

## LIMITS OF NEGATIVE GROUP DELAY PHENOMENON IN LINEAR CAUSAL MEDIA

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**Abstract**—Asymptotic limits of Negative Group Delay (NGD) in linear causal media satisfying Kramers-Kronig relations are investigated. Even though there is no limit on the NGD-bandwidth product of a linear medium, it is shown that the out-of-band to center frequency amplitude ratio, or out-of-band gain, increases with the NGD-bandwidth product, and is proportional to the amplitude of undesired transients when waveforms with defined “turn on/off” times propagate in the media. The optimal dispersion characteristic exhibiting NGD, which maximizes the NGD-bandwidth product as a function of the out-of-band gain, is obtained through Kramers-Kronig relations. It is shown that the NGD-bandwidth product has an upper asymptotic limit proportional to the square root of the logarithm of the maximum out-of-band gain. The derived NGD-bandwidth upper asymptotic limit of the optimally engineered causal dispersion characteristic is validated with two examples of physical media, a Lorentzian dielectric medium, and an artificially fabricated loaded transmission line medium.

### 1. INTRODUCTION

Negative group velocity, and consequently NGD, is an example of abnormal wave propagation phenomena, which also include superluminal [1], backward wave propagation (negative refractive index [2]), and simultaneous negative phase and group velocity [3]. Media exhibiting NGD behavior cause the output peak of a well-behaved wave packet or a pulse, to precede the input peak. This phenomenon is achieved through pulse reshaping and does not violate

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causality, since the initial transient of the pulse is still positively delayed and propagates at a speed not exceeding the speed of light in vacuum [4]. NGD phenomena occur within limited frequency bands of media exhibiting anomalous dispersion, and are also accompanied with absorption. Anomalous dispersion phenomena not only do not violate the relativistic causality requirements, but they must exist within some frequency bands for all dispersive media [5] as a consequence of Kramers-Kronig relations, which are applicable to all physically realizable, causal linear systems. Propagation of electromagnetic waves through a medium with anomalous dispersion was initially studied by Sommerfeld and Brillouin [6]. They considered a semi-infinite sinusoid waveform, with a defined “turn on” point in time, propagating through a medium with Lorentzian dispersion. Clear identification and definition of phase, group, energy and “front” velocities were made, showing that the “front” velocity is always positive and exactly luminal under all circumstances, thus satisfying relativistic causality. Therefore, the group velocity does not equate with the “front” velocity in such media, but merely characterizes the propagation of distinct features of a well-behaved wave packet, such as pulse maximum. The difference between the “front” velocity and group velocity was recently demonstrated in a medium with a slow group velocity [7].

In this paper, we extend a part of the work presented in Ref. [5]. The work there presents a proof showing that within a frequency band of abnormal propagation (such as NGD propagation), the magnitude of a causal medium response has a minimum. The work presented here attempts to quantify the relationship between the achieved NGD-bandwidth product and the maximum out-of-band gain. The maximum out-of-band gain is a ratio between the amplitude response maximum (which occurs at a frequency outside of the NGD bandwidth) and the amplitude response minimum (which occurs at the center frequency).

When a smooth temporal waveform is truncated (defined “turn on/off” times are introduced), the waveform’s frequency-domain spectrum widens, and a part of it extends into the out-of-band region. This part of the waveform’s spectrum corresponds to the fastest changing parts of the waveform (in the vicinity of “turn on/off” times). When such waveforms with a finite temporal width are propagated through a medium exhibiting an out-of-band gain, an undesired distortion in the output response is observed in the vicinity of “turn on/off” times [8]. Such a distorted transient response will follow any points of discontinuity in the waveform or any of its derivatives, not just the “turn on/off” times. The out-of-band gain is proportional to the medium’s transient amplitude response, and therefore it is an

undesired trade-off quantity accompanying any NGD phenomenon.

We investigate asymptotic NGD limits as a function of the maximum out-of-band gain for gain-compensated causal linear media. For passive causal linear media, the same asymptotic limits will apply, however, as a function of the in-band attenuation in this case. The asymptotic limits are derived via Kramers-Kronig relations, which relate the real and imaginary parts of a linear causal system response, or equivalently its phase and magnitude response. Since the NGD phenomenon is a consequence of a system's phase response, in this paper we start with several assumed phase response characteristics exhibiting NGD phenomenon. Then we apply Kramers-Kronig relations to determine corresponding causal amplitude responses for each phase characteristic. The optimally engineered causal characteristic is identified, which exhibits the largest NGD to out-of-band gain ratio.

