Progress In Electromagnetics Research, PIER 13, 169–241, 1996

UNIFORM ASYMPTOTIC ANALYSIS OF GUIDED MODES OF GRADED-INDEX OPTICAL FIBERS WITH EVEN POLYNOMIAL PROFILE CENTER CORES

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1. Introduction

When we evaluate propagation characteristics of guided modes of the graded-index optical fibers, we need to solve the vector wave equation in the inhomogeneous region [1–5]. In general, it is difficult to get an analytic solution of the vector wave equation that governs the guided modes of graded-index optical fibers. The inherent difficulty encountered in the derivation of an analytic solution of the vector wave equations comes from the analytic nature of the vector wave equation about longitudinal components of electromagnetic fields [1,2]. Accordingly, we usually use main frames to design such optical fibers with the aid of numerical methods [6–10]. Numerical methods are general method for designing such optical fibers, but time consuming. On the other hand, analytic solutions are attractive for giving intuitive insights for the wave propagation in the optical fiber that make it easy to understand the dependence of the guided modes on the refractive-index profile and for a design theory to develop broadband graded-index optical fibers. Moreover, analytic solutions allow us to use a desktop computer for analyzing propagation characteristics of the graded-index optical fibers as well as the step-index optical fibers.Therefore many researchers have managed to construct the analytic solution of the vector wave equation: for the vector wave equation about the longitudinal components of the electromagnetic field, we cite references [1,2,11–15] and for the vector wave equation about the transverse components, we refer to references [3,16–20].

The purpose of this paper is to present a unified formulation for analyzing the guided modes of the graded-index optical fibers and the step-index optical fibers. To do that, we show that the electromagnetic field of the guided modes can be expressed in terms of the solution of the vector wave equation about the radial functions that link to the transverse electric fields or the transverse magnetic fields [19] and derive an analytic solution of the vector wave equation. The vector wave equation employed here can be obtained by rearranging the vector wave equation with respect to the transverse electric fields [19–21]. Using the conventional perturbation method [22], from this equation we can construct an analytic solution of the vector wave equation, since the unperturbed equation has an analytic solution when we restrict the refractive-index profile of the optical fiber to an even polynomial profile by which the typical refractive-index profiles developed by industries such as near parabolic, Gaussian, and squared hyperbolic tangent profiles are described. As a result, we can represent the electromagnetic field of the guided modes of the graded-index optical fiber in a similar way as those of the step-index optical fiber. The unperturbed equation of the vector wave equation is a scalar wave equation that holds under the weak guidance approximation. Even for the even polynomial refractive-index profiles, we cannot get an exact and analytic solution of the scalar wave equation, although we get an exact and analytic solution for the square-law profile. Here we construct an asymptotic solution of the scalar wave equation [23]. It is noted that this is appropriate for the perturbation, but analytic solutions obtained until now do not take such forms for the perturbation [24–28]. The method of solution is a related equation method [29,30]. Applying a pair of transformations, the Langer transformation and the Liouville transformation, to the scalar wave equation several times, we approximately get a related equation whose solution can be expressed in analytic form [23, 30]. Therefore we have an approximate solution of the scalar wave equation but not exact one. The algorithm for constructing such solution is recursive so that we can easily obtain an analytic solution with the aid of an algebraic computer code such as “REDUCE” [31]. The analytic solution is a uniform asymptotic solution expressed by the Laguerre polynomials or the Kummer function [32]. Using the perturbation method, finally we obtain an analytic solution of the vector wave equation in which we use the recursive formula and the addition theorem about the Laguerre polynomials [32]. Results show that the first-order perturbed solution plays a dominant part of the vector corrections to the scalar wave solution, as indicated in the vector correction of the square-law optical fiber [20]. The accuracy of the uniform asymptotic solution of the vector wave equation depends on the accuracy of the uniform asymptotic solution of the scalar wave equation. When we use an M-th order perturbed solution in the analysis, we need an N-th order asymptotic solution whose order N satisfies the constraint such that $2N > 2M + 3$ for the dominant mode. Therefore we carefully check the accuracy of the uniform asymptotic solutions. After that we evaluate the propagation characteristics of the guided modes of graded-index optical fibers with even polynomial profile cores. Finally we numerically examine accuracies of the asymptotic solutions for both the scalar and the vector equations by checking normalized propagation constants and waveguide dispersions and so on [33–34]. First we check the convergence of the uniform asymptotic solution. The accuracy checks of the scalar wave solutions assert an existence of a best asymptotic solution among asymptotic solutions for a fixed asymptotic parameter, that is a well known nature of the asymptotic solution [35, 36]. As a matter of fact, the best asymptotic solution in the single mode region keeps its accuracy in the multimode region. Numerical comparison between the scalar wave solution and the vector wave solution indicates the effectiveness of the weak guidance approximation. On the other hand, the WKB solution that corresponds to the first-order scalar wave solution involves relatively large errors in the single mode region. Comparison between the first- and second-order perturbed solutions shows that the former plays a dominant part of the vector correction to the scalar wave solution and for a large refractive-index difference in the inhomogeneous region we need to use the vector wave solution as shown in the numerical example. Finally we design a dispersion-shifted optical fiber with an even polynomial profile core [37–42] after evaluating effects of the refractive-index profile on the cutoff frequencies, waveguide dispersions, mode field diameters, and bending losses. Numerical calculation can be carried out by using a desktop computer.

2. Electromagnetic Fields of the Guided Modes

Here we consider an axially symmetric optical fiber and use the cylindrical coordinate system (r, θ, z). We assume that the guided modes propagate along the z-axis according to $\exp\left[j\left(\omega t + n\theta - \beta z\right)\right]$ where β is a propagation constant. Referring to [19], for the hybrid modes $(n = 1,\ 2,\ 3, \ldots)$ we have a following vector wave equation, that is, a second order coupled differential equation in the inhomogeneous region:

$$\begin{aligned}
&\phi_1'' + \phi_1'/r + \left(k^2\left(b - h(r)\right) - (n^2+1)/r^2\right)\phi_1 + 2f_a\phi_1 \\
&\quad + \left(2n/r^2 + f_b - f_c\right)\phi_2 = 0 \\
&\phi_2'' + \phi_2'/r + \left(k^2\left(b - h(r)\right) - (n^2+1)/r^2\right)\phi_2 \\
&\quad + \left(2n/r^2 + f_b + f_c\right)\phi_1 = 0
\end{aligned} \tag{1}$$

In Equation (1), the primes denote derivatives about r, b and $h(r)$ are related to the propagation constant and the refractive-index as

$$\beta = k(1-b)^{1/2}\ , \quad k = k_0 n_0 \tag{2}$$

and

$$n(r) = n_0\left(1 - h(r)\right)^{1/2} \tag{3}$$

respectively, where k_0 and n_0 are the wavenumber in the vacuum and the refractive-index at $r = 0$, and f_a, f_b, and f_c are expressed in terms of $h(r)$ and its derivatives such as

$$\begin{aligned} f_a &= [h'(r)/r\,(1-h(r)) - h''(r)/\,(1-h(r)) \\ &\quad - 3\left(h'(r)/\,(1-h(r))\right)^2/2]\Big/4 \\ f_b &= \left(n/r^2\right)\left[1/\,(1-h(r))^{1/2} + (1-h(r))^{1/2} - 2\right] \\ &\quad - (n/2r)h'(r)/\,(1-h(r))^{3/2} \\ f_c &= \left(n/r^2\right)\left[1/\,(1-h(r))^{1/2} - (1-h(r))^{1/2}\right] \\ &\quad - (n/2r)h'(r)/\,(1-h(r))^{3/2} \end{aligned} \tag{4}$$

Then the electromagnetic fields for the hybrid modes in the inhomogeneous region can be expressed by the solution of Equation (1) in the following form [19]:

$$\begin{aligned} E_r &= j\phi_1/\,(1-h(r))^{1/2} \\ E_\theta &= -\phi_2 \\ E_Z &= (1/\beta)[(n/r)\left(\phi_1{}' + \left(1/r - \left\{h'(r)/\,(1-h(r))\right\}/2\right)\phi_1\right) \\ &\qquad /\,(1-h(r))^{1/2} - (n/r)\phi_2] \\ H_r &= -(1/\omega\mu_0\beta)\left[(n/r)\phi_1/\,(1-h(r))^{1/2} + \left(\beta^2 + (n/r)^2\right)\phi_2\right] \\ H_\theta &= (j/\omega\mu_0\beta)\left[k^2\left((1-h(r)) - (n/r)^2\right)\phi_1/\,(1-h(r))^{1/2}\right. \\ &\qquad \left. -(n/r)\left(\phi_2{}' + (1/r)\phi_2\right)\right] \\ H_Z &= (j/\omega\mu_0)\left[(n/r)\phi_1/\,(1-h(r))^{1/2} - \left(\phi_2{}' + (1/r)\phi_2\right)\right] \end{aligned} \tag{5}$$

For the transverse magnetic fields we also get a similar formulation for the hybrid modes [19]. It is noted that the vector wave equation given by Equation (1) is easier to solve than the vector wave equation about the longitudinal components of the electromagnetic field [1,2].

For TE- and TM-modes, we can easily get a field representation about the electromagnetic field of the guided modes. For TE-modes, we have

$$\begin{aligned} E_\theta &= \phi \\ H_r &= -(\beta/\omega\mu_0)\phi \\ H_Z &= j(1/\omega\mu_0)\left(\phi' + (1/r)\phi\right) \end{aligned} \tag{6}$$

where ϕ satisfies

$$\phi'' + \phi'/r + \left(k^2\left(b - h(r)\right) - 1/r^2\right)\phi = 0 \tag{7}$$

For the TM-modes, we get

$$\begin{aligned} E_r &= j\psi/\left(1 - h(r)\right)^{1/2} \\ E_Z &= (1/\beta)\left(\psi' + \left(1/r - \left\{h'(r)/\left(1 - h(r)\right)\right\}/2\right)\psi\right) \\ &\qquad /\left(1 - h(r)\right)^{1/2} \\ H_\theta &= j\left(\beta/\omega\mu_0\right)\left(1 - h(r)\right)^{1/2}\psi/(1 - b) \end{aligned} \tag{8}$$

where ψ fulfills

$$\psi'' + \psi'/r + \left(k^2\left(b - h(r)\right) - 1/r^2\right)\psi + 2f_a\psi = 0 \tag{9}$$

In the homogeneous region, the electromagnetic fields of the guided modes can be represented in terms of the Bessel functions and modified Bessel functions. So the problem is reduced to solve the vector wave Equation (1) and Equations (7) and (9) for a given $h(r)$.

3. Formal Perturbed Solution of the Vector Wave Equation

Here we show a formal solution of the vector wave equation (1) and a solution of Equation (9). First we solve Equation (1). Adding and subtracting the upper part and the lower part of Equation (1), we have a following equation such that

$$\begin{aligned} &\Phi'' + \Phi'/r + \left(k^2\,(b - h(r)) - (n-1)^2/r^2\right)\Phi + (f_a + f_b)\,\Phi \\ &\quad + (f_a + f_c)\,\Psi = 0 \\ &\Psi'' + \Psi'/r + \left(k^2\,(b - h(r)) - (n+1)^2/r^2\right)\Psi + (f_a - f_b)\,\Psi \\ &\quad + (f_a - f_c)\Phi = 0 \end{aligned} \tag{10}$$

where $\Phi = \phi_1 + \phi_2$ and $\Psi = \phi_1 - \phi_2$. Changing a variable $u = r/d$ where d is a normalization constant and ε is a small positive parameter defined by $\varepsilon = 1/kd$, we have

$$\begin{aligned} &\left[D_{n-1} + b - n\varepsilon^2\right]\Phi + H_{11}\Phi + H_{12}\Psi = 0 \\ &\left[D_{n+1} + b + n\varepsilon^2\right]\Psi + H_{21}\Phi + H_{22}\Psi = 0 \end{aligned} \tag{10$'$}$$

where $D_m{}'$s are second-order differential operators defined by

$$D_m = \varepsilon^2\left[d^2/du^2 + (1/u)d/du - m^2/u^2\right] - h(u)\ , \quad m = n-1\ ,\ n+1$$

In Equation $(10')$, $H_{mn}(m,\ n = 1,\ 2)$ take small values, provided that the absolute value of $h(u)$ is smaller smaller than 1. This assumption holds for typical graded-index optical fibers. In fact, let us consider a refractive-index profile $h(r)$ given by

$$\begin{aligned} h(r) = (r/d)^2 - a_2(r/d)^4 + a_3(r/d)^6 - a_4(r/d)^8 + \ldots \\ + (-1)^{L-1}a_L(r/d)^{2L} \end{aligned} \tag{11}$$

where $a_L{}'$s are appropriate constants and L is a positive integer. Then we have

$$
\begin{aligned}
H_{11} &= -\varepsilon^2 \left[((5n+6)/4 - 2a_2(n+1))\, u^2 \right. \\
&\quad \left. + ((13n+24)/8 - (4a_2 - 3a_3)\,(n+2))\, u^4 + \ldots \right] \\
H_{12} &= -\varepsilon^2 \left[((2n+3)/2 - a_2(n+2))\, u^2 \right. \\
&\quad \left. + (3(n+2)/2 - a_2(7n+16)/2 + 2a_3(n+3))\, u^4 + \ldots \right] \\
H_{22} &= \varepsilon^2 \left[((5n-6)/4 - 2a_2(n-1))\, u^2 \right. \\
&\quad \left. + ((13n-24)/8 - (4a_2 - 3a_3)\,(n-2))\, u^4 + \ldots \right] \\
H_{21} &= \varepsilon^2 \left[((2n-3)/2 - a_2(n-2))\, u^2 \right. \\
&\quad \left. + (3(n-2)/2 - a_2(7n-16)/2 + 2a_3(n-3))\, u^4 + \ldots \right]
\end{aligned} \tag{12}
$$

These quantities become small, provided that kd is greater greater than 1. Accordingly, we can solve Equation $(10')$ by the perturbation method. Hereafter we consider the refractive-index profile given by Equation (12), since typical refractive-index profiles can be described by Equation (12). When $h(r)$ is described by Equation (11), the vector wave equation can be classified into two cases according to whether $b - h(u)$ has a pair of turning points or not.

First we consider the case that $b - h(u)$ has a pair of simple turning points. The corresponding scalar wave equation to Equation $(10')$ is

$$
\begin{aligned}
\left[D_{n-1} + E_{n-1,\,p} \right] U_{n-1,\,p} = 0 \\
\left[D_{n+1} + E_{n+1,\,p} \right] U_{n+1,\,p} = 0
\end{aligned}
$$

where $E_{n,\,p}$ and $U_{n,\,p}$ are the eigenvalue and the eigenfunction of the scalar wave equation, respectively. Now let us solve Equation $(10)'$ by the perturbation method. According to the usual way about the perturbation method [22], we replace Equation $(10')$ by

$$
\begin{aligned}
\left[D_{n-1} + b - n\varepsilon^2 \right] \Phi + \lambda \left(H_{11}\Phi + H_{12}\Psi \right) = 0 \\
\left[D_{n+1} + b + n\varepsilon^2 \right] \Psi + \lambda \left(H_{21}\Phi + H_{22}\Psi \right) = 0
\end{aligned} \tag{14}
$$

where λ is a parameter such that $0 < \lambda \leq 1$. Next we expand Φ, Ψ, and b into power series of λ such that

$$\begin{aligned}\Phi &= \Phi^{(0)} + \lambda\Phi^{(1)} + \lambda^2\Phi^{(2)} + \dots \\ \Psi &= \Psi^{(0)} + \lambda\Psi^{(1)} + \lambda^2\Psi^{(2)} + \dots\end{aligned} \tag{15}$$

and

$$b = b^{(0)} + \lambda b^{(1)} + \lambda^2 b^{(2)} + \dots \tag{16}$$

Substituting Equations (15) and (16) into Equation (14) and equating the coefficients of equal power of λ, we have

$$\begin{aligned}\left[D_{n-1} + b^{(0)} - n\varepsilon^2\right]\Phi^{(0)} &= 0 \\ \left[D_{n+1} + b^{(0)} + n\varepsilon^2\right]\Psi^{(0)} &= 0\end{aligned} \tag{17}$$

$$\begin{aligned}\left[D_{n-1} + b^{(0)} - n\varepsilon^2\right]\Phi^{(1)} + H_{11}\Phi^{(0)} + H_{12}\Psi^{(0)} + b^{(1)}\Phi^{(0)} &= 0 \\ \left[D_{n+1} + b^{(0)} + n\varepsilon^2\right]\Psi^{(1)} + H_{21}\Phi^{(0)} + H_{22}\Psi^{(0)} + b^{(1)}\Psi^{(0)} &= 0\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\left[D_{n-1} + b^{(0)} - n\varepsilon^2\right]\Phi^{(2)} + H_{11}\Phi^{(1)} + H_{12}\Psi^{(1)} \\ &\quad + b^{(1)}\Phi^{(1)} + b^{(2)}\Phi^{(0)} = 0 \\ &\left[D_{n+1} + b^{(0)} + n\varepsilon^2\right]\Psi^{(2)} + H_{21}\Phi^{(1)} + H_{22}\Psi^{(1)} \\ &\quad + b^{(1)}\Psi^{(1)} + b^{(2)}\Psi^{(0)} = 0\end{aligned} \tag{19}$$

and so on. Solving these equations and setting λ equal to 1, we can get a perturbed solution of Equation $(10')$, provided that we have an analytic solution of the unperturbed equation, that is, the scalar wave equation can be solved in a closed form; from Equations (17), (18), and (19) we obtain the zero-, first-, and second-order solutions.

Now let us derive formal solutions. Multiplying Equations (13) and (17) by $\Phi^{(0)}$ and $U_{\nu,\,p}(\nu = n-1, n+1)$, integrating them from 0 to infinity, and subtracting them, we have

$$\left(b^{(0)} - E_{n-1,\,m} - n\varepsilon^2\right)\int_0^\infty u\Phi^{(0)}U_{n-1,\,m}du = 0 \tag{20a}$$

and

$$\left(b^{(0)} - E_{n+1,\,p} + n\varepsilon^2\right)\int_0^\infty u\Psi^{(0)}U_{n+1,\,p}du = 0 \tag{20b}$$

On the other hand, it follows from Equation (13) that the orthogonality relations such that

$$\int_0^\infty uU_{\nu,\,m}U_{\nu,\,p}du = \delta_{m,\,p}\int_0^\infty u(U_{\nu,\,m})^2du, \quad \nu = n-1,\ n+1 \tag{21}$$

where $\delta_{m,\,p}$ is the Kronecker's delta. From Equations (20) and (21), we obtain

$$\Phi^{(0)} = U_{n-1,\,m} \quad , \quad \Psi^{(0)} = 0 \quad , \quad \text{and} \quad b^{(0)} = E_{n-1,\,m} + n\varepsilon^2 \tag{22}$$

where we assume that $E_{n-1,\,m} - E_{n+1,\,p} + 2n\varepsilon^2$ is not equal to zero for any p. Similarly, we have another solution of Equation (17) such that

$$\Psi^{(0)} = U_{n+1,\,m} \quad , \quad \Phi^{(0)} = 0 \quad , \quad \text{and} \quad b^{(0)} = E_{n+1,\,m} - n\varepsilon^2 \tag{23}$$

where we assume that $E_{n+1,\,m} - E_{n-1,\,p} - 2n\varepsilon^2 = 0$ for any p. It should be noted that the HE- and EH-modes follow from Equations (22) and (23), respectively.

