = (4\pi/c)\vec{z}_0 \times \vec{n} \circ \vec{J}_s(\vec{\chi}) \end{aligned} \tag{23}$$

These relations constitute boundary conditions to be satisfied by scalar potentials on the interface supporting the impressed surface currents. Similar relations with the zero right-hand sides hold at any source-free interface. Within the present context, an interface formally represents a surface where at least one of the coefficients in the equations (18) exhibits a discontinuity.

With the substitution of (13), the impedance boundary condition (6) transforms into a pair of relations linking scalar potentials and their derivatives at $z = +0$:

$$\begin{aligned} &\Big[ik_0 a_{ss} + b_{ss}\partial_z\Big]\mathcal{H}(\vec{\chi}, z) + \Big[ik_0 a_{sp} + b_{sp}\partial_z\Big]\mathcal{E}(\vec{\chi}, z) = 0 \\ &\Big[ik_0 a_{ps} + b_{ps}\partial_z\Big]\mathcal{H}(\vec{\chi}, z) + \Big[ik_0 a_{pp} + b_{pp}\partial_z\Big]\mathcal{E}(\vec{\chi}, z) = 0 \end{aligned} \tag{24}$$

Here it is implied that $\vec{J}(\vec{\chi}, +0) = 0$, $\vec{M}(\vec{\chi}, +0) = 0$. The definition of the dimensionless coefficients $a_{\lambda\nu}$, $b_{\lambda\nu}$, $(\lambda, \nu = p, s)$ is given in Appendix B. Thus, a reduction of the vectorial EM problem to a pair of coupled differential equations involving two scalar potentials has been achieved.

On taking into account the explicit expressions for the coefficients encountered in (14)–(17) and (19)–(23), one can easily verify that the scalarized representation (13) has the property, and the equations (18) and (23) preserve their form under the following transformations:

$$\begin{aligned} &\mathcal{E} \to \mathcal{H}, \quad \mathcal{H} \to -\mathcal{E} \\ &\vec{J} \to \vec{M}, \quad \vec{J}_s \to \vec{M}_s, \quad \vec{M} \to -\vec{J}, \quad \vec{M}_s \to -\vec{J}_s \\ &\hat{\varepsilon} \leftrightarrow \hat{\mu}, \quad \hat{\xi} \leftrightarrow \hat{\zeta} \end{aligned} \tag{25}$$

The proposed formulation provides a unified description of a discrete-layered and smoothly-inhomogeneous medium and is quite general in the sense that it embraces the basic EM problems for a regular medium, that is of wave excitation, natural mode propagation and plane-wave diffraction. The inspection of coefficients in Appendices A and B, which figure in the scalarized representation (13) as well as in the equations (18), (23) and (24), shows that they are completely independent of a particular coordinate system which is used to specify the dyadic quantities $\hat{\varepsilon}, \hat{\mu}, \hat{\xi}, \hat{\zeta}$ and $\hat{L}(\vec{\chi})$. Within the matrix formulations used in the past, the analytical investigation of the problem, as well as the interpretation of the results obtained, is a complicated task. As it is shown below, the scalarized problem lends itself pretty easily to further analytical investigation. For an isotropic plane-layered medium, the results of this Section coincide with those previously obtained [16].

3. Dyadic Green's Functions

In this Section we are confronted with the derivation and analysis of the spectral dyadic Green's functions $\hat{G}_{\alpha\beta}(\vec{\chi}, z, z')$, $(\alpha, \beta = e, m)$, for the volume sources under the assumption that all interfaces are source-free (i.e., $\vec{J}_s = \vec{M}_s \equiv 0$) and no impressed sources residing at $z = +\infty$ exist. The solution to the corresponding excitation problem can be easily achieved with aid of $\hat{G}_{\alpha\beta}(\vec{\chi}, z, z')$ in the following form:

$$\begin{aligned}
&\vec{E}(\vec{\chi},z) \\
&\quad = \int_0^{+\infty} dz' \left[\hat{G}_{ee}(\vec{\chi}, z, z') \circ \vec{J}(\vec{\chi}, z') + \hat{G}_{em}(\vec{\chi}, z, z') \circ \vec{M}(\vec{\chi}, z')\right] \\
&\vec{H}(\vec{\chi},z) \\
&\quad = \int_0^{+\infty} dz' \left[\hat{G}_{me}(\vec{\chi}, z, z') \circ \vec{J}(\vec{\chi}, z') + \hat{G}_{mm}(\vec{\chi}, z, z') \circ \vec{M}(\vec{\chi}, z')\right]
\end{aligned} \tag{26}$$

The spatial dyadic Green's functions $\hat{G}_{\alpha\beta}(\vec{R}, \vec{R}')$ are connected to their spectral counterparts via inverse double Fourier transform:

$$\hat{G}_{\alpha\beta}(\vec{R}, \vec{R}') = (2\pi)^{-2} \int d\vec{\chi} \exp\left[i\vec{\chi} \circ (\vec{r} - \vec{r}')\right] \hat{G}_{\alpha\beta}(\vec{\chi}, z, z') \tag{27}$$

Below, an efficient analytical technique of deriving the spectral dyadic Green's functions is presented, their singular behavior at a source plane is revealed, and the useful expressions for the spectral and spatial Green's functions in case of a point source positioned inside a bianisotropic slab are obtained.

We start from the equations (18), (23) and (24) for scalar potentials with the source-free boundary conditions at the interfaces: $\vec{J}_s = \vec{M}_s \equiv 0$. The solution $\mathcal{H}, \mathcal{E}$ to the corresponding excitation problem should exhibit an outgoing wave behavior with $z \to +\infty$. Suppose that we have a pair of solutions $\mathcal{H}(\vec{\chi}, z) \equiv G_{s\nu}(\vec{\chi}, z, z')$, $\mathcal{E}(\vec{\chi}, z) \equiv G_{p\nu}(\vec{\chi}, z, z')$, $(\nu = p, s)$, to the abovementioned problem with the source terms

$$\begin{aligned}
q_p(\vec{\chi}, z) = (c/4\pi)\delta(z - z'), \quad q_s(\vec{\chi}, z) \equiv 0, \quad (\nu = p) \\
q_s(\vec{\chi}, z) = (c/4\pi)\delta(z - z'), \quad q_p(\vec{\chi}, z) \equiv 0, \quad (\nu = s)
\end{aligned} \tag{28}$$

where $\delta(z-z')$ is the 1D Dirac delta function. From now on the scalar Green's functions $G_{\lambda\nu}(\vec{\chi},z,z')$, $(\lambda,\nu = p,s)$, are assumed to be at our disposal in the explicit form. Owing to linearity of the problem for $\mathcal{H},\mathcal{E}$ the following representation of its solution in case of arbitrary sources $q_{p,s}$ is valid:

$$\begin{aligned}
&\mathcal{E}(\vec{\chi},z)\\
&\quad = (4\pi/c)\int_0^{+\infty} dz'\Big[G_{pp}(\vec{\chi},z,z')q_p(\vec{\chi},z') + G_{ps}(\vec{\chi},z,z')q_s(\vec{\chi},z')\Big]\\
&\mathcal{H}(\vec{\chi},z)\\
&\quad = (4\pi/c)\int_0^{+\infty} dz'\Big[G_{sp}(\vec{\chi},z,z')q_p(\vec{\chi},z') + G_{ss}(\vec{\chi},z,z')q_s(\vec{\chi},z')\Big]
\end{aligned} \tag{29}$$