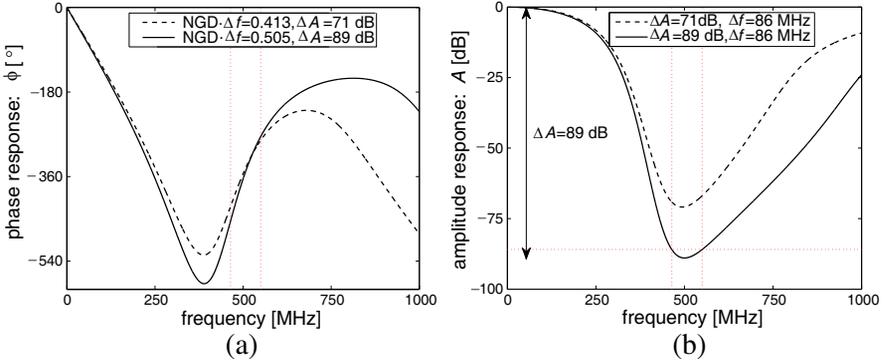
## 2. NGD AND TRANSIENTS IN LORENTZIAN DIELECTRIC MEDIUM

The refractive index of a single-resonance passive Lorentzian dielectric medium is given by [6]

$$n(\omega) = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2 + 2j\delta\omega - \omega^2}}, \quad (1)$$

where  $\omega_0$  and  $\omega_p$  are the medium's resonance and plasma frequencies, respectively, and  $\delta$  is the damping factor which is related to medium's absorption bandwidth around the resonance. Further, the form of expression (1) is chosen such that it corresponds to an amplitude response  $A(\omega) = \exp(\text{Im}\{\omega \cdot n(\omega) \cdot l/c\})$ , and a phase response given by  $\phi(\omega) = \text{Re}\{-\omega \cdot n(\omega) \cdot l/c\}$ , for a medium with a physical length  $l$ . Group delay within a narrow-band region around a frequency  $\omega$  is given by the negative slope of the phase characteristic as  $\tau_g(\omega) = -d\phi(\omega)/d\omega$ .

In order to demonstrate the relationship between NGD and the accompanying absorption, amplitude and phase characteristics of two Lorentzian media are shown in Fig. 1. The considered examples have their parameters chosen such that they both exhibit a  $360^\circ$  phase shift at the center frequency of 500 MHz, and a 3-dB amplitude swing within a bandwidth of 86 MHz. However, their absorptions at the center frequency are different (71 dB and 89 dB respectively), which in turn yields different phase characteristic slopes, and therefore group delays ( $-4.8$  ns and  $-5.87$  ns, respectively). Due to the inverse relationship of a time-domain pulse width and the frequency bandwidth within which



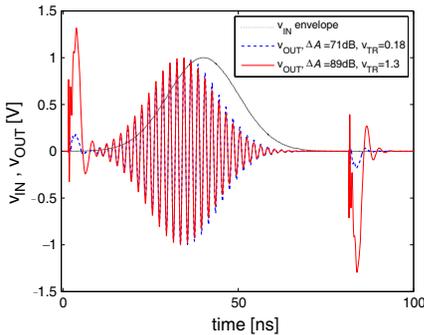
**Figure 1.** (a) Phase and (b) amplitude response characteristics of two Lorentzian media, both having a center frequency of 500 MHz, 3-dB amplitude bandwidth of 86 MHz, physical lengths corresponding to a  $360^\circ$  phase at  $\omega_0$ , their respective center frequency loss of 71 dB and 89 dB, and the corresponding center frequency group delays of  $-4.8$  ns and  $-5.87$  ns.

most of the pulse power is located, the NGD-bandwidth product is a metric quantity which provides a measure of the NGD relative to the width of a propagated pulse [9]. The examples shown in Fig. 1 exhibit maximum NGD-bandwidth products of 2.59 and 3.17, respectively ( $\text{NGD}_{MAX}\Delta\omega = t_g\Delta\omega = -\tau_g(\omega_0)\Delta\omega$ ). The out-of-band gain value,  $\Delta A$ , also shown in Fig. 1(b) for the two cases, is the ratio between the largest (occurs at extremes frequencies outside the bandwidth) and smallest (occurs at the center frequency) amplitude characteristic values.

