CHARACTERISTICS OF GUIDED MODES IN UNIAXIAL CHIRAL CIRCULAR WAVEGUIDES

J.-F. Dong* and J. Li

Institute of Optical Fiber Communication and Network Technology, Ningbo University, Ningbo 315211, China

Abstract—The characteristics of guided modes in the circular waveguide consist of uniaxial chiral medium have been investigated. The characteristic equation of guided modes is derived. The dispersion curves and energy flux of guided modes for three kinds of uniaxial chiral media are presented. Unusual dispersion characteristics and negative energy flux are found, i.e., backward wave is supported in the uniaxial chiral waveguide.

1. INTRODUCTION

In recent years, chiral metamaterials have attracted much attention because the negative refractive index can be realized in the chiral metamaterials [1–9] and because a chiral slab with negative refractive index can be used as a perfect lens which providing subwavelength resolution for circularly polarized waves [10, 11]. Surface polaritons [12] and Goos-Hanchen shift [13] at the surface of chiral negative refractive media have been studied. Waveguides consisting of chiral metamaterials with negative refractive indices, such as slab, grounded slab, parallel-plate waveguide and fiber, have been investigated theoretically [14–17]. A special case of chiral negative refractive index medium, termed as chiral nihility [1] in which the permittivity and permeability are simultaneously zero, has also intensively explored [18–30]. Especially, planar and circular open [18, 20–24, 29] or closed [30] waveguides containing chiral nihility have been studied. However, these studies focus on the isotropic chiral medium. Usually, uniaxially anisotropic chiral media are quite easy to be realized artificially [31]. Recently, Cheng and Cui [31, 32] investigated negative refractions in

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* Corresponding author: Jian-Feng Dong (dongjianfeng@nbu.edu.cn).
uniaxially anisotropic chiral media. They found that the condition to realize the negative refraction in uniaxial chiral media can be quite loose. Mahmoud and Viitanen [33] considered propagation of waves in uniaxial chiral circular waveguide with boundary condition of hard surface. However, the possibility of negative electromagnetic parameter and negative refractive index has not been discussed in [33]. Guided modes in open circular waveguides (fibers) consist of isotropic chiral media [17, 34, 35] or negative index materials [36] have been studied in the literature. Slab or planar waveguides consisting of uniaxial anisotropic non-chiral media have also been examined [37, 38]. In our previous papers, novel characteristics of guided modes in the isotropic chiral negative refractive index fiber [17] and chiral nihility fiber [22, 23] have been investigated. In this paper, we extend these studies into open circular waveguides consisting of uniaxial chiral media with negative electromagnetic parameters. Firstly, we derive the characteristic equation of guided modes, then present numerical results of low-order guided modes for three kinds of uniaxial chiral media: I \( \varepsilon_t > 0, \varepsilon_z > 0 \), II \( \varepsilon_t < 0, \varepsilon_z > 0 \), and III \( \varepsilon_t > 0, \varepsilon_z < 0 \), among which there are no counterparts in isotropic chiral case and relatively easily realized (see Fig. 1 in [31]), and we also discuss the effects of chirality parameter on dispersion characteristics and energy flux.

2. MODAL CHARACTERISTIC EQUATION

Consider the circular waveguide in which the core is uniaxial chiral medium and cladding is conventional material. The radius of core is \( a \). The cladding is assumed to extend infinitely. Here we adopt the cylindrical coordinate system \((r, \varphi, z)\) and time-harmonic field with \( \exp(j\omega t) \).

