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FOCUSING PROPERTIES OF MAXWELL'S FISH EYE MEDIUM

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Abstract—The existence of drains in the field of the point source both for the spherical cavity with perfectly conducting boundary, filled with a homogeneous medium, and for unbounded Maxwell's fish eye (MFE) are rigorously proved. The existence of all class of generalized Green functions for unbounded MFE medium is established. The Green function describing the perfect focusing is found in this class. The same result for the MFE lens is obtained.

1. INTRODUCTION

Maxwell's fish eye medium (MFEM) is an unbounded, spherically symmetric, inhomogeneous medium with the refractive index

$$n_{fe}(r) = \frac{2n_0\rho^2}{r^2+\rho^2}. \tag{1}$$

Within the approximation of geometrical optics [1], Maxwell found the perfect focusing properties of this medium, the trajectories of all rays generating from arbitrary point source $\mathbf{r}'$, come to the point of its image $\mathbf{r}'' = -\rho^2\mathbf{r}'/r'^2$, $|\mathbf{r}'| = r'$.

In the pioneering paper on scalar wave processes in the MFE media [2] Demkov and Ostrovsky proved the unique transformation properties of the corresponding Helmholtz equation, by means of which a three-dimensional Green function in the closed form was obtained. Developing the ideas of this work, Szmytkowski [3] obtained the closed form Green function for the scalar Helmholtz equation in the N-dimensional MFE medium ($N \geq 2$).

The interest in this research area increased rapidly with Leonhardt et al.'s papers [4–8]. It is stated there that MFE-medium is able to

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provide the perfect image resolution which is free of known Abbe's limitation [9]. In other words, this medium properties unfolded by Maxwell are also valid when passing from geometrical and optical representation to the wave description. Leonhardt also suggested a more practical modification of the finite Maxwell's fish eye lens (MFEL) bounded by perfectly conducting sphere. The obtained analytical expressions for the field of point source in the MFE lens (i.e., the Green function in two [4] and three dimensions) have singularities both at the point source $\mathbf{r}'$ and at point if its image $\mathbf{r}''$. The latter is a so-called active point drain for energy withdrawal.

The suggestion to justify the possibility of perfect focusing with such an unusual object like point drain initiated a set of critiques [10–14] which claim the super-resolving focusing developed in [4, 5] is merely a consequence of a special drain location, and not a fundamental property of MFEL. The most detailed one is the paper by Merlin [12]. The author considers the spherical cavity with perfectly conducting boundary filled with a homogeneous medium to be a good analogue of MFEL. Although this statement remains doubtful, Merlin presents an interesting internal excitation problem's stationary solution. The case when the source is a radial electric dipole in the center of a sphere is considered in [12]; here the following simple expression for the Hertz potential of the total field $\Pi = kp_0(e^{ikr}/kr + iA\sin kr/kr)$ is obtained, $k = \omega/c$, A is a known constant. Using this representation for Π, Merlin concludes that there is no drain in this cavity. Now, how a harmonic source $\Pi_0 = p_0 e^{ikr}/r$, pumping electromagnetic energy in the limited closed domain, can produce a stationary field? Leonhardt and Philbin [5] emphasize this fact, 'in order to maintain a stationary regime we must supplement the source by a drain at the image'. It is easy to separate the outgoing and incoming waves and to find an average energy flux for each of them in the above expression for the potential. They appear to be equal to each other, which ensure the stationarity of solution in [12]. This means that there is not only a source of outgoing spherical waves, but also the drain for the incoming spherical waves in the center of the sphere.

In this paper, we present the ascertainment of drains role, when constructing three-dimensional Green functions of unbounded MFE medium and MFE lens. The structure of the paper is as follows. In Section 2, we investigate the field of radial electrical dipole located arbitrarily in the spherical cavity with homogeneous filling. The existence of a whole class of generalized Green functions of the Helmholtz equation for an unbounded MFE medium is proved and also the Green function describing the perfect focusing is found in the main Section 3. The similar Green function for MFEL is obtained in the last section.

2. VECTOR GREEN FUNCTION FOR A SPHERICAL CAVITY WITH HOMOGENEOUS FILLING

Let us consider the following generalization of the model [12]. We investigate boundary value problem for Maxwell's equations in the spherical cavity $0 < r < a$ with perfectly conducting boundary in the spherical coordinate system $\vec{r} = (r, \theta, \varphi)$. Radial electrical dipole $\vec{J} = I\tilde{l}\delta(\vec{r})e^{-i\omega t}\vec{z}_0$ is located at the point $\vec{r} = (b, 0, 0)$. The field of this source and the secondary field can be written as an expansion by Hansen vector harmonics [15]. We can obtain the total field

$$\vec{E} = \vec{E}^i + \vec{E}^s, \quad \vec{H} = \vec{H}^i + \vec{H}^s \tag{2}$$

using boundary conditions at $r = a$.

$$\vec{E}^i = \frac{G}{b^2}\sqrt{\frac{\mu}{\varepsilon}}\sum_{n=1}^{\infty} g_n^i \vec{N}_{0n}^{(+)}(kr, \theta, \varphi), \quad \vec{H}^s = \frac{-iG}{b^2}\sum_{n=1}^{\infty} g_n^i \vec{M}_{0n}^{(+)}(kr, \theta, \varphi), \tag{3}$$

$$\vec{E}^s = \frac{-G}{b^2}\sqrt{\frac{\mu}{\varepsilon}}\sum_{n=1}^{\infty} g_n^s \vec{N}_{0n}^{(0)}(kr, \theta, \varphi), \quad \vec{H}^s = \frac{iG}{b^2}\sum_{n=1}^{\infty} g_n^s \vec{M}_{0n}^{(0)}(kr, \theta, \varphi), \tag{4}$$

where $G = I\tilde{l}/2\sqrt{\pi}$, $g_n^i = \sqrt{n(n+1)(2n+1)}kbj_n(kb)$, $g_n^s = g_n^i A_n$, $A_n = \left.\frac{\partial(xh_n^{(1)}(x))/\partial x}{\partial(xj_n(x))/\partial x}\right|_{x=ka}$. Hanson vector harmonics, by definition [15] equals to