Calculation of the HE-modes starts from Equation (22). Now we expand the first-order correction terms $\Phi^{(1)}$ and $\Psi^{(1)}$ into power series of $U_{n-1,\,q}$ and $U_{n+1,\,q}$, respectively such as

$$\Phi^{(1)} = \sum A_q{}^{(1)}U_{n-1,\,q} \tag{24a}$$

and

$$\Psi^{(1)} = \sum B_q{}^{(1)}U_{n+1,\,q} \tag{24b}$$

where $A_q{}^{(1)}$ and $B_q{}^{(1)}$ are unknown coefficients to be determined. Substituting Equations (22) and (24) into Equation (18) and using the orthogonality relation Equation (21), we obtain formal solutions of $A_q{}^{(1)}$ and $B_q{}^{(1)}$ such as

$$\begin{aligned} A_q{}^{(1)} &= -\left[\int_0^\infty uH_{11}U_{n-1,\,m}U_{n-1,\,q}du \Big/ \int_0^\infty u(U_{n-1,\,q})^2 du\right] \\ &\quad \Big/ (E_{n-1,\,m} - E_{n-1,\,q}) \quad , \qquad q \neq m \\ B_q{}^{(1)} &= -\left[\int_0^\infty uH_{21}U_{n-1,\,m}U_{n+1,\,q}du \Big/ \int_0^\infty u(U_{n+1,\,q})^2 du\right] \\ &\quad \Big/ (E_{n-1,\,m} - E_{n+1,\,q} + 2n\varepsilon^2) \end{aligned} \tag{25}$$

and

$$b^{(1)} = -\int_0^\infty uH_{11}\left(U_{n-1,\,m}\right)^2 du \Big/ \int_0^\infty u\left(U_{n-1,\,m}\right)^2 du \tag{26}$$

Now we calculate the degenerate case that $E_{n-1,\,m} - E_{n+1,\,p} + 2n\varepsilon^2$ is zero. Let us represent $\Psi^{(0)}$ as

$$\Psi^{(0)} = AU_{n+1,\,p}$$

where A is unknown constant. Then from Equation (18), for some p we have

$$\begin{aligned} &\left(\int_0^\infty u\left(U_{n-1,\,m}\right)^2 du\right) b^{(1)} + \int_0^\infty uH_{11}\left(U_{n-1,\,m}\right)^2 du \\ &\quad + \left(\int_0^\infty uH_{12}U_{n+1,\,p}U_{n-1,\,m}du\right) A = 0 \\ &\left[\left(\int_0^\infty u\left(U_{n+1,\,p}\right)^2 du\right) b^{(1)} + \int_0^\infty uH_{21}\left(U_{n+1,\,p}\right)^2 du\right] A \\ &\quad + \int_0^\infty uH_{22}U_{n-1,\,m}U_{n+1,\,p}du = 0 \end{aligned} \tag{26$'$}$$

and

$$\begin{aligned} A_q{}^{(1)} &= -\left[\left(\int_0^\infty uH_{11}U_{n-1,\,m}U_{n-1,\,q}du \right.\right. \\ &\qquad \left.\left. + \left(\int_0^\infty uH_{12}U_{n+1,\,p}U_{n-1,\,q}du\right) A\right) \Big/ \int_0^\infty u\left(U_{n-1,\,q}\right)^2 du\right] \\ &\quad \Big/ (E_{n-1,\,m} - E_{n-1,\,q}) \quad , \qquad q \neq m \end{aligned}$$

$$B_q{}^{(1)} = -\left[\left(\left(\int_0^\infty uH_{22}U_{n-1,\,m}U_{n+1,\,q}du\right.\right.\right. \tag{25$'$}$$

$$\left.\left.+\left(\int_0^\infty uH_{21}U_{n+1,\,p}U_{n+1,\,q}du\right)A\right)\Big/\int_0^\infty u\left(U_{n+1,\,q}\right)^2 du\right]$$

$$\Big/\left(E_{n-1,\,m} - E_{n+1,\,q} + 2n\varepsilon^2\right) \quad , \qquad\qquad q = p$$

From Equation (26′) we obtain A and $b^{(1)}$ and from Equation (25′) we have $A_q{}^{(1)}$ and $B_q{}^{(1)}$ $(q = p)$. It should be noted that $B_p{}^{(1)}$ can be determined by the second-order perturbation together with $b^{(2)}$.

Next we show the second-order perturbation. To do that, we expand $\Phi^{(2)}$ and $\Psi^{(2)}$ such as

$$\Phi^{(2)} = \sum A_q{}^{(2)}U_{n-1,\,q} \tag{27a}$$

and

$$\Psi^{(2)} = \sum B_q{}^{(2)}U_{n+1,\,q} \tag{27b}$$

Substituting Equation (27) into Equation (19), we have

$$\begin{aligned}
A_q{}^{(2)} = &-\left[\left(\sum{}' A_p{}^{(1)}\int_0^\infty uH_{11}U_{n-1,\,p}U_{n-1,\,q}du\right.\right.\\
&\left.+\sum B_p{}^{(1)}\int_0^\infty uH_{12}U_{n+1,\,p}U_{n-1,\,q}du\right)\\
&\left.\Big/\int_0^\infty u(U_{n-1,\,q})^2du + A_q{}^{(1)}b^{(1)}\right]\\
&\Big/(E_{n-1,\,m} - E_{n-1,\,q})\\
B_q{}^{(2)} = &-\left[\left(\sum{}' B_p{}^{(1)}\int_0^\infty uH_{22}U_{n+1,\,p}U_{n+1,\,q}du\right.\right.\\
&\left.+\sum A_p{}^{(1)}\int_0^\infty uH_{21}U_{n-1,\,p}U_{n+1,\,q}du\right)\\
&\left.\Big/\int_0^\infty u\left(U_{n+1,\,q}\right)^2 du + B_q{}^{(1)}b^{(1)}\right]\\
&\Big/\left(E_{n-1,\,m} - E_{n+1,\,q} + 2n\varepsilon^2\right)
\end{aligned} \tag{28}$$

and

$$b^{(2)} = -\left[\sum{}' A_p{}^{(1)} \int_0^\infty uH_{11}U_{n-1,\,p}U_{n-1,\,m}du \right.$$
$$\left. + \sum B_p{}^{(1)} \int_0^\infty uH_{12}U_{n+1,\,p}U_{n-1,\,m}du \right] \bigg/ \int_0^\infty u\left(U_{n-1,\,m}\right)^2 du \tag{29}$$

Next we calculate the degenerate case. For this case, we have a simultaneous equation with respect to $b^{(2)}$ and $B_p{}^{(1)}$ that is a similar equation as Equation $(25')$ such that

$$\left(\int_0^\infty u\left(U_{n-1,\,m}\right)^2 du\right) b^{(2)} + \left(\int_0^\infty uH_{12}U_{n+1,\,p}U_{n-1,\,m}du\right) B_p{}^{(1)}$$
$$+ \sum{}' B_q{}^{(1)} \int_0^\infty uH_{12}U_{n+1,\,q}U_{n-1,\,m}du$$
$$+ \sum{}' A_p{}^{(1)} \int_0^\infty uH_{11}U_{n-1,\,p}U_{n-1,\,m}du = 0 \tag{29$'$}$$
$$\left(\int_0^\infty u\left(U_{n+1,\,p}\right)^2 du\right) Ab^{(2)} + \left(\int_0^\infty uH_{21}\left(U_{n+1,\,p}\right)^2 du\right) B_p{}^{(1)}$$
$$+ \sum{}' B_q{}^{(1)} \int_0^\infty uH_{21}U_{n+1,\,q}U_{n+1,\,p}du$$
$$+ \sum{}' A_q{}^{(1)} \int_0^\infty uH_{22}U_{n-1,\,q}U_{n+1,\,p}du = 0$$

Solving this equation, we have $b^{(2)}$ and $B_p{}^{(1)}$. Using these results, we can calculate $A_q{}^{(2)}$ and $B_q{}^{(2)}$ except for $q = p$. It is noted that $A_p{}^{(2)}$ can be calculated in the third-order perturbation. Here we do not calculate more higher-order perturbational solutions, because their contributions are not significant for vector corrections.

Next we calculate the EH-modes. From Equation (23), we have

$$\Psi^{(1)} = \sum A_q{}^{(1)} U_{n+1,\,q} \tag{30a}$$

and

$$\Phi^{(1)} = \sum B_q{}^{(1)} U_{n-1,\,q} \tag{30b}$$

where $A_q{}^{(1)}$ and $B_q{}^{(1)}$ are given by

$$
\begin{aligned}
A_q{}^{(1)} &= -\left[\int_0^\infty uH_{22}U_{n+1,\,m}U_{n+1,\,q}du \Big/ \int_0^\infty u\left(U_{n+1,\,q}\right)^2 du\right] \\
&\quad \Big/ \left(E_{n+1,\,m}-E_{n+1,\,q}\right) \\
B_q{}^{(1)} &= -\left[\int_0^\infty uH_{12}U_{n+1,\,m}U_{n-1,\,q}du \Big/ \int_0^\infty u\left(U_{n-1,\,q}\right)^2 du\right] \\
&\quad \Big/ \left(E_{n+1,\,m}-E_{n-1,\,q}-2n\varepsilon^2\right)
\end{aligned} \tag{31}
$$

and

$$
b^{(1)} = -\int_0^\infty uH_{22}\left(U_{n+1,\,m}\right)^2 du \Big/ \int_0^\infty u\left(U_{n+1,\,m}\right)^2 du \tag{32}
$$

For the degenerate case [20] that $E_{n+1,\,m}-E_{n-1,\,q}-2n\varepsilon^2$ is zero, $\Phi^{(0)}$ is determined by the first-order perturbation. We express $\Phi^{(0)}$ as

$$
\Phi^{(0)} = BU_{n-1,\,q}
$$

where B is unknown constant. Then from Equation (18), for some q we have

$$
\begin{aligned}
&\left[\left(\int_0^\infty u\left(U_{n-1,\,q}\right)^2 du\right) b^{(1)} + \int_0^\infty uH_{11}\left(U_{n-1,\,q}\right)^2 du\right] B \\
&\quad + uH_{12}U_{n+1,\,m}U_{n-1,\,q}du = 0 \\
&\left(\int_0^\infty u\left(U_{n+1,\,m}\right)^2 du\right) b^{(1)} + \int_0^\infty uH_{21}\left(U_{n+1,\,m}\right)^2 du \\
&\quad + \left(\int_0^\infty uH_{22}U_{n-1,\,q}U_{n+1,\,m}du\right) B = 0
\end{aligned} \tag{32$'$}
$$

and

$$
\begin{aligned}
A_p{}^{(1)} = -\Bigg[\Bigg(&\int_0^\infty uH_{21}U_{n+1,\,m}U_{n+1,\,p}du \\
&+ \left(\int_0^\infty uH_{22}U_{n-1,\,q}U_{n+1,\,p}du\right) B\Bigg) \Big/ \int_0^\infty u\left(U_{n+1,\,p}\right)^2 du\Bigg] \\
&\Big/ \left(E_{n+1,\,m}-E_{n+1,\,p}\right)
\end{aligned} \tag{31$'$}
$$

$$B_p{}^{(1)} = -\left[\left(\left(\int_0^\infty uH_{12}U_{n+1,\,m}U_{n-1,\,p}du\right.\right.\right.$$
$$\left.\left.+\left(\int_0^\infty uH_{11}U_{n-1,\,q}U_{n-1,\,p}du\right)B\right)\bigg/\int_0^\infty u\left(U_{n-1,\,p}\right)^2 du\right]$$
$$\bigg/\left(E_{n+1,\,m} - E_{n-1,\,p} - 2n\varepsilon^2\right)\ , \qquad p = q$$

From Equation (32′) we obtain B and $b^{(1)}$ and from Equation (31′) we have $A_p{}^{(1)}$ and $B_p{}^{(1)}$ except for q. It is noted that $B_q{}^{(1)}$ can be determined by the second-order perturbation.

In a similar way, we can show the second-order perturbation. Results are as follows:

$$\Psi^{(2)} = \sum A_p{}^{(2)}U_{n+1,\,p} \tag{33a}$$

and

$$\Phi^{(2)} = \sum B_p{}^{(2)}U_{n-1,\,p} \tag{33b}$$

Substituting Equation (33) into Equation (19), we have

$$\begin{aligned}
A_p{}^{(2)} = -&\left[\left(\sum{}' A_r{}^{(1)}\int_0^\infty uH_{22}U_{n+1,\,r}U_{n+1,\,p}du\right.\right.\\
&\left.+\sum B_r{}^{(1)}\int_0^\infty uH_{21}U_{n-1,\,r}U_{n+1,\,p}du\right)\\
&\left.\bigg/\int_0^\infty u\left(U_{n+1,\,p}\right)^2 du + A_p{}^{(1)}b^{(1)}\right]\\
&\bigg/\left(E_{n+1,\,m} - E_{n+1,\,p}\right)\\
B_p{}^{(2)} = -&\left[\left(\sum{}' B_r{}^{(1)}\int_0^\infty uH_{12}U_{n+1,\,r}U_{n-1,\,p}du\right.\right.\\
&\left.+\sum A_r{}^{(1)}\int_0^\infty uH_{11}U_{n-1,\,r}U_{n-1,\,p}du\right)\\
&\left.\bigg/\int_0^\infty u\left(U_{n-1,\,p}\right)^2 du + B_p{}^{(1)}b^{(1)}\right]\\
&\bigg/\left(E_{n+1,\,m} - E_{n-1,\,p} - 2n\varepsilon^2\right)
\end{aligned} \tag{34}$$

and

$$b^{(2)} = -\left[\sum{}' A_p{}^{(1)} \int_0^\infty uH_{22}U_{n+1,\,p}U_{n+1,\,m}du \right. \tag{35}$$
$$\left. + \sum B_p{}^{(1)} \int_0^\infty uH_{21}U_{n-1,\,p}U_{n+1,\,m}du \right] \Big/ \int_0^\infty u\left(U_{n+1,\,m}\right)^2 du$$

Next we calculate the degenerate case. For this case, we have a simultaneous equation with respect to $b^{(2)}$ and $B_q{}^{(1)}$ that is a similar equation as Equation $(25')$ such that

$$\left(\int_0^\infty u\left(U_{n+1,\,m}\right)^2 du\right) b^{(2)} + \left(\int_0^\infty uH_{22}U_{n-1,\,q}U_{n+1,\,m}du\right) B_q{}^{(1)}$$
$$+ \sum{}' B_r{}^{(1)} \int_0^\infty uH_{22}U_{n-1,\,r}U_{n+1,\,m}du$$
$$+ \sum{}' A_r{}^{(1)} \int_0^\infty uH_{21}U_{n+1,\,r}U_{n+1,\,m}du = 0 \tag{35$'$}$$
$$\left(\int_0^\infty u\left(U_{n-1,\,q}\right)^2 du\right) Bb^{(2)} + \left(\int_0^\infty uH_{11}\left(U_{n-1,\,q}\right)^2 du\right) B_q{}^{(1)}$$
$$+ \sum{}' B_r{}^{(1)} \int_0^\infty uH_{12}U_{n+1,\,r}U_{n-1,\,q}du$$
$$+ \sum{}' A_r{}^{(1)} \int_0^\infty uH_{12}U_{n+1,\,r}U_{n-1,\,q}du = 0$$

From Equation $(35')$, we have $b^{(2)}$ and $B_q{}^{(1)}$ Using these results, we can calculate $A_r{}^{(2)}$ and $B_r{}^{(2)}$ except for $r = q$. It is noted that $A_q{}^{(2)}$ can be calculated in the third-order perturbation.

Finally we calculate the TM-modes. After changing the variable $u = r/d$, from Equation (9) we have

$$[D + b]\,\psi + \lambda H\psi = 0 \tag{36}$$

where λ is a positive constant as shown in Equation (14), and D and H are given by

$$D = \varepsilon^2\left[d^2/du^2 + (1/u)d/du - 1/u^2\right] - h(u) \tag{37}$$

and

$$H = -\varepsilon^2 \left[(3 - 4a_2)\, u^2 + (6 - 4\,(4a_2 - 3a_3))\, u^4 + \ldots\right] \tag{38}$$

respectively. Now we expand ψ and b into power series of λ such that

$$\psi = \psi^{(0)} + \lambda\psi^{(1)} + \lambda^2\psi^{(2)} + \ldots \tag{39}$$

and

$$b = b^{(0)} + \lambda b^{(1)} + \lambda^2 b^{(2)} + \ldots \tag{40}$$

Substituting Equations (39) and (40) into Equation (36) and equating the coefficients of the equal power of λ, we have

$$\left[D + b^{(0)}\right] \psi^{(0)} = 0 \tag{41}$$

$$\left[D + b^{(0)}\right] \psi^{(1)} + H\psi^{(0)} + b^{(1)}\psi^{(0)} = 0 \tag{42}$$

and

$$\left[D + b^{(0)}\right] \psi^{(2)} + H\psi^{(1)} + b^{(1)}\psi^{(1)} + b^{(2)}\psi^{(0)} = 0 \tag{43}$$

The corresponding scalar wave equation is the lower side of Equation (13):

$$[D + E_{1,\,m}]\, U_{1,\,m} = 0 \tag{44}$$

From Equation (44), we have the orthogonality relation

$$\int_0^\infty uU_{1,\,m}U_{1,\,p}du = \delta_{m,\,p} \int_0^\infty u\,(U_{1,\,m})^2\, du \tag{45}$$

From Equations (41) and (44), we obtain

$$\left(b^{(0)} - E_{1,\,m}\right) \int_0^\infty u\psi^{(0)}U_{1,\,m}du = 0 \tag{46}$$

Thus we can calculate the eigenvalue and the eigenfunction of the TM-modes. Results are as follows:

$$\begin{aligned} b^{(0)} &= E_{1,\,m} \\ \psi^{(0)} &= U_{1,\,m} \end{aligned} \tag{47}$$

$$b^{(1)} = -\int_0^\infty uH\left(U_{1,\,m}\right)^2 du \Big/ \int_0^\infty u\left(U_{1,\,m}\right)^2 du$$
$$\psi^{(1)} = \Sigma {A_q}^{(1)} U_{1,\,q} \tag{48}$$

with

$$A_q{}^{(1)} = -\left[\int_0^\infty uHU_{1,\,m}U_{1,\,q}du \Big/ \int_0^\infty u\left(U_{1,\,m}\right)^2 du\right] \Big/ \left(E_{1,\,m} - E_{1,\,q}\right)\;, \quad q \neq m \tag{49}$$

and

$$b^{(2)} = -\left[\Sigma' {A_q}^{(1)} \int_0^\infty uHU_{1,\,m}U_{1,\,q}du \Big/ \int_0^\infty u\left(U_{1,\,m}\right)^2 du + {A_m}^{(1)} b^{(1)}\right]$$
$$\psi^{(2)} = \Sigma {A_q}^{(2)} U_{1,\,q} \tag{50}$$

with

$$A_q{}^{(2)} = -\left[\Sigma {A_p}^{(1)} \int_0^\infty uHU_{1,\,p}U_{1,\,q}du \Big/ \int_0^\infty u\left(U_{1,\,m}\right)^2 du + {A_q}^{(1)} b^{(1)}\right] \Big/ \left(E_{1,\,m} - E_{1,\,q}\right) \tag{51}$$

Here we do not show more higher-order terms. These results are formal solutions of Equation (1) and Equation (9) in the case that $b - h(u)$ has simple zeros.