The constitutive relations in the uniaxial chiral medium are [39]:

\[
D = [\varepsilon_t \vec{I}_t + \varepsilon_z \hat{z} \hat{z}] \cdot E - j\kappa \sqrt{\mu_0 \varepsilon_0} \hat{z} \cdot H \quad (1)
\]

\[
B = [\mu_t \vec{I}_t + \mu_z \hat{z} \hat{z}] \cdot H + j\kappa \sqrt{\mu_0 \varepsilon_0} \hat{z} \cdot E \quad (2)
\]

where \( \varepsilon_t (\mu_t) \) and \( \varepsilon_z (\mu_z) \) are permittivity (permeability) of the uniaxial chiral medium in transversal and longitudinal direction, respectively; \( \varepsilon_0 \) and \( \mu_0 \) are permittivity and permeability of free space. \( \kappa \) is the chirality parameter, which describes electromagnetic coupling. \( \hat{z} \) is a unit vector along the waveguide axis (longitudinal direction) and \( \vec{I}_t = \hat{x} \hat{x} + \hat{y} \hat{y} \).
By separating the modal fields in the waveguide into transversal and longitudinal electromagnetic field components, we obtain:

\[ E = (E_t + E_z \hat{z}) \exp(-j\beta z) \] (3)

\[ H = (H_t + H_z \hat{z}) \exp(-j\beta z) \] (4)

where \( \beta \) is the longitudinal propagation constant. According to Maxwell’s equations and above constitutive relations, the relationships between the transversal and longitudinal electromagnetic field components can be derived as follows:

\[ E_t = -j \frac{\beta}{\lambda^2} \nabla_t E_z - j \frac{\omega \mu_t}{\lambda^2} \nabla_t H_z \times \hat{z} \] (5)

\[ H_t = -j \frac{\beta}{\lambda^2} \nabla_t H_z - j \frac{\omega \varepsilon_t}{\lambda^2} \hat{z} \times \nabla_t E_z \] (6)

where \( \nabla_t = \nabla - \hat{z} \frac{\partial}{\partial z} \), \( \lambda^2 = \frac{\omega^2}{\mu_t \varepsilon_t} - \beta^2 \), and longitudinal electromagnetic field components satisfy wave equations [33]:

\[ \begin{bmatrix} \nabla_t^2 E_z \\ \nabla_t^2 H_z \end{bmatrix} - k_c^2 \begin{bmatrix} \varepsilon_z/\varepsilon_t & -j\kappa \sqrt{\varepsilon_0 \mu_0}/\varepsilon_t \\ j\kappa \sqrt{\varepsilon_0 \mu_0}/\mu_t & \mu_z/\mu_t \end{bmatrix} \begin{bmatrix} E_z \\ H_z \end{bmatrix} = 0 \] (7)

By finding the eigenvectors and eigenvalues of the \( 2 \times 2 \) matrix in Equation (7), one can obtain the modal field structure [33]. The eigenvalues \( k_c^2 \) of \( 2 \times 2 \) matrix in Equation (7) have two values:

\[ k_c^2 \pm c = \frac{\lambda^2}{2} \left[ \varepsilon_z/\varepsilon_t + \mu_z/\mu_t \pm \sqrt{\left( \varepsilon_z/\varepsilon_t - \mu_z/\mu_t \right)^2 + 4\kappa^2 \varepsilon_0 \mu_0/\varepsilon_t \mu_t} \right] \] (8)

The corresponding eigenfunctions are given by [33]:

\[ (E_z, H_z) = \left( E_z, j \frac{\alpha}{\eta_l} E_z \right) \] (9)

where \( \alpha = (k_c^2 - \varepsilon_z/\varepsilon_t) \sqrt{\varepsilon_t \mu_t/(\kappa \sqrt{\varepsilon_0 \mu_0})}, \eta_l = \sqrt{\mu_t/\varepsilon_t} \).

Thus the wave equation of the longitudinal electromagnetic field component \( E_z \) becomes

\[ \nabla_t^2 E_z + k_c^2 E_z = 0 \] (10)