$$\vec{M}_{mn}(kr,\theta,\varphi) = \gamma_{mn} z_n(kr)\,\vec{C}_{mn}(\theta,\varphi), \tag{5}$$

$$\vec{N}_{mn}(kr,\theta,\varphi) = \gamma_{mn}\left\{\frac{n(n+1)}{kr} z_n(kr)\vec{P}_{mn}(\theta,\varphi) + \frac{1}{kr}\frac{d}{dr} r z_n(kr)\vec{B}_{mn}(\theta,\varphi)\right\}, \tag{6}$$

where

$$\gamma_{mn}^2 = \frac{(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}, \quad \vec{C}_{mn}(\theta,\varphi) = \nabla\times\left[\vec{r}P_n^m(\cos\theta)e^{im\varphi}\right],$$

$$\vec{P}_{mn}(\theta,\varphi) = \vec{r}P_n^m(\cos\theta)\,e^{im\varphi}, \quad \vec{B}_{mn}(\theta,\varphi) = r\nabla\left[P_n^m(\cos\theta)\,e^{im\varphi}\right],$$

$z_n(x)$ denotes the spherical Bessel functions: $j_n(xt) = \sqrt{\pi/2x}J_{n+1/2}(x)/x$, $h_n^{(1,2)}(x) = \sqrt{\pi/2x}H_{n+1/2}^{(1,2)}(x)/x$, $P_n^m(\cos\theta)$ are the Legendre functions. Superscript (0) in $\vec{M}_{0n}^{(0)}$, $\vec{N}_{0n}^{(0)}$ means that $z_n(x) = j_n(x)$ (in [15] symbols $Rg\vec{M}_{mn}$, $Rg\vec{N}_{mn}$ are used for this). In case when $z_n(x) = h_n^{(1,2)}(kr)$, symbols $\vec{M}_{0n}^{(+,-)}$, $\vec{N}_{0n}^{(+,-)}$ are used correspondingly.

Let us consider the electromagnetic energy currents in the neighborhood of point source. For this it is required to change coordinate system and pass to $\vec{r}' = (r', \theta', \varphi')$ connected with the former one by parallel shift $\vec{r} = \vec{r}_0 + \vec{r}'$. Here $\vec{r}_0 = (b, 0, 0)$ in the original coordinate system, z is the axis of both systems. The translation addition theorem for regular vector wave functions in general $\vec{r}_1 = \vec{r}_{12} + \vec{r}_2$, where $\vec{r}_j = (r_j, \theta_j, \varphi_j)$ is [15]

$$\begin{bmatrix} \vec{M}^{(0)}_{mn} \\ \vec{N}^{(0)}_{mn} \end{bmatrix}(kr_1, \theta_1, \varphi_1) = \sum_{\nu=1}^{\infty}\sum_{\mu=-\nu}^{\nu}\left\{\begin{bmatrix} A^{(0)}_{\mu\nu mn} \\ B^{(0)}_{\mu\nu mn} \end{bmatrix}(kr_{12}, \theta_{12}, \varphi_{12})\vec{M}^{(0)}_{\mu\nu}(kr_2, \theta_2, \varphi_2) + \begin{bmatrix} B^{(0)}_{\mu\nu mn} \\ A^{(0)}_{\mu\nu mn} \end{bmatrix}(kr_{12}, \theta_{12}, \varphi_{12})\,\vec{N}^{(0)}_{\mu\nu}(kr_2, \theta_2, \varphi_2)\right\}. \quad (7)$$

In our case, $\theta_{12} = 0$ therefore (see [16]) only coefficients $A^{(0)}_{\mu\nu mn}$, $B^{(0)}_{\mu\nu mn}$, $\mu = m = 0$ are non-zero,

$$A^{(0)}_{0\nu 0n}(kb, 0, 0) = \frac{\gamma_{0n}}{\gamma_{0\nu}} \sum_{p=|n-\nu|}^{n+\nu} (2p+1) \begin{pmatrix} n & \nu & p \\ 0 & 0 & 0 \end{pmatrix}^2 \times i^{\nu-n+p} \frac{(2\nu+1)}{2\nu(\nu+1)} [n(n+1)+\nu(\nu+1)-p(p+1)] j_p(kb), \quad (8)$$

$$B^{(0)}_{0\nu 0n}(kb, 0, 0) = -\frac{\gamma_{0n}}{\gamma_{0\nu}} \sum_{p=|n-\nu|}^{n+\nu} (2p+1) \begin{pmatrix} n & \nu & p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & \nu & p-1 \\ 0 & 0 & 0 \end{pmatrix} \times i^{\nu-n+p+1} \frac{(2\nu+1)}{2\nu(\nu+1)} [(n+\nu+1+p)(n+\nu+1-p) (p+n-\nu)(p-n+\nu)]^{\frac{1}{2}} j_p(kb). \quad (9)$$

At the new coordinate system, the field is given by formulas

$$\vec{E}^s = \frac{-G}{b^2}\sqrt{\frac{\mu}{\varepsilon}} \sum_{n=1}^{\infty} g_n \sum_{\nu=1}^{\infty} \left[B^{(0)}_{0\nu 0n}\vec{M}^{(0)}_{0\nu}(kr', \theta', \varphi') + A^{(0)}_{0\nu 0n}\vec{N}^{(0)}_{0\nu}(kr', \theta', \varphi')\right], \quad (10)$$

$$\vec{H}^s = \frac{iG}{b^2} \sum_{n=1}^{\infty} g_n \sum_{\nu=1}^{\infty} \left[A^{(0)}_{0\nu 0n}\vec{M}^{(0)}_{0\nu}(kr', \theta', \varphi') + B^{(0)}_{0\nu 0n}\vec{N}^{(0)}_{0\nu}(kr', \theta', \varphi')\right], \quad (11)$$

$$\vec{E}^i = \sqrt{\frac{2}{3}} G k^2 \sqrt{\frac{\mu}{\varepsilon}} \vec{N}^{(+)}_{01}(kr', \theta', \varphi'), \quad (12)$$

$$\vec{H}^i = -i\sqrt{\frac{2}{3}} G k^2 \vec{M}^{(+)}_{01}(kr', \theta', \varphi'). \quad (13)$$