We are now in a position to develop the spectral Green's dyads $\hat{G}_{\alpha\beta}(\vec{\chi},z,z')$. On assigning the distributional sense [25] to the differentiation with respect to z and the integration over z', interchange them and take into account the definition (19),(20) of $q_{p,s}$. When comparing the result with representation (26), it can be easily seen that

$$\begin{aligned}
\hat{G}_{ee}(\vec{\chi},z,z') &= (4\pi i/ck_0)\,\hat{\alpha}_{ee}(\vec{n},z')\,\delta(z-z')\\
&\quad + (4\pi i k_0/c)\Big[\vec{v}_e(\tilde{\vec{v}}_e G_{pp} - \tilde{\vec{w}}_e G_{ps}) + \vec{w}_e(\tilde{\vec{w}}_e G_{ss} - \tilde{\vec{v}}_e G_{sp})\Big],
\end{aligned} \tag{30}$$

$$\begin{aligned}
\hat{G}_{em}(\vec{\chi},z,z') &= -(4\pi i/ck_0)\,\hat{\alpha}_{em}(\vec{n},z')\,\delta(z-z')\\
&\quad + (4\pi i k_0/c)\Big[\vec{v}_e(\tilde{\vec{v}}_m G_{ps} + \tilde{\vec{w}}_m G_{pp}) - \vec{w}_e(\tilde{\vec{v}}_m G_{ss} + \tilde{\vec{w}}_m G_{sp})\Big]\\
\hat{G}_{me}(\vec{\chi},z,z') &= (4\pi i/ck_0)\,\hat{\alpha}_{me}(\vec{n},z')\,\delta(z-z')\\
&\quad + (4\pi i k_0/c)\Big[\vec{v}_m(\tilde{\vec{v}}_e G_{sp} - \tilde{\vec{w}}_e G_{ss}) + \vec{w}_m(\tilde{\vec{v}}_e G_{pp} - \tilde{\vec{w}}_e G_{ps})\Big]\\
\hat{G}_{mm}(\vec{\chi},z,z') &= (4\pi i/ck_0)\,\hat{\alpha}_{mm}(\vec{n},z')\,\delta(z-z')\\
&\quad + (4\pi i k_0/c)\Big[\vec{v}_m(\tilde{\vec{v}}_m G_{ss} + \tilde{\vec{w}}_m G_{sp}) + \vec{w}_m(\tilde{\vec{v}}_m G_{ps} + \tilde{\vec{w}}_m G_{pp})\Big]
\end{aligned}$$

Here $\hat{\alpha}_{ee},\ldots\alpha_{mm}$, $\vec{w}_\beta$ and $\vec{v}_\beta$ are introduced in (14)–(17), $\tilde{\vec{w}}_\beta$ and $\tilde{\vec{v}}_\beta$ are vector differential operators acting on z' and depending on $\chi,\vec{n}$:

$$\tilde{\vec{w}}_e = \vec{n}b'_l + \vec{z}_0 b'_z + k_0{}^{-1}\Big[\chi(\vec{n}p'_{zl} - \vec{z}_0 p'_{ll}) + i(\vec{n}p'_{zz} - \vec{z}_0 p'_{lz})\partial'_z\Big] \tag{31}$$

$$\begin{aligned}\tilde{\vec{v}}_e &= -\vec{z}_0 \times \vec{n} + \vec{n}f'_l + \vec{z}_0 f'_z \\ &\quad + k_0{}^{-1}\Big[\chi(\vec{n}s'_{zl} - \vec{z}_0 s'_{ll}) + i(\vec{n}s'_{zz} - \vec{z}_0 s'_{lz})\partial'_z\Big]\end{aligned}$$

$$\begin{aligned}\tilde{\vec{w}}_e &\to \tilde{\vec{w}}_m, \quad (b_\sigma \to c_\sigma, \quad p_{\sigma\tau} \to q_{\sigma\tau}) \\ \tilde{\vec{v}}_e &\to \tilde{\vec{v}}_m, \quad (f_\sigma \to d_\sigma, \quad s_{\sigma\tau} \to r_{\sigma\tau})\end{aligned} \tag{32}$$

In these relations and later we employ a convention that for an arbitrary function $f(z)$ of variable z the notation f' means the function $f(z')$ of variable z' (but not the derivative of f!), $\partial'_z = \partial/\partial z'$. The formulae (30) constitute a compact scalarized representation for the spectral dyadic Green's functions through four scalar Green's functions. They should be supported with the method of deriving the latter. The scalar Green's functions can be expressed through a proper set of linearly independent solutions to a source-free problem for $\mathcal{H}, \mathcal{E}$ in full analogy with the case of an isotropic plane-layered medium [12]. We defer further discussion of this item. Obviously, it is a key to successful application of the relations (30).

It is important to emphasize that the derivatives ∂_z, ∂'_z appearing in (30) are understood in the distributional sense [25]. This is only essential with regard to the quantities $\partial^2 G_{\lambda\nu}/\partial z \partial z'$, Each of them contains the usual derivative signified by proper index and the Dirac delta function:

$$\frac{\partial^2 G_{\lambda\nu}(\vec{\chi}, z, z')}{\partial z \partial z'} = \frac{\partial^2 G_{\lambda\nu}(\vec{\chi}, z, z')}{\partial z \partial z'}_{|usu} - \delta(z - z') h_{\lambda\nu}(\vec{\chi}, z') \tag{33}$$

where

$$h_{\lambda\nu}(\vec{\chi}, z') = \frac{\partial G_{\lambda\nu}(\vec{\chi}, z, z')}{\partial z}\Bigg|_{z=z'-0}^{z=z'+0} \tag{34}$$

This claim is easily verified by applying a well-known rule [25] of differentiating a function of the variable z' with a step-like discontinuity at $z' = z$. If we integrate the differential equations for $\mathcal{H}(\vec{\chi}, z) \equiv G_{s\nu}(\vec{\chi}, z, z')$, $\mathcal{E}(\vec{\chi}, z) \equiv G_{p\nu}(\vec{\chi}, z, z')$, $(\nu = p, s)$, over z from $z' - \eta$ to $z' + \eta$, $(\eta > 0)$, and let $\eta \to 0$ then we obtain a system of linear algebraic equations for $h_{s\nu}, h_{p\nu}$. A solution for $h_{s\nu}, h_{p\nu}$ is

$$\begin{aligned} h_{ss}(\vec{\chi}, z) &= Q_{zz}\,\Delta/\tilde{\Delta}, \qquad & h_{ps}(\vec{\chi}, z) &= S_{zz}\,\Delta/\tilde{\Delta} \\ h_{sp}(\vec{\chi}, z) &= -R_{zz}\,\Delta/\tilde{\Delta}, \qquad & h_{pp}(\vec{\chi}, z) &= P_{zz}\,\Delta/\tilde{\Delta} \end{aligned} \tag{35}$$

The quantities in the right-hand side of these expressions are defined in Appendix A. On collecting delta-functions in (30) one arrives at a conclusion that each of the spectral Green's dyads contains a regular term labeled with the index rg and a singular term proportional to $\delta(z - z')$:

$$\hat{G}_{\alpha\beta}(\vec{\chi}, z, z') = \hat{G}^{rg}_{\alpha\beta}(\vec{\chi}, z, z') + \ \vec{z}_0\vec{z}_0\,\delta(z - z')\,A_{\alpha\beta}(z')(4\pi/ik_0c) \tag{36}$$

$(\alpha, \beta = e, m)$. The regular constituent $\hat{G}^{rg}_{\alpha\beta}(\vec{\chi}, z, z')$ is formally obtainable from the expressions (30) for $\hat{G}_{\alpha\beta}(\vec{\chi}, z, z')$ after letting out the first terms with the Dirac deltas and replacing, in the remaining terms, the generalized derivatives with the usual ones. The quantity $\hat{G}^{rg}_{\alpha\beta}(\vec{\chi}, z, z')$ is not defined at $z = z'$ although it has finite limiting values at $z \to z' \pm 0$. The quantities $A_{\alpha\beta}$ are given by

$$\begin{aligned} A_{ee}(z') &= \mu_{zz}(z')/a(z'), \qquad & A_{em}(z') &= -\xi_{zz}(z')/a(z') \\ A_{me}(z') &= \zeta_{zz}(z')/a(z'), \qquad & A_{mm}(z') &= \varepsilon_{zz}(z')/a(z') \end{aligned} \tag{37}$$

$$a(z') = \varepsilon_{zz}(z')\mu_{zz}(z') + \xi_{zz}(z')\zeta_{zz}(z')$$

In these formulae $\mu_{zz}, \ldots, \zeta_{zz}$ are the components of the respective constitutive dyads $\hat{\mu}, \ldots, \hat{\zeta}$ along the axis of stratification (see Appendix A). We should point out that the regular and singular terms in (36) are defined in a unique way in obvious contrast to the spatial dyadic Green's functions (for references concerning magnetodielectric media see e.g. [26]). Physically, the relations in (36) reveal singular behavior of the spectral Green's dyads at a source plane. From a computational viewpoint, they enable one to evaluate explicitly a contribution due to the singular terms in (26), thus reducing the remaining operation to integration in usual sense which can be performed by standard numerical means.