In the cylindrical coordinate system \( (r, \varphi, z) \), the solution of the longitudinal electromagnetic field component in the core has the form \( E_z = J_m(k_c r) \exp(jm\varphi) \). Then the total longitudinal electromagnetic field components in the core can be written as

\[ E_{z1} = [A_m J_m(k_{c+} r) + B_m J_m(k_{c-} r)] \exp(jm\varphi) \exp(-j\beta z) \] (11a)

\[ H_{z1} = \frac{j}{\eta_t} [A_m \alpha_+ J_m(k_{c+} r) + B_m \alpha_- J_m(k_{c-} r)] \exp(jm\varphi) \exp(-j\beta z) \] (11b)
The transversal electromagnetic field components can be derived from Equations (5), (6) as:

\[
E_{r1} = \begin{cases} 
A_m & \left[ \frac{jmk_t}{\lambda^2 r} \alpha_+ J_m(k_{c+r}) - \frac{j\beta k_c+J_m'(k_{c+r})}{\lambda^2} \right] \\
B_m & \left[ \frac{jmk_t}{\lambda^2 r} \alpha_- J_m(k_{c-r}) - \frac{j\beta k_c-J_m'(k_{c-r})}{\lambda^2} \right]
\end{cases} 
\exp(jm\varphi) \exp(-j\beta z) 
\]

\[
E_{\varphi 1} = \begin{cases} 
A_m & \left[ \frac{m\beta}{\lambda^2 r} J_m(k_{c+r}) - \frac{k_l k_c+J_m'(k_{c+r})}{\lambda^2} \right] \\
B_m & \left[ \frac{m\beta}{\lambda^2 r} J_m(k_{c-r}) - \frac{k_l k_c-J_m'(k_{c-r})}{\lambda^2} \right]
\end{cases} 
\exp(jm\varphi) \exp(-j\beta z) 
\]

\[
H_{r1} = \begin{cases} 
A_m & \left[ \frac{1}{\lambda^2 \eta_t} \left(- \frac{k_l k_c+J_m'(k_{c+r})}{r} + \beta k_c+J_m'(k_{c+r}) \right) \right] \\
B_m & \left[ \frac{1}{\lambda^2 \eta_t} \left(- \frac{k_l k_c+J_m'(k_{c-r})}{r} + \beta k_c-J_m'(k_{c-r}) \right) \right]
\end{cases} 
\exp(jm\varphi) \exp(-j\beta z) 
\]

\[
H_{\varphi 1} = \begin{cases} 
A_m & \left[ \frac{1}{\lambda^2 \eta_t} \left( \frac{j\beta m\alpha_+}{r} J_m(k_{c+r}) - k_l k_c+J_m'(k_{c+r}) \right) \right] \\
B_m & \left[ \frac{1}{\lambda^2 \eta_t} \left( \frac{j\beta m\alpha_-}{r} J_m(k_{c-r}) - k_l k_c-J_m'(k_{c-r}) \right) \right]
\end{cases} 
\exp(jm\varphi) \exp(-j\beta z) 
\]

where \( A_m, B_m \) are constants, \( J_m(\cdot) \) the Bessel function of first kind, \( J_m'(\cdot) \) the differentiation with respect to argument, and \( m \) a positive or negative integer specifying the azimuthal field dependence.

In the cladding, the electromagnetic fields can be obtained as [22]:

\[
\begin{align*}
E_{z2} &= C_m K_m(\tau_{2r}) \exp(jm\varphi) \exp(-j\beta z) \\
H_{z2} &= \frac{1}{\eta_2} D_m K_m(\tau_{2r}) \exp(jm\varphi) \exp(-j\beta z) \\
E_{r2} &= \left[ \frac{j\beta}{\tau_2} C_m K_m'(\tau_{2r}) - \frac{imk_2}{\tau_2^2} D_m K_m(\tau_{2r}) \right] \exp(jm\varphi) \exp(-j\beta z) \\
H_{r2} &= \left[ \frac{mk_2}{\eta_2^2 \tau_2^2} C_m K_m(\tau_{2r}) - \frac{\beta}{\eta_2^2 \tau_2} D_m K_m'(\tau_{2r}) \right] \exp(jm\varphi) \exp(-j\beta z) \\
E_{\varphi 2} &= \left[ -\frac{m\beta}{\tau_2^2} C_m K_m(\tau_{2r}) + \frac{k_2}{\tau_2^2} D_m K_m'(\tau_{2r}) \right] \exp(jm\varphi) \exp(-j\beta z) \\
H_{\varphi 2} &= \left[ \frac{jm\beta}{\eta_2^2 \tau_2^2} C_m K_m'(\tau_{2r}) - \frac{k_2}{\eta_2^2 \tau_2} D_m K_m(\tau_{2r}) \right] \exp(jm\varphi) \exp(-j\beta z)
\end{align*}
\]
\( \tau_2 = \sqrt{\beta^2 - k_2^2} \) the transverse attenuation factor in the cladding; 
\( k_2 = \omega \sqrt{\mu_2 \varepsilon_2} \), and \( \eta_2 = \sqrt{\mu_2 / \varepsilon_2} \) the wavenumber and wave impedance in the cladding, respectively.