$\vec{M}^{(0)}_{0\nu} = \frac{1}{2}\vec{M}^{(+)}_{0\nu} + \frac{1}{2}\vec{M}^{(-)}_{0\nu}$, $\vec{N}^{(0)}_{0\nu} = \frac{1}{2}\vec{N}^{(+)}_{0\nu} + \frac{1}{2}\vec{N}^{(-)}_{0\nu}$ are the regular functions, and whole field is the sum of outgoing waves from the point source location

$$\vec{E}^{(+)} = \frac{G}{b^2}\sqrt{\frac{\mu}{\varepsilon}}\left\{\sqrt{\frac{2}{3}}(kb)^2\,\vec{N}^{(+)}_{01}\left(kr',\theta',\varphi'\right) - \frac{1}{2}\sum_{\nu=1}^{\infty}\vec{M}^{(+)}_{0\nu}\left(kr',\theta',\varphi'\right)\right.$$
$$\left.\sum_{n=1}^{\infty} g_n B^{(0)}_{0\nu 0n} - \frac{1}{2}\sum_{\nu=1}^{\infty}\vec{N}^{(+)}_{0\nu}\left(kr',\theta',\varphi'\right)\sum_{n=1}^{\infty} g_n A^{(0)}_{0\nu 0n}\right\}, \tag{14}$$

$$\vec{H}^{(+)} = i\frac{G}{b^2}\left\{-\sqrt{\frac{2}{3}}(kb)^2\,\vec{M}^{(+)}_{01}\left(kr',\theta',\varphi'\right) + \frac{1}{2}\sum_{\nu=1}^{\infty}\vec{M}^{(+)}_{0\nu}\left(kr',\theta',\varphi'\right)\right.$$
$$\left.\sum_{n=1}^{\infty} g_n A^{(0)}_{0\nu 0n} + \frac{1}{2}\sum_{\nu=1}^{\infty}\vec{N}^{(+)}_{0\nu}\left(kr',\theta',\varphi'\right)\sum_{n=1}^{\infty} g_n B^{(0)}_{0\nu 0n}\right\} \tag{15}$$

and incoming into this point

$$\begin{aligned}\vec{E}^{(-)} &= \frac{G}{b^2}\sqrt{\frac{\mu}{\varepsilon}}\left\{-\frac{1}{2}\sum_{\nu=1}^{\infty}\vec{M}^{(-)}_{0\nu}\left(kr',\theta',\varphi'\right)\sum_{n=1}^{\infty} g_n B^{(0)}_{0\nu 0n}\right.\\ &\qquad\left.-\frac{1}{2}\sum_{\nu=1}^{\infty}\vec{N}^{(-)}_{0\nu}\left(kr',\theta',\varphi'\right)\sum_{n=1}^{\infty} g_n A^{(0)}_{0\nu 0n}\right\},\\ \vec{H}^{(-)} &= i\frac{G}{b^2}\left\{\frac{1}{2}\sum_{\nu=1}^{\infty}\vec{M}^{(-)}_{0\nu}\left(kr',\theta',\varphi'\right)\sum_{n=1}^{\infty} g_n A^{(0)}_{0\nu 0n}\right.\\ &\qquad\left.+\frac{1}{2}\sum_{\nu=1}^{\infty}\vec{N}^{(-)}_{0\nu}\left(kr',\theta',\varphi'\right)\sum_{n=1}^{\infty} g_n B^{(0)}_{0\nu 0n}\right\}\end{aligned} \tag{16}$$

We can use the following properties of vector functions (5), (6)

$$\vec{M}_{0n}\times\vec{M}_{0\nu}=0, \quad \left[\vec{N}_{0n}\times\vec{N}_{0\nu}\right]_{r'}=0,$$
$$\left[\vec{M}_{0n}\times\vec{N}_{0\nu}\right]_{r'} = f_n\left(r'\right)\tilde{q}_\nu\left(r'\right)\frac{d}{d\theta'}P_n\left(\cos\theta'\right)\frac{d}{d\theta'}P_\nu\left(\cos\theta'\right), \tag{17}$$
$$\left[\vec{N}_{0n}\times\vec{M}_{0\nu}\right]_{r'} = -f_\nu\left(r'\right)\tilde{q}_n\left(r'\right)\frac{d}{d\theta'}P_n\left(\cos\theta'\right)\frac{d}{d\theta'}P_\nu\left(\cos\theta'\right), \tag{18}$$

where $f_n(r') = \gamma_{0n} z_n(kr')$, $\tilde{q}_n = \frac{\gamma_{0n}}{kr'}\frac{d}{dr'}r' z_n(kr')$ and use the orthogonality

$$\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \frac{dP_\nu\left(\cos\theta'\right)}{d\theta'}\frac{dP_{\bar{\nu}}\left(\cos\theta'\right)}{d\theta'} = \delta_\nu \delta_{\bar{\nu}} \gamma_{0\nu}^{-2} \tag{19}$$

to find the average time of forward $P^{(+)}$ and reverse $P^{(-)}$ of energy currents through the sphere with arbitrary fixed radius $r' < a$

$$P^{(\pm)}=\oint \left\langle \vec{S}^{(\pm)} \right\rangle \hat{r} ds = \frac{1}{2}\mathrm{Re} r'^2 \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \left[\vec{E}^{(\pm)} \times \vec{H}^{(\pm)*}\right]_{r'} \tag{20}$$

$$P^{(+)}=\frac{-G^2}{2b^4}\sqrt{\frac{\mu}{\varepsilon}} r'^2 \mathrm{Im}\left\{4\pi\frac{2}{3}\left[-\frac{2}{3}(kb)^4 f_1^{(+)*}\tilde{q}_1^{(+)}+\sqrt{\frac{2}{3}}(kb)^2 f_1^{(+)*}\tilde{q}_1^{(+)}\mathrm{Re}\sum_{n=1}^{\infty} g_n A_{010n}^{(0)}\right]\right.$$
$$\left.+\pi\sum_{\nu=1}^{\infty}\left|\sum_{n=1}^{\infty} g_n B_{0\nu0n}^{(0)}\right|^2 \frac{\nu(\nu+1)}{2\nu+1} f_\nu^{(+)}\tilde{q}_\nu^{(+)*}-\pi\sum_{\nu=1}^{\infty}\left|\sum_{n=1}^{\infty} g_n A_{0\nu0n}^{(0)}\right|^2 \frac{\nu(\nu+1)}{2\nu+1} f_\nu^{(+)*}\tilde{q}_\nu^{(+)}\right\}, \tag{21}$$