We now proceed with the development of the dyadic Green's functions in the practically important case where an inhomogeneous bianisotropic medium occupies a slab $0 < z < b$ of finite thickness b, the halfspace $b < z < +\infty$ is filled with an isotropic magnetodielectric characterized by constant permittivity ε and permeability μ, the source point is located within the slab $(0 < z' < b)$, and the observation point is situated in the adjacent halfspace $(b < z < +\infty)$ — see Figure 2.

Accounting for the suitable form of the operators in (14) and (15) and that of $G_{\lambda\nu}(\vec{\chi}, z, z')$ leads to the following expressions for the spectral dyadic Green's functions:

$$\hat{G}_{e\beta}(\vec{\chi}, z, z') = (4\pi i/c)\Big[\vec{z}_0 \times \vec{n}\vec{a}_p^{\beta}(\vec{\chi}, z') \\ + (\gamma\vec{n} - \vec{z}_0\chi)\, \vec{a}_s^{\beta}(\vec{\chi}, z')/k_0\varepsilon\Big] \exp\Big[i\gamma(z-b)\Big] \tag{38}$$

$$\hat{G}_{m\beta}(\vec{\chi}, z, z') = (4\pi i/c)\Big[\vec{z}_0 \times \vec{n}\vec{a}_s^{\beta}(\vec{\chi}, z') \\ + (\vec{z}_0\chi - \vec{n}\gamma)\, \vec{a}_p^{\beta}(\vec{\chi}, z')/k_0\mu\Big] \exp\Big[i\gamma(z-b)\Big] \\ (0 < z' < b < z < +\infty)$$

Here

$$\beta = e, m, \quad \gamma \equiv \gamma(\chi) = (k^2 - \chi^2)^{1/2}, \quad k^2 = k_0^2\epsilon\mu \tag{39}$$

and the branch of the square root in the definition of γ is chosen to satisfy the condition $0 \le \arg\gamma < \pi$. The vectors $\vec{a}_{\lambda}^{\beta}$ have the dimensionless Cartesian coordinates. Physically, they determine the $p-$, or horizontally polarized constituent $(\vec{a}_p^{\beta})$ and the $s-$, or vertically polarized constituent $(\vec{a}_s^{\beta})$ of the EM field created in the adjacent halfspace by the source of the electric $(\beta = e)$ or the magnetic $(\beta = m)$ type immersed into the bianisotropic slab. In the explicit form they are given as

$$\vec{a}_{\lambda}^{e}(\vec{\chi}, z')/k_0 = \tilde{\vec{v}}_e G_{\lambda p}(\vec{\chi}, b-0, z') - \tilde{\vec{w}}_e G_{\lambda s}(\vec{\chi}, b-0, z') \\ \vec{a}_{\lambda}^{m}(\vec{\chi}, z')/k_0 = \tilde{\vec{v}}_m G_{\lambda s}(\vec{\chi}, b-0, z') + \tilde{\vec{w}}_m G_{\lambda p}(\vec{\chi}, b-0, z'), \quad (\lambda = p, s) \tag{40}$$

The approximate analytical form of $\hat{G}_{\alpha\beta}(\vec{R}, \vec{R}')$ for an observation point in the lossless adjacent space and the point source within the bianisotropic slab may be found in the usual manner by the asymptotic evaluation of an integral in (27) via the saddle point method [12]. With this purpose, introduce a spherical coordinate system L, θ, φ with origin at the point $\vec{R}_0 = (x', y', b)$ on the outer boundary of the slab (see Figure 2):

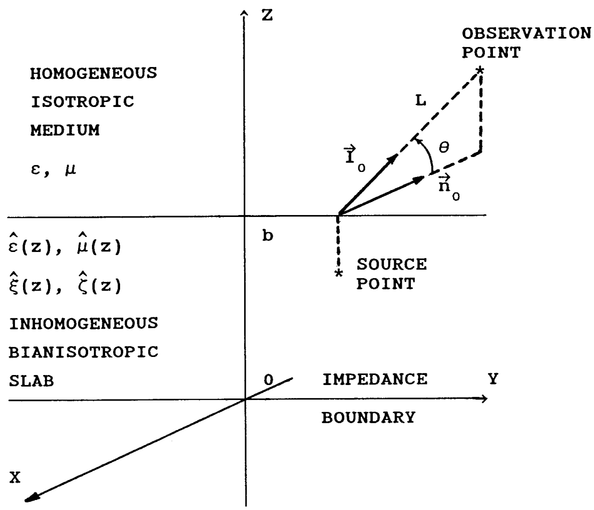

Figure 2. Geometry for the radiation from a slab.

$$x - x' = L\cos\theta\cos\varphi, \quad y - y' = L\cos\theta\sin\varphi, \quad z - b = L\sin\theta \quad (41)$$

Here L is the distance between the origin $\vec{R}_0$ and the observation point $\vec{R}$: $L = [(r - r')^2 + (z - b)^2]^{1/2}$, θ and ϕ are, respectively, the sloping angle and the azimuth angle of the observation point. The orthonormal basis set $\vec{l}_0, \vec{\theta}_0, \vec{\phi}_0$ is defined by the following expressions:

$$\begin{aligned} &\vec{l}_0(\vec{R}) = \left[\vec{r} - \vec{r}' + \vec{z}_0(z - b)\right]/L = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ sin\theta) \\ &\vec{\theta}_0(\vec{R}) = \vec{l}_0 \times \vec{\varphi}, \qquad \vec{\varphi}_0(\vec{r}) = \vec{z}_0 \times \vec{n}_0 \end{aligned} \quad (42)$$

where

$$\vec{n}_0(\vec{r}) = (\vec{r} - \vec{r}')/|\vec{r} - \vec{r}'| = (\cos\varphi,\ \sin\varphi,\ 0) \quad (43)$$

On assuming that $kL \gg 1$, $0 < \theta \leq \pi/2$ and neglecting the natural mode contribution in the radiated field (i.e., the contribution due to eigenmodes and leaky modes of the bianisotropic slab) from (27) and (38) after standard manipulations [12] one gets the desired result:

$$\begin{aligned}
&\hat{G}_{e\beta}(\vec{R},\vec{R}') \\
&\qquad = -w\vec{l}_0(\vec{R}) \times \hat{G}_{m\beta}(\vec{R},\vec{R}') \\
&\qquad = \sin\theta\Big[\vec{\varphi}_0\, \vec{a}_p^{\beta}(\vec{\chi}_0,z') - w\vec{\theta}_0\, \vec{a}_s^{\beta}(\vec{\chi}_0,z')\Big] \exp(ikL)(2k/cL) \\
&\hat{G}_{m\beta}(\vec{R},\vec{R}') \\
&\qquad = w^{-1}\vec{l}_0(\vec{R}) \times \hat{G}_{e\beta}(\vec{R},\vec{R}') \\
&\qquad = \sin\theta\Big[\vec{\varphi}_0\, \vec{a}_s^{\beta}(\vec{\chi}_0,z') + w^{-1}\vec{\theta}_0\, \vec{a}_p^{\beta}(\vec{\chi}_0,z')\Big] \exp(ikL)(2k/cL)
\end{aligned} \tag{44}$$

$(\beta = e, m)$. In (44) $w = (\mu/\varepsilon)^{1/2}$ is the impedance of the adjacent medium, $\vec{\chi}_0 = \chi_0\vec{n}_0$, $\vec{n}_0$ is defined in (43) and $\chi_0 = k\cos\theta$. The quantities $\vec{a}_\lambda^\beta$ retain here the same physical contents as in the preceding relations (38).