According to four boundary conditions (continuity of the tangential electromagnetic field components \( E_z, E_\varphi, H_z, H_\varphi \)) at interface \( r = a \), the following equation can be derived:

\[
\begin{bmatrix}
J_m(u_+) & J_m(u_-) & -K_m(v) & 0 \\
\alpha_a J_m(u_+) & \alpha_a J_m(u_-) & 0 & -\frac{\eta_2}{\eta_a} K_m(v) \\
a_{31} & a_{32} & \frac{m\beta}{\tau_2^2 a} K_m(v) & -\frac{k_2}{\tau_2^2} K_m'(v) \\
a_{41} & a_{42} & -\frac{\eta_2}{\eta_a} k_2^2 K_m(v) & \frac{\eta_2}{\eta_a} \frac{m\beta}{\tau_2^2 a} K_m(v)
\end{bmatrix}
\begin{bmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{bmatrix} = 0 \quad (13)
\]

where \( a_{31} = \frac{m\beta}{\chi^2 a} J_m(u_+) - \frac{\omega \mu k + \alpha k}{\eta \chi^2 a} J_m'(u_+) \), \( a_{32} = \frac{m\beta}{\chi^2 a} J_m(u_-) - \frac{\omega \mu k - \alpha k}{\eta \chi^2 a} J_m'(u_-) \), \( a_{41} = \frac{m\beta a}{\chi^2 a} J_m(u_+) - \frac{\omega \varepsilon \mu k + \alpha k}{\chi^2 a} J_m'(u_+) \), \( a_{42} = \frac{m\beta a}{\chi^2 a} J_m(u_-) - \frac{\omega \varepsilon \mu k - \alpha k}{\chi^2 a} J_m'(u_-) \), and \( u_\pm = k_{\pm a}, \quad v = \tau_2 a \). The characteristic equation of guided modes is simply given as determinant of \( 4 \times 4 \) matrix in (13) equal to zero.

Energy flux along the \( z \)-axis in the waveguide is defined by:

\[
S_z = \frac{1}{2} \text{Re}(E \times H^*) \cdot \hat{z} = \frac{1}{2} \text{Re}(E_r H_\varphi^* - E_\varphi H_r^*) \quad (14)
\]

Power in the core (\( P_1 \)) and cladding (\( P_2 \)) are the integration of the energy flux \( S_z \) and \( S_{z1} \) and \( S_{z2} \), respectively:

\[
P_1 = \int_0^{2\pi} \int_0^a r S_{z1} dr d\varphi = 2\pi \int_0^a r S_{z1} dr \quad (15a)
\]

\[
P_2 = \int_0^{2\pi} \int_a^{\infty} r S_{z2} dr d\varphi = 2\pi \int_a^{\infty} r S_{z2} dr \quad (15b)
\]

The normalized power is defined as [36]

\[
P = \frac{P_1 + P_2}{|P_1| + |P_2|} \quad (16)
\]

3. NUMERICAL RESULTS AND DISCUSSION

The longitudinal propagation constant \( \beta \) can be calculated numerically from the characteristic equation, and relationships of constants \( A_m, B_m, C_m, \) and \( D_m \) in the formulas of electromagnetic fields can be derived from Equation (13). Thus all electromagnetic fields components, the energy flow distribution and normalized power can be obtained. In this section, we will present the dispersion curves, energy flux of guided modes for three kinds of uniaxial chiral media [31, 32]:
\[ \varepsilon_t > 0, \varepsilon_z > 0; \varepsilon_t < 0, \varepsilon_z > 0; \text{and} \ \varepsilon_t > 0, \varepsilon_z < 0. \] Here we assume \( \mu_t = \mu_z = \mu_0 \), use normalized frequency \( k_0 a \), and focus on the lower-order \( (m = -1, 0, 1) \) guided modes.