$$P^{(-)}=\frac{-\pi G^2}{2b^4}\sqrt{\frac{\mu}{\varepsilon}} r'^2 \mathrm{Im}\left\{\sum_{\nu=1}^{\infty}\left|\sum_{n=1}^{\infty} g_n B_{0\nu0n}^{(0)}\right|^2 \frac{\nu(\nu+1)}{2\nu+1} f_\nu^{(-)}\tilde{q}_\nu^{(-)*}\right.$$
$$\left.-\sum_{\nu=1}^{\infty}\left|\sum_{n=1}^{\infty} g_n A_{0\nu0n}^{(0)}\right|^2 \frac{\nu(\nu+1)}{2\nu+1} f_\nu^{(-)*}\tilde{q}_\nu^{(-)}\right\} \tag{22}$$

where $f_\nu^{(\pm)}(r') = \gamma_{0\nu} h_\nu^{\binom{1}{2}}(kr')$, $\tilde{q}_\nu^{(\pm)} = \frac{\gamma_{0\nu}}{kr'}\frac{d}{dr'} r' h_\nu^{\binom{1}{2}}(kr')$.

We can write down the coefficient in explicit form

$$A_{010n}^{(0)} = \frac{\gamma_{0n}}{\gamma_{01}}\left\{(2n-1)(n+1)\begin{pmatrix} n & 1 & n-1 \\ 0 & 0 & 0 \end{pmatrix}^2 j_{n-1}(kb)\right.$$
$$\left.+(2n+3)\, n \begin{pmatrix} n & 1 & n+1 \\ 0 & 0 & 0 \end{pmatrix}^2 j_{n+1}(kb)\right\}. \tag{23}$$

There $3jm$-Wigner symbols are easily calculated [17]

$$\begin{pmatrix} n & 1 & n+1 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^{n+1}}{\sqrt{2(n+1)+1}} C_{n010}^{n+1,0} = \frac{(-1)^{n+1}(n+1)}{\sqrt{(2n+3)(2n+1)(n+1)}}, \tag{24}$$

$$\begin{pmatrix} n & 1 & n-1 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^{n-1}}{\sqrt{2(n-1)+1}} C_{n010}^{n-1,0} = \frac{(-1)^n n}{\sqrt{(2n+1)(2n-1)n}} \tag{25}$$

therefore

$$A_{010n}^{(0)} = \frac{3}{2}\frac{\gamma_{0n}}{\gamma_{01}}\frac{n(n+1)}{kb} j_n(kb),$$
$$\sum_{n=1}^{\infty} g_n^i A_{010n}^{(0)} = \sqrt{\frac{3}{2}}\sum_{n=1}^{\infty} n(n+1)(2n+1)\, j_n^2(kb), \tag{26}$$

where $C_{n010}^{n\pm1,0}$ are Clebsch-Gordan coefficients. We use the well-known second order Neumann's series expansion [18] to calculate this row.

Then

$$\sum_{n=1}^{\infty} n\,(n+1)\,(2n+1)\,j_n^2\,(kb) = \frac{2}{3}\,(kb)^2\,. \quad (27)$$

Therefore, the first two summands in the curly brackets in (21) are

$$\frac{8\pi}{3} f_1^{(+)*} \tilde{q}_1^{(+)} \left[-\frac{2}{3}\,(kb)^4 + \sqrt{\frac{2}{3}}\,(kb)^2 \sum_{n=1}^{\infty} g_n^{(0)} A_{010n}^{(0)} \right] = 0. \quad (28)$$

Finally, we find from (21), (22) the expression

$$P^{(+)} = -P^{(-)} = \frac{1}{8}\sqrt{\frac{\mu}{\varepsilon}}\,\frac{G^2}{k^2 b^4} \sum_{\nu=1}^{\infty} \left(\left| \sum_{n=1}^{\infty} g_n A_{0\nu 0n}^{(0)} \right|^2 + \left| \sum_{n=1}^{\infty} g_n B_{0\nu 0n}^{(0)} \right|^2 \right), \quad (29)$$

i.e., the current of electromagnetic energy $P^{(+)}$ propagating from the dipole location point is compensated by equal current $P^{(-)}$ absorbing at the same point.

Thus, the stationary field has one singular point in a bounded domain filled with a homogeneous medium. This point simulates both the point source of direct outgoing waves and the point absorber of the inverse incoming waves equal to it by power. We can consider this unique singular point of the field as a point convertor of incoming spherical waves into outgoing spherical waves. Here the radial dipole was considered but the same results take place for the cross-dipole and, consequently, for an arbitrarily oriented electrical point dipole.

3. 3D GREEN FUNCTION FOR MAXWELL'S FISH EYE MEDIUM

It is required to find in R^3 the solution of the Helmholtz equation

$$\left[\nabla^2 + \frac{4n_0^2 k^2 \rho^4}{\left(r^2+\rho^2\right)^2} \right] G_\nu\left(\mathbf{r}, \mathbf{r}'\right) = \delta\left(\mathbf{r} - \mathbf{r}'\right), \quad (30)$$

satisfying of certain additional conditions, that will be stated later.