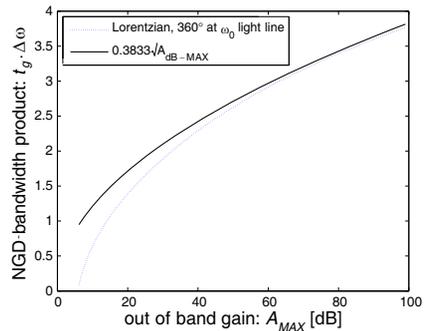
A time-domain interpretation of the relationship between the NGD-bandwidth product and the absorption (center frequency amplitude response) for Lorentzian dielectric media is demonstrated in Fig. 2. The waveform used in this example is a 500 MHz sinusoidal carrier, amplitude modulated by a Gaussian pulse with a standard deviation of 10 ns, which is turned-on/off at carrier zero-crossings 40 ns away from the pulse peak. Most of the waveform frequency spectrum falls within the 86 MHz bandwidth of the two example media from Fig. 1, ensuring a low distortion of the steady-state part of the waveform. The delay due to physical lengths of the two media is 2 ns, corresponding to a  $360^\circ$  phase shift at 500 MHz. In order to make the output waveforms magnitudes comparable to the input ones, the amplitude characteristics of the two media are scaled so the input and output peak amplitudes are equal. Consequently, the

amplitude characteristics in Fig. 1(b) are scaled by 70.5 dB and 88.5 dB for the two cases, respectively. Note that the scaling factor required to normalize the peak amplitude of the output pulse is slightly different from the maximum out-of-band gain. Now amplitude response values at extreme frequencies away from the resonance are increased by the same amount. In the examples shown in Fig. 2, as the out-of-band gain is increased from 71 dB to 89 dB (a factor of 7.94 in magnitude), the transient amplitude past the turn-off point is increased by a factor of 7.22. Therefore, an approximately proportional relationship between the out-of-band gain,  $\Delta A$ , and the transient amplitude is demonstrated for this particular waveform. Different waveforms with defined turn-on/off times will cause different transient amplitudes, but an increase in the out-of-band gain will always result in an increase in the transient amplitude. Even though the transient amplitude has increased substantially, the NGD-bandwidth product has increased only by a factor of 1.22, between the two examples in Fig. 2.

The examples shown in Figs. 1(a) and 1(b) demonstrate the relationship between the NGD-bandwidth product and the out-of-band gain in the frequency domain, which translates into a time-domain relationship between the NGD-to-pulse-width ratio and the transient amplitude depicted in Fig. 2, for waveforms with defined turn-on/off times (or more broadly, for waveforms with discontinuities in the waveform function and/or its derivatives).



**Figure 2.** Time-domain responses of two gain-compensated Lorentzian media (both outputs scaled by 70.5 dB and 88.5 dB accordingly), to a Gaussian modulated sinusoidal waveform.



**Figure 3.** Center frequency NGD-bandwidth product as a function of out-of-band gain  $A_{MAX}$ , for a Lorentzian medium with  $\delta = 0.05\omega_0$ , and medium length corresponding to a 360° light-line phase at  $\omega_0$ .

By repeating the exercise captured in Figs. 1 and 2 for Lorentzian media with different parameters, NGD-bandwidth product as a function of the out-of-band gain (or equivalently, center frequency absorption) can be obtained, as shown in Fig. 3. An asymptotic upper limit for this functional relationship was numerically obtained, and given by

$$\text{NGD}_{MAX} \cdot \Delta\omega = t_g \cdot \Delta\omega \leq 0.3833\sqrt{\Delta A_{\text{dB}}}, \quad (2)$$

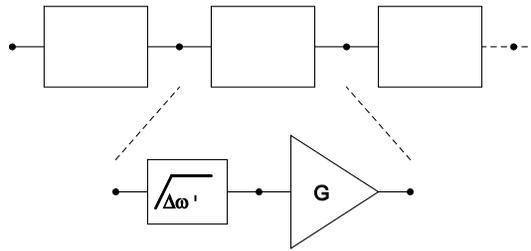
where  $\Delta A_{\text{dB}}$  is the maximum out-of-band gain (ratio of the maximum and minimum values of the amplitude characteristic, given in decibels). For the example shown in Fig. 2, the square root ratio of the respective maximum out-of-band gains given in decibels is  $(89 \text{ dB}/71 \text{ dB})^{1/2} \approx 1.12$ , which is close to the observed 1.22 factor of increase in NGD.

Now that the asymptotic limit for the NGD-bandwidth product as a function of the out-of-band gain has been demonstrated for a Lorentzian medium, the same functional relationship will be explored for artificially fabricated media, in particular for filter-based distributed circuits' media, and then for a linear causal medium with an optimally engineered dispersion characteristic.