The synthesis of the dyadic Green's functions in case where both the source point and the observation point are located in the adjacent halfspace can be easily achieved by utilizing the dyadic impedance $\hat{L}(\vec{\chi}, b)$ for the outer boundary of the slab which is developed in Section 5 and by subsequent use of the dyadic Green's functions for a halfspace bounded by an anisotropic plane with the nonlocal impedance [27,28]. (Note that the results of [28] should be slightly modified for our purposes to account for the dependence of the impedance on $\vec{\chi}$), We omit the analysis of this and other variants in the location of the observation and the source points for brevity's sake.

By construction, the expressions for the dyadic Green's functions in this section are applicable to 3D sources localized outside the interfaces. Hence, they contain the terms which play no part in case of a planar excitation. Besides, when the location of the excitation is in the interface, say $z = z_s$, a limiting operation $z' \to z_s \pm 0$ should be performed. It is more convenient to account for the planar excitation in an interface from the outset. This idea is implemented in the next section.

4. Planar Excitation in the Interface

We return to the general model of a plane-layered bianisotropic medium outlined in Introduction and consider the excitation problem for $\vec{E}(\vec{\chi}, z)$, $\vec{H}(\vec{\chi}, z)$ with a planar excitation in the interface. It is assumed that the volume sources are absent $(\vec{J} = \vec{M} \equiv 0)$, and the only sources available are the surface sources $\vec{J}_s$, $\vec{M}_s$ streaming over the interface $z = z_s$. A proper solution for the vector field amplitudes which exhibits an outgoing wave behavior with $z \to +\infty$ can be obtained via the following relation:

$$\begin{aligned}\vec{E}(\vec{\chi}, z) &= \hat{G}^s_{ee}(\vec{\chi}, z) \circ \vec{J}_s(\vec{\chi}) + \hat{G}^s_{em}(\vec{\chi}, z) \circ \vec{M}_s(\vec{\chi}) \\ \vec{H}(\vec{\chi}, z) &= \hat{G}^s_{me}(\vec{\chi}, z) \circ \vec{J}_s(\vec{\chi}) + \hat{G}^s_{mm}(\vec{\chi}, z) \circ \vec{M}_s(\vec{\chi})\end{aligned} \tag{45}$$

where $\hat{G}^s_{\alpha\beta}(\vec{\chi}, z)$, $(\alpha, \beta = e, m)$ are the spectral dyadic Green's functions accounting for planar sources at the interface $z = z_s$. If the inverse Fourier transformation is applied to $\hat{G}^s_{\alpha\beta}(\vec{\chi}, z)$, one gets the spatial counterpart

$$\hat{G}^s_{\alpha\beta}(\vec{R}, \vec{r}') = (2\pi)^{-2} \int d\vec{\chi} \exp\left[i\vec{\chi} \circ (\vec{r} - \vec{r}')\right] \hat{G}^s_{\alpha\beta}(\vec{\chi}, z') \tag{46}$$

The knowledge of $\hat{G}^s_{\alpha\beta}(\vec{\chi}, z)$, $\hat{G}^s_{\alpha\beta}(\vec{R}, \vec{r}')$ is of prior importance in the study of EM problems for bianisotropic structures loaded by strips, patches, slots etc.

The aim of this Section is to develop, for a general bianisotropic structure, the dyadic Green's functions $\hat{G}^s_{\alpha\beta}(\vec{\chi}, z)$ in the spectral domain and to evaluate the spatial Green's functions in the far zone of a point source located in the interface within a bianisotropic slab.

The main tools in our analysis are two linearly independent solutions $\mathcal{H} = \mathcal{H}^-_j, \mathcal{E} = \mathcal{E}^-_j, (j = 1, 2)$, to source-free $(q_{p,s} \equiv 0,\ \vec{J}_s = \vec{M}_s \equiv 0)$ equations (18), (23) which obey boundary conditions (24), and the other two linearly independent solutions $\mathcal{H} = \mathcal{H}^+_j, \mathcal{E} = \mathcal{E}^+_j, (j = 1, 2)$, to the same equations which depict an outgoing wave at $z \to +\infty$. It should be remarked that the functions $\mathcal{H}^-_j, \mathcal{E}^-_j$ are not expected to satisfy the radiation condition with $z \to +\infty$. Similarly, $\mathcal{H}^+_j, \mathcal{E}^+_j$ in general do not obey boundary conditions (24) at $z = +0$. On inserting the functions $\mathcal{H}^\rho_j, \mathcal{E}^\rho_j, (\rho = +, -)$, into the source-free representation (13) one arrives at the vector functions

$$\begin{aligned}\vec{E}^\rho_j(\vec{\chi}, z) &= \vec{v}_e\, \mathcal{E}^\rho_j(\vec{\chi}, z) - \vec{w}_e\, \mathcal{H}^\rho_j(\vec{\chi}, z) \\ \vec{H}^\rho_j(\vec{\chi}, z) &= \vec{v}_m\, \mathcal{H}^\rho_j(\vec{\chi}, z) + \vec{w}_m\, \mathcal{E}^\rho_j(\vec{\chi}, z)\end{aligned} \tag{47}$$

$(j = 1, 2)$, to be used further on. For our purposes, the knowledge of $\mathcal{H}_j^\rho, \mathcal{E}_j^\rho$ over the respective intervals $0 < z < z_s, (\rho = -)$, and $z_s < z < +\infty, (\rho = +)$, is sufficient. With these definitions at hand, one can express $\hat{G}_{\alpha\beta}(\vec{\chi}, z)$ in a concise form as:

$$\begin{aligned}
\hat{G}^s_{e\beta}(\vec{\chi}, z) &= \pm(4\pi/c) \sum_{j=1}^{2} \vec{E}_j^\pm(\vec{\chi}, z)\, Q_{\beta j}^\pm(\vec{\chi}) \\
\hat{G}^s_{m\beta}(\vec{\chi}, z) &= \pm(4\pi/c) \sum_{j=1}^{2} \vec{H}_j^\pm(\vec{\chi}, z)\, Q_{\beta j}^\pm(\vec{\chi})
\end{aligned} \tag{48}$$

where the signs $+, -$ refer to the cases $z > z_s$ and $z < z_s$, respectively. For arbitrary $\vec{Q}_{\beta j}^\rho$ the quantities $\vec{E}(\vec{\chi}, z)$, $\vec{H}(\vec{\chi}, z)$ in (45) satisfy all the equations for vector field amplitudes except for the boundary condition (5) at a source-carrying interface. For (5) to hold, the quantities $\vec{Q}_{\beta j}^\rho$ must be appropriately chosen. We relegate their explicit representations to Appendix C.