First, as in [3] we find the solution for the case when the source is located in the symmetry center of the medium ($\mathbf{r}' = 0$). Then Equation (30) with zero right-hand side transforms to the equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{4\nu\,(\nu+1)\,\rho^2}{\left(r^2+\rho^2\right)^2} \right] G_\nu\,(r,0) = 0, \quad (31)$$

where $2\nu + 1 = \sqrt{1 + 4k^2 n_0^2 \rho^2}$. The substitution

$$G_\nu(r, 0) = \sqrt{\frac{\rho}{r}} F(\xi), \quad \xi = \frac{r^2 - \rho^2}{r^2 + \rho^2}, \quad (-1 \le \xi \le 1) \tag{32}$$

leads (31) to the Legendre differential equation

$$\left[\left(1 - \xi^2\right) \frac{d^2}{d\xi^2} - 2\xi \frac{d}{d\xi} + \nu(\nu + 1) - \frac{1/4}{1 - \xi^2}\right] F(\xi) = 0. \tag{33}$$

The general solution of this equation is suitable written as a linear combination of associated the third kind Legendre functions $R_\nu^{\pm 1/2}(\xi)$ [3]. Then the general solution of Equation (31) is following

$$G_\nu(r, 0) = A\sqrt{\frac{\rho}{r}} R_\nu^{-1/2}(\xi) + B\sqrt{\frac{\rho}{r}} R_\nu^{1/2}(\xi), \tag{34}$$

where

$$R_\nu^{\pm 1/2}(\xi) = Q_\nu^{\pm 1/2}(\xi) + i\frac{\pi}{2} P_\nu^{\pm 1/2}(\xi)$$
$$= \sqrt{\frac{\pi}{2}} \begin{Bmatrix} i \\ -(\nu+1/2)^{-1} \end{Bmatrix} \left(1 - \xi^2\right)^{-1/4} \exp\{\pm i(\nu+1/2) \arccos \xi\}, \tag{35}$$

$(0 \le \arccos \xi \le \pi,\ -1 \le \xi \le 1)$, $P_\nu^{\pm 1/2}(\xi)$, $Q_\nu^{\pm 1/2}(\xi)$ are associated Legendre functions of first and second kind at the range $(-1 \le \xi \le 1)$.

We can perform the following inversion transformation [2, 3] for obtaining the Green function with arbitrary position of the point source

$$\mathbf{r} \mapsto \frac{\rho^2 + a^2}{|\mathbf{r} - \mathbf{a}|^2}(\mathbf{r} - \mathbf{a}) + \mathbf{a}, \quad G_\nu(r, 0) \mapsto \frac{C}{|\mathbf{r} - \mathbf{a}|} G_\nu\left(\left|\frac{\rho^2 + a^2}{|\mathbf{r} - \mathbf{a}|^2}(\mathbf{r} - \mathbf{a}) + \mathbf{a}\right|, 0\right), \tag{36}$$

with the inversion center of the sphere at the point $\mathbf{a} = -\mathbf{r}'\rho^2/r'^2$ and radius $\sqrt{\rho^2 + a^2}$. Equation (30) is invariant relatively to such transformation for arbitrary ρ and $\mathbf{a}$. Considering that

$$r \mapsto \frac{\rho^2 |\mathbf{r} - \mathbf{r}'|}{r' \left|\mathbf{r} + \mathbf{r}'\rho^2/r'^2\right|}, \quad \xi = \frac{r^2 - \rho^2}{r^2 + \rho^2} \mapsto \chi = -1 + \frac{2\rho^2 (\mathbf{r} - \mathbf{r}')^2}{(r^2 + \rho^2)(r'^2 + \rho^2)}, \tag{37}$$

$2 \arccos x = \arccos(2x^2 - 1)$, $(0 \le x \le 1)$ [19], we obtain the general solution of the Equation (30) $(C = 1)$

$$G_\nu(r, \mathbf{0}) \mapsto G_\nu\left(\mathbf{r}, \mathbf{r}'\right), \tag{38}$$

$$G_\nu\left(\mathbf{r}, \mathbf{r}'\right) = F_\rho\left(\mathbf{r}, \mathbf{r}'\right)\Big(-(\nu+1/2)^{-1} \tilde{A} \exp\{-i(2\nu + 1) \arccos x\}$$
$$+ i\tilde{B} \exp\{i(2\nu + 1) \arccos x\}\Big) \tag{39}$$

where

$$\begin{aligned} &\tilde{A}=A\sqrt{\pi}/2\rho, \quad \tilde{B}=B\sqrt{\pi}/2\rho, \\ &x=\rho\left|\mathbf{r}-\mathbf{r}'\right|/\sqrt{(r^2+\rho^2)\,(r'^2+\rho^2)}, \quad (0\leq x\leq 1)\,, \\ F_\rho\left(\mathbf{r},\mathbf{r}'\right)=&\frac{\sqrt{(r^2+\rho^2)\,(r'^2+\rho^2)}}{\left|\mathbf{r}-\mathbf{r}'\right|\left|\mathbf{r}+\mathbf{r}'\rho^2/r'^2\right|}. \end{aligned} \tag{40}$$

It follows that the required function with arbitrary $\tilde{A}$, $\tilde{B}$ that has two singular points $\mathbf{r}'$ and $\mathbf{r}'' = -\rho^2\mathbf{r}'/r'^2$. The unknown coefficients $\tilde{A}$, $\tilde{B}$ allow to apply the mentioned above additional conditions on the function $G_\nu(\mathbf{r},\mathbf{r}')$ that determine its behavior near these points. Let us obtain the dominant terms of the expansion (39) at the neighborhoods of this singular points:

$$G_\nu\left(\mathbf{r},\mathbf{r}'\right) = \left[G'_+\left(\mathbf{r},\mathbf{r}'\right) + G'_-\left(\mathbf{r},\mathbf{r}'\right)\right]\left(1+O\left(\left|\mathbf{r}-\mathbf{r}'\right|\right)\right) \tag{41}$$

at the neighborhood of point $\mathbf{r}'$, considering $\arccos x = \pi/2 - x + \ldots$, where

$$\begin{aligned} G'_\pm\left(\mathbf{r},\mathbf{r}'\right) &= A'_\pm\frac{e^{\pm i\kappa'|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \quad A'_+ = \frac{-r'\tilde{A}}{\nu+1/2}e^{-i\pi(\nu+1/2)}, \\ A'_- &= ir'\tilde{B}e^{i\pi(\nu+1/2)}, \quad \kappa' = \frac{(2\nu+1)\,\rho}{(\rho^2+r'^2)}. \end{aligned}$$

and

$$G_\nu\left(\mathbf{r},\mathbf{r}'\right) = \left[G''_+\left(\mathbf{r},\mathbf{r}'\right) + G''_-\left(\mathbf{r},\mathbf{r}'\right)\right]\left(1+O\left(\left|\mathbf{r}-\mathbf{r}''\right|\right)\right) \tag{42}$$