### 3. NGD VS. OUT-OF-BAND GAIN RELATIONSHIP IN DISTRIBUTED MEDIA

Artificial media fabricated using lumped and distributed circuit elements can be used to produce the NGD effect. One of the earliest circuits exhibiting NGD at microwave frequencies was a synthesizer proposed by Lucyszyn et al. [10]. Since this initial work, several passive NGD circuits, mostly based on series or parallel RLC resonators, have been reported. As with any media exhibiting anomalous dispersion, these circuits exhibit large attenuation for any reasonable negative delay [11, 12]. The attenuation can be compensated by cascading active elements with RLC resonators [9, 13–15]. At baseband frequencies (zero center frequency), a gain-compensated NGD effect can be achieved by cascading active elements with RC filters [16]. However, gain-compensation at the NGD center frequency also amplifies the out-of-band part of the amplitude response above the in-band level, since the amplitude transfer function of a causal NGD medium has a minimum at the center frequency [5].

An example of a one-dimensional medium exhibiting NGD is a cascaded gain-compensated high-pass distributed filter, as schematically shown in Fig. 4. The transfer function of an  $N$ -stage circuit of Fig. 4, with gain factor chosen to yield an overall unity gain



**Figure 4.** Schematic of a generic baseband NGD distributed media, comprised of ideally matched cascaded gain-compensated high-pass filters.

at  $\omega = 0$ , and ignoring the physical length, can be derived as

$$H_N(\omega) = A_{MAX} \left[ \frac{\omega - j\Delta\omega'/2}{\omega - jA_{MAX}^{1/N}\Delta\omega'/2} \right]^N, \quad (3a)$$

$$\Delta\omega' = \Delta\omega \sqrt{\frac{1 - 2^{1/N}/A_{MAX}^{2/N}}{2^{1/N} - 1}}, \quad (3b)$$

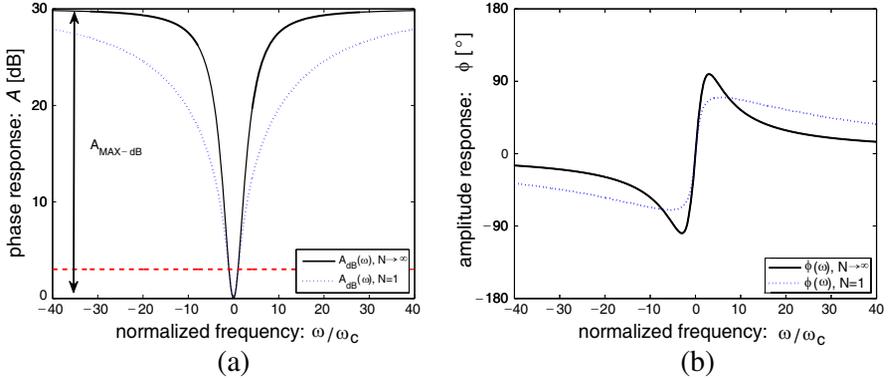
where  $\Delta\omega$  is the 3 dB bandwidth and  $A_{MAX}$  the maximum out-of-band gain of the cascaded circuit.  $A_{MAX}$  is an arbitrary parameter with a finite value. As  $\omega \rightarrow \infty$  in (3a), the transfer function out-of-band gain will approach this maximum but finite value, regardless of the number of stages,  $N$ . The number of stages determines the distribution of the finite  $A_{MAX}$  among individual stages, such that their individual out-of-band gain values are  $A_{MAX}^{1/N}$ . Individual stage bandwidth,  $\Delta\omega'$ , is scaled according to (3b), so that the overall  $N$ -stage bandwidth,  $\Delta\omega$ , remains constant [9], regardless of the number of stages,  $N$ .

The amplitude and phase response for a single-stage case is shown in Fig. 5. As the number of stages tends to infinity (distributed medium), while keeping the overall bandwidth,  $\Delta\omega$ , and the maximum out-of-band gain,  $A_{MAX}$ , constant, the overall transfer function can be derived as

$$H_\infty(\omega) = A_{MAX} \exp \left[ \ln(A_{MAX}) \frac{ja\Delta\omega/2}{\omega - ja\Delta\omega/2} \right], \quad (4a)$$

$$a = \sqrt{\frac{2 \ln(A_{MAX})}{\ln 2} - 1}. \quad (4b)$$

The amplitude and phase responses for this distributed medium,  $N \rightarrow \infty$ , are also shown in Figs. 5(a) and 5(b), respectively, and they