For TE-modes, we have

$$b = E_{1,\,m}$$
$$\phi = U_{1,\,m} \tag{52}$$

Next we consider the case that $b - h(u)$ is always positive. In this case, the solutions of Equation $(10')$ can be obtained by using variation of parameters [36]. It is apparent that Equation (17) has a solution

$$\Phi^{(0)} = 0 \;\text{ and }\; \Psi^{(0)} = 0 \tag{53}$$

or

$$\Psi^{(0)} = 0 \;\text{ and }\; \Phi^{(0)} = 0 \tag{54}$$

Now we show a perturbed solution in the case that we start from Equation (53). Using the two independent solutions of Equation (17) $\Phi_1^{(0)}$ and $\Phi_2^{(0)}$ we can get a first-order solution of Equation (18) that vanishes at infinity in the form as

$$\begin{aligned}\Phi_1^{(1)} = F\Phi_1^{(0)} &+ \left[\int_0^\infty \left(u\left(b^{(1)} + H_{11}\right)\Phi_1^{(0)}\Phi_2^{(0)}\Big/W\right)du\right]\Phi_1^{(0)} \\ &- \left[\int_0^\infty \left(u\left(b^{(1)} + H_{11}\right)\left(\Phi_1^{(0)}\right)^2\Big/W\right)du\right]\Phi_2^{(0)} , \end{aligned} \tag{55}$$

$$W = \Phi_1^{(0)}\Phi'^{(0)}_2 - \Phi_2^{(0)}\Phi'^{(0)}_1$$

$$\begin{aligned}\Psi_1^{(1)} = &\left[\int_0^\infty \left(uH_{21}\Phi_1^{(0)}\Psi_2^{(0)}\Big/V\right)du\right]\Phi_1^{(0)} \\ &- \left[\int_0^\infty \left(uH_{21}\Phi_1^{(0)}\Psi_1^{(0)}\Big/V\right)du\right]\Phi_2^{(0)}\end{aligned} \tag{56}$$

$$V = \Psi_1^{(0)}\Psi'^{(0)}_2 - \Psi_2^{(0)}\Psi'^{(0)}_1$$

where F is an arbitrary constant. Similarly, starting from Equation (54) we have

$$\begin{aligned}\Psi_1^{(1)} = G\Psi_1^{(0)} &+ \left[\int_0^\infty \left(u\left(b^{(1)} + H_{22}\right)\Psi_1^{(0)}\Psi_2^{(0)}\Big/V\right)du\right]\Psi_1^{(0)} \\ &- \left[\int_0^\infty \left(u\left(b^{(1)} + H_{22}\right)\left(\Psi_1^{(0)}\right)^2\Big/V\right)du\right]\Psi_2^{(0)}\end{aligned} \tag{57}$$

$$\begin{aligned}\Phi_1^{(1)} = &\left[\int_0^\infty \left(uH_{12}\Psi_1^{(0)}\Phi_2^{(0)}\Big/V\right)du\right]\Psi_1^{(0)} \\ &- \left[\int_0^\infty \left(uH_{12}\Psi_1^{(0)}\Phi_1^{(0)}\Big/V\right)du\right]\Psi_2^{(0)}\end{aligned} \tag{58}$$

where G is an arbitrary constant. Here we do not show the more higher-order terms.

Thus we get formal solutions of the vector wave equation for the guided modes of the graded-index optical fiber. Accordingly the problem is reduced to obtain a closed form solution of the scalar wave equation for even polynomial refractive-index profiles. Fortunately we have a uniform asymptotic solution for the guided modes in inhomogeneous slab waveguides with even polynomial refractive-index profile where we use a related equation method [43]. Next section we devote to derive a uniform asymptotic solution of the scalar wave equation by using the related equation method.

4. Uniform Asymptotic Solution of the Scalar Wave Equation

Here we only consider the graded-index optical fiber with the even polynomial profile core such as

$$h(r) = (r/d)^2 - a_2(r/d)^4 + a_3(r/d)^6 + \ldots + (-1)^{L-1}a_L(r/d)^{2L} \quad (11)$$

where a_p's are appropriate constants, L is a positive integer, and "d" is a normalization constant. If we set $a_p = 1/p!$ in Equation (11), the permittivity (see Equation (3)) is an approximate Gaussian profile. Setting $a_2 = 2/3$, $a_3 = 17/45$, $a_4 = 62/315$, and so on in Equation (11), we get an approximate squared hyperbolic tangent profile. Similarly we can represent near parabolic profiles by using Equation (11). Thus the even polynomial profile is an actual model of the refractive-index profile of the graded-index optical fibers.

Now let us solve Equation (13) that covers equations for the TE- and TM-modes. The method of solution used here is a related equation method. Rewriting all variables in Equation (13) by the quantities with the subscript "1", then we have

$$\begin{aligned}&\left[\varepsilon^2\left(d^2/du_1{}^2 + (1/u_1)\,d/du_1 - \nu^2/u_1{}^2\right) + Q_1\left(u_1\right)\right]U_1\left(u_1\right) = 0\ ,\\ &\qquad \nu = n-1,\ n+1\end{aligned} \quad (59)$$

where we set $U_1(u_1) = U_{\nu,m}(u)$, $u_1 = u$, and

$$\begin{aligned}Q_1(u_1) &= E_{\nu,\,m} - h\left(u_1\right)\\ &= E_{\nu,\,m} - \left(u_1\right)^2 - a_2\left(u_1\right)^4 + a_3\left(u_1\right)^6 + \cdots\\ &\quad + (-1)^{L-1}a_L\left(u_1\right)^{2L}\end{aligned} \quad (60)$$

The positive or negative value of $Q_1(u_1)$-function classifies the property of the solution into an oscillatory one and decaying or increasing one in one direction. Now we need to construct two kind of solutions of Equation (59); solution in the case that $Q_1(u_1)$-function has two simple zeros and solution the case that $Q_1(u_1)$-function is always positive.

(I) *Solution in the case that* $Q_1(u_1)$ *-function has two simple zeros*

First we define a pair of transformations, the Langer transformation and the Liouville transformation [24]. They are defined by

$$du_{p+1}/du_p = \left(Q_p\left(u_p\right)\right)^{1/2} \Big/ \left(I_p\left(1-{u_{p+1}}^2\right)^{1/2}\right) , \quad \text{for } p=1,\ 2,\dots,\ N \tag{61}$$

and

$$U_{p+1}\left(u_{p+1}\right) = \left(\left(u_{p+1}/u_p\right) du_p/du_{p+1}\right)^{-1/2} U_p\left(u_p\right) , \quad \text{for } p=1,\ 2,\dots,\ N \tag{62}$$

respectively. In Equation (62), $Q_p(u_p)$ and I_p are given by

$$Q_p(u_p) = \begin{cases} E_{\nu,m} - h(u_1) , & \text{for } p=1 \\ {I_{p-1}}^2\left(1-{u_p}^2\right) - \varepsilon^2 R_{p-1}(u_p), & \text{for } p=2,\ 3,\dots,\ N+1 \end{cases} \tag{63}$$

with

$$\begin{aligned} R_{p-1}(u_p) = {} & \left(\nu^2/{u_p}^2\right)\left[\left(\left(u_p/u_{p-1}\right) du_{p-1}/du_p\right)^2 - 1\right] \\ & + \left(\left(u_p/u_{p-1}\right) du_{p-1}/du_p\right)^{1/2}\left(d^2/{du_p}^2 + \left(1/u_p\right) d/du_p\right) \\ & \times \left(\left(u_p/u_{p-1}\right) du_{p-1}/du_p\right)^{-1/2} \qquad \text{for } p=2,\ 3,\dots,\ N+1 \end{aligned} \tag{64}$$

and

$$I_p = (\pi/4)\int_0^{\xi_p} \left(Q_p(u_p)\right)^{1/2} du_p \qquad \text{for } p=1,\ 2,\dots,\ N \tag{65}$$

where ξ_p is a simple zero of $Q_p(u_p)$. Now we briefly discuss the analytic property of the Langer transformation. This transforms u_p to an independent variable u_{p+1}. The mapping considered here is the one that makes the domains $0 < u_p < \xi_p$ and $\xi_p < u_p$ correspond one-to-one to the domains $0 < u_{p+1} < 1$ and $1 < u_{p+1}$, respectively. If $Q_p(u_p)$ and $dQ_p(u_p)/du_p$ are continuously differentiable and nonzero at ξ_p, respectively, then du_p/du_{p+1} takes a nonvanishing positive value and u_{p+1} is continuously differentiable about u_p. Hence the Langer transformation and the Liouville transformation have no singularity in the whole domain considered here. As a consequence, we get a uniform asymptotic solution of Equation (59).

Hereafter we show an algorithm for calculating a uniform asymptotic solution of Equation (59). Application of a pair of transformations Equations (61) and (62) to Equation (59) N times results in

$$\Big[d^2/du_{N+1}{}^2 + (1/u_{N+1})\, d/du_{N+1} + (I_N/\varepsilon)^2 \left(1 - u_{N+1}{}^2\right) - \nu^2/u_{N+1}{}^2 - R_N\left(u_{N+1}\right)\Big]\, U_{N+1}\left(u_{N+1}\right) = 0\ , \qquad \nu = n-1,\ n+1 \tag{66}$$

If $R_N(u_{N+1}) = 0$, we have a related equation such as

$$\Big[d^2/du_{N+1}{}^2 + (1/u_{N+1})\, d/du_{N+1} + (I_N/\varepsilon)^2 \left(1 - u_{N+1}{}^2\right) - \nu^2/u_{n+1}{}^2\Big]\, U_{N+1}\left(u_{N+1}\right) = 0\ , \qquad \nu = n-1,\ n+1 \tag{67}$$

Provided that

$$I_N = s\varepsilon, \quad s = 4m+2\nu+2, \quad \nu = n-1,\ n+1, \quad m = 0,\ 1,\ 2, \ldots \tag{68}$$

one solution of the related equation Equation (67) that vanishes at infinity can be expressed in the following form:

$$U_{N+1}(u_{N+1}) = \left(su_{N+1}{}^2\right)^{\nu/2} \exp\left[-su_{N+1}{}^2/2\right] F\left(-m,\ \nu+1;\ su_{N+1}{}^2\right) \tag{69}$$

where $F(m, n; Z)$ is the Kummer function [32] and the index "m" denotes a mode number about the radial direction. Rewriting Equation (68), we have

$$\int_0^{\xi_N} \left(Q_N\left(u_N\right)\right)^{1/2} du_N = (\pi/4)s\varepsilon \tag{70}$$

This is a quantum condition for determining the guided modes of the uncladded graded-index optical fiber. From Equations (62) and (69), we have

$$\begin{aligned} &U_{N+1}\left(u_{N+1}\right) \\ &= \left(\prod_{p=1}^{N}\left(u_p/u_{p+1}\right) du_{p+1}/du_p\right)^{-1/2} \times \left(s{u_{N+1}}^2\right)^{\nu/2} \\ &\times \exp\left[-s{u_{N+1}}^2/2\right] F\left(-m,\ \nu+1;\ s{u_{N+1}}^2\right) \end{aligned} \tag{71}$$

In Equation (71), we use an abbreviation such as $U_1(u_1) = U_{N+1}(u_{N+1}(u_1))$, because u_{N+1} can be represented by the power series of u_1 as shown in the algorithm mentioned later. It should be noted that the related equation method generates an analytic solution of Equation (59) for the even polynomial refractive-index profiles as expected in the calculation of the guided modes of the inhomogeneous slab waveguide [43].

Now let us express $u_{N+1}(u_1)$ in terms of the power series of u_1. First we assume that $Q_p(u_p)$ is an even polynomial of u_p and ξ_p is a simple zero of $Q_p(u_p)$. Then $Q_p(u_p)$ can be written in the form:

$$Q_p\left(u_p\right) = \left({\xi_p}^2 - {u_p}^2\right) g_p\left(u_p\right)$$

where $g_p(u_p)$ is represented by a power series of ${u_p}^2$. Therefore we can calculate I_p with the aid of the integral formula. Consequently we have a series solution of Equation (61) in the form:

$$u_P = c_{p0}u_{p+1} + c_{p1}{u_{p+1}}^3 + c_{p2}{u_{p+1}}^5 + \ldots \tag{72}$$

It should be noted that u_p has no constant term, because the origin of u_p corresponds to that of u_{p+1}. Using Equation (72), we can calculate $R_p(u_{p+1})$ in the power series of u_{p+1}. As a result, $Q_{p+1}(u_{p+1})$ can be represented by a power series of ${u_{p+1}}^2$. Conversely, u_{p+1} can be expressed by a power series of u_p. For the even polynomial profile core, $Q_1(u_1)$ is an even polynomial of u_1 so that $Q_2(u_2)$ can be written by a power series of ${u_2}^2$. Therefore recursive substituting of power series representations reaches the final result, that is, u_{N+1} can be represented by the power series of u_1. From Equation (60), we have

$$E_{\nu,\,m} = {\xi_1}^2 - a_2{\xi_1}^4 + a_3{\xi_1}^6 - \ldots + (-1)^{L-1}a_L{\xi_1}^{2L} \tag{73}$$

The relation Equation (72) hold for ξ_p and ξ_{p+1}. Recursive substitution about zero points results in the eigenvalue of b. These calculation can be done by using the algebraic computer code such as “REDUCE” [31].

Even for the even polynomial refractive-index profile, $R_p(u_{p+1})$ for any p, in general, is not identically zero. Only an approximate solution is available; the approximate solution is a uniform asymptotic solution with a small parameter “ε”. So we discuss the error of the approximate solution obtained here. To do so, let us estimate the error of $R_N(u_{N+1})$. For the refractive-index profile given by Equation (11), we have a following result (see Appendix A):

$$R_N\left(u_{N+1}\right) = \mathrm{O}\left(\varepsilon^{2N-1}\right) \tag{74}$$

where O( ) is the Landau's O-symbol [36]. We can derive this conclusion in the derivation process of the N-th order asymptotic solution, although this is straightforward but tedious. Let a solution derived from the related equation obtained by N time transformation call an N-th order approximate solution and set it ${U_{\nu,\,m}}^{(N)}$ and ${E_{\nu,\,m}}^{(N)}$. Then we have following relations between the exact solution and the approximate one such that

$$U_{\nu,\,m}(u) = {U_{\nu,\,m}}^{(N)}(u) + \mathrm{O}\left(\varepsilon^{2N-1}\right) \tag{75}$$

and

$$E_{\nu,\,m} = {E_{\nu,\,m}}^{(N)} + \mathrm{O}\left(\varepsilon^{2N}\right) \tag{76}$$

where we set $u = u_1$.

$$\begin{aligned} {U_{\nu,\,m}}^{(N)}(u) = {} & \left((u/u_{N+1}(u))\, du_{N+1}(u)/du\right)^{-1/2}\left(su_{N+1}(u)^2\right)^{\nu/2} \\ & \times \left\{\exp\left(-su_{N+1}(u)^2/2\right)\right\} F\left(-m,\ \nu+1;\ su_{N+1}(u)^2\right) \end{aligned} \tag{77}$$

with

$$u_{N+1}(u) = \Sigma\left(d_p\left(u/(s\varepsilon)^{1/2}\right)^{2p-1}\right) \tag{78}$$

and

$$d_p = TD_pS^t, \quad p = 0,\ 1,\ 2, \ldots,\ 2N - 1 \tag{79}$$

where D_p is a $(2N-2) \times (2n-2)$ square matrix whose elements are shown in part in the appendix A, S^t is a transpose of S, and T and S are column vectors defined by $T = (1, s\varepsilon, (s\varepsilon)^2, \ldots, (s\varepsilon)^{2N-2})$ and $S = (1, 1/s^2, 1/s^4, \ldots, 1/s^{4N-2})$. It should be noted that the elements d_{pqr} of D_p for $r = 1,\ 2, \ldots,\ p-1$ or $q = 1,\ 2, \ldots,\ 2r - q - 1$ are identically zero. The N-th order eigenvalue is given by

$$E_{\nu,\,m}{}^{(N)} = d_0(s\varepsilon) \tag{80}$$

These results [23, 28] involve more terms than those of the guided mode of the slab waveguide [43] and more higher-order terms than those in [24–27].

Here we prove relations Equation (68) and Equation (69) by referring to the technique as shown in [44]. It is noted that $U_{\nu,\,m}(u_{N+1})$ and $U_{\nu,\,m}{}^{(N)}(u_{N+1})$ satisfy

$$\begin{aligned}&\Big[d^2/du_{N+1}{}^2 + (1/u_{N+1})\, d/du_{N+1} + (I_N/\varepsilon)^2 \left(1 - u_{N+1}{}^2\right) \\ &\quad - \nu^2/u_{N+1}{}^2 - R_N\left(u_{N+1}\right)\Big]\, U_{\nu,\ m}\left(u_{N+1}\right) = 0\end{aligned} \tag{66}$$

and

$$\begin{aligned}&\left[d^2/du_{N+1}{}^2 + (1/u_{N+1})\, d/du_{N+1} + \left(I_N{}^{(N)}/\varepsilon\right)^2 \left(1 - u_{N+1}{}^2\right)\right. \\ &\quad \left. -\nu^2/u_{N+1}{}^2\right] U_{\nu,\ m}{}^{(N)}\left(u_{N+1}\right) = 0\end{aligned} \tag{81}$$

respectively. Multiplying Equations (66) and (81) by $U_{\nu,\,m}{}^{(N)}(u_{N+1})$ and $U_{\nu,\,m}(u_{N+1})$, integrating them over $(0, 1-\delta)$, and subtracting them, we have

$$\begin{aligned}&\left[(I_N)^2 - \left(I_N{}^{(N)}\right)^2\right] \\ &\times \int_0^{1-\delta} \left(1 - u_{N+1}{}^2\right) u_{N+1} U_{\nu,\,m}\left(u_{N+1}\right) U_{\nu,\,m}{}^{(N)}\left(u_{N+1}\right) du_{N+1} \\ &\quad = \varepsilon^2 \int_0^{1-\delta} u_{N+1} R_N\left(u_{N+1}\right) U_{\nu,\,m}\left(u_{N+1}\right) U_{\nu,\,m}{}^{(N)}(u_{N+1}) du_{N+1}\end{aligned} \tag{82}$$

where δ is a small positive parameter. It is noted that the asymptotic behavior of the solution of Equation (81) can be written as

$$\begin{aligned} &U_{\nu,\,m}{}^{(N)}\left(u_{N+1}\right) \\ &\quad= \left[4\Big/\left(\pi\left(1-{u_{N+1}}^2\right)^{1/2}\right)^{1/2}\right] \\ &\qquad \times\left[\cos\left(\left({I_N}^{(N)}/\varepsilon\right)\int_{-1}^{u_{N+1}}\left(1-{u_{N+1}}^2\right)^{1/2}du_{N+1}-\pi/4\right)\right] \end{aligned} \tag{83}$$

Then we have

$$\int_0^{1-\delta} u_{N+1}\left(U_{\nu,\,m}{}^{(N)}\left(u_{N+1}\right)\right)^2 du_{N+1} = 1+\mathrm{O}\left(\delta\right) \tag{84}$$

We also have

$$\int_0^{1-\delta} u_{N+1}\left(U_{\nu,\,m}\left(u_{N+1}\right)\right)^2 du_{N+1} = 1+\mathrm{O}\left(\delta\right) \tag{84$'$}$$

Using these results and Equation (83), from Equation (82) we have

$$\begin{aligned} &\left[{I_N}^2-{I_N}^{(N)2}\right] = \varepsilon^2\int_0^{1-\delta}\left\{u_{N+1}\left(R_N\left(u_{N+1}\right)\Big/\left(1-{u_{N+1}}^2\right)^{1/2}\right)\right. \\ &\quad\left.\times\left[\sin^2\left(\left({I_N}^{(N)}/\varepsilon\right)\int_{-1}^{u_{N+1}}\left(1-{u_{N+1}}^2\right)^{1/2}du_{N+1}+\pi/4\right)\right]\right\}du_{N+1} \\ &\quad\Big/\int_0^{1-\delta}\left\{\left(u_{N+1}/(1-{u_{N+1}}^2)^{1/2})\right)\right. \\ &\quad\left.\times\left[\sin^2\left(({I_N}^{(N)}/\varepsilon)\int_{-1}^{u_{N+1}}(1-{u_{N+1}}^2)^{1/2}du_{N+1}+\pi/4\right)\right]\right\}du_{N+1} \\ &\qquad = \varepsilon^2\int_0^{1-\delta}u_{N+1}R_N\left(u_{N+1}\right)du_{N+1}+\mathrm{O}\left(\delta\right) \end{aligned} \tag{85}$$

It follows from Equations (74) and (85) that as δ tends to zero we have

$${I_N}^2 = \left({I_N}^{(N)}\right)^2+\mathrm{O}\left(\varepsilon^{2N+1}\right)$$

Since $I_N = \mathrm{O}(\varepsilon)$, we have following result as

$$I_N = {I_N}^{(N)} + \mathrm{O}\left(\varepsilon^{2N}\right)$$

The above relation directly reaches the final estimation Equation (76). In fact, from the definition of the turning point, ${\xi_{N+1}}^{(N)}$ in u_{N+1} space, we have $({\xi_{N+1}}^{(N)})^2 = 1$. From Equation (90) with $p = N+1$, the N-th order approximate turning point satisfies a relation as

$$(\xi_{N+1})^2 = \left({\xi_{N+1}}^{(N)}\right)^2 + \mathrm{O}\left(\varepsilon^{2N-1}\right)$$

It is apparent that this relation holds for ${\xi_p}^{(N)}$, $p = 2,\ 3,\ \ldots,\ N+1$ Since $(\xi_1)^2 = \mathrm{O}(\varepsilon)$, we have

$$(\xi_1)^2 = \left(\xi_1^{(N)}\right)^2 + \mathrm{O}\left(\varepsilon^{2N}\right)$$

Therefore from Equation (73) we have the final estimation Equation (76).