at the neighborhood of point $\mathbf{r}''$, considering

$$\begin{aligned} \arccos x &= \arcsin\sqrt{1-x^2} = \left|\mathbf{r}-\mathbf{r}''\right|/\rho\left(\rho^2/r'^2+1\right)+O\left(\left|\mathbf{r}-\mathbf{r}''\right|^2\right), \\ x &= 1-\frac{1}{2}\left|\mathbf{r}-\mathbf{r}''\right|^2\rho^{-2}\left(\rho^2/r'^2+1\right)^{-2}+O\left(\left|\mathbf{r}-\mathbf{r}''\right|^3\right), \end{aligned}$$

where $G''_\pm(\mathbf{r},\mathbf{r}')=A''_\pm\frac{e^{\pm i\kappa''|\mathbf{r}-\mathbf{r}''|}}{|\mathbf{r}-\mathbf{r}''|}$, $A''_+=i\rho\tilde{B}$, $A''_-=\frac{-\rho\tilde{A}}{\nu+1/2}$, $\kappa''=\frac{(2\nu+1)r'^2}{\rho(\rho^2+r'^2)}$.

As we can see from these relations, the field at the neighborhood of each singular points $\mathbf{r}'(\mathbf{r}'')$ is the sum of the outgoing $G'_+(G''_+)$ and incoming $G'_-(G''_-)$ spherical waves. The energy currents of these waves across the sphere $S'(S'')$ with small radius centered at a point $\mathbf{r}'(\mathbf{r}'')$ are

$$Q'_\pm = \mathrm{Im}\int\limits_{S'} G'^*_\pm\left(\mathbf{r},\mathbf{r}'\right)\frac{\partial}{\partial r}G'_\pm\left(\mathbf{r},\mathbf{r}'\right)d\sigma \tag{43}$$

and similarly for $Q''_\pm$. They equals to

$$Q'_+=-Q''_-=\frac{8\pi\rho r'^2\left|\tilde{A}\right|^2}{(\nu+1/2)\left(\rho^2+r'^2\right)}, \quad Q''_+=-Q'_-=\frac{8\pi(\nu+1/2)\rho r'^2\left|\tilde{B}\right|^2}{(\rho^2+r'^2)}. \tag{44}$$

Thus, outgoing spherical wave G'_+ from the source is transformed by MFE medium into the spherical wave G''_- that incomes to the point $\mathbf{r}''$. Outgoing spherical wave G''_+ from this point is transformed into the spherical wave G'_- that incomes to the point $\mathbf{r}'$. Equalities (44) $Q'_+ + Q''_- = Q''_+ + Q'_- = 0$ mean that there is no energy transport to infinity. Therefore, the presence of drain extracting the energy from the system is necessary for providing stability of the field. Drains can be located only at the singular points $\mathbf{r}'$ and $\mathbf{r}''$ of the field. In general, it is located in each of these points. Formula $|Q'_-| = Q''_+ \leq |Q''_-| = Q'_+$ takes place and, correspondingly, for the coefficients in (39) the relation $|\tilde{B}| \leq |\tilde{A}|/(\nu + 1/2)$ is satisfied. We express δ with equality

$$i\tilde{B} = \delta\tilde{A}/\left(\nu + 1/2\right). \tag{45}$$

Then in general case $0 < |\delta| < 1$, and $\mathbf{r}'$, and $\mathbf{r}''$ are the drain. Only when $\delta = 0$ and $|\delta| = 1$ one drain exists. Let us consider these special cases. The value of the parameter δ of the considered the Green function further will be specified by the notation $G_\nu^{(\delta)}(\mathbf{r}, \mathbf{r}')$.

The following requirements on the function $G_\nu(\mathbf{r}, \mathbf{r}')$ are specified in [3]:

$$G_\nu\left(\mathbf{r}, \mathbf{r}'\right) \to -\frac{1}{4\pi\left|\mathbf{r} - \mathbf{r}'\right|}, \text{ when } \mathbf{r} \to \mathbf{r}'; \tag{46a}$$

$$\text{the regularity in its point } \mathbf{r}'' \tag{46b}$$

Furthermore, in [3] the Green function decrease is required when $\mathbf{r} \to \infty$, but this requirement is automatically implemented in expression (11). From the expressions (41) and (42) we obtain the value of unknowns

$$\frac{\tilde{A}}{\nu + 1/2} = i\tilde{B} = \frac{-1}{8i\pi r' \cos\pi\nu}, \tag{47}$$

using the condition (46), hence, $\delta = 1$. As a result the conditions (46) lead to the Green function

$$G_\nu^{(1)}\left(\mathbf{r}, \mathbf{r}'\right) = \frac{-1}{4\pi r' \cos\pi\nu} F_\rho\left(\mathbf{r}, \mathbf{r}'\right) \sin\left[(2\nu + 1)\arccos x\right], \tag{48}$$

of papers [2, 3]. As we can see from (41), at the neighborhood $\mathbf{r}'$ the field described by this Green function is the sum of outgoing and incoming spherical waves with the same module amplitude values. Thus, $\mathbf{r}'$ are both the point source and the energy point drain. The field at the neighborhood $\mathbf{r}''$ is finite; i.e., there is no perfect focusing.

In another alternate, the Green function can be specified by the condition (46a) and by the requirement of absence of incoming waves at the neighborhood of the point source $\mathbf{r}'$. Then

$$\tilde{B}=0, \quad \frac{\tilde{A}}{\nu+1/2}=\frac{e^{i\pi(\nu+1/2)}}{4\pi r'}, \quad \delta=0, \tag{49}$$

$$G_\nu^{(0)}\left(\mathbf{r},\mathbf{r}'\right)=\frac{-e^{i\pi(\nu+1/2)}}{4\pi r'}F_\rho\left(\mathbf{r},\mathbf{r}'\right)\exp\left[-i\left(2\nu+1\right)\arccos x\right]. \tag{50}$$

At the neighborhood of $\mathbf{r}'$

$$G_\nu^{(0)}\left(\mathbf{r},\mathbf{r}'\right)=-\frac{e^{i\kappa'|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}\left(1+O\left(\left|\mathbf{r}-\mathbf{r}'\right|\right)\right), \tag{51}$$

at the neighborhood of $\mathbf{r}''$

$$G_\nu^{(0)}\left(\mathbf{r},\mathbf{r}'\right)=-\frac{\rho e^{i\pi(\nu+1/2)}e^{-i\kappa''|\mathbf{r}-\mathbf{r}''|}}{4\pi r'\left|\mathbf{r}-\mathbf{r}''\right|}\left(1+O\left(\left|\mathbf{r}-\mathbf{r}''\right|\right)\right), \tag{52}$$

i.e., $\mathbf{r}''$ denotes energy point drain. Thus, the point source and drain are separated in this case.