**Figure 5.** Single-stage and distributed gain compensated high-pass filter, (a) amplitude response and (b) phase response for  $A_{MAX-dB} = 30$  dB (gain chosen so the amplitude is unity at  $\omega = 0$ ).

can be determined from (4a) as

$$A_{\infty}(\omega) = A_{MAX} \exp \left[ -\ln(A_{MAX}) \frac{(a\Delta\omega/2)^2}{\omega^2 + (a\Delta\omega/2)^2} \right], \quad (5a)$$

$$\phi_{\infty}(\omega) = \ln(A_{MAX}) \frac{\omega (a\Delta\omega/2)}{\omega^2 + (a\Delta\omega/2)^2}. \quad (5b)$$

The group delay of the medium transfer function is given by  $\tau_g = -d\phi/d\omega$ . The largest NGD occurs at the center frequency, where the phase response has the largest positive slope. For the distributed case it is given by

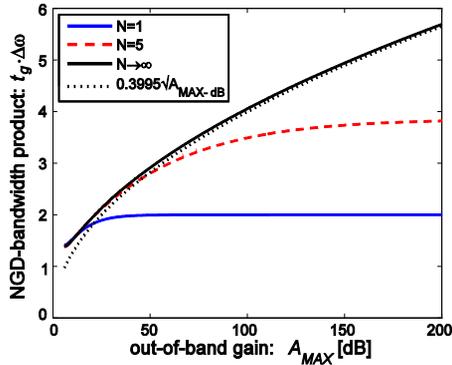
$$t_g = -\tau_g(0) = \left. \frac{d\phi}{d\omega} \right|_{\omega=0} = \frac{2}{\Delta\omega} \frac{\ln(A_{MAX})}{a}. \quad (6)$$

The NGD-bandwidth product is a measure of the NGD to applied pulse temporal width ratio (i.e., how much pulse can be negatively delayed with respect to its width). For the distributed case it is given by

$$t_g \cdot \Delta\omega = \frac{\sqrt{2 \ln(2)} \ln(A_{MAX})}{\sqrt{\ln(A_{MAX}) - \ln(\sqrt{2})}} \approx \sqrt{2 \ln(2)} \sqrt{\ln(A_{MAX})}, \quad (7)$$

$$t_g \cdot \Delta\omega \approx \sqrt{\frac{\ln(2) \ln(10)}{10}} \sqrt{A_{MAX-dB}} \approx 0.3995 \sqrt{A_{MAX-dB}}. \quad (8)$$

Expression (8) shows that the asymptotic limit of the NGD-bandwidth product is a square root function of the maximum out-of-band gain, for



**Figure 6.** Center frequency NGD-bandwidth product as a function of out-of-band gain  $A_{MAX}$ , for different number of stages,  $N$ .

this particular medium. For a given out-of-band gain, the distributed case of this medium ( $N \rightarrow \infty$ ) exhibits the largest NGD, as evident from Fig. 6. Comparing expression (8) to the Lorentzian medium asymptotic limit expression (2), the same functional relationship is observed with a slightly larger proportionality factor in the filter-based medium case.

The asymptotic limit given by (8) applies to the particular distributed medium presented in this section. The same limit also applies to resonator-based NGD circuits [9], which essentially exhibit stopband filter behavior centered at a non-zero frequency. The objective in this paper is to derive the asymptotic limit of the NGD-bandwidth product as a function of the maximum out-of-band gain for an optimally engineered linear causal medium. This is performed by considering several examples of the medium’s phase response. Asymptotic NGD limits for a piece-wise 1st order phase response of a linear and causal medium will be derived in the following sections, and compared to expressions (2) and (8).

#### 4. AMPLITUDE AND PHASE RESPONSE RELATIONSHIP IN CAUSAL MEDIA

Physically realizable, causal systems can have an output response to an applied input signal only at times following the instance that input signal is applied. The frequency-domain transfer function of a system, and its output response to an impulse delta function located at  $t = 0$ , can be respectively written as

$$H(\omega) = P(\omega) + jQ(\omega) = A(\omega) \exp [j\phi(\omega)], \tag{9a}$$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \exp(j\omega t) d\omega. \quad (9b)$$