Next we show the relation Equation (75). Subtracting Equation (67) from Equation (81), we have

$$\begin{aligned}
&\left[\left(d^2/d{u_{N+1}}^2 + (1/u_{N+1})\, d/du_{N+1} + (I_N/\varepsilon)^2\right)\left(1 - {u_{N+1}}^2\right)\right] \\
&\quad \times \Delta U_{\nu,\, m}\left(u_{N+1}\right) \\
&\qquad = \left[R_N(u_{N+1}) - \left((I_N/\varepsilon)^2 - \left({I_N}^{(N)}/\varepsilon)^2\right)\right)\left(1 - {u_{N+1}}^2\right)\right] \\
&\qquad\quad \times U_{\nu,\, m}(u_{N+1})
\end{aligned} \tag{86}$$

where $\Delta U_{\nu,\, m}(u_{N+1})$ is defined by

$$\Delta U_{\nu,\, m}\left(u_{N+1}\right) = U_{\nu,\, m}\left(u_{N+1}\right) - {U_{\nu,\, m}}^{(N)}\left(u_{N+1}\right)$$

Hence by variation of parameters [36] we get

$$\begin{aligned}
U_{\nu,\, m}\left(u_{N+1}\right) &= {U_{\nu,\, m}}^{(N)}\left(u_{N+1}\right) + \mathrm{O}\left(\varepsilon^{2N+1}\right) \\
&\quad + \mathrm{O}\left((I_N/\varepsilon)^2 - \left({I_N}^{(N)}/\varepsilon\right)^2\right)
\end{aligned} \tag{87}$$

From this estimation, we get to the final result Equation (75).

(II) *Solution in the case of positive* $Q_1(u_1)$

Here we derive an approximate solution of Equation (60) [30]. Again we use a pair of transformations: Langer transformation and Liouville transformation. The related equation in this case is the Bessel's differential equation. The transformation pairs are as follows:

$$du_{p+1}/du_p = (Q_p(u_p))^{1/2}, \qquad \text{for} \;\; p = 1,\; 2, \ldots,\; N \tag{88}$$

and

$$U_{p+1}(u_{p+1}) = ((u_{p+1}/u_p)\, du_p/du_{p+1})^{-1/2}\, U_p(u_p)\;, \quad \text{for} \;\; p = 1,\; 2, \ldots,\; N \tag{89}$$

where

$$Q_p(u_p) = \begin{cases} E_{\nu,m} - h(u_1)\;, & \text{for} \;\; p = 1 \\ 1 - {u_p}^2 - \varepsilon^2 R_{p-1}(u_p)\;, & \text{for} \;\; p = 2,\; 3, \cdots,\; N+1 \end{cases} \tag{90}$$

with

$$\begin{aligned} R_{p-1}(u_p) = {} & \left(\nu^2/{u_p}^2\right)\left[((u_p/u_{p-1})\, du_{p-1}/du_p)^2 - 1\right] \\ & + ((u_{p+1}/u_p)\, du_p/du_{p+1})^{1/2}\left(d^2/{du_p}^2 + (1/u_p)\, d/du_p\right) \\ & \times ((u_p/u_{p-1})\,/du_{p-1}/du_p)^{-1/2}\;, \\ & \text{for} \;\; p = 2,\; 3, \ldots,\; N+1 \end{aligned} \tag{91}$$

Applying Equations (88) and (89) to Equation (59) N times, we have

$$\begin{aligned} & \left[d^2/{du_{N+1}}^2 + (1/u_{N+1})\, d/du_{N+1} + 1 - {u_{N+1}}^2 - \nu^2/{u_{N+1}}^2 \right. \\ & \left. -R_N(u_{N+1})\right] U_{N+1}(u_{N+1}) = 0\;, \qquad \nu = n-1,\; n+1 \end{aligned} \tag{92}$$

If $R_N(u_{N+l}) = 0,$ we have a related equation such as

$$\begin{aligned} & \left[d^2/{du_{N+1}}^2 + (1/u_{N+1})\, d/du_{N+1} + 1 - {u_{N+1}}^2 - \nu^2/{u_{N+1}}^2\right] \\ & \times U_{N+1}(u_{N+1}) = 0\;, \qquad \nu = n-1,\; n+1 \end{aligned} \tag{93}$$

Since a regular solution of Equation (93) at the origin is the Bessel function $J_\nu(u)$ and Equation (89) holds for each $p,$ we obtain a formal solution of Equation (59) such as

$$u_{N+1}{}^{(N)}(u) = [(u/u_{N+1}(u))\, du_{N+1}(u)/du]^{-1/2}\, J_\nu\,(u_{N+1}(u)) \qquad (94)$$

with

$$u_{N+1}(u) = \Sigma \left(f_p\,(E_{\nu,\,m}/\varepsilon)^{1/2}\, u \right)^{2p-1} \qquad (95)$$

and

$$f_p = GF_pK^t\ , \quad p = 0,\ 1,\ 2, \ldots,\ 2N-1 \qquad (96)$$

where F_p is a $(2N-2) \times (2n-2)$ square matrix, K^t is a transpose of K, and G and K are column vectors defined by $G = (1,\ E_{\nu,\,m}, \ldots,\ (E_{\nu,\,m})^{N-1})$ and $K = (1,\ (\varepsilon/E_{\nu,\,m})^2,\ (\varepsilon/E_{\nu,\,m})^4, \ldots,\ (\varepsilon/E_{\nu,\,m})^{2N-2})$. It should be noted that the elements f_{pqr} of F_p for $r = 1, 2, \ldots,\ p-1$ or $q = p,\ p+1, \ldots,\ N$ except for $f_{111} = 1$ are identically zero and are shown in [28]. Since Equation (94) holds in this case, we have

$$U_{\nu,\,m}(u) = U_{\nu,\,m}{}^{(N)}(u) + \mathrm{O}\left(\varepsilon^{2N-1}\right) \qquad (97)$$

This solution is an asymptotic solution too. For the even polynomial profile core, we construct two kinds of uniform asymptotic solutions. It is noted that the first-order asymptotic solution corresponds to the conventional WKB solution without uniformity.

Lastly, we show the N-th order solutions of the normalized propagation constant for two profiles. The Gaussian profile $h(r) = 2\Delta[1 - \exp(-r^2/d^2)]$ and the squared hyperbolic tangent profile $h(r) = 2\Delta \tanh^2(r/d)$ are given by replacing a_q $(q = 1,\ 2, \ldots,\ 2L-1)$ by $2\Delta/q!$ and $a_1 = \alpha(2\Delta)$, $a_2 = (2/3)\alpha^2(2\Delta)$, $a_3 = (17/45)\alpha^3(2\Delta), \ldots$ in Equation (11) with $\alpha = (1/2)\ln[(1 + (1 - 1/e)^{1/2})/(1 - (1 - 1/e)^{1/2})]$ where e is the base of the natural logarithm., respectively. The N-th order normalized propagation constant is defined by

$$B_s{}^{(N)} = 1 - E_{\nu,\,m}{}^{(N)}/2\Delta\ , \qquad \nu = n-1 \ \text{ or } \ n+1 \qquad (98)$$

For the Gaussian profile the third order solution is given by

$$
\begin{aligned}
&B_s{}^{(3)}\\
=&1 - s/V + \left((3/16)s^2 - (1/4)\nu^2 + 1/4\right) \Big/ V^2\\
&+ \left((11/768)s^3 - (1/64)s\nu^2 + (53/576)s\right) \Big/ V^3\\
&+ \left((85/24576)s^4 - (1/1024)s^2\nu^2 - (5/512)\nu^4\right.\\
&\left.+(1/3072)s^2 + (9/256)\nu^2 - 13/512\right) \Big/ V^4\\
&+ \left((20479/3932160)s^5 - (576/32768)s^3\nu^2 + (407/49152)s\nu^4\right.\\
&\left.+(37541/294912)s^3 - (5159/24576)s\nu^2 - (288673/737280)s\right) \Big/ V^5\\
&+ \dots\\
=&B_s{}^{(1)} + (1/4)/V^2 + ((53/576)s)/V^3\\
&+ \left((1/3072)s^2 + (9/256)\nu^2 - 13/512\right) \Big/ V^4\\
&+ \left((37541/294912)s^3 - (5159/24576)s\nu^2 - (288673/737280)s\right) \Big/ V^5\\
&+ \dots
\end{aligned}
\tag{99}
$$

$$
s = 4m + 2\nu + 4 \ , \qquad V = kd(2\Delta)^{1/2}
$$

where the first order solution, that corresponds to the WKB solution, is given by

$$
\begin{aligned}
&B_s{}^{(1)}\\
&=1 - s/V + \left((3/16)s^2 - (1/4)\nu^2\right) \Big/ V^2\\
&\quad + \left((11/768)s^3 - (1/64)s\nu^2\right) \Big/ V^3\\
&\quad + \left((85/24576)s^4 - (1/1024)s^2\nu^2 - (5/512)\nu^4\right) \Big/ V^4\\
&\quad + \left((20479/3932160)s^5 - (576/32768)s^3\nu^2 + (407/49152)s\nu^4\right) \Big/ V^5\\
&\quad + \dots
\end{aligned}
\tag{100}
$$

For the squared hyperbolic tangent profile, we have

$$
\begin{aligned}
& {B_s}^{(3)} \\
=& 1 - s/V + \left((1/4)s^2 - (1/3)\nu^2 + 1/3\right) \Big/ V^2 \\
& + \left((1/30)\nu^2 + (191/540)s\right) \Big/ V^3 \\
& + \left((4/315)s^2\nu^2 - (16/945)\nu^4 - (1/315)s^2 + (4/189)\nu^2 - 4/945\right) \Big/ V^4 \\
& + \left((73937/1569600)s^5 - (18749/91560)s^3\nu^2 + (710267/4120200)s\nu^4\right. \\
& + (134077/91560)s^3 - (2824369/1030050)s\nu^2 \\
& \left. + (10042849/2060100)s\right) / V^5 + \ldots \\
=& 7{B_s}^{(1)} + (1/3)/V^2 + (191/540)s/V^3 \\
& + \left(-(1/315)s^2 + (4/189)\nu^2 - 4/945\right) \Big/ V^4 \\
& + \left((134077/91560)s^3 - (2824369/1030050)s\nu^2\right. \\
& \left. + (10042849/2060100)s\right) / V^5 + \ldots
\end{aligned}
\tag{101}
$$

where the corresponding WKR solution is given by

$$
\begin{aligned}
{B_s}^{(1)} = 1 - s/V &+ \left((1/4)s^2 - (1/3)\nu^2\right) \Big/ V^2 + \left((1/30)s\nu^2\right) \Big/ V^3 \\
&+ \left((4/315)s^2\nu^2 - (16/945)\nu^4\right) \Big/ V^4 \\
&+ \left((73937/1569600)s^5 - (18749/91560)s^3\nu^2\right. \\
&\left. + (710267/4120200)s\nu^4\right) / V^5 + \ldots
\end{aligned}
\tag{102}
$$

It follows from Equations (99)–(102) that the third order asymptotic solution raises the accuracy of the WKB solution with the error of order $1/V^2$ and corrects it up to the term with order $1/V^5$. Thus the WKB solution involves a relatively large error in the low frequency region.

5. Uniform Asymptotic Solution of the Vector Wave Equation

Using the results of the sections 3 and 4, we calculate a uniform asymptotic solution of the vector wave equation with the aid of the perturbation method.

Before calculating the perturbed solution, we first discuss the M-th order perturbed solution constructed by the N-th order asymptotic solution from the point of the calculation error. We examine the HE-modes. Formal solutions in section 3 indicate that the order of error of the perturbed solution can be estimated by evaluating definite integrals such that

$$\int_0^\infty u^{2R-1} U_{\nu,\,m} U_{\mu,\,p} du \Big/ \int_o^\infty u (U_{\nu,\,m})^2 du \ , \qquad R = 1,\ 2, \ldots ; \qquad \nu,\ \mu = n-1,\ n+1$$

Considering Equation (71) and changing a variable u into $v = u^2/\varepsilon$, we have

$$\int_o^\infty u^{2R-1} U_{\nu,\,m} U_{\mu,\,p} du \Big/ \int_o^\infty u \left(U_{\nu,\,m}\right)^2 du = \varepsilon^{R-1} \int_o^\infty v^{R-1} U_{\nu,\,m} U_{\mu,\,p} dv \Big/ \int_o^\infty \left(U_{\nu,\,m}\right)^2 dv \tag{103}$$

For the HE-modes, the eigenvalue $E_{\nu,\,m}$ takes the value of $\mathrm{O}(\varepsilon)$ for the even polynomial profile as shown in Equation (80). Therefore, from Equations (25), (26), and (103), we have

$$A_q{}^{(1)} = \mathrm{O}\left(\varepsilon^2\right) \ , \qquad m = 0,\ 1, 2, \ldots$$

$$B_q{}^{(1)} = \begin{cases} \mathrm{O}\left(\varepsilon^2\right) \ , & m = 0 \\ \mathrm{O}\left(\varepsilon\right) \ , & m = 1,\ 2,\ 3, \ldots \end{cases}$$

and

$$b^{(1)} = \mathrm{O}\left(\varepsilon^3\right) \ , \qquad m = 0,\ 1,\ 2, \ldots$$

Using these results we also have

$$A_q{}^{(2)} = \mathrm{O}\left(\varepsilon^3\right) \ , \qquad m = 0,\ 1,\ 2,\ldots$$

$$B_q{}^{(2)} = \begin{cases} \mathrm{O}\left(\varepsilon^3\right) \ , & m = 0 \\ \mathrm{O}\left(\varepsilon^2\right) \ , & m = 1,\ 2,\ 3,\ldots \end{cases}$$

and

$$b^{(2)} = \begin{cases} \mathrm{O}\left(\varepsilon^5\right) \ , & m = 0 \\ \mathrm{O}\left(\varepsilon^4\right) \ , & m = 1,\ 2,\ 3,\ldots \end{cases}$$

For higher-order corrections, we also have similar estimations. As a result, we have a final estimation such that for the dominant mode $(m = 0)$,

$$\begin{aligned} \Phi &= \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \ldots + \Phi^{(M)} + \mathrm{O}\left(\varepsilon^{2M+2}\right) \\ \Psi &= \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \ldots + \Psi^{(M)} + \mathrm{O}\left(\varepsilon^{2M+2}\right) \end{aligned} \tag{104}$$

and

$$b = b^{(0)} + b^{(1)} + b^{(2)} + \ldots + b^{(M)} + \mathrm{O}\left(\varepsilon^{2M+3}\right) \tag{105}$$

and for the remaining HE-modes $(m = 1,\ 2,\ 3,\ldots)$,

$$\begin{aligned} \Phi &= \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \ldots + \Phi^{(M)} + \mathrm{O}\left(\varepsilon^{M+2}\right) \\ \Psi &= \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \ldots + \Psi^{(M)} + \mathrm{O}\left(\varepsilon^{M+1}\right) \end{aligned} \tag{104$'$}$$

and

$$b = b^{(0)} + b^{(1)} + b^{(2)} + \cdots + b^{(M)} + \mathrm{O}\left(\varepsilon^{M+3}\right) \tag{105$'$}$$

Similar estimations for the EH-modes $(m = 0, 1, 2, \ldots)$ are as follows:

$$\begin{aligned} \Psi &= \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \ldots + \Psi^{(M)} + \mathrm{O}\left(\varepsilon^{M+1}\right) \\ \Phi &= \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \ldots + \Phi^{(M)} + \mathrm{O}\left(\varepsilon^{M+2}\right) \end{aligned} \tag{106}$$

and

$$b = b^{(0)} + b^{(1)} + b^{(2)} + \cdots + b^{(M)} + \mathrm{O}\left(\varepsilon^{M+3}\right) \tag{107}$$

It is apparent from Equation (104) and (105) that the first order perturbed solution is an excellent one for the HE_{11}-mode.