As a concluding step, let us consider an inhomogeneous bianisotropic slab which is adjacent to an isotropic homogeneous magnetodielectric halfspace. This model has been described earlier in conjunction with formulae (38). In the present case of a source-carrying interface, one has for the observation point in the adjacent halfspace $(b < z < +\infty)$:

$$\begin{aligned}
&\hat{G}^s_{e\beta}(\vec{\chi}, z) \\
&\quad = (4\pi/c)\Big[\vec{z}_0 \times \vec{n}\, \vec{b}_p^\beta(\vec{\chi}) + (\vec{n}\gamma - \vec{z}_0\chi)\, \vec{b}_s^\beta(\vec{\chi})/k_0\varepsilon\Big] \exp\Big[i\gamma(z-b)\Big] \\
&\hat{G}^s_{m\beta}(\vec{\chi}, z) \\
&\quad = (4\pi/c)\Big[\vec{z}_0 \times \vec{n}\, \vec{b}_s^\beta(\vec{\chi}) + (\vec{z}_0\chi - \vec{n}\gamma)\, \vec{b}_p^\beta(\vec{\chi})/k_0\mu\Big] \exp\Big[i\gamma(z-b)\Big]
\end{aligned} \tag{49}$$

Here $\beta = e, m, \vec{b}_{p,s}^\beta$ are vectors with the dimensionless Cartesian coordinates, γ is given by (39). The quantities $\vec{b}_p^\beta$ and $\vec{b}_s^\beta$ determine, respectively, the p- (horizontally) and the s- (vertically) polarized constituents of the EM field which is excited in the adjacent medium by the tangential surface currents of the electric $\beta = e)$ or the magnetic $(\beta = m)$ type distributed over the interface $z = z_s$. In the explicit form they read as :

$$
\begin{aligned}
\vec{b}_p^{\beta}(\vec{\chi}) &= \sum_{j=1}^{2} \mathcal{E}_j^{+}(\vec{\chi}, b+0) Q_{\beta j}^{+}(\vec{\chi}) \\
\vec{b}_s^{\beta}(\vec{\chi}) &= \sum_{j=1}^{2} \mathcal{H}_j^{+}(\vec{\chi}, b+0) Q_{\beta j}^{+}(\vec{\chi})
\end{aligned}
\tag{50}
$$

$(\beta = e, m)$. As to spatial Green's functions $\hat{G}_{\alpha\beta}(\vec{R}, \vec{r}')$, we can evaluate integrals in the right-hand side of (46) via the saddle point method [12] when the observation point lies in the lossless adjacent medium $(0 < \theta \leq \pi/2)$ and $kL \gg 1$. We use here the same notations as earlier in connection with the formulae (41)–(44). In the case under consideration the spatial Green's functions have the following asymptotic behavior:

$$
\begin{aligned}
\hat{G}_{e\beta}^{s}(\vec{R}, \vec{r}') &= -w\vec{l}_0(\vec{R}) \times \hat{G}_{m\beta}^{s}(\vec{R}, \vec{r}') \\
&= \sin\theta \left[\vec{\varphi}_0\, \vec{b}_p^{\beta}(\vec{\chi}_0) - w\vec{\theta}_0 \vec{b}_s^{\beta}(\vec{\chi}_0)\right] \exp(ikL)(2k/icL) \\
\hat{G}_{m\beta}^{s}(\vec{R}, \vec{r}') &= w^{-1}\vec{l}_0(\vec{R}) \times \hat{G}_{e\beta}^{s}(\vec{R}, \vec{r}') \\
&= \sin\theta \left[\vec{\varphi}_0\, \vec{b}_s^{\beta}(\vec{\chi}_0) + w^{-1}\vec{\theta}_0 \vec{b}_p^{\beta}(\vec{\chi}_0)\right] \exp(ikL)(2k/icL)
\end{aligned}
\tag{51}
$$

$(\beta = e, m)$. The preceding physical interpretation of the quantities $\vec{b}_\lambda^{\beta}$ holds here as well with the clear reservation that this time one has to deal with the point source located at $\vec{R}' = \vec{R}_0 = (x', y', z_s)$.

The key element of the proposed scheme as well as that from the next Section is the set of proper solutions $\mathcal{H}_j^{\pm}, \mathcal{E}_j^{\pm}$ to a source-free problem. In inhomogeneous regions they can be found analytically with the aid of WKBJ approximation or numerically via finite difference or finite element techniques. In homogeneous regions both the aforementioned numerical methods as well as the standard analytical technique for solving a system of ordinary differential equations with constant coefficients are applicable.

5. Plane-wave Diffraction and Natural Mode Propagation Problems

The goal of the present Section is to provide analytical representations to the solutions of the aforementioned problems involving an inhomogeneous bianisotropic slab. This is achieved in a concise and elegant manner by invoking the concept of the non-local impedance for a penetrable interface (see, e.g., [18,19]).

We begin with an assumption that there exist no impressed sources within the region $0 < z < z_s$. Then the scalar potentials which describe the EM field due to impressed sources localized in the region $z_s \le z < +\infty$ are representable in the ranges $0 < z < z_s$ as

$$\begin{aligned} \mathcal{E}(\vec{\chi}, z) &= T_1 \mathcal{E}_1^-(\vec{\chi}, z) + T_2 \mathcal{E}_2^-(\vec{\chi}, z) \\ \mathcal{H}(\vec{\chi}, z) &= T_1 \mathcal{H}_1^-(\vec{\chi}, z) + T_2 \mathcal{H}_2^-(\vec{\chi}, z) \end{aligned} \tag{52}$$

In these expressions, the functions $\mathcal{E}_j^-(\vec{\chi}, z)$, $\mathcal{H}_j^-(\vec{\chi}, z)$ have been introduced in Section 4, $T_j \equiv T_j(\chi)$ are the coefficients determined by the impressed sources in the region $z_s \le z < +\infty, (j = 1, 2)$. On excluding these coefficients from the conjunction conditions (23) at $z = z_s$ one arrives at the "one-sided" boundary conditions for scalar potentials (and their first derivatives with respect to z) at $z = z + 0$. These relations are equivalent to an impedance boundary condition for the vector field amplitudes at $z = z_s + 0$

$$\vec{E}_\tau - \hat{L}(\vec{\chi}, z_s) \circ \left[\vec{z}_0 \times \vec{H}_\tau - (4\pi/c)\vec{J}_s \right] = (4\pi/c)\vec{M}_s \tag{53}$$

with the equivalent dyadic impedance

$$\hat{L}(\chi, z_s) = \sum_{\sigma,\tau=1,t} \vec{a}_\sigma \vec{a}_\tau L_{\sigma\tau}(\vec{\chi}, z_s) \tag{54}$$

The coordinates $L_{\sigma\tau}(\vec{\chi}, z_s)$ of the dyadic impedance $\hat{L}(\vec{\chi}, z_s)$ in the basis set $\vec{a}_l, \vec{a}_t$ have the following compact form:

$$L_{\sigma\tau}(\vec{\chi}, z_s) = \left. \frac{E_{\sigma 1}^-(\vec{\chi}, z) H_{\tau 2}^-(\vec{\chi}, z) - E_{\sigma 2}^-(\vec{\chi}, z) H_{\tau 1}^-(\vec{\chi}, z)}{H_{l1}^-(\vec{\chi}, z) H_{t2}^-(\vec{\chi}, z) - H_{l2}^-(\vec{\chi}, z) H_{t1}^-(\vec{\chi}, z)} \right|_{z=z_s-0} \tag{55}$$

where $\sigma, \tau = l, t,$ and the quantities $E^{-}_{\sigma j},\ H^{-}_{\sigma j},\ \ (j = l, 2)$, are defined in Appendix C. The dependence of $L(\vec{\chi}, z_s)$ on $\vec{\chi}$ manifested in (54), (55) means that the penetrable boundary $z = z_s$ can be treated as an impedance boundary possessing spatial dispersion [18,19]. Note that $\hat{L}(\vec{\chi}, z_s)$ is determined by the properties of the medium in the region $0 < z < z_s$ and does not depend upon the medium's properties in the region $z_s < z < +\infty$.