The causality condition and consequently the Kramers-Kronig relations are given by [17]

$$h(t < 0) = 0, \quad (10a)$$

$$P(\omega) = \frac{1}{\pi} V.P. \int_{-\infty}^{\infty} \frac{Q(\nu)}{\nu - \omega} d\nu, \quad (10b)$$

$$Q(\omega) = -\frac{1}{\pi} V.P. \int_{-\infty}^{\infty} \frac{P(\nu)}{\nu - \omega} d\nu, \quad (10c)$$

where *V.P.* denotes Cauchy principal value. The real and imaginary parts of the transfer function are not independent, and one can be determined from the other, using (10b), (10c). Kramers-Kronig relations can be rearranged to express the dependency between the transfer function amplitude and phase response, as [18]

$$A_{dB}(\omega) = 20 \log |A(\omega)| = -\frac{20}{\pi \ln 10} V.P. \int_{-\infty}^{\infty} \frac{\phi(\nu)}{\nu - \omega} d\nu. \quad (11)$$

For a given phase response of a causal system, and therefore a given NGD characteristic, its amplitude response can be determined by (11), up to a constant (an arbitrary frequency-invariant amplification or attenuation can be added).

In order to simplify the analysis, a transfer function phase characteristic exhibiting NGD within the signal baseband (around  $\omega = 0$ ) is chosen for the cases explored in the following sections. However, the obtained results and conclusions can be extended to NGD phenomena occurring within an arbitrary frequency band. As an example, the filter-based case presented in Section 3 (centered at zero frequency) yields the same asymptotic limit as the one reported in [9], which applies to filter-based cases centered at a non-zero frequency. Furthermore, a zero-length medium is considered, in order to further simplify the analysis. The finite medium length can easily be taken into account by adding a constant positive delay at the end of the analysis.

## 5. OPTIMALLY ENGINEERED PHASE RESPONSE

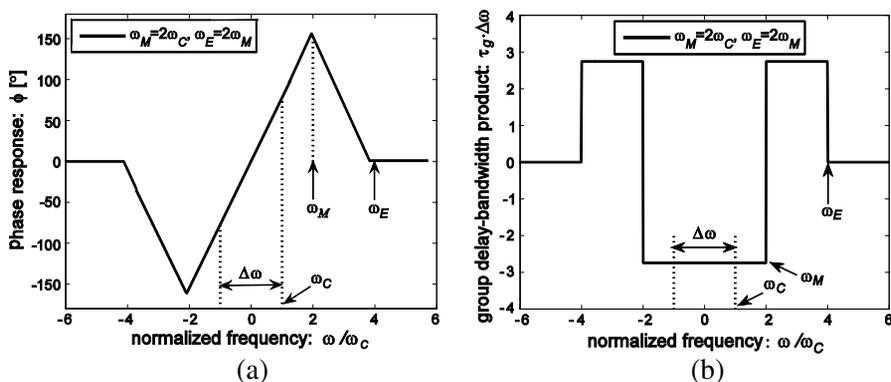
### 5.1. Causal Medium with a Piece-wise Linear Phase Response

As a first example, a piece-wise linear phase response of a medium is chosen, and given by

$$\phi(\omega) = \begin{cases} t_g\omega, & 0 \leq \omega < \omega_M \\ (\omega - \omega_E) \frac{t_g\omega_M}{\omega_M - \omega_E}, & \omega_M \leq \omega \leq \omega_E \\ 0, & \omega \geq \omega_E \end{cases}, \quad (12)$$

and  $\phi(-\omega) = -\phi(\omega)$ . Here  $t_g$  is the slope of the phase response (NGD) at the center frequency,  $\omega_M$  is the frequency where the slope changes sign, and  $\omega_E$  is where the phase characteristic reaches zero and stays zero beyond this frequency. The phase response is an odd function around  $\omega = 0$ , and it is shown in Fig. 7 along with the corresponding group delay plot.

The phase characteristic has a zero-value for high frequencies, which is equivalent to a phase characteristic approaching a constant-slope line (“light-line”) if a non-zero length medium was added to the model, and is the case for all physical media. Therefore, the phase function in Fig. 7 is a realistic piece-wise 1st order approximation for a feasible system exhibiting NGD. A constant NGD is achieved over the entire frequency band  $(-\omega_M, \omega_M)$ . For a more general discussion we will let the NGD bandwidth be defined within a smaller band



**Figure 7.** Piece-wise linear (a) phase response and corresponding (b) group-delay-bandwidth product for  $\omega_M = 2$ ,  $\omega_E = 2\omega_M$  and  $\omega_C = 0.5\omega_M$ .

$\Delta\omega = 2\pi\Delta f = (-\omega_C, \omega_C)$  and let the bandwidth be solely determined by a 3 dB swing in the system amplitude response in this case.