Unfortunately we have an exact solution of the unperturbed equation even for the even polynomial refractive-index profile as shown in section 4. An available analytic solution is a uniform asymptotic solution. Therefore we need to estimate errors in the corrected terms generated by using an approximate solution. Here we only show it for the HE-modes. From Equations (22), (75), and (76), we have

$$\begin{aligned} \Phi^{(0)} &= U_{\nu,\,m}{}^{(N)}(u) + \mathrm{O}\left(\varepsilon^{2N-1}\right) \\ \Psi^{(0)} &= 0 \end{aligned} \tag{108}$$

and

$$b^{(0)} = E_{\nu,\,m}{}^{(N)} + n\varepsilon^2 + \mathrm{O}\left(\varepsilon^{2N}\right) \tag{109}$$

Next let us evaluate Equation (24). To do that, we rewrite it such as

$$\begin{aligned} \Phi^{(1)} &= \sum A_q{}^{(1)} U_{n-1,\,q}{}^{(N)} \\ &= \sum \Big[A_q'^{(1)} U_{n-1,\,q}{}^{(N)} + A_q'^{(1)} \left(U_{n-1,\,q} - U_{n-1,\,q}{}^{(N)} \right) \\ &\quad + (A_q{}^{(1)} - A_q'^{(1)}) U_{n-1,\,q} \end{aligned}$$

where $A_q'^{(1)}$ can be determined by replacing $E_{\nu,\,q}$ and $U_{\nu,\,q}$ by $E_{\nu,\,q}{}^{(N)}$ and $U_{\nu,\,q}{}^{(N)}$ in Equation (25). On the other hand, from Equation (75) we have

$$\begin{aligned} &\int_0^\infty v^{R-1} U_{\nu,\,m} U_{\mu,\,p} dv \Big/ \int_0^\infty (U_{\nu,\,m})^2 dv \\ &= \int_0^\infty v^{R-1} U_{\nu,\,m}{}^{(N)} U_{\mu,\,p}{}^{(N)} dv \\ &\Big/ \int_0^\infty (U_{\nu,\,m}{}^{(N)})^2 dv + \mathrm{O}\left(\varepsilon^{2N-1}\right) \\ &\nu,\ \mu = n-1,\ n+1 \end{aligned} \tag{110}$$

Accordingly, from Equations (25) we have $A_q{}^{(1)} - A_q'^{(1)} = \mathrm{O}(\varepsilon^{2N+1})$ Thus we have following estimations as

$$\Phi^{(1)} = \sum {A'_q}^{(1)} {U_{n-1,\,q}}^{(N)} + \mathrm{O}\left(\varepsilon^{2N+1}\right)$$
$$\Psi^{(1)} = \sum {B'_q}^{(1)} {U_{n+1,\,q}}^{(N)} + \begin{cases} \mathrm{O}\left(\varepsilon^{2N+1}\right) & \text{for } m = 0, \\ \mathrm{O}\left(\varepsilon^{2N}\right) & \text{for } m = 1, 2, \ldots \end{cases} \tag{111}$$

and

$$b^{(1)} = b'^{(1)} + \mathrm{O}\left(\varepsilon^{2N+2}\right) \tag{112}$$

where ${B'_q}^{(1)}$ and $b'^{(1)}$ are given by replacing $E_{\nu,\,q}$ and $U_{\nu,\,q}$ by ${E_{\nu,\,q}}^{(N)}$ and ${U_{\nu,\,q}}^{(N)}$ in Equations (25) and (26), respectively. Thus the error generated in the first-order perturbation is less than that of the zero-th order solution. This fact holds in higher-order perturbations. For EH-modes and TM-modes, we also reach this same conclusion. Therefore for the calculation of the M-th order perturbed solution we use an N-th order asymptotic solution whose approximation order satisfies the following condition as

$$\begin{aligned} &2N > 2M + 3, \quad \text{for HE} - \text{mode} \quad \text{or} \\ &2N > M + 3\;, \quad \text{for other modes} \end{aligned} \tag{113}$$

In the actual calculation, we search an optimum approximation order by examining the asymptotic nature of the uniform asymptotic solution among the orders that fulfill the constraint (113) numerically.

At this stage we calculate definite integrals as shown in Equation (110) and obtain perturbed solutions. Using the addition theorem and the recursive formula about the Laguerre polynomials, because the Kummer function can be written in terms of the Laguerre polynomials (see Appendix B), and expanding remaining terms in Equation (77) into a power series of u, we can represent the N-th order asymptotic solution in terms of the eigenfunctions of the scalar wave equation for the square-law medium. Those eigenfunctions are defined by

$$\begin{aligned} &V_{n-1,\,m}(v) \\ &\quad = [\Gamma(m+1)/(\Gamma(m+n)]\, v^{n-1} \exp(-v/2) {L_m}^{(n-1)}(v) \end{aligned} \tag{114a}$$

and

$$\begin{aligned} &V_{n+1,\,m}(v) \\ &\quad = [\Gamma(m+1)/(\Gamma(m+n+2)]\, v^{n+1} \exp(-v/2) {L_m}^{(n+1)}(v) \end{aligned} \tag{114b}$$

Here we use again $v = u^2/\varepsilon$ and $L_m{}^{(n)}(v)$ is the Laguerre polynomials. It is noted that the eigenfunctions mentioned above coincide with those of Equation (77) by setting the coefficients of the refractive-index profile to zero. We also have same orthogonality relations as Equation (21):

$$\int_0^\infty V_{\nu,\,m} V_{\nu,\,p} dv = \delta_{m,\,p}, \quad \nu = n-1, n+1 \tag{21'}$$

Using the recursive formula about the Laguerre polynomials, we have following formulas as

$$\begin{aligned} &V_{n-1,\,m}(v) \\ &\quad = \left[(m+n+1)^{1/2} V_{n,\,m+1}(v) - (m+1)^{1/2} V_{n,\,m}(v)\right] \Big/ v^{1/2} \end{aligned} \tag{115}$$

and

$$\begin{aligned} &V_{n+1,\,m}(v) \\ &\quad = \left[(m+n+1)^{1/2} V_{n,\,m}(v) - (m+1)^{1/2} V_{n,\,m+1}(v)\right] \Big/ v^{1/2} \end{aligned} \tag{116}$$

Using these formulas, we can calculate definite integrals as shown in Equation (103). In the actual calculation, we use an algebraic computer code such as "REDUCE" [31] operating on the personal computer. Thus we can calculate uniform asymptotic solutions of the vector wave equation Equation (10) as well as the solution for the TM-mode.

The uniform asymptotic solution of Equation (10) for the HE-modes up to the first-order correction terms are given as follows:

$$b = b^{(0)} + b^{(1)} \tag{117}$$

and

$$\begin{aligned} \Phi &= \Phi^{(0)} + \Phi^{(1)} \\ \Psi &= \Psi^{(1)} \end{aligned} \tag{118}$$

where

$$
\begin{aligned}
& b^{(0)} \\
=& (4m+2n)\varepsilon + \left(n - (3/8)a_2(4m+2n)^2 + ((n-1)^2-1)a_2/2\right)\varepsilon^2 \\
& + \left[((5/16)a_3 - (17/64){a_2}^2)(4m+2n)^3 + \left((7/4)a_3 - (19/16){a_2}^2\right.\right. \\
& \left.\left. -((3/4)a_3 - (9/16){a_2}^2)(n-1)^2\right)(4m+2n)\right]\varepsilon^3 \\
& - \left[\left((375/1024){a_2}^2 - (165/256)a_2a_3 + (35/128)a_4\right)(4m+2n)^4\right. \\
& - \left((129/128){a_2}^2 - (63/32)a_2a_3 + (15/16)a_4\right)(4m+2n)^2\nu^2 \\
& + \left((11/64){a_2}^2 - (9/16)a_2a_3 + (3/8)a_4\right)\nu^4 \\
& + \left((1377/384){a_2}^2 - (237/32)a_2a_3 + (65/16)a_4\right)(4m+2n)^2 \\
& - \left((71/32){a_2}^2 - (45/8)a_2a_3 + (15/4)a_4\right)\nu^2 \\
& \left. +(131/64){a_2}^2 - (81/16)a_2a_3 + (27/8)a_4\right]\varepsilon^4 + \ldots
\end{aligned} \tag{117a}
$$

$$
\Phi^{(0)} = {U_{n-1,\,m}}^{(N)}(u_{N+1}(u)) \tag{118a}
$$

where

$$
\begin{aligned}
& u_{N+1}(u) \\
=& (1/(4m+2n))\left[((1-(3/16)a_2)((4m+2n)\varepsilon) + ((5/32)a_3\right. \\
& -(77/512){a_2}^2\big) \times ((4m+2n)\varepsilon)^2 + \left((2/3)a_3 - (71/192){a_2}^2\right. \\
& \left. - \left((1/6)a_3 - (11/192){a_2}^2\right)(n-1)^2\right)\varepsilon^2 + \ldots\big)((4m+2n)\varepsilon)^{1/2}u \\
& + \left((-(1/8)a_2)(\varepsilon/(4m+2n)) + \left((5/48)a_3 - (17/192){a_2}^2\right)\varepsilon^2 + \ldots\right) \\
& \times \left(((4m+2n)\varepsilon)^{1/2}u\right)^3 + \left(\left((1/12)a_3 - (11/384){a_2}^2\right)\right. \\
& \left.\times\left(\varepsilon^2/(4m+2n)^4\right) + \ldots\right)\left(((4m+2n)\varepsilon)^{1/2}u\right)^5 + \ldots\Big]
\end{aligned}
$$

$$
\begin{aligned}
b'^{(1)} =& ((5n+6)/4 - 2(n+1)a_2)(2m+n)\varepsilon^3 \\
& + ((13n+24)/8 - (11n+26)a_2/4 \\
& +3(n+2)a_3 - 2(n+1){a_2}^2) \\
& \times (6m^2+6mn+n^2+n)\varepsilon^4 + \ldots
\end{aligned} \tag{117b}
$$

$$\Phi^{(1)} = \sum A_q{'}^{(1)} U_{n-1,\,q}{}^{(N)}\left(u_{N+1}(u)\right) \tag{118b}$$

with

$$\begin{aligned}
A_{m+2}{}^{(1)} = &-\left[\left((13n+24)/8-(37n+70)a_2/8+3(n+2)a_3\right.\right.\\
&\left.\left.+(n+1){a_2}^2\right)\varepsilon^3/8+\ldots\right]\left[(m+1)(m+2)\right.\\
&\left.\times(m+n)(m+n+1)\right]^{1/2}, \qquad m=0,\ 1,\ 2,\ldots\\
A_{m+1}{}^{(1)} = &\left[\left((5n+6)/8-(n+1)a_2\right)\varepsilon^2/2\right.\\
&+\left\{\left((13n+24)/16-(n+2)\left(2a_2-3a_3/2\right)\right)(2m+n+1)\right.\\
&\left.+\left((5n+6)a_2/8-(n+1){a_2}^2\right)(22m+11n+5)/8\right\}\varepsilon^3\\
&\left.+\ldots\right]\left[(m+1)(m+n)\right]^{1/2}, \qquad m=0,\ 1,\ 2,\ldots\\
A_{m-1}{}^{(1)} = &\left[-\left((5n+6)/8-(n+1)a_2\right)\varepsilon^2/2\right.\\
&+\left\{\left((13n+24)/16-(n+2)\left(2a_2-3a_3/2\right)\right)(2m+n-1)\right.\\
&\left.+\left((5n+6)a_2/8-(n+1){a_2}^2\right)(22m+11n-5)/8\right\}\varepsilon^3\\
&\left.+\ldots\right]\left[(m+1)(m+n-1)\right]^{1/2}, \qquad m=1,\ 2,\ 3,\ldots\\
A_{m-2}{}^{(1)} = &\left[\left((13n+24)/8-(37n+70)a_2/8+3(n+2)a_3\right.\right.\\
&\left.\left.+(n+1){a_2}^2\right)\varepsilon^3/8+\ldots\right]\left[(m-1)m(m+n-1)\right.\\
&\left.\times(m+n-2)\right]^{1/2}, \qquad m=2,\ 3,\ 4,\ldots
\end{aligned}$$

$$\Psi^{(1)} = \sum A_q{}^{(1)} U_{n+1,\,q}{}^{(N)} \tag{118c}$$

with

$$\begin{aligned}
B_{m+1}{}^{(1)} = &-\left[\left\{3(n-2)/4-(16n-35)a_2/8+(n-3)a_3\right.\right.\\
&\left.\left.+(n-2){a_2}^2/4\right\}\varepsilon^3/4+\ldots\right]\left[(m+1)(m+2)\right.\\
&\left.\times(m+n+1)(m+n+2)\right]^{1/2}, \qquad m=0,\ 1,\ 2,\ldots
\end{aligned}$$

$$
\begin{aligned}
B_m{}^{(1)} = & \Big[\left((2n-3)/2-(n-2)a_2\right)\varepsilon^2/4 \\
& + \{n\left((2n-3)/2-(n-2)a_2\right)/8 \\
& + (3(n-2)/4-(7n-16)a_2/4 \\
& +(n-3)a_3)\,(4m+n+2)/2 \\
& + \left((2n-3)a_2/2-(n-2){a_2}^2\right)(22m+8n+5)/16\Big\}\,\varepsilon^3 \\
& +\ldots]\times[(m+n)(m+n+1)]^{1/2}\ , \qquad m=0,\ 1,\ 2,\ldots \\
B_{m-1}{}^{(1)} = & \{1/\left(2n(1-a_2)\right)\} \\
& \times[(2n-3-2(n-2)a_2)\,\varepsilon \\
& +3\,\{3(n-2)/2-(5n-13)a_2/2 \\
& -2(n-3)a_2a_3-(n-2){a_2}^2\Big\} \\
& \times(2m+n)\varepsilon^2 \\
& +\{(3(5n-12)/2-3(17n-12)a_2/2 \\
& +20(n-3)a_3+2(15n-16){a_2}^2\Big) \\
& \times(5m^2+5mn+n^2+1) \\
& +3\left((2n-3)a_3/2-(n-2)a_2a_3\right) \\
& \times(5m^2+5mn++n^2-n) \\
& +\Big\{3(n-2)a_2/2-(17n-16){a_2}^2/2+2(n-3)a_2a_3\Big\} \\
& \times(136m^2+136mn+29n^2+20)/4 \\
& +\left((2n-3){a_2}^2/2-(n-2){a_2}^3\right) \\
& \times\left(680m^2+680mn+157n^2+36n+100\right)/16\Big\}\,\varepsilon^3 \\
& +\ldots]\,[m(m+n)]^{1/2}\ , \qquad m=1,\ 2,\ 3,\ldots \quad \text{for } a_2=1 \\
B_{m-2}{}^{(1)} = & \Big[-\left((2n-3)/2-(n-2)a_2\right)\varepsilon^2/4 \\
& +\{n\left((2n-3)/2-(n-2)a_2\right)/8 \\
& -(3(n-2)/4-(7n-16)a_2/4 \\
& -(n-3)a_3/2)\,(4m+3n-2)/2 \\
& -\left((2n-3)a_2/2-(n-2){a_2}^2\right)(22m+14n-5)/16\Big\}\,\varepsilon^3
\end{aligned}
$$

$$+\ldots]\,[m(m-1)]^{1/2}\ , \qquad m=2,\ 3,\ 4,\ldots$$

$$\begin{aligned} B_{m-3}{}^{(1)} =& [\{3(n-2)/4-(16n-35)a_2/8 \\ & +(n-3)a_3+(n-2)a_2{}^2/4\}\,\varepsilon^3/4+\ldots] \\ & \times[(m-2)(m-1)m(m+n-1)]^{1/2}\ , \quad m=3,\ 4,\ 5,\ldots \end{aligned}$$

Results for the EH-modes are as follows:

$$b=b^{(0)}+b^{(1)} \tag{119}$$

and

$$\begin{aligned} \Psi &= \Psi^{(0)}+\Psi^{(1)} \\ \Phi &= \Phi^{(1)} \end{aligned} \tag{120}$$

where

$$\begin{aligned} b^{(0)} =& (4m+2n+4)\varepsilon+\big(-n-(3/8)a_2(4m+2n+4)^2 \\ & +(1/2)\left((n+1)^2-1\right)a_2\big)\,\varepsilon^2 \\ & +\big[\left((5/16)a_3-(17/64)a_2{}^2\right)(4m+2n+4)^3 \\ & +\big((7/4)a_3-(19/16)a_2{}^2 \\ & -\left((3/4)a_3-(9/16)a_2{}^2\right)(n+1)^2\big)(4m+2n+4)\big]\,\varepsilon^3 \\ & -\big[\left((375/1024)a_2{}^2-(165/256)a_2a_3+(35/128)a_4\right) \\ & \times(4m+2n+4)^4-\left((129/128)a_2{}^2-(63/32)a_2a_3+(15/16)a_4\right) \\ & \times(4m+2n+4)^2\nu^2+\left((11/64)a_2{}^2-(9/16)a_2a_3+(3/8)a_4\right)\nu^4 \\ & +\left((1377/384)a_2{}^2-(237/32)a_2a_3+(65/16)a_4\right)(4m+2n+4)^2 \\ & -\left((71/32)a_2{}^2-(45/8)a_2a_3+(15/4)a_4\right)\nu^2 \\ & +(131/64)a_2{}^2-(81/16)a_2a_3+(27/8)a_4\big]\,\varepsilon^4+\ldots \end{aligned} \tag{119a}$$

$$\Psi^{(0)}=U_{n+1,\,m}{}^{(N)}(u_{N+1}(u)) \tag{120a}$$

where

$$
\begin{aligned}
& u_{N+1}(u) \\
=& (1/(4m+2n+4)) \\
& \times \Big[\big((1-(3/16)a_2)\,((4m+2n+4)\varepsilon) + \big((5/32)a_3 - (77/512){a_2}^2\big) \\
& \times ((4m+2n+4)\varepsilon)^2 \\
& + \big((2/3)a_3 - (71/192){a_2}^2 - \big((1/6)a_3 - (11/192){a_2}^2\big)(n+1)^2\big)\,\varepsilon^2 \\
& +\ldots)\,((4m+2n+4)\varepsilon)^{1/2}\,u \\
& + ((-(1/8)a_2)\,(\varepsilon/(4m+2n+4)) \\
& + \big((5/48)a_3 - (17/192){a_2}^2\big)\,\varepsilon^2 + \ldots\big) \\
& \times \Big(((4m+2n+4)\varepsilon)^{1/2}\,u\Big)^3 \\
& + \Big(\big((1/12)a_3 - (11/384){a_2}^2\big)\,\big(\varepsilon^2/(4m+2n+4)^4\big) + \ldots\Big) \\
& \times \Big(((4m+2n+4)\varepsilon)^{1/2}\,u\Big)^5 + \ldots\Big]
\end{aligned}
\tag{119b}
$$

$$
\begin{aligned}
b^{(1)} =& -((5n-6)/4 - 2(n-1)a_2)\,(2m+n+2)\varepsilon^3 \\
& - \big((13n-24)/8 + (11n-26)a_2/4 + 3(n-2)a_3 - 2(n-1){a_2}^2\big) \\
& \times (6m^2 + 6mn + 12m + n^2 + 5n + 6)\varepsilon^4 + \ldots
\end{aligned}
$$

$$
\Psi^{(1)} = \sum {A_q}^{(1)} {U_{n-1,\,q}}^{(N)}(u_{N+1}(u)) \tag{120b}
$$

with

$$
\begin{aligned}
{A_{m+2}}^{(1)} =& [((13n-24)/8 - (37n-70)a_2/8 + 3(n-2)a_3 \\
& +(n-1){a_2}^2\big)\,\varepsilon^3/8 + \ldots\big]\,[(m+1)(m+2) \\
& \times (m+n)(m+n+1)]^{1/2}\ , \qquad m = 0,\ 1,\ 2, \ldots \\
{A_{m+1}}^{(1)} =& -\,\big[((5n-6)/8 - (n-1)a_2)\,\varepsilon^2/2 \\
& + \{((13n-24)/16 - (n-2)\,(2a_2 - 3a_3/2))\,(2m+n+3) \\
& + \big((5n-6)a_2/8 - (n-1){a_2}^2\big)\,(22m+11n+27)/8\big\}\,\varepsilon^3
\end{aligned}
$$

$$+\ldots]\left[(m+1)(m+n+2)\right]^{1/2}, \qquad m=0,\ 1,\ 2,\ldots$$

$$\begin{aligned}
{A_{m-1}}^{(1)} =& \Big[\left((5n-6)/8-(n-1)a_2\right)\varepsilon^2/2 \\
&+\{((13n-24)/16-(n-2)\left(2a_2-3a_3/2\right))\left(2m+n+1\right) \\
&+\left((5n-6)a_2/8-(n-1){a_2}^2\right)\left(22m+11n+17\right)/8\Big\}\varepsilon^3 \\
&+\ldots]\left[m(m+n+1)\right]^{1/2}, \qquad m=1,\ 2,\ 3,\ldots \\
{A_{m-2}}^{(1)} =& -\big[((13n-24)/8-(37n-70)a_2/8+3(n-2)a_3 \\
&+(n-1){a_2}^2\big)\varepsilon^3/8+\ldots\big]\left[(m-1)m(m+n)\right. \\
&\left.\times(m+n+1)\right]^{1/2}, \qquad m=2,\ 3,\ 4,\ldots
\end{aligned}$$