### 3.1. Case I: \( \varepsilon_t > 0, \varepsilon_z > 0 \)

We choose \( \varepsilon_t = 1.5\varepsilon_0, \varepsilon_z = 2.5\varepsilon_0 \). For small and middle value of chirality parameter \( \kappa \), the dispersion curves are similar as isotropic chiral circular waveguides [34,35], as shown in Fig. 1(a) for \( \kappa = 0.5 \). The normalized propagation constants \( \beta/k_0 \) of \( H_{-11} \) mode (dashed curve) and \( H_{11} \) mode (dash-dotted curve) are different, and increase as normalized frequency \( k_0 a \) increases. The fundamental mode is \( H_{-11} \) mode. For larger value of chirality parameter \( \kappa \), another type of mode (label as \( H_{mns} \)) with negative slope shape appears, as shown in Fig. 1(b) for \( \kappa = 2.0 \). The normalized propagation constants \( \beta/k_0 \) of these modes (\( H_{01s}, H_{-11s}, H_{11s} \) modes) are always larger than \( \sqrt{\varepsilon_t/\varepsilon_0} = 1.225 \). For a fixed normalized frequency \( k_0 a \), the value of \( \beta/k_0 \) of \( H_{01s} \) mode (solid curve) is smaller than those of \( H_{-11s} \) and \( H_{11s} \) modes.

In order to clearly investigate the propagation of electromagnetic wave in the waveguide, the energy flux \( S_z \) of guided modes in radial direction is examined. Fig. 2 shows normalized energy flux \( S_z \) of \( H_{01}, H_{-11}, H_{11} \) modes for \( \kappa = 0.5 \) at normalized frequency \( k_0 a = 3 \). They are all positive in the core and cladding. The maximum of energy flux \( S_z \) is located in the middle of the core for \( H_{01} \) mode and in the center of the core for \( H_{-11} \) and \( H_{11} \) modes. However, for another type of mode, there are negative energy flux \( S_z \) near the center for \( H_{-11s} \) mode and in the middle for \( H_{11s} \) mode (see Fig. 3). The energy flux \( S_z \)
in the core changes sign for $H_{-11s}$ mode and changes sign two times for $H_{11s}$ mode at $k_0a = 2$. For $H_{11s}$ mode, at small normalized frequency $k_0a$, the energy flux $S_z$ becomes negative near interface, even in the cladding (Fig. 4). This is a novel phenomenon.

It is found from calculation that the normalized power $P$ of all guided modes (including $H_{mns}$ modes) are positive, although power in the cladding can be negative in some cases (for example in Fig. 4, the value of normalized power is smaller than one), thus they are forward waves, which means that the flow of the power is parallel to wave vector propagation direction.

3.2. Case II: $\varepsilon_t < 0$, $\varepsilon_z > 0$

We choose $\varepsilon_t = -\varepsilon_0$, $\varepsilon_z = 2\varepsilon_0$. Fig. 5 shows the dispersion curves of $H_{01s}$, $H_{-11s}$, $H_{11s}$, $H_{02s}$, $H_{-12s}$, $H_{12s}$ modes for $\kappa = 0.5$ and $\kappa = 1.4$. The normalized propagation constants $\beta/k_0$ of all guided modes decrease monotonically as $k_0a$ increases, i.e., dispersion curves
Figure 3. Normalized energy flux $S_z$ of guided modes for $\kappa = 2.0$ at $k_0a = 2.0$.