The amplitude response obtained through integration using (11) is shown in Fig. 8, and given by

$$A_{\text{dB}}(\omega) = (t_g\Delta\omega) \frac{10}{\pi \ln 10} \frac{\omega_M}{\omega_C} b(\omega), \quad (13a)$$

where

$$b(\omega) = \frac{2\omega_E}{\omega_E - \omega_M} \ln \left( \frac{\omega_E}{\omega_M} \right) + \frac{\omega_E + |\omega|}{\omega_E - \omega_M} \ln \left( \frac{\omega_M + |\omega|}{\omega_E + |\omega|} \right) \\ + \frac{|\omega|}{\omega_M} \ln \left( \frac{\omega_M + |\omega|}{|\omega_M - |\omega||} \right) + \frac{\omega_E - |\omega|}{\omega_E - \omega_M} \ln \left( \left| \frac{\omega_M - |\omega|}{\omega_E - |\omega|} \right| \right). \quad (13b)$$

The integration constant is chosen so that  $A_{\text{dB}}(0) = 0$ . By setting the 3 dB bandwidth of the obtained amplitude characteristic to be at  $\omega = \pm\omega_C$ , the negative group delay at the center frequency  $t_g$  is uniquely defined as

$$t_g \cdot \Delta\omega = -\tau_g(0) \cdot \Delta\omega = \frac{\omega_C}{\omega_M} \frac{\pi \ln 2}{b(\omega_C)}. \quad (14)$$

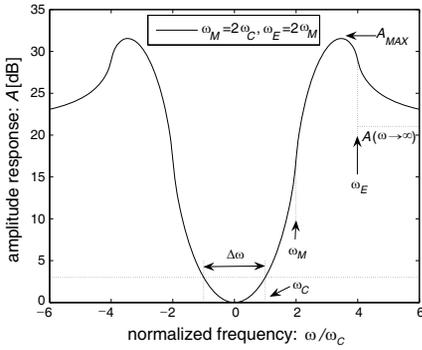
The out-of-band gain as  $\omega \rightarrow \infty$  can be obtained from (13a) as

$$A_{\text{dB}}(\omega \rightarrow \infty) = (t_g \cdot \Delta\omega) \frac{\omega_M}{\omega_C} \frac{10}{\pi \ln 10} \frac{2\omega_E}{\omega_E - \omega_M} \ln \left( \frac{\omega_E}{\omega_M} \right). \quad (15)$$

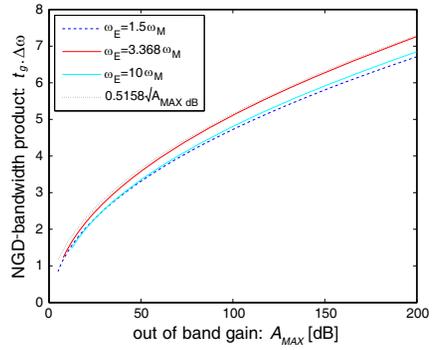
Selecting smaller values for  $\omega_M$  in (14) yields higher values for the NGD at the center frequency. However, this comes at the expense of larger out-of-band gain  $A_{\text{dB}}(\infty)$ . Large out-of-band gain is undesired, since it amplifies transients when signals with defined turn-on and/or turn-off times are applied [8].

A family of NGD-bandwidth product versus  $A_{\text{dB}}(\infty)$  functions can be obtained from expressions (14) and (15), where discrete values of  $\omega_E$  correspond to different functions. In this case,  $\omega_M$  is an independent parameter which is continuously swept, and substituted into (14) and (15). By plotting such obtained family of functions, it can be shown that for any given out-of-band gain, the maximum NGD is achieved when  $\omega_E \rightarrow \omega_M$ . Moreover, we can then find the asymptotic relationship between the maximum NGD and out-of-band gain, for  $\omega_E \rightarrow \omega_M$  and large values of  $\omega_M/\omega_C$  (large out-of-band gain). First, expression (14) can be simplified in this case as

$$t_g \cdot \Delta\omega \approx \pi (\ln 2) \frac{\omega_M}{\omega_C} \frac{\omega_E}{\omega_E + \omega_M}. \quad (16)$$



**Figure 8.** Amplitude response satisfying Kramers-Kronig relations for the given phase response, Fig. 7, with  $\omega_M = 2$ ,  $\omega_E = 2\omega_M$  and  $\omega_C = 0.5\omega_M$ .



**Figure 9.** Center frequency NGD-bandwidth product as a function of out-of-band gain  $A_{MAX}$ , for different values of  $\omega_E$ .