Interestingly, the limiting process of the case of homogeneous medium ($\rho\to\infty$) is performed only for Green function $G_\nu^{(0)}(\mathbf{r},\mathbf{r}')$. This Green function and its two-dimensional analogue are used in the Leonhardt's concept for justifying the possibility of developing devices with a super-resolution.

Another Green function that has advantages over these two mentioned can be considered. In contrast to the function $G_\nu^{(1)}(\mathbf{r},\mathbf{r}')$ the perfect focusing $\mathbf{r}''$ takes place, herewith, in contrast to the function $G_\nu^{(0)}(\mathbf{r},\mathbf{r}')$ there is no drain at this point. For obtaining such Green function, we require that condition (46a) is satisfied, and the amplitudes of the forward and backward waves at the neighborhood $\mathbf{r}''$ of (42) are identical. Then from (41), (42) and (39) we find formulas

$$\frac{-\tilde{A}}{\nu+1/2}=i\tilde{B}=\frac{1}{8\pi r'\sin\pi\nu}, \quad \delta=-1, \tag{53}$$

$$G_\nu^{(-1)}\left(\mathbf{r},\mathbf{r}'\right)=\frac{1}{4\pi r'\sin\pi\nu}F_\rho\left(\mathbf{r},\mathbf{r}'\right)\cos\left[\left(2\nu+1\right)\arccos x\right]. \tag{54}$$

Here, $\mathbf{r}'$ is both the point source and energy drain as in the case of the function $G_\nu^{(1)}(\mathbf{r},\mathbf{r}')$,

$$G_\nu^{(-1)}(\mathbf{r},\mathbf{r}')=-\frac{e^{-i\pi(\nu+1/2)}}{8\pi\sin\pi\nu}\left\{\frac{e^{i\kappa'|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}+e^{i\pi(2\nu+1)}\frac{e^{-i\kappa'|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right\}\left(1+O\left(\left|\mathbf{r}-\mathbf{r}'\right|\right)\right). \tag{55}$$

The field at the neighborhood of $\mathbf{r}''$ is infinity, i.e., there is perfect focusing

$$G_\nu^{(-1)}(\mathbf{r},\mathbf{r}')=-\frac{\rho}{8\pi r'\sin\pi\nu}\left\{\frac{e^{-i\kappa''|\mathbf{r}-\mathbf{r}''|}}{|\mathbf{r}-\mathbf{r}''|}+\frac{e^{i\kappa''|\mathbf{r}-\mathbf{r}''|}}{|\mathbf{r}-\mathbf{r}''|}\right\}\left(1+O\left(\left|\mathbf{r}-\mathbf{r}''\right|\right)\right). \tag{56}$$

In this case, all energy incoming with straight spherical wave from the source to the point $\mathbf{r}''$ returns back with the inverse spherical wave. This alternate indicates that the perfect focusing in the MFE media is possible when the condition (55) is satisfied, i.e., there is a phase matching between the source of straight and the drain of inverse waves. A similar result is obtained for all $\delta = e^{i\alpha}$, $0 < \alpha < 2\pi$.

There are drains at all three alternates — the Green function does not exist without drain. The reason for this is that though the problem is formulated for R^3, it is the internal boundary-value problem. The spectral problem is analytically solved by the parameter ν and it is shown that the spectrum is purely discrete in paper [2]. The range $r \gg \rho$, where $n\,(r) \sim 0$ plays role of boundary in the MFE medium. The reflection coefficient from it equals to unity. It can be seen via the representations of the field at the neighborhood of points $\mathbf{r}'$ and $\mathbf{r}''$, that the total energy current through covering profile of two spheres equals to zero (see (44)). (A similar result takes place for the one-dimensional Helmholtz equation at the case of an inhomogeneous medium, called Epstein's transition layer [20]). Thus, all generated energy returns back to the region of finite r, and therefore, the stationary state is possible only if the drain presents there. In the general case $(0 < |\delta| < 1)$ both the point source and the point of its image can be the drain. It depends on the problem statement.

The Green function of a homogeneous medium is uniquely determined by (46a) at the point source and by the condition at infinity, excluding waves incoming from infinity. The Green function of MFE medium is uniquely determined by the same condition (46a) at the point source, but instead of the condition at infinity the condition $i\tilde{B} = \delta\tilde{A}/(\nu + 1/2)$ is necessary. That is, in contrast to a homogeneous medium, the whole collection of the Green functions $G_\nu^{(\delta)}(\mathbf{r}, \mathbf{r}')$ with arbitrary parameter $\delta (0 \leq |\delta| \leq 1)$ determining the distribution of energy between two drains are physically meaningful at the MFE medium.

4. 3D GREEN FUNCTION FOR MAXWELL'S LENS

Let us discuss the Dirichlet problem in the Maxwell's fish eye medium (1) at the region bounded by the sphere S_ρ of radius ρ centered by the coordinate origin. The aim is to find the Green function for the boundary value problem with one of the alternates of the additional conditions considered above. In this section we discuss the alternate when $\delta = -1$. In addition to the Green function $G_\nu^{(-1)}(\mathbf{r}, \mathbf{r}')$ of infinite medium (54), it is necessary to find a function $\tilde{G}_\nu^{(-1)}(\mathbf{r}, \mathbf{r}')$ describing the influence of the boundary for solving this problem.