From now on we take that the interface $z = z_s$ separates an inhomogeneous bianisotropic slab $(0 < z < z_s)$ and an isotropic homogeneous magnetodielectric halfspace $(z_s < z < +\infty)$. This model coincides with the one that has been described earlier in connection with the formulae (38) if one sets $z_s = b$. In what follows we will use the previous notations referring to this structure without repetition of their definitions. Also, we shall consider a source-free problem, with the impressed sources absent: $\vec{J} = \vec{M} \equiv 0$. Then the scalar potentials pertaining to the region $b < z < +\infty$ can be written as

$$\begin{aligned} \mathcal{E}(\vec{\chi}, z) &= A_p \exp\Big[-i\gamma(z-b)\Big] + B_p \exp\Big[i\gamma(z-b)\Big] \\ \mathcal{H}(\vec{\chi}, z) &= A_s \exp\Big[-i\gamma(z-b)\Big] + B_s \exp\Big[i\gamma(z-b)\Big] \end{aligned} \tag{56}$$

Here $A_{p,s}$ are (given) complex coefficients which are determined by the infinitely remote sources residing at $z = +\infty$, and $B_{p,s}$ are the unknown coefficients which determine the wave reflected from the slab. Inserting (56) into formulae (13) and using (9) leads to a representation of the total EM field , in the halfspace $b < z < +\infty$ as a superposition of an incident plane wave

$$\vec{E}_{in}(\vec{R}) \ = \ \vec{e}_{in} \exp(i\vec{k}_{in} \circ \vec{R}), \quad \vec{H}_{in}(\vec{R}) \ = \ \vec{h}_{in} \exp(i\vec{k}_{in} \circ \vec{R}) \tag{57}$$

and a reflected plane wave

$$\vec{E}_{r}(\vec{R}) \ = \ \vec{e}_{r} \exp(i\vec{k}_{r} \circ \vec{R}), \quad \vec{H}_{r}(\vec{R}) \ = \ \vec{h}_{r} \exp(i\vec{k}_{r} \circ \vec{R}) \tag{58}$$

with the amplitudes

$$\vec{e}_{in} \ = \ (\vec{a}_t A_p - \vec{k}_{in} \times \vec{a}_t A_s / k_0 \varepsilon) \exp(i\gamma b) \tag{59}$$

$$\begin{aligned} \vec{h}_{in} &= (\vec{a}_t A_s + \vec{k}_{in} \times \vec{a}_t A_p / k_0 \mu) \exp(i\gamma b) \qquad (60) \\ \vec{e}_{r} &= (\vec{a}_t B_p - \vec{k}_{r} \times \vec{a}_t B_s / k_0 \varepsilon) \exp(i\gamma b) \\ \vec{h}_{r} &= (\vec{a}_t B_s + \vec{k}_{r} \times \vec{a}_t B_p / k_0 \mu) \exp(i\gamma b) \end{aligned}$$

Here $\vec{k}_{in}$ and $\vec{k}_r$ are, respectively, the 3D wave vectors of the incident and the reflected plane waves: $k_{in,r} = \vec{\chi} \mp \vec{z}_0 \gamma$. It is clear from (59), (60) that, physically, $A_p(B_p)$ and $A_s(B_s)$ determine the p- and the s- polarized constituents for the incident (reflected) plane wave.

The quantities $B_{p,s}$ are found by imposing the source-free $(\vec{J}_s = \vec{M}_s \equiv 0)$ boundary condition (53) to the total field at $z = b(= z_s)$. In case where

$$\begin{aligned}\Delta_r(\vec{\chi}) & \\ \equiv & \left[\gamma + \varepsilon k_0 L_{ll}(\vec{\chi}, b)\right]\left[\gamma L_{tt}(\vec{\chi}, b) + k_0 \mu\right] - \varepsilon\gamma k_0 L_{lt}(\vec{\chi}, b) L_{tl}(\vec{\chi}, b) \\ \neq & \, 0\end{aligned} \tag{61}$$

the solution to $B_{p,s}$ can be expressed through $A_{p,s}$ as

$$B_p = R_{pp} A_p + R_{ps} A_s, \quad B_s = R_{sp} A_p + R_{ss} A_s \tag{62}$$

$$R_{\lambda\nu} \equiv R_{\lambda\nu}(\vec{\chi}) \equiv \Delta_{\lambda\nu}(\vec{\chi})/\Delta_r(\vec{\chi}), \qquad (\lambda, \nu = p, s) \tag{63}$$

$$\begin{aligned}\Delta_{pp}(\vec{\chi}) &= (\gamma + \varepsilon k_0 L_{ll})(\gamma L_{tt} - \mu k_0) - \varepsilon\gamma k_0 L_{lt} L_{tl} \\ \Delta_{ss}(\vec{\chi}) &= (\gamma - \varepsilon k_0 L_{ll})(\gamma L_{tt} + \mu k_0) + \mu\gamma k_0 L_{lt} L_{tl} \\ \Delta_{ps}(\vec{\chi}) &= -2\mu\gamma k_0 L_{tl}, \quad \Delta_{sp}(\vec{\chi}) = 2\varepsilon\gamma\ k_0 L_{lt}\end{aligned} \tag{64}$$

$L_{\sigma\tau} \equiv L_{\sigma\tau}(\chi, b)$, $(\sigma, \tau = l, t)$. The quantity $R_{\lambda\nu}(\vec{\chi})$ is the complex reflection coefficient for a ν-polarized incident wave into a λ-polarized constituent in the reflected wave, $(\lambda, \nu = p, s)$. In case where

$$\Delta_r(\chi) = 0 \tag{65}$$

the solution for $B_{p,s}$ exists only for a zero incident field $(A_p = A_s = 0)$. This time $B_{p,s}$ should obey one of the two equivalent relations

$$\begin{aligned}\left[\gamma + \varepsilon k_0 L_{ll}(\chi, b)\right] B_s + \varepsilon\gamma\ L_{lt}(\chi, b) B_p/\mu = 0 \\ \mu k_0 L_{tl}(\chi, b) B_s + \left[\gamma L_{tt}(\chi, b) + \mu k_0\right] B_p = 0\end{aligned} \tag{66}$$

and are otherwise arbitrary. The total field in the halfspace $b < z < +\infty$ coincides now with $\vec{E}_r, \vec{H}_r$, defined by (58) and (60). It should be emphasized that owing to a condition $0 < \arg\gamma < \pi$ implied in the definition (39) of a multivalued function $\gamma\ (\chi)$ of complex variable χ (and similar condition imposed on the multivalued function $\gamma_c(\chi)$ in (B5), Appendix B, in case where the halfspace $-\infty < z < 0$ is filled with an isotropic homogeneous magnetodielectric medium) any root $\vec{\chi}$ of the dispersion equation (65) corresponds to a wave traveling and exponentially decaying from the slab. Such wave represents an eigenmode of the slab [12]. However, one may consider a multivalued function $\Delta_r(\vec{\chi}) \equiv \Delta_r(\chi, \vec{n})$ of complex variable χ on the other sheets of its Riemann surface where $-\pi < \arg \gamma < 0$. Any root $\chi \equiv \chi(\vec{n})$ of the dispersion equation $\Delta_r(\chi, \vec{n}) = 0$ which is located on one of these new (irregular) sheets of the Riemann surface of $\Delta_r(\chi, \vec{n})$ corresponds to a non-spectral, or leaky wave. It exponentially grows from the slab since for such a root one has $Im\gamma < 0(and/or Im\gamma_c < 0)$. In conformity with expectations, the wavenumbers $\chi \equiv \chi(\vec{n})$ corresponding to natural modes of the slab (i.e. to the eigenmodes and the leaky modes) are the poles for the reflection coefficients (63). It is interesting to note that owing to the dependence of Δ_r on $\vec{n}$ the type of a natural mode (i.e. an eigenmode or a leaky mode), as well as the very fact of its existence may depend upon the direction of propagation given by $\vec{n}$.