and

$$\Phi^{(1)}=\sum {B_q}^{(1)}{U_{n+1,\,q}}^{(N)} \tag{120c}$$

with

$$\begin{aligned}
{B_{m+3}}^{(1)} =& [\{3(n+2)/4-(16n+35)a_2/8+(n+3)a_3 \\
&+(n+2){a_2}^2/4\big\}\varepsilon^3/4+\ldots\big]\left[(m+1)(m+2)\right. \\
&\left.\times(m+3)(m+n+2)\right]^{1/2}, \qquad m=0,\ 1,\ 2,\ldots \\
{B_{m+2}}^{(1)} =& -\Big[\left((2n+3)/2-(n+2)a_2\right)\varepsilon^2/4 \\
&+\{n\left((2n+3)/2-(n+2)a_2\right)/8 \\
&-\left(3(n+2)/4-(7n+16)a_2/4+(n+3)a_3\right) \\
&\times\ (4m+3n+6)/2-\left((2n+3)a_2/2-(n+2){a_2}^2\right) \\
&\times\ (22m+14n+27)/16\}\,\varepsilon^3 \\
&+\ldots]\times\left[(m+1)(m+2)\right]^{1/2}, \qquad m=0,\ 1,\ 2,\ldots \\
{B_{m+1}}^{(1)} =& \{1/\left(2n(1-a_2)\right)\} \\
&\times\left[(2n+3-2(n+2)a_2\right)\varepsilon \\
&+3\,\{3(n+2)/2-(5n+13)a_2/2 \\
&-2(n+3)a_2a_3-(n+2){a_2}^2\big\} \\
&\times(2m+n+2)\varepsilon^2 \\
&+\{(3(5n+12)/2-3(17n+12)a_2/2
\end{aligned}$$

$$
\begin{aligned}
&+20(n+3)a_3+2(15n+16){a_2}^2\big)\\
&\times(5m^2+5mn+10m+n^2+5n+6)\\
&+3\left((22n+3)a_3/2-(n+2)a_2a_3\right)\\
&\times(5m^2+5mn+10m+n^2+6n+5)\\
&+\big(3(n+2)a_2/2-(17n+16){a_2}^2/2+2(n+3)a_2a_3\big)\\
&\times(136m^2+136mn+272m+29n^2+136n+156)/4\\
&+\big((2n+3){a_2}^2/2-(n+2){a_2}^3\big)\\
&\times\big(680m+680mn+1360m\\
&+157n^2+644n+780\big)/16\big\}\,\varepsilon^3\\
&+\ldots]\,[m(m+n)]^{1/2}\ , \qquad m=0,\ 1,\ 2,\ldots \quad \text{for} \quad a_2=1
\end{aligned}
$$

$$
\begin{aligned}
{B_m}^{(1)}=&\big[((2n+3)/2-(n+2)a_2)\,\varepsilon^2/4\\
&+\{n\,((2n+3)/2-(n+2)a_2)\,/8\\
&-(3(n+2)/4-(7n+16)a_2/4\\
&+(n+3)a_3)\,(4m+n+2)/2\\
&+\big((2n+3)a_2/2-(n+2){a_2}^2\big)\,(22m+8n+17)/16\big\}\,\varepsilon^3\\
&+\ldots]\,[(m+n)m+n+1)]^{1/2}\ , \qquad m=0,\ 1,\ 2,\ldots
\end{aligned}
$$

$$
\begin{aligned}
{B_{m-1}}^{(1)}=&-[\{3(n+2)/4-(16n+35)a_2/8+(n+3)a_3\\
&+(n+2){a_2}^2/4\big\}\,\varepsilon^3/4+\ldots\big]\,[m(m+n-1)\\
&\times(m+n)(m+n+1)]^{1/2}\ , \qquad m=1,\ 2,\ 3,\ldots
\end{aligned}
$$

Finally we show the vector wave solution of the TM-modes:

$$b=b^{(0)}+b^{(1)} \tag{121}$$

and

$$\psi=\psi^{(0)}+\psi^{(1)} \tag{122}$$

where

$$\begin{aligned} b^{(0)} &= 4(m+1)\varepsilon - 6a_2\left((m+1)\varepsilon\right)^2 \\ &\quad + \left[\left(20a_3 - 17{a_2}^2\right)(m+1)^3 + \left(4a_3 - (3/2){a_2}^2\right)(m+1)\right]\varepsilon^3 \\ &\quad - \left[\left((375/4){a_2}^2 - 165a_2a_3 + 70a_4\right)(m+1)^4 \right. \\ &\quad \left. + \left((165/4){a_2}^2 - 87a_2a_3 + 50a_4\right)(m+1)^2\right]\varepsilon^4 + \dots \end{aligned} \tag{121a}$$

$$\Psi^{(0)} = {U_{1,\,m}}^{(N)}(u_{N+1}(u)) \tag{122a}$$

where

$$\begin{aligned} &u_{N+1}(u) \\ =&(1/(4m+4)) \\ &\times [((1 - (3/16)a_2)\left((4m+4)\varepsilon\right) \\ &+ \left((5/32)a_3 - (77/512){a_2}^2\right)\left((4m+4)\varepsilon\right)^2 \\ &+ \left((2/3)a_3 - (71/192){a_2}^2 - \left((1/6)a_3 - (11/192){a_2}^2\right)\right) \\ &+ \dots)\left((4m+4)\varepsilon\right)^{1/2}u \\ &+ ((-(1/8)a_2)\left(\varepsilon/(4m+4)\right) \\ &+ \left((5/48)a_3 - (17/192){a_2}^2\right)\varepsilon^2 + \dots) \\ &\times \left(\left((4m+4)\varepsilon\right)^{1/2}u\right)^3 \\ &+ \left(\left((1/12)a_3 - (11/384){a_2}^2\right)\left(\varepsilon^2/(4m+4)^4\right) + \dots\right) \\ &\times \left.\left(\left((4m+4)\varepsilon\right)^{1/2}u\right)^5 + \dots\right] \end{aligned}$$

$$\begin{aligned} b^{(1)} = &\left[(6 - 8a_2)(m+1)\varepsilon^3 + \left(36 - 78a_2 + 72a_3 - 24{a_2}^2\right)\right. \\ &\left.(m+1)^2\varepsilon^4 + \dots\right] \end{aligned} \tag{121b}$$

$$\psi^{(1)} = \sum {A_q}^{(1)}{U_{1,\,q}}^{(N)} \tag{122b}$$

with

$$A_{m+2}{}^{(1)} = -\left[\left(3 - 35a_2/4 + 6a_3 + a_2{}^2\right)(m+2)\varepsilon^3/4 + \ldots\right] \times \left[(m+1)(m+3)\right]^{1/2}, \qquad m = 0,\ 1,\ 2,\ldots$$

$$A_{m+1}{}^{(1)} = \left[(3/4 - a_2)\,\varepsilon^2 + \left(3(2m+3) - (190m+303)a_2/16 + 6(2m+3)a_3 - (22m+27)a_2{}^2/4\right)\varepsilon^3 + \ldots\right] \times \left[(m+1)(m+2)\right]^{1/2}, \qquad m = 0,\ 1,\ 2,\ldots$$

$$A_{m-1}{}^{(1)} = \left[(-3/4 + a_2)\,\varepsilon^2 - \left(3(2m+1) - (190m+77)a_2/16 + 6(2m+1)a_3 - (22m+17)a_2{}^2)/4\right)\varepsilon^3 + \ldots\right] \times \left[m(m+1)\right]^{1/2}, \qquad m = 1,\ 2,\ 3\ldots$$

$$A_{m-2}{}^{(1)} = \left[\left(3 - 35a_2/4 + 6a_3 + a_2{}^2\right)m\varepsilon^3/4 + \ldots\right] \times \left[(m+1)(m-1)\right]^{1/2}, \qquad m = 2,\ 3,\ 4,\ldots$$

Higher-order perturbed solutions can be calculated in this way. We do not list them here, because of their complexity.

For the case that $b - h(u)$ is always positive, for the vector solution that starts from the case Equation (53), two independent solution of the zero-th order equation can be expressed by using Equation (94). The result is as follows:

$$\Phi_1{}^{(0)}(u) = \left[(u/u_{N+1}(u))\, du_{N+1}(u)/du\right]^{-1/2} J_\nu\left(u_{N+1}(u)\right) \quad (123)$$

and

$$\Phi_2{}^{(0)}(u) = \left[(u/u_{N+1}(u))\, du_{N+1}(u)/du\right]^{-1/2} Y_\nu\left(u_{N+1}(u)\right) \quad (124)$$

with

$$u_{N+1}(u) = \sum \left(f_p \left(b^{(0)} + n\varepsilon^2\right)^{1/2} u\Big/\varepsilon\right)^{2p-1} \quad (125)$$

where $J_\nu(u)$ and $Y_\nu(u)$ are the Bessel function and the Neumann function, respectively. Starting from Equation (54), we have

$$\Psi_1{}^{(0)}(u) = \left[(u/u_{N+1}(u))\, du_{N+1}(u)/du\right]^{-1/2} J_\nu\left(u_{N+1}(u)\right) \quad (126)$$

and

$$\Psi_2{}^{(0)}(u) = \left[(u/u_{N+1}(u))\, du_{N+1}(u)/du\right]^{-1/2} Y_\nu\left(u_{N+1}(u)\right) \quad (127)$$

with

$$u_{N+1}(u) = \sum \left(f_p \left(b^{(0)} - n\varepsilon^2 \right)^{1/2} u \Big/ \varepsilon \right)^{2p-1} \quad (128)$$

From Equations (55)–(58) and Equations (123)–(128), we can obtain the first-order solutions for the hybrid modes analytically where we use the addition theorem and the indefinite integral formulas about the Bessel function and the Neumann function .

Lastly we briefly discuss the weak guidance approximation. It follows from Equations (117)–(120) that the scalar wave solutions for the eigenvalue and the eigenfunction involve only the error of order ε^3, and order ε^2 for the dominant mode or order ε for remaining modes, respectively. Therefore the scalar wave approximation provides a highly accurate solution even for the dominant modes, provided that the relative refractive-index difference is small. It should be noted that vector corrections is effective for the refractive-index whose gradient is large.

6. Characteristic Equations for the Guided Modes

Here we calculate propagation characteristics of the guided modes of the graded-index optical fiber with even polynomial profile cores; normalized propagation constants, waveguide dispersions, mode field diameters,and bending losses. To do that, we consider a graded-index fiber as shown in Fig. 1 whose refractive-index profile is described by

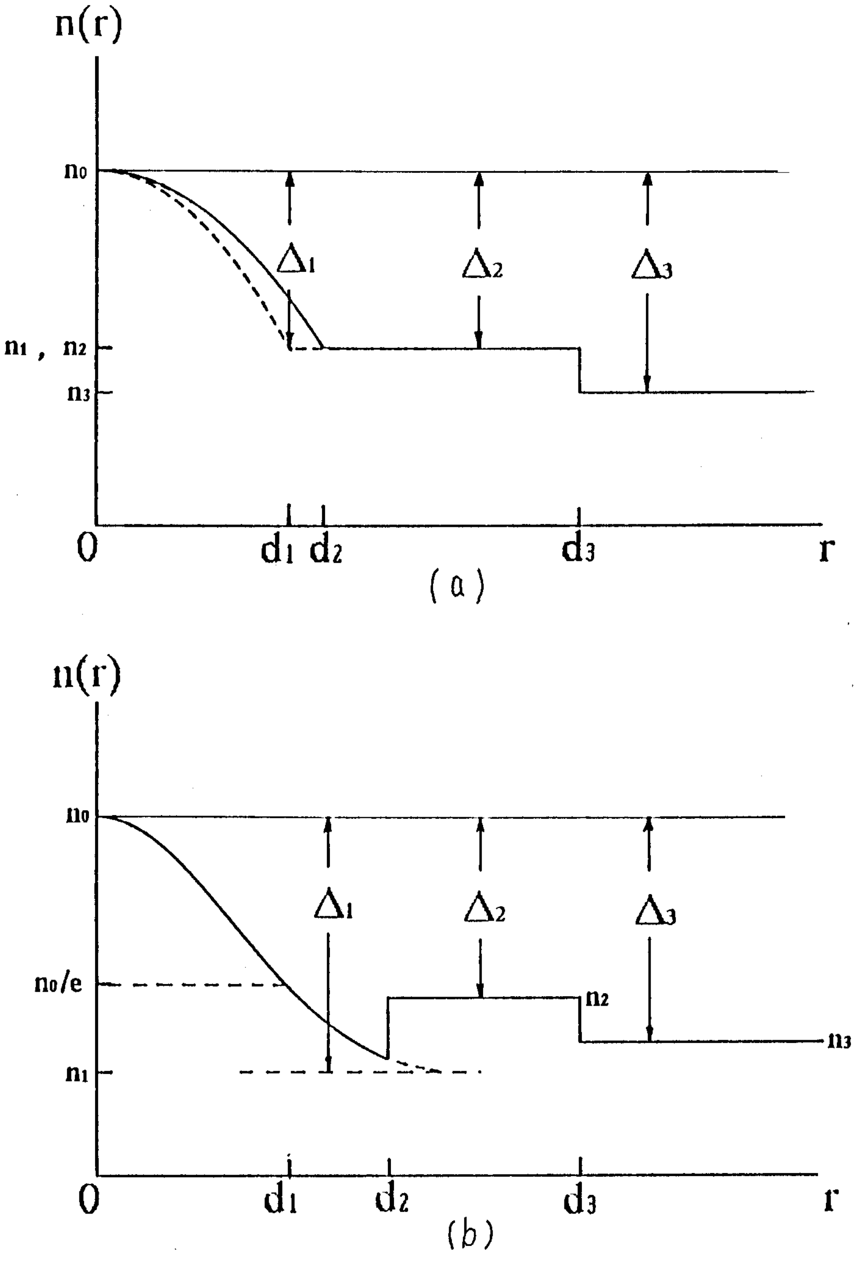

Figure 1. Refractive index profiles (a) Near parabolic profile designated by solid line and parabolic profile denoted by dotted line, (b) Gaussian profile.

$$n(r) = \begin{cases} n_0 \left(1 - h(r)\right)^{1/2} \ , & 0 < r < d_2 \\ n_2 \ , & d_2 < r < d_3 \\ n_3 \ , & d_3 < r \end{cases} \tag{129}$$

where n_0, n_2, n_3, d_2, and d_3 are positive constants. In each region, we can express the electromagnetic field for the guided modes in terms of the analytic solutions. Here we only show the leading components from which all remaining components are derived by differentiation.

In the inhomogeneous core region, we have a pair of solutions of Equation $(10')$ as shown in section 3. For the case that $b - h(r)$ has simple zeros, it follows from the derivation process of the solution of Equation $(10')$ that a pair of candidate solutions of Equation (1) are $C(\Phi + \Psi)$ and $D(\Phi - \Psi)$ where C and D are finally determined by the boundary condition. Therefore we can express E_r and E_θ such that

$$\begin{aligned} E_r =& jC(\Phi + \Psi) / \left(1 - h(r)\right)^{1/2} \\ E_\theta =& - D(\Phi - \Psi) \end{aligned} \tag{130}$$

In Equation (130), the radial mode number "m" in Φ and Ψ in section 5 is replaced by the noninteger mode index "μ" defined by $m + \Delta\mu$ where $\Delta\mu$ is an unknown quantity that represents a cladding effect on the modal field [11, 20]. Note that C and D always satisfy Equation (130) for any μ. Second we derive the field expression of the guided modes in the case that $b - h(r)$ is always positive. The leading functions expressed by the zero-th order solution are given by

$$\begin{aligned} E_r =& jC\Phi^{(0)} / \left(1 - h(r)\right)^{1/2} \\ E_\theta =& - D\Phi^{(0)} \end{aligned} \tag{131}$$

where $\Phi^{(0)}$ is given by Equation (123) or Equation (126). The leading functions represented by first-order corrected vector solutions are given by

$$\begin{aligned} E_r =& jC \left(\Phi^{(0)} + C' \left(\Phi^{(1)} + \Psi^{(1)}\right)\right) / (1 - h(r))^{1/2} \\ E_\theta =& - D \left(\Phi^{(0)} + D' \left(\Phi^{(1)} - \Psi^{(1)}\right)\right) \end{aligned} \tag{132}$$

where $\Phi^{(1)}$ and $\Psi^{(1)}$ are given by Equations (55) and (56) or Equations (57) and (58), C' and D' are unknown constants, and C and D are same constants as those shown in Equation (130). Thus we obtain an expression of the leading functions for deriving the electromagnetic field of the guided modes in the inhomogeneous core region.

In the homogeneous regions such as the second core region and the cladding region, we have following representation of the leading functions. In the middle region we have

$$\begin{aligned} E_Z &= EJ_n(\gamma_2 r) + FY_n(\gamma_2 r) \ , \\ H_Z &= GJ_n(\gamma_2 r) + HY_n(\gamma_2 r) \ , \\ \gamma_2 &= k(b - 2\Delta_2)^{1/2} \ , \\ 2\Delta_2 &= \left(n_0{}^2 - n_2{}^2\right)/n_0{}^2 \end{aligned} \tag{133}$$

where this region acts as the core character and E, F, G, and H are unknown constants. In the range where the mode has cladding characters, we have

$$\begin{aligned} E_Z &= EI_n(\gamma_2 r) + FK_n(\gamma_2 r) \ , \\ H_Z &= GI_n(\gamma_2 r) + HK_n(\gamma_2 r) \ , \\ \gamma_2 &= k(2\Delta_2 - b)^{1/2} \end{aligned} \tag{134}$$

where $I_n(r)$ and $K_n(r)$ are modified Bessel functions. Since we only consider the guided modes here, for Equation (133) we have, in the cladding region,

$$\begin{aligned} E_Z &= \left[\left\{EJ_n(\gamma_2 d_3) + FY_n(\gamma_2 d_3)\right\}/K_n(\gamma_3 d_3)\right] K_n(\gamma_3 r) \ , \\ H_Z &= \left[\left\{GJ_n(\gamma_2 d_3) + HY_n(\gamma_2 d_3)\right\}/K_n(\gamma_3 d_3)\right] K_n(\gamma_3 r) \ , \\ \gamma_3 &= k(2\Delta_3 - b)^{1/2} \ , \\ 2\Delta_3 &= (n_0{}^2 - n_3{}^2)/n_0{}^2 \end{aligned} \tag{135}$$

and for Equation (134), we get

$$\begin{aligned} E_Z &= \left[\left\{EI_n(\gamma_2 d_3) + FK_n(\gamma_2 d_3)\right\}/K_n(\gamma_3 d_3)\right] K_n(\gamma_3 r) \ , \\ H_Z &= \left[\left\{GI_n(\gamma_2 d_3) + HK_n(\gamma_2 d_3)\right\}/K_n(\gamma_3 d_3)\right] K_n(\gamma_3 r) \ , \\ \gamma_3 &= k(2\Delta_3 - b)^{1/2} \end{aligned} \tag{136}$$

where we use the continuity condition of E_Z and H_Z at $r = d_3$.