We can apply transformation (36) when $\mathbf{a} = 0$, $C = \rho$ to the function (54). Then

$$G_\nu^{(-1)}\left(\mathbf{r},\mathbf{r}'\right) \mapsto \tilde{G}_\nu^{(-1)}\left(\mathbf{r},\mathbf{r}'\right), \tag{57}$$

$$\tilde{G}_\nu^{(-1)}\left(\mathbf{r},\mathbf{r}'\right) = \frac{1}{4\pi r' \sin\pi\nu}\tilde{F}_\rho\left(\mathbf{r},\mathbf{r}'\right)\cos\left\{(2\nu+1)\arccos\left(\tilde{x}\right)\right\}, \tag{58}$$

where

$$\tilde{F}_\rho\left(\mathbf{r},\mathbf{r}'\right) = \frac{\sqrt{\left(r^2+\rho^2\right)\left(r'^2+\rho^2\right)}}{\left|\mathbf{r}\rho^2/r^2 - \mathbf{r}'\right|\left|\mathbf{r}+\mathbf{r}'r^2/r'^2\right|}, \quad \tilde{x} = \frac{r\left|\mathbf{r}\frac{\rho^2}{r^2} - \mathbf{r}'\right|}{\sqrt{\left(\rho^2+r^2\right)\left(\rho^2+r'^2\right)}}. \tag{59}$$

The function

$$\bar{G}_\nu^{(-1)}\left(\mathbf{r},\mathbf{r}'\right) = G_\nu^{(-1)}\left(\mathbf{r},\mathbf{r}'\right) - \tilde{G}_\nu^{(-1)}\left(\mathbf{r},\mathbf{r}'\right), \tag{60}$$

where $G_\nu^{(-1)}(\mathbf{r},\mathbf{r}')$ determined by Equation (54) satisfies to Equation (30) and boundary condition $\bar{G}_\nu^{(-1)}(\mathbf{r},\mathbf{r}') = 0$ on the sphere S_ρ. Let us consider $\bar{G}_\nu^{(-1)}(\mathbf{r},\mathbf{r}')$ at the neighborhood of singular points locating inside of S_ρ: $\mathbf{r} = \mathbf{r}'$ and $\mathbf{r} = -\mathbf{r}'$.

The behavior of $\bar{G}_\nu^{(-1)}(\mathbf{r},\mathbf{r}')$ at the neighborhood of the point $\mathbf{r} = \mathbf{r}'$ is defined by the first term in (60) and therefore coincides with the expression (55).

The behavior of $\bar{G}_\nu^{(-1)}(\mathbf{r},\mathbf{r}')$ in the neighborhood of the point $\mathbf{r} = -\mathbf{r}'$ is defined by the second term in (60).

From the expressions

$$\tilde{x} = \left[1 - \frac{\rho^2\left(\mathbf{r}+\mathbf{r}'\right)^2}{2r'^2\left(\rho^2+r'^2\right)}\right]\left(1 + O\left(\left|\mathbf{r}+\mathbf{r}'\right|\right)\right),$$

$$\arccos\tilde{x} = \frac{\rho\left|\mathbf{r}+\mathbf{r}'\right|}{r'\sqrt{\rho^2+r'^2}}\left(1 + O\left(\left|\mathbf{r}+\mathbf{r}'\right|\right)\right),$$

we obtain

$$\bar{G}_\nu^{(-1)}\left(\mathbf{r},\mathbf{r}'\right) = \frac{1}{4\pi\sin\pi\nu}\frac{1}{\left|\mathbf{r}+\mathbf{r}'\right|}\cos\left\{\frac{(2\nu+1)\rho}{r'\sqrt{\rho^2+r'^2}}\left|\mathbf{r}+\mathbf{r}'\right|\right\}\left(1+O\left(\left|\mathbf{r}+\mathbf{r}'\right|\right)\right). \tag{61}$$

Finally, at the neighborhood of the point $\mathbf{r} = -\mathbf{r}'$ the dominant term of the total field expansion is the sum of incoming and outgoing spherical waves of same amplitude.

Consequently, not only infinite Maxwell's fish eye medium (1), but the MFE lens (i.e., modified alternate of Leonhard) both provides the perfect focusing. It is achieved only when drain for the incoming spherical waves is placed in the point source location, and phase matching (55) between the drain and the source of outgoing waves is performed.

5. CONCLUSIONS

Following conclusions can be made on the analysis above. Merlin's denying the presence of drain at stationary fields at limited regions is erroneous. As noted in Introduction, this fact follows from the solution obtained by him if we present it as a sum of incoming and outgoing waves. It is shown in Section 2 in more detail. The statement by Leonhardt about possibility of perfect focusing in the Maxwell's fish eye medium is true, but the proof of it is wrong. His Green function does not prove this, because the possibility of the perfect focusing is assumed by the property of MFE medium without any auxiliary features, in this case without drain of the image. In [6] the authors state, 'Maxwell's fish eye thus makes a perfect lens with point-like resolution for electromagnetic waves, but only when such waves are detected by perfect point detectors'. That is, the perfect imaging is possible only when drain (i.e., ideal observer) is present. In other words, the Green function obtained by Leonhardt describes both the source and the field, and the observer.

In standard definition, Green function of the Helmholtz equation characterizes the stationary field with spaced dots of source and drain of energy. If we use this definition for the MFE medium, we receive the case when $\delta = 0$, considered by Leonhardt. The alternates $\delta = 1$ of papers [2, 3] and $\delta = -1$ of Section 3 of this work are out of range of the classical definition of the Green function. In order to include them into consideration, it is required to generalize the concept of Green function to the case of singular points, that combines the properties of both the source and the drain. Ordinary emission condition (there are no incoming waves from infinity) have to be replaced by the condition (45), which determines the portion of energy $|\delta|$ incoming from the point of image to the point source. Both the singular points of the Green function have the following physical interpretation. In general case $0 < |\delta| \leq 1$, this singular points are the point convertors of the incoming spherical waves into the outgoing spherical waves with a certain phase shift. By such definition, we should take into consideration not a single Green function $G_\nu^{(0)}(\vec{r}, \vec{r}')$ of Helmholtz equation for a Maxwell's fish eye medium but a whole set of Green functions $G_\nu^{(\delta)}(\vec{r}, \vec{r}')$, instead. The Green function describing perfect focusing exists only within this extended definition, i.e., $G_\nu^{(-1)}(\vec{r}, \vec{r}')$ for the MFE medium and $\bar{G}_\nu^{(-1)}(\vec{r}, \vec{r}')$ for MFE lens, respectively.

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