With the coefficients $B_{p,s}$ at hand, one can utilize the continuity condition for scalar potentials at $z = b$ to derive a system of two equations with respect to $\mathrm{T}_{1,2}$:

$$\begin{aligned} T_1\mathcal{E}_1^-(\vec{\chi}, b-0) + T_2\mathcal{E}_2^-(\vec{\chi}, b-0) &= A_p + B_p \\ T_1\mathcal{H}_1^-(\vec{\chi}, b-0) + T_2\mathcal{H}_2^-(\vec{\chi}, b-0) &= A_s + B_s \end{aligned} \tag{67}$$

It enables one to determine the coefficients $T_{1,2}$ appearing in (52) both for a plane-wave diffraction problem and a natural mode propagation problem $(A_p = A_s = 0)$.

6. Conclusions

We have developed a method for reducing a bulky EM problem in an arbitrarily layered generally bianisotropic medium through a proper choice of scalar potentials. The basic equations of this method are independent of a Cartesian coordinate system in which the constitutive parameters of a medium are specified, and are distinguished for their concise and highly symmetric form. The scalarized formulations for an excitation, plane-wave diffraction and natural mode propagation problems are derived. The main emphasis of the present paper is on the analytical details. Further improvement in the analysis as well as the examples of numerical investigation via finite difference method will be reported in future publications.

Appendix A

Anticipating further needs, let us introduce the components of the constitutive dyads $\hat{\eta} = \hat{\varepsilon}, \hat{\mu}, \hat{\xi}$ and $\hat{\zeta}$ in the basis set (10) by

$$\eta_{\sigma\tau} = \vec{a}_\sigma \circ \hat{\eta} \circ \hat{a}_\tau, \qquad (\sigma, \tau = l, t, z) \tag{A1}$$

For brevity, we will suppress the dependence of all quantities on $\vec{n}, z$ if no confusion arises. It should be noted that, in fact, η_{zz} does not depend on $\vec{n}$ and is a function of the z variable only: $\eta_{zz} = \eta_{zz}(z)$.

By using these notations, the functions $P_{\sigma\tau}$, $Q_{\sigma\tau}$, $R_{\sigma\tau}$ and $S_{\sigma\tau}$, $(\sigma, \tau = l, z)$, of arguments $\vec{n}, z$ are defined as

$$\begin{aligned} P_{\sigma\tau} = \varepsilon_{\sigma\tau}(\mu_{ll}\mu_{zz} - \mu_{zl}\mu_{lz}) + \xi_{\sigma l}(\mu_{zz}\zeta_{l\tau} - \mu_{lz}\zeta_{z\tau}) \\ + \xi_{\sigma z}(\mu_{ll}\zeta_{z\tau} - \mu_{zl}\zeta_{l\tau}) \end{aligned} \tag{A2}$$

$$\begin{aligned} &P_{\sigma\tau} \to R_{\sigma\tau}, \quad (\varepsilon \leftrightarrow \zeta, \quad \mu \leftrightarrow \xi) \\ &R_{\sigma\tau} \to S_{\sigma\tau}, \quad P_{\sigma\tau} \to Q_{\sigma\tau}, \quad (\varepsilon \leftrightarrow \mu, \quad \zeta \leftrightarrow \xi) \end{aligned} \tag{A3}$$

Then the quantities $p_{\sigma\tau}$, $q_{\sigma\tau}$, $r_{\sigma\tau}$ and $s_{\sigma\tau}$, $(\sigma, \tau = l, z)$ encountered in Sections 2,3 are obtainable from the respective quantities designated by a capital letter after dividing by $\Delta = \Delta(\vec{\chi}) : p_{\sigma\tau} = P_{\sigma\tau}/\Delta$ etc. The quantity Δ is defined in (A7), and $\tilde{\Delta}$ from formulae (35) is given as

$$\tilde{\Delta} = P_{zz}Q_{zz} + R_{zz}S_{zz} \tag{A4}$$

Letting a bar under the quantity signify a matrix, we now introduce a 3×3 matrix $\underline{\eta} = [\eta_{\sigma\tau}]$ composed of the components $\eta_{\sigma\tau}, (\sigma, \tau = l, t, z)$ of the dyads $\hat{\eta} = \hat{\varepsilon},\ \hat{\mu},\ \hat{\xi},\ \hat{\zeta}$ in the basis set (10), and a 6×6 matrix

$$\underline{\pi} = \begin{bmatrix} \underline{\mu} & -\underline{\zeta} \\ \underline{\xi} & \underline{\varepsilon} \end{bmatrix} \tag{A5}$$

Let us designate by π_{rs}^{pq} the determinant of a matrix obtainable from $\underline{\pi}$ via striking out p-th and q-th columns and the r-th and s-th rows,

$(p, q, r, s = 1, ...6)$. With this notation in mind, define the following functions of variables $\vec{n}, z$:

$$\begin{array}{llll} B_l = \pi^{25}_{45}, & B_z = -\pi^{25}_{56}, & C_l = \pi^{25}_{12}, & C_z = -\pi^{25}_{23} \\ D_l = \pi^{25}_{15}, & D_z = \pi^{25}_{35}, & F_l = \pi^{25}_{24}, & F_z = \pi^{25}_{26} \\ T_l = -\pi^{45}_{25}, & T_z = \pi^{56}_{25}, & U_l = -\pi^{12}_{25}, & U_z = \pi^{23}_{25} \\ V_l = -\pi^{24}_{25}, & V_z = -\pi^{26}_{25}, & W_l = -\pi^{15}_{25}, & W_z = -\pi^{35}_{25} \end{array} \tag{A6}$$

$$\Delta = \pi^{25}_{25} \tag{A7}$$

Then the quantities $b_\sigma, c_\sigma, d_\sigma, f_\sigma, t_\sigma, u_\sigma, v_\sigma$ and w_σ, $(\sigma = z, l)$, used in the present paper follow from the respective quantities designated by the respective capital letters after normalization by Δ : $b_\sigma = B_\sigma/\Delta$ etc.

In full analogy with the preceding convention let π^p_r denote the determinant of a matrix obtainable from $\underline{\pi}$ by striking out p-th column and r-th row, $(p, r = 1, ...6)$. We define at this point the following functions of $\vec{n}, z$:

$$\delta_\varepsilon = \pi^2_2/\Delta, \quad \delta_\mu = \pi^5_5/\Delta, \quad \delta_\zeta = \pi^2_5/\Delta, \quad \delta_\xi = -\pi^6_2/\Delta \tag{A8}$$

They figure in relations (21),(22).

It is important to note the following symmetry properties of the just introduced coefficients :

$$\Delta \to \Delta \text{ when } \varepsilon \leftrightarrow \mu, \quad \xi \leftrightarrow \zeta \quad \text{or when} \quad l \leftrightarrow z \tag{A9}$$

$$\begin{array}{c} p_{\sigma\tau} \to q_{\sigma\tau}, \quad r_{\sigma\tau} \to s_{\sigma\tau}, \quad b_\sigma \to c_\sigma, \quad d_\sigma \to f_\sigma, \quad t_\sigma \to u_\sigma, \quad v_\sigma \to w_\sigma \\ \text{when} \quad \varepsilon \leftrightarrow \mu, \quad \xi \leftrightarrow \zeta, \qquad (\sigma, \tau = l, z) \end{array} \tag{A10.1}$$

$$\delta_\varepsilon \to \delta_\mu, \quad \delta_\xi \to \delta_\zeta \text{ when } \varepsilon \leftrightarrow \mu, \quad \xi \leftrightarrow \zeta \tag{A10.2}$$

$$\begin{array}{llll} p_{\sigma\tau} \to p_{\tau\sigma}, & q_{\sigma\tau} \to q_{\tau\sigma}, & r_{\sigma\tau} \to r_{\tau\sigma}, & s_{\sigma\tau} \to s_{\tau\sigma} \\ b_\sigma \to b_\tau, & c_\sigma \to c_\tau, & d_\sigma \to d_\tau, & f_\sigma \to f_\tau \\ t_\sigma \to t_\tau, & u_\sigma \to u_\tau, & v_\sigma \to v_\tau, & w_\sigma \to w_\tau \end{array} \tag{A11.1}$$

$$\text{when } l \leftrightarrow z, \quad (\sigma, \tau = l, z)$$

$$\delta_\varepsilon \to \delta_\varepsilon, \delta_\mu \to \delta_\mu, \delta_\xi \to \delta_\xi, \delta_\zeta \to \delta_\zeta \qquad \text{when } l \leftrightarrow z \tag{A11.2}$$