Thus we get the electromagnetic field representation in the whole domain. First we consider the case that $b - h(r)$ has simple zeros. Matching the boundary conditions at $r = d_3$ and $r = d_3$, we have a characteristic equation for determining the noninteger mode index $\Delta\mu$ or μ from which we can calculate the propagation constant and the amplitude of the guided modes; numerical differentiation of the propagation constant results in the group delays and the waveguide dispersions. Second we consider the case that $b - h(r)$ is always positive. Using the zero-th order solution, we can determine $b^{(0)}$ and the ratio of C and D. If we use the first-order perturbed solutions, we get $b^{(1)}$ and the ratio of C' and D'. Using the higher-order perturbational solutions, we have more precise solutions of the guided modes. With regard to the calculation of the propagation characteristics, we use the method of numerical differentiation too. This is a standard formulation of the guided modes of the graded-index optical fiber where the leading functions are solutions of a coupled second-order differential equation, and a unified formulation of the problem for determining the guided mode of optical fibers.

Here we show an example for determining the guided mode of the graded-index optical fiber with the even polynomial profile core and compare the vector wave analysis and the scalar wave one. For simplicity's sake, we assume that $n_3 = n_2$. Then we have

$$\begin{aligned} E_Z =& EK_n(\gamma_2 r) \ , \\ H_Z =& FK_n(\gamma_2 r) \ , \\ & \gamma_2 = k(2\Delta_2 - b)^{1/2} \ , \\ & r > d_2 \end{aligned} \tag{137}$$

where E and F are unknown constants. Matching the boundary condition at $r = d_2$ for the case that $b - h(r)$ has simple zeros, we have a following characteristic equation:

$$\begin{aligned} &(1 - 2\Delta_2)\, n^2 \left[\left({X_n}^2 - 1 \right) \Big/ (\gamma_2 d_2)^2 \right] \\ &\quad \times \left({d_2}^2 S(d_2)\, T(d_2) - n^2 \left(\Phi(d_2)^2 - \Psi(d_2)^2 \right) \right) \\ &\quad - n \left\{ (1 - 2\Delta_2) \left(\Phi(d_2) - \Psi(d_2) \right) \left(d_2 X_n S(d_2) - n \left(\Phi(d_2) + \Psi(d_2) \right) \right) \right. \end{aligned}$$

$$+\left(1-h\left(d_2\right)\right)\left(\Phi\left(d_2\right)+\Psi\left(d_2\right)\right)\left(d_2 X_n T\left(d_2\right)-n\left(\Phi\left(d_2\right)-\Psi\left(d_2\right)\right)\right)$$
$$+\left(1-h\left(d_2\right)\right)\left(\Phi\left(d_2\right)^2-\Psi\left(d_2\right)^2\right)\Big\}\left(\gamma_2 d_2\right)^2=0 \tag{138}$$

where X_n, $S(r)$, and $T(r)$ are defined by

$$X_n=-1-\left(\gamma_2 d_2/n\right)K_{n-1}\left(\gamma_2 d_2\right)/K_n\left(\gamma_2 d_2\right) \tag{139}$$

and

$$\begin{aligned} S(r)&=\Phi^{'}(r)+\Psi^{'}(r)+\left\{1/r-(1/2)h^{'}(r)/(1-h(r))\right\}(\Phi(r)+\Psi(r)) \\ T(r)&=\Phi^{'}(r)-\Psi^{'}(r)+\left(\Phi(r)-\Psi(r)\right)/r \end{aligned} \tag{140}$$

Simultaneously we have

$$\begin{aligned} D/C=n&\left\{d_2 S\left(d_2\right)-nX_n\left(\Phi\left(d_2\right)+\Psi\left(d_2\right)\right)\right\} \\ &\Big/\left[\left\{nd_2 X_n T\left(d_2\right)-\left(n^2+\left(\gamma_2 d_2\right)^2\right)\left(\Phi\left(d_2\right)-\Psi\left(d_2\right)\right)\right\}\right. \\ &\qquad\left.\left(1-h\left(d_2\right)\right)^{1/2}\right] \end{aligned} \tag{141}$$

where we also have ratios E/C and F/C which we do not show here. Under the weak guidance approximation, we have $1-h(d_2)=1$ and $1-2\Delta_2=1$. Then from Equation (138) we obtain

$$\begin{aligned} &\left\{\left(X_n+1\right)\left(d_2 S\left(d_2\right)-n\Phi^{(0)}\left(d_2\right)\right)-\left(\gamma_2 d_2\right)^2\Phi^{(0)}\left(d_2\right)\right\} \\ &\times\left\{\left(X_n-1\right)\left(d_2 S\left(d_2\right)+n\Psi^{(0)}\left(d_2\right)\right)-\left(\gamma_2 d_2\right)^2\Psi^{(0)}\left(d_2\right)\right\}=0 \end{aligned} \tag{142}$$

where $\Phi^{(0)}$ and $\Psi^{(0)}$ are solutions of the scalar wave equation and $S(r)$ is defined by

$$S(r)=\begin{cases}\Phi'^{(0)}+\Phi^{(0)}/r \\ \Psi'^{(0)}+\Psi^{(0)}/r\end{cases} \tag{143}$$

Then Equation (141) becomes

$$D/C = \begin{cases} 1, \quad \text{for } \left(\gamma_2 d_2\right)^2 \Phi^{(0)}\left(d_2\right) \\ \qquad = \left(X_n - 1\right)\left(d_2 S\left(d_2\right) + n\Phi^{(0)}\left(d_2\right)\right) \\ -1, \quad \text{for } \left(\gamma_2 d_2\right)^2 \Psi^{(0)}\left(d_2\right) \\ \qquad = \left(X_n + 1\right)\left(d_2 S\left(d_2\right) - n\Psi^{(0)}\left(d_2\right)\right) \end{cases} \tag{144}$$

Under the weak guidance approximation, the characteristic equation is reduced to a well-known simplified one.

7. Accuracy Checks of the Uniform Asymptotic Solutions

Here we check the accuracy of the uniform asymptotic solution by evaluating the propagation characteristics of the guided modes of the graded-index optical fiber of which refractive-index profile is given by Eq. (129). First we define a normalized frequency, a normalized propagation constant, and a normalized waveguide dispersion by

$$V = kd_3\left(2\Delta_3\right)^{1/2} \tag{145}$$

$$B = 1 - b/2\Delta_3 \tag{146}$$

and

$$(c/n_0)\sigma = \left(k_0/n_0\right)d^2\beta/dk_o^2 \tag{147}$$

respectively. In Equation (147), c is the light velocity in vacuum.

First we check the accuracy of the asymptotic solution of the scalar wave equation. The normalized parameters calculated by the $N-$th order uniform asymptotic solution can be defined by

$$B_s{}^{(N)} = 1 - E_{\nu,m}{}^{(N)}/2\Delta_3 \tag{148}$$

and

$$(c/n_0)\sigma_s{}^{(N)} = (k_0/n_0)\, d^2 \left[k_0 n_0 \left(1 - E_{\nu,\,m}{}^{(N)} \right)^{1/2} \right] / dk_o^2 \qquad (149)$$

For a fixed V, the asymptotic solution, in general, may diverge as the approximation order N tends to infinity. Therefore we need to determine a best approximation order for a given V. In order to determine a best approximation order for a fixed V, we evaluate following two quantities as

$$\Delta B_s = \left| B - B_s{}^{(N)} \right| \qquad (150)$$

and

$$\Delta \sigma_s = (c/n_0) \left| \sigma - \sigma_s{}^{(N)} \right| \qquad (151)$$

Since in general we have not an exact solution, we test the accuracy of the asymptotic solution through the quantities [25, 34] as

$$\Delta B_s{}^{(p)} = \left| B_s{}^{(p)} - B_s{}^{(p-1)} \right| , \qquad p = 2,\ 3, \ldots,\ N \qquad (152)$$

and

$$\Delta \sigma_s{}^{(p)} = (c/n_0) \left| \sigma_s{}^{(p)} - \sigma_s{}^{(p-1)} \right| , \qquad p = 2,\ 3, \ldots,\ N \qquad (153)$$

Now we numerically estimate the accuracy of the uniform asymptotic solutions. Fig. 2 shows that the asymptotic solutions do not converge as the approximation order N of the asymptotic solution increases for a fixed V. In order to calculate a solution of the vector wave equation using the perturbation method, we need accurate solutions of the scalar wave equation for the guided mode in the single mode region that satisfy the condition Equation (112). Here we numerically show that the uniform asymptotic solution obtained here attains a prescribed accuracy for higher V than some V_0 in the single-mode region. For a fixed V and a $h(r)$ given by Equation (11), we have some p that minimizes the quantities (152) and (153) for the asymptotic solutions and consider it as a best solution among the asymptotic solutions. When we determine an optimum approximation order, we use a series solution of the scalar wave equation as a reference solution. Exactly speaking, an optimum approximation order p depends on V and the truncation size L of Equation (11). For V greater than V_0 the optimum solution does not depend on the refractive-index profile as shown in Fig. 2. Therefore the numerical calculation can be done for the refractive-index with a parameter L that equals to the optimum approximation order. It follows from the comparison with the reference solution obtained by the conventional series solution that even for the waveguide dispersion we obtain 3 significant figures for V greater than V_0 in the single mode region the fifth order uniform asymptotic solution as shown in Tables 1 and 2 . It is noted that the convergence rate of a conventional series solution is very slow. As shown in Fig. 3, it is noted that the WKB solution involves significant error in the single mode region.

Next we examine the accuracy of the vector corrected solution and check the accuracy of the scalar wave approximation by estimating corrected terms of the vector solution. To do so, we define two quantities: normalized propagation constants ${B_v}^{(P)}$ and the waveguide dispersions $(c/n_0){\sigma_v}^{(p)}$ $(p = 0,\ 1,\ 2, \ldots)$; ${B_v}^{(P)}$ and $(c/n_0){\sigma_v}^{(p)}$ mean values calculated by the pth order vector solution and in the case $p = 0$, they are values calculated by the optimized scalar wave solution. Results are shown in Tables 3 and 4. These tables show that the first order vector correction term plays a dominant role in the vector correction as expected in the analytical result and also show that the weak guidance approximation well works except for the low frequency region and the near cutoff region. As the relative refractive-index difference becomes large, the difference between the solution of the scalar wave equation and of the vector wave equation becomes significant, in particular, on the waveguide dispersion. as shown in Fig. 4. Therefore we have to use the first order vector corrected solution to evaluate the propagation characteristics of the guided modes of the graded-index optical fiber. In the next section we design a dispersion-shifted optical fiber with an even polynomial refractive-index profile center core.

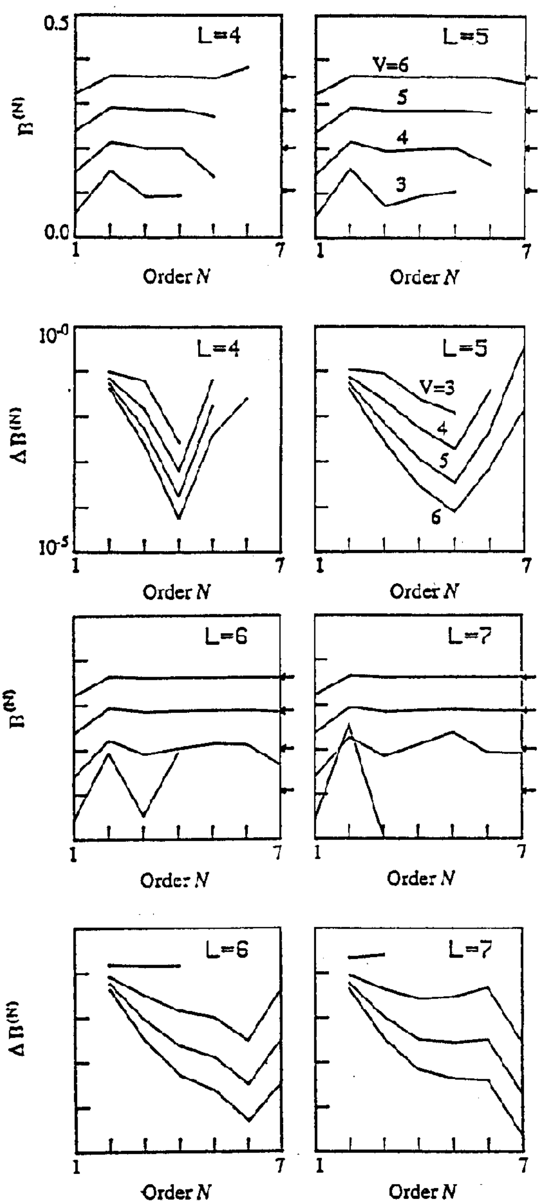

Figure 2. Convergence behavior of asymptotic solution for the Gaussian profile, $\Delta_3 = 0.008,\ Ra_1 = 0.4,\ Ra_2 = 0.6,\ R\Delta_1 = 1.0,\ R\Delta_2 = 0.8.$

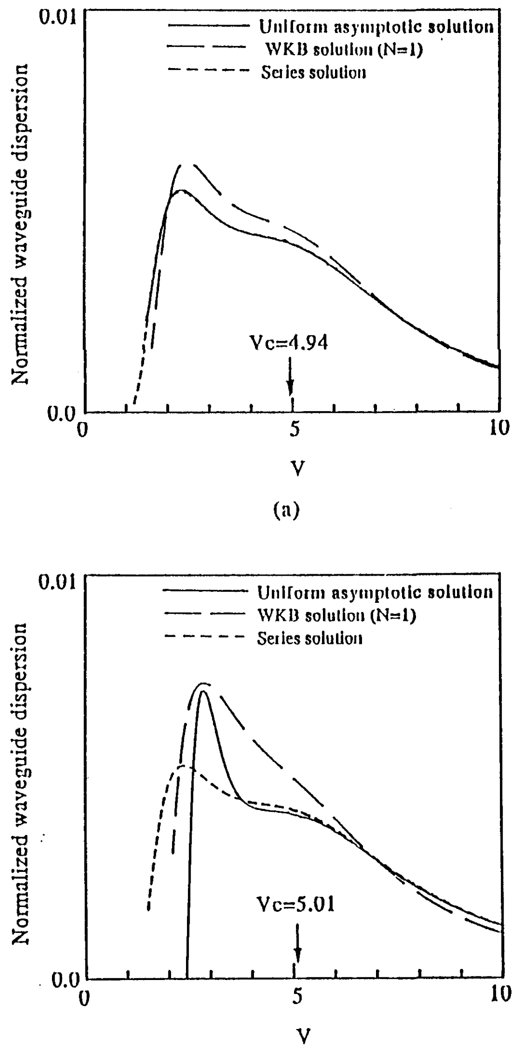

Figure 3. **Normalized waveguide dispersions versus normalized frequency (a) Near parabolic profile,** $\Delta_3 = 0.008$, $Ra_1 = 0.4$, $Ra_2 = 0.47$, $R\Delta_1 = 0.8$, $R\Delta_2 = 0.8$, **(b) Gaussian profile,** $\Delta_3 = 0.008$, $Ra_1 = 0.4$, $Ra_2 = 0.6$, $R\Delta_1 = 1.0$, $R\Delta_2 = 0.8$.

Figure 4. Normalized waveguide dispersions versus normalized frequency for the Gaussian profile, $Ra_1 = 0.4,\ Ra_2 = 0.6,\ R\Delta_2 = 0.8.$

(a) Near parabolic profile

$\Delta_3 = 0.8\%, \quad Ra_1 = 0.4, \quad Ra_2 = 0.47, \quad R\Delta_1 = 0.8, \quad Ra_2 = 0.8$

L	V	$B^{(5)}$	B	ΔB
	6.0	3.80×10^{-1}	3.80×10^{-1}	3.04×10^{-7}
	5.0	3.04×10^{-1}	3.04×10^{-1}	9.86×10^{-7}
5	4.0	2.16×10^{-1}	2.16×10^{-1}	1.71×10^{-6}
	3.0	1.16×10^{-1}	1.16×10^{-1}	2.94×10^{-5}
	2.0	2.39×10^{-2}	2.33×10^{-2}	6.96×10^{-4}
	1.5	4.25×10^{-3}	2.29×10^{-3}	1.96×10^{-3}

(b) Gaussian profile

$\Delta_3 = 0.8\%, \quad Ra_1 = 0.4, \quad Ra_2 = 0.6, \quad R\Delta_1 = 1.0, \quad Ra_2 = 0.8$

L	V	$B^{(5)}$	B	ΔB
	6.0	3.61×10^{-1}	3.61×10^{-1}	6.97×10^{-5}
	5.0	2.87×10^{-1}	2.87×10^{-1}	2.91×10^{-4}
5	4.5	2.46×10^{-1}	2.46×10^{-1}	5.95×10^{-4}
	4.0	2.01×10^{-1}	2.03×10^{-1}	1.19×10^{-3}
	3.5	1.54×10^{-1}	1.56×10^{-1}	2.08×10^{-3}
	3.0	1.06×10^{-1}	1.07×10^{-1}	1.86×10^{-3}

Table 1. Uniform asymptotic solution and series solution of the scalar wave equation for the normalized propagation constant.

I: Uniform asymptotic solution.

II: Series solution.

$E_1 = |(I - II)/II| \times 100\%$

(a) Near parabolic profile

$\Delta_3 = 0.8\%, \quad Ra_1 = 0.4, \quad Ra_2 = 0.47, \quad R\Delta_1 = 0.8, \quad Ra_2 = 0.8$

V	I	II	E_1
6.0	3.64×10^{-3}	3.64×10^{-3}	2.60×10^{-3}
5.0	4.21×10^{-3}	4.21×10^{-3}	1.77×10^{-3}
4.0	4.47×10^{-3}	4.47×10^{-3}	3.60×10^{-2}
3.0	5.02×10^{-3}	5.00×10^{-3}	4.15×10^{-1}
2.0	5.09×10^{-3}	5.05×10^{-3}	9.06×10^{-1}
1.5	2.32×10^{-3}	1.95×10^{-3}	$1.87 \times 10^{+1}$

(b) Gaussian profile

$\Delta_3 = 0.8\%, \quad Ra_1 = 0.4, \quad Ra_2 = 0.6, \quad R\Delta_1 = 1.0, \quad Ra_2 = 0.8$

V	I	II	E_1
6.0	3.65×10^{-3}	3.68×10^{-3}	8.43×10^{-1}
5.0	4.09×10^{-3}	4.17×10^{-3}	2.10
4.5	4.16×10^{-3}	4.28×10^{-3}	2.94
4.0	4.27×10^{-3}	4.36×10^{-3}	2.24
3.5	4.87×10^{-3}	4.52×10^{-3}	7.68
3.0	6.77×10^{-3}	4.86×10^{-3}	$3.94 \times 10^{+1}$

Table 2. Uniform asymptotic solution and series solution of the scalar wave equation for the normalized waveguide dispersion.