Figure 4. Normalized energy flux $S_z$ of $H_{11s}$ mode for $\kappa = 2.0$ at $k_0a = 0.5$. 
Figure 5. Dispersion curves of $H_{01s}$, $H_{-11s}$, $H_{11s}$, $H_{02s}$, $H_{-12s}$, $H_{12s}$ modes for different chirality parameters.

Figure 6. Normalized energy flux $S_z$ of guided modes for $\kappa = 0.5$ at $k_0a = 1.0$. 
of guided modes have negative slope. $\beta/k_0$ are almost the same for different signs of $m$ ($H_{-11s}$ and $H_{11s}$ modes, $H_{-12s}$ and $H_{12s}$ modes). The dispersion curves are similar even when $\kappa$ approaches zero. When $\kappa$ is very large, no solution of characteristic equation can be found.

The normalized energy flux $S_z$ of all guided modes at $k_0a = 1.0$ for $\kappa = 0.5$ and $\kappa = 1.4$ are plotted in Fig. 6 and Fig. 7. The energy flux $S_z$ of all guided modes is negative in the core and positive in the cladding for $\kappa = 0.5$. However, for $\kappa = 1.4$, they are distinctly different. For $H_{-11s}$ mode, $S_z$ is positive near the center and negative near the interface in the core. For $H_{11s}$ mode, there are three regions, near the center and interface, $S_z$ are negative, and between these two regions, $S_z$ is positive.

It is found that the normalized power $P$ for all guided modes are negative, and the absolute value is smaller than one, which means that the power is negative in the core and positive in the cladding.
Thus these guided modes are backward waves, which means that the flow of the power is antiparallel to wave vector propagation direction, even if chirality parameter $\kappa$ is very small or zero. This phenomenon is consistent with the result in the slab of uniaxial anisotropic media [37, 38]. This feature may have potential application in phase compensation or coupling devices.

3.3. Case III: $\varepsilon_t > 0$, $\varepsilon_z < 0$

We choose $\varepsilon_t = 2\varepsilon_0$, $\varepsilon_z = -\varepsilon_0$. Fig. 8 shows dispersion curves of guided modes for different chirality parameters $\kappa = 0.5$ and $\kappa = 2.0$. The shapes of dispersion curves are similar as Fig. 1(b). There are other types of guided modes with negative slope shape even for $\kappa$ approach zero. For chirality parameter $\kappa = 0.5$ and $\kappa = 2.0$ and smaller normalized frequency $k_0a$, the energy flux $S_z$ of guided modes are also similar as in Fig. 3 and Fig. 4. In this case, for all guided modes (including $H_{mns}$ modes with negative slope curves), the normalized power $P$ are always positive, thus they are also forward waves.

4. CONCLUSION

The characteristics of guided modes in the circular waveguide consist of uniaxial chiral medium have been investigated theoretically. The characteristic equation of guided modes is obtained. Numerical results for three kinds of uniaxial chiral media: I $\varepsilon_t > 0$, $\varepsilon_z > 0$, II $\varepsilon_t < 0$, $\varepsilon_z > 0$, and III $\varepsilon_t > 0$, $\varepsilon_z < 0$ are presented. Effects of the chirality parameter on dispersion curves and energy flux of guided modes are discussed. Abnormal dispersion characteristics with negative slope
curves and negative energy flux in the core are found, i.e., backward wave is supported in the uniaxial chiral waveguide (even for chirality parameter approach zero) for $\varepsilon_t < 0, \varepsilon_z > 0$. For $\varepsilon_t > 0, \varepsilon_z > 0$ and $\varepsilon_t > 0, \varepsilon_z < 0$, there is negative energy flux even in the cladding. The results presented here will be helpful for potential applications in novel waveguide devices such as phase compensation or coupling devices. It is noted that we have neglected dispersion and losses of chiral media as done in [31–33]. However, usually chiral media are dispersive and lossy. It is a further work to study dispersive and lossy chiral waveguides in future.

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