For an isotropic medium with the permittivity $\varepsilon(z)$ and permeability $\mu(z)$ one has:

$$\begin{aligned} &P_{zz} = P_{ll} = \varepsilon\mu^2, \quad &&Q_{zz} = Q_{ll} = \mu\varepsilon^2, \quad &&\Delta = \varepsilon^2\mu^2 \\ &p_{zz} = p_{ll} = \varepsilon^{-1}, \quad &&q_{zz} = q_{ll} = \mu^{-1}, \quad &&\tilde{\Delta} = \varepsilon^3\mu^3 \\ &\delta_\varepsilon = \varepsilon, &&\delta_\mu = \varepsilon \end{aligned} \tag{A12}$$

The remaining quantities defined by formulae (A2), (A3), (A5) and (A6) equal identically zero.

Appendix B

For coefficients $a_{\lambda\nu}(\chi), b_{\lambda\nu}(\chi)$ from (24), $(\lambda, \nu = p, s)$, the explicit expressions are:

$$\begin{aligned} a_{pp} &= 1 + (\chi q_{lz}/k_0 - u_l)L_{tt} \\ a_{ps} &= L_{tl} - (\chi s_{lz}/k_0 + w_l)L_{tt} \\ a_{sp} &= v_l - u_l L_{lt} + \chi(r_{lz} + q_{lz}L_{lt})/k_0 \\ a_{ss} &= L_{ll} - w_l L_{lt} - t_l + \chi(p_{lz} - s_{lz}L_{lt})/k_0 \end{aligned} \tag{B1}$$

$$\begin{aligned} b_{pp} &= q_{zz}L_{tt} \quad, \qquad &b_{ps} &= -s_{zz}L_{tt} \\ b_{sp} &= r_{zz} + q_{qq}L_{lt}, \qquad &b_{ss} &= p_{zz} - s_{zz}L_{lt} \end{aligned} \tag{B2}$$

Here the variable z entering the right-hand side in (Bl), (B2) should be taken at $z = +0$,

$$L_{\sigma\tau} \equiv L_{\sigma\tau}(\vec{\chi}) = a_\sigma \circ \hat{L}(\vec{\chi}) \circ \vec{a}_\tau, \qquad (\sigma,\ \tau = l, t) \tag{B3}$$

For the case where the halfspace $-\infty < z < 0$ is filled with an isotropic homogeneous magnetodielectric medium of permittivity ε_c and permeability μ_c, one has (see e.g. [12,19]) :

$$L_{lt} = L_{tl} = 0, \quad L_{ll} = \gamma_c/k_0\varepsilon, \quad L_{tt} = k_0\mu_c/\gamma_c \tag{B4}$$

In these relations

$$\gamma_c \equiv \gamma_c(\chi) = (k_0^2 \varepsilon_c \mu_c - \chi^2)^{1/2}, \qquad (0 \leq \arg \gamma_c < \pi) \tag{B5}$$

For the case of a statistically rough interface in an arbitrarily layered medium, the equivalent dyadic impedance $\hat{L}(\vec{\chi})$ has been found in [21].

Appendix C

The quantities $\vec{Q}^{\rho}_{\beta j}(\vec{\chi})$ used in Section 4 can be expanded over the vectors $\vec{a}_l = \vec{n}$ and $a_t = \vec{z}_0 \times \vec{n}$ in the following way:

$$\vec{Q}^{\rho}_{\beta j}(\vec{\chi}) = \vec{a}_t \, r^{\rho}_{\beta j}(\vec{\chi}) - \vec{a}_l \, s^{\rho}_{\beta j}(\vec{\chi}) \tag{C1}$$

where $\beta = e, m, j = 1, 2,$ and $\rho = \pm$. To express the coefficients in the right-hand side of (Cl), let us introduce the projections $E^{\rho}_{\sigma j}, H^{\rho}_{\sigma j}$ of the vector field amplitudes $\vec{E}^{\rho}_j$, $\vec{H}^{\rho}_j$ from (47) onto the vectors $\vec{a}_\sigma$, $(\sigma = l, t, z)$:

$$E^{\rho}_{\sigma j}(\vec{\chi}, z) = \vec{a}_\sigma \circ \vec{E}^{\rho}_j(\vec{\chi}, z), \quad H^{\rho}_{\sigma j}(\vec{\chi}, z) = \vec{a}_\sigma \circ \vec{H}^{\rho}_j(\vec{\chi}, z) \tag{C2}$$

In particular, by construction $E^{\rho}_{tj} \equiv \mathcal{E}^{\rho}_j$, $H^{\rho}_{tj} \equiv \mathcal{H}^{\rho}_j$, where the functions $\mathcal{E}^{\rho}_j$, $\mathcal{H}^{\rho}_j$ are introduced in Section 4 in connection with formulae (47). We will speak of the values $E^{\rho}_{\sigma j}(\vec{\chi}, z)$, $H^{\rho}_{\sigma j}(\vec{\chi}, z)$ at $z = z_s$ in the sense that these are the limiting values of the respective functions with $z \to z_s \pm 0$ for $\rho = \pm$. Last, we introduce a determinant $\Delta_s \equiv \Delta_s(\vec{\chi})$ of the matrix

$$\begin{bmatrix} H^+_{t1} & H^+_{t2} & H^-_{t1} & H^-_{t2} \\ E^+_{t1} & E^+_{t2} & E^-_{t1} & E^-_{t2} \\ H^+_{l1} & H^+_{l2} & H^-_{l1} & H^-_{l2} \\ E^+_{l1} & E^+_{l2} & E^-_{l1} & E^-_{l2} \end{bmatrix} \tag{C3}$$

composed of the values of the quantities (C2) at $z = z_s$. Then the coefficient $r^+_{el}(\vec{\chi})$ is defined by the expression

$$\begin{aligned} r^+_{el}\Delta_s = {} & H^+_{t2}\,(E^-_{t1}\,E^-_{l2} \,-\, E^-_{t2}\,E^-_{l1}) + H^-_{t1}\,(E^-_{t2}\,E^+_{l2} \,-\, E^+_{t2}\,E^-_{l2}) \\ & + H^-_{t2}\,(E^+_{t2}\,E^-_{l1} \,-\, E^-_{t1}\,E^+_{l2}) \end{aligned} \tag{C4}$$

where the terms on the right-hand side are taken at $z = z_s$. The remaining coefficients are obtainable from the above expressions via the following transformations:

$$\begin{aligned} & r_{e1}^{+} \to r_{e2}^{+}, && (\text{“}1\text{”} \leftrightarrow \text{“}2\text{”}) \\ & r_{ej}^{+} \to r_{mj}^{+}, && (E \to H, \quad H \to -E) \\ & r_{\beta j}^{+} \to r_{\beta j}^{-}, && (\text{“}+\text{”} \leftrightarrow \text{“}-\text{”}) \\ & r_{\beta j}^{\pm} \to s_{\beta j}^{\pm}, && (\text{“}l\text{”} \leftrightarrow \text{“}t\text{”}) \end{aligned} \tag{C5}$$

Here $\beta = e, m$, $j = 1, 2$, and "..." signifies the quantity which plays the part of an index.

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