I: Uniform asymptotic solution $(N = L = 5)$.
II: 1st order vector solution.
III: 2nd order vector solution.

$$E_1 = |(I - II)/II| \times 100\%,$$
$$E_2 = |(II - III)/III| \times 100\%$$

(a) Near parabolic profile

$\Delta_3 = 0.8\%, \quad Ra_1 = 0.4, \quad Ra_2 = 0.47, \quad R\Delta_1 = 0.8, \quad Ra_2 = 0.8$

V	I	II	III	E_1	E_2
6.0	3.80×10^{-1}	3.79×10^{-1}	3.79×10^{-1}	3.13×10^{-1}	1.60×10^{-4}
5.0	3.04×10^{-1}	3.03×10^{-1}	3.03×10^{-1}	4.41×10^{-1}	3.91×10^{-4}
4.0	2.16×10^{-1}	2.14×10^{-1}	2.14×10^{-1}	6.04×10^{-1}	1.05×10^{-3}
3.0	1.16×10^{-1}	1.15×10^{-1}	1.15×10^{-1}	7.20×10^{-1}	3.34×10^{-3}
2.5	6.56×10^{-2}	6.52×10^{-2}	6.52×10^{-2}	6.45×10^{-1}	6.54×10^{-3}
2.0	2.39×10^{-2}	2.39×10^{-2}	2.39×10^{-2}	2.79×10^{-2}	1.39×10^{-2}

(b) Gaussian profile

$\Delta_3 = 0.8\%, \quad Ra_1 = 0.4, \quad Ra_2 = 0.6, \quad R\Delta_1 = 1.0, \quad Ra_2 = 0.8$

V	I	II	III	E_1	E_2
6.0	3.61×10^{-1}	3.60×10^{-1}	3.60×10^{-1}	2.16×10^{-1}	5.38×10^{-4}
5.5	3.25×10^{-1}	3.25×10^{-1}	3.25×10^{-1}	2.75×10^{-1}	7.73×10^{-4}
5.0	2.87×10^{-1}	2.86×10^{-1}	2.86×10^{-1}	2.68×10^{-1}	1.10×10^{-3}
4.5	2.46×10^{-1}	2.45×10^{-1}	2.45×10^{-1}	2.26×10^{-1}	1.54×10^{-3}
4.0	2.01×10^{-1}	2.01×10^{-1}	2.01×10^{-1}	1.23×10^{-1}	1.99×10^{-3}
3.5	1.54×10^{-1}	1.54×10^{-1}	1.54×10^{-1}	7.69×10^{-2}	1.85×10^{-3}

Table 3. **Normalized propagation constants calculated by vector solutions together with the scalar solution.**

I: Uniform asymptotic solution $(N = L = 5)$.
II: 1st order vector solution.
III: 2nd order vector solution.

$$E_1 = |(I - II)/II| \times 100\%,$$
$$E_2 = |(II - III)/III| \times 100\%$$

(a) Near parabolic profile

$\Delta_3 = 0.8\%, \quad Ra_1 = 0.4, \quad Ra_2 = 0.47, \quad R\Delta_1 = 0.8, \quad Ra_2 = 0.8$

V	I	II	III	E_1	E_2
6.0	3.64×10^{-3}	3.66×10^{-3}	3.66×10^{-3}	6.42×10^{-1}	1.16×10^{-3}
5.0	4.21×10^{-3}	4.25×10^{-3}	4.25×10^{-3}	1.02	8.73×10^{-4}
4.0	4.47×10^{-3}	4.51×10^{-3}	4.51×10^{-3}	9.50×10^{-1}	1.98×10^{-4}
3.0	5.02×10^{-3}	5.02×10^{-3}	5.02×10^{-3}	6.95×10^{-3}	3.12×10^{-3}
2.5	5.49×10^{-3}	5.45×10^{-3}	5.45×10^{-3}	7.75×10^{-1}	4.97×10^{-3}
2.0	5.09×10^{-3}	5.04×10^{-3}	5.04×10^{-3}	1.11	4.05×10^{-3}

(b) Gaussian profile

$\Delta_3 = 0.8\%, \quad Ra_1 = 0.4, \quad Ra_2 = 0.6, \quad R\Delta_1 = 1.0, \quad Ra_2 = 0.8$

V	I	II	III	E_1	E_2
6.0	3.65×10^{-3}	3.71×10^{-3}	3.71×10^{-3}	1.50	7.86×10^{-4}
5.5	3.92×10^{-3}	3.98×10^{-3}	3.98×10^{-3}	1.42	6.61×10^{-4}
5.0	4.09×10^{-3}	4.13×10^{-3}	4.13×10^{-3}	1.16	3.01×10^{-3}
4.5	4.16×10^{-3}	4.18×10^{-3}	4.18×10^{-3}	5.60×10^{-1}	7.01×10^{-3}
4.0	4.27×10^{-3}	4.25×10^{-3}	4.25×10^{-3}	2.97×10^{-1}	1.45×10^{-2}
3.5	4.87×10^{-3}	4.84×10^{-3}	4.84×10^{-3}	5.58×10^{-1}	2.41×10^{-2}

Table 4. Normalized waveguide dispersions calculated by vector solutions together with the scalar solution.

8. Optimization of Parameters of a Dispersion-Shifted Optical Fiber with an Even Polynomial Refractive-Index Center Core

Here we also consider a graded-index optical fiber whose refractive-index profile is given by Equation (133) as shown in Fig. 1. In order to design a dispersion-shifted optical fiber [37–42], we optimize fiber parameters under the constraints such that the chromatic dispersion is zero at $1.55\mu m$, the bending loss is less than a prescribed value, and the spot size of the fundamental mode makes as large as possible. To design such optical fibers, we may make use of refractive index data for fiber materials [48–49] which is conventionally expressed in terms of a three-term Sellmeier equation. Under the condition with zero dispersion at $1.55\mu m$, we calculate a macro-bending loss evaluated by the formula given in reference [50] and a mode field diameter is defined by the Petermann's second definition [51]. Now we define two kind of parameters, that specify refractive-index distribution of the optical fiber considered here, as core radius ratios

$$Ra1 = d_1/d_3 \ , \qquad Ra2 = d_2/d_3$$

and relative refractive-index difference ratios

$$R\Delta 1 = \Delta_1/\Delta_3 \ , \qquad R\Delta 2 = \Delta_2/\Delta_3$$

respectively. Under the constraint that the chromatic dispersion slope is less than $60[ps/km/nm/\mu m]$, we examine structural parameters of the optical fibers having zero dispersion at $1.55\mu m$. It follows from Fig. 5 and Table 5 that the graded-index optical fiber with even polynomial center core may be a good dispersion-shifted optical fiber. It is also apparent from Fig. 6 that the chromatic dispersion slopes of two kinds of dispersion-shifted dual shape core optical fibers can be kept within the prescribed value $60[ps/km/nm/\mu m]$ over wide wavelength range centered at $1.55\mu m$ for refractive-index profiles of the center core from the square-law profile to the Gaussian profile. This result shows that the chromatic dispersion slopes less than a prescribed value are realizable for a relatively wide class of the refractive-index profiles of the center core after optimizing other structural parameters as shown in Fig. 6. Taking this fact into account, we show a design example of parabolic core dispersion-shifted optical fibers as shown in

Fig. 7 [52]. Parameters of the designed square-law optical fibers are shown in Table 6. We can conclude that dual shape core optical fibers are more appropriate for the dispersion-shifted optical fiber than the conventional optical fibers. It should be noted that these simulation can be done by using a desktop computer. Thus the formulation presented here involves the parameter about the refractive-index profile in its analytic solution and allows us to use a personal computer that makes the design of some wide class of graded-index optical fibers to be convenient [52]. A remaining important task is to examine the dispersion sensitivity due to the parameter change to design dispersion-shifted optical fibers [41]. Although we do not carry out this task, we can do it in the near future.

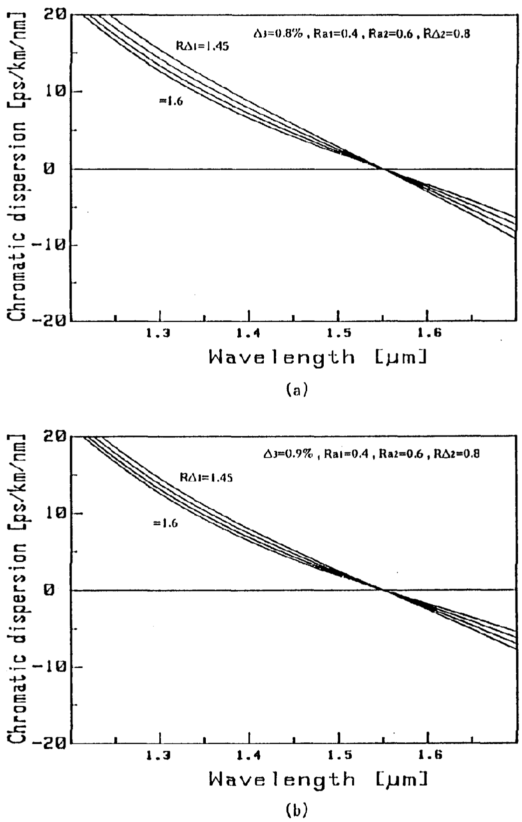

Figure 5. Chromatic dispersions versus wavelength for the Gaussian profile, (a): $\Delta_3 = 0.008$, $Ra_1 = 0.4$, $Ra_2 = 0.6$, $R\Delta_2 = 0.9$. **(b):** $\Delta_3 = 0.009$, $Ra_1 = 0.4$, $Ra_2 = 0.6$ $R\Delta_2 = 0.9$.

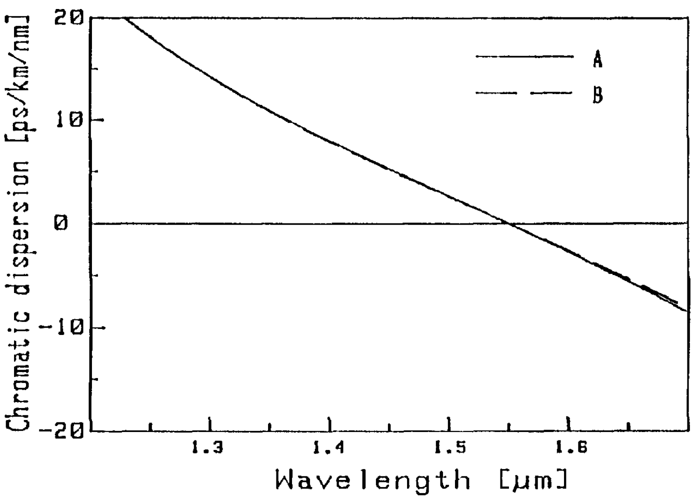

Figure 6. **Chromatic dispersions versus wavelength for the Gaussian profile, A:** $\Delta_3 = 0.008,\ Ra_1 = 0.4,\ Ra_2 = 0.4,\ R\Delta_1 = 1.5,\ R\Delta_2 = 0.8.$ **B:** $\Delta_3 = 0.008,\ Ra_1 = 0.4,\ Ra_2 = 0.6,\ R\Delta_1 = 1.5,\ R\Delta_2 = 0.8.$

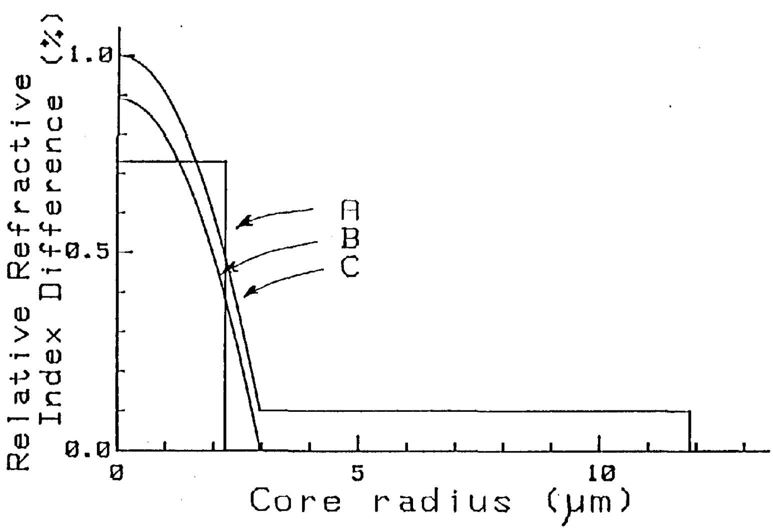

Figure 7. **Refractive-index profiles for zero dispersion at** $1.55\mu m$ **with the mode field diameter** $7\mu m$.

I: Core radius (μm)

II: Bending loss at $\phi = 20$mm (dB/m)

III: Modefield diameter (μm)

		$\Delta_3 = 0.8\%$			$\Delta_3 = 0.9\%$		
Ra_1	$R\Delta_1$	I	II	III	I	II	III
0.4	1.4	8.12	10.99	8.01	8.26	0.19	7.32
	1.45	8.55	7.76	7.80	8.52	0.21	7.20
	1.5	8.82	7.44	7.64	8.70	0.26	7.09
	1.55	9.03	8.24	7.52	8.85	0.36	6.99
	1.6	9.20	9.90	7.40	8.98	0.52	6.90
0.3	1.2	9.80	14.33	7.87	9.68	0.59	7.28
	1.25	10.12	16.91	7.80	9.91	0.93	7.13
	1.3	10.36	22.70	7.51	10.09	1.57	7.00
	1.35	10.55	32.35	7.37	10.23	2.76	6.88
	1.4	10.71	47.30	7.24	10.36	4.92	6.77

$Ra_2 = 0.6$, $R\Delta_2 = 0.8$

Table 5. Core radius for zero-dispersion, macro-bending loss, and mode field diameter at 1.55μm.

A: step index profile
B: parabolic index profile
C: dual shape core optical fiber

Profile	A	B	C
Core rqadius (μm)	2.25	2.967	11.87
Rerative refractive index difference (%)	0.73	0.89	1.005
Other parameters	—	—	$R\Delta = 0.9$, $Ra = 0.25$
macro-bending loss at $\phi = 20$mm (dB/m)	2.1	1.3	0.0005
dispersion slope (ps/km/nm/μm)	47	52	53

Table 6. Designed waveguide parameters for zero-dispersion at 1.55μm **with mode field diameter** 7μm.

9. Conclusion

We present an analytic solution of the vector wave equation for the guided modes of graded-index optical fibers with an even polynomial profile center core in a form of a uniform asymptotic solution. Accordingly, a formulation of the problem for determining the guided modes of graded-index optical fibers is as simple as that for step-index optical fibers. Moreover, an amount of computation is considerably reduced so that we can use a personal computer instead of main frames for designing dispersion-shifted optical fibers by controlling several parameters. As a numerical example, we design a dispersion-shifted optical fiber with an even polynomial refractive-index center core.

Remaining important problems are to construct analytic solutions of the vector wave equations for other refractive-index profiles considered here and to design a dispersion-flattened graded-index optical fiber.

APPENDIX A: ESTIMATION OF $R_N(u_{N+l})$

For the even polynomial refractive-index profile given by Equation (11), we can express Q_p as a power series of u_p :

$$Q_p\left(u_p\right) = b_{p0} + b_{p1}{u_p}^2 + b_{p2}{u_p}^4 + \dots, \qquad \text{for } p = 1,\ 2, \dots,\ N \quad \text{(A1)}$$

Substituting Equation (A1) into Equations (61) and (65), we have a series solution of Equation (61) as shown in Equation (72) as

$$u_p = c_{p0}u_{p+1} + c_{p1}{u_{p+1}}^3 + c_{p2}{u_{p+1}}^5 + \dots, \quad \text{for } p = 1,\ 2, \dots,\ N \quad \text{(A2)}$$

where c_{pq} $(q = 0,\ 1, \dots)$ depends on b_{pr}′s. Substituting Equation (A2) into Equation (64), we get

$$R_p\left(u_{p+1}\right) = e_{p0} + e_{p1}{u_{p+1}}^2 + e_{p2}{u_{p+1}}^4 + \dots, \quad \text{for } p = 1,\ 2, \dots,\ N \quad \text{(A3)}$$

where e_{pq} $(q = 0,\ 1, \dots)$ depends on b_{pr}′s. From Equations (63) and (64) together with Equation (A1), we have

$$b_{p+1\ q} = \begin{cases} {I_p}^2 - \varepsilon^2 e_{pq}, & \text{for } q = 0,\ 1 \\ -\varepsilon^2 e_{pq}, & \text{for } q = 2,\ 3,\ 4, \dots \end{cases} \quad \text{(A4)}$$

Since $E_{\nu,m} = \mathrm{O}(\varepsilon)$, we have $I_1 = 0(\varepsilon)$ and

$$b_{1q} = \begin{cases} \mathrm{O}\left(\varepsilon\right), & \text{for } q = 0,\ 1 \\ \mathrm{O}\left(1\right), & \text{for } q = 2,\ 3,\ 4, \dots \end{cases}$$

Using the estimation shown above, after a straight forward manipulation we have

$$e_{1q} = \mathrm{O}\left(\varepsilon^{q+1}\right) \ , \quad \text{for} \ \ q = 0, 1, 2, \ldots$$

From Equation (A4), we have

$$b_{2q} = \begin{cases} \mathrm{O}\left(\varepsilon^{2}\right) \ , & \text{for} \ \ q = 0, \ 1 \\ \mathrm{O}\left(\varepsilon^{3+q}\right) \ , & \text{for} \ \ q = 2, \ 3, \ 4, \ldots \end{cases}$$

Here we assume that

$$I_p = \mathrm{O}\left(\varepsilon\right) \ , \qquad \text{for} \ \ p = 2, \ 3, \ldots, \ N$$

and

$$e_{pq} = \mathrm{O}\left(\varepsilon^{2P-1+q}\right) \ , \qquad \text{for} \ \ q = 0, \ 1, \ 2, \ldots$$

Then from Equation (A4), we obtain

$$b_{p+1q} = \begin{cases} \mathrm{O}\left(\varepsilon^{2}\right) \ , & \text{for} \ \ q = 0, \ 1 \\ \mathrm{O}\left(\varepsilon^{2p+1+q}\right) \ , & \text{for} \ \ q = 2, \ 3, \ 4, \ldots \end{cases}$$

and

$$e_{p+1q} = \mathrm{O}\left(\varepsilon^{2p+1+q}\right) \ , \qquad \text{for} \ \ q = 0, \ 1, \ 2, \ldots$$

Consequently, from the mathematical reduction we have

$$e_{Nq} = \mathrm{O}\left(\varepsilon^{2N-1+q}\right) \ , \qquad \text{for} \ \ q = 0, \ 1, \ 2, \ldots \tag{A5}$$

From this result, we finally have the estimation Equation (74).

APPENDIX B: Recursive Formula and Addition Theorem about the Laguerre Polynomials

Here we write down the recursive formula and the addition theorem about the Laguerre polynomials. The recursive formula and the addition theorem are glven as

$$L_n{}^{(\alpha)}(u) = \sum \left[\Gamma(\alpha - \beta + r)/(\Gamma(r+1)\Gamma(\alpha - \beta))\right] L_{n-r}{}^{(\beta)}(u) \tag{B1}$$

and

$$L_n{}^{(\alpha)}(u+w) = \exp(v) \sum \left[(-1)^m / (\Gamma(m+1)\right] L_{n-r}{}^{(\alpha+m)}(u) w^m \quad \text{(B2)}$$

Since w can be represented by even polynomials of u and u is greater greater than w for small parameter ε, each term of Equation (77) can be expressed in terms of $L_n{}^{(\alpha)}(u')$s of which coefficients are a power series of u. Accordingly we can use integral formulas appeared in the square-law which are derived from Equations $(21')$, (115), and (116) . Finally we reach following result as

$$\begin{aligned} &\int_0^\infty v^{R-1} U_{p,\,q}{}^{(N)} U_{p,\,r}{}^{(N)} dv \\ &= D^{(0)}(p,\ q,\ r;\ R) + D^{(1)}(p,\ q,\ r;\ R) + D^{(2)}(p,\ q,\ r;\ R) + \ldots, \\ &\qquad p,\ q,\ r = 0,\ 1,\ 2, \ldots; \qquad R = 1,\ 2,\ 3, \ldots \end{aligned} \quad \text{(B3)}$$

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