Substituting  $\omega_M/\omega_C$  from (16) into (15), and by letting  $\omega_E \rightarrow \omega_M$ , we obtain the asymptotic upper limit for NGD as a function of  $A_{dB}(\infty)$  as

$$t_g \cdot \Delta\omega \leq \pi \sqrt{\frac{(\ln 2)(\ln 10)}{40}} \sqrt{A_{dB}(\infty)} \approx 0.6275 \sqrt{A_{dB}(\infty)}. \quad (17)$$

However, for the piece-wise linear phase response, the maximum value of the out-of-band gain,  $A_{MAX}$ , is always higher than the  $A_{dB}(\infty)$  value, as shown in Fig. 8. Finding the zeros of the first derivative of (13a) involves solving a transcendental equation, which also depends on the  $\omega_E/\omega_M$  ratio. By numerically solving this equation, and manipulating (13a), it can be shown that the maximum NGD/ $A_{MAX}$  ratio is obtained when  $\omega_E \approx 3.368\omega_M$ . Now we can combine (13a) and (14) to produce NGD-bandwidth product versus  $A_{MAX}$  curves, which are shown in Fig. 9. The upper asymptotic limit in this case is evaluated as

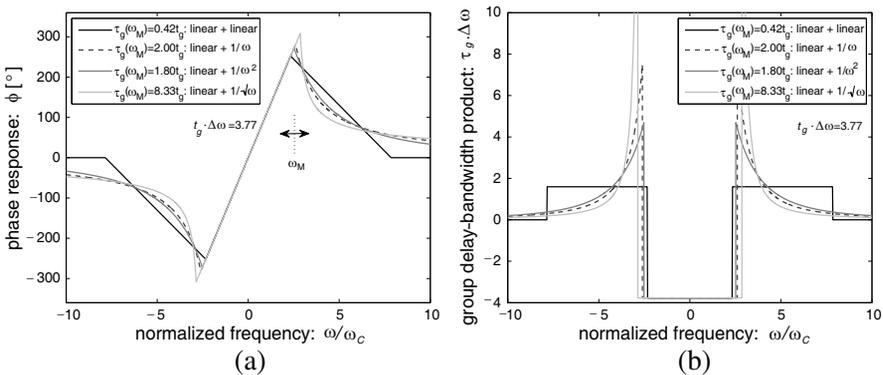
$$t_g \cdot \Delta\omega \leq 0.5158 \sqrt{A_{MAX-dB}}. \quad (18)$$

Comparing expression (18) to expression (8), we can see that the derived upper asymptotic limit for this linear and causal medium, given by its 1st order phase approximation, has the same square root form, and its coefficient is larger (by 29%). The upper asymptotic limit given by (18) does not contradict the limit (8), derived for the particular medium in Section 3, nor the limit (2), which was numerically obtained for a Lorentzian medium in Section 2.

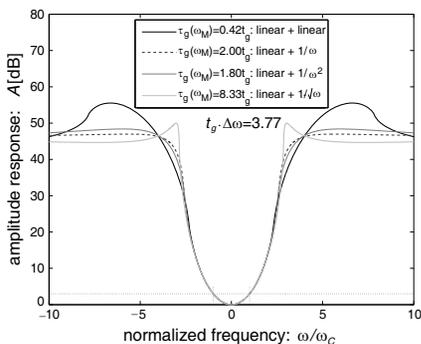
### 5.2. Causal Medium with a Reciprocal Decay of the Phase Response

As evident from (5b), the phase response of the medium presented in Section 3 has a reciprocal function decay,  $(1/\omega)$ , for high frequencies. The 1st order approximation of a general linear causal medium phase response from Section 5.1, however, has a linear decay past the phase slope reversal point,  $\omega_M$ . In this section, we will examine media with the same linear phase response around the center frequency, but with different order reciprocal phase decays,  $(1/\omega^k)$ , past the  $\omega_M$  frequency. The decay functions parameters are chosen such that the phase characteristics are continuous at the reversal point,  $\omega_M$ , and the decay slopes at the same point,  $\tau_g(\omega_M)$ , are related to the center frequency slope,  $t_g$ , such that the  $NGD/A_{MAX}$  ratio is maximized in each case. Fig. 10 shows phase and corresponding group delay responses for the reciprocal phase decay cases of orders  $k = 1/2$ ,  $k = 1$ ,  $k = 2$  and  $k \rightarrow \infty$  (equivalent to a linear decay). For comparison purposes, each case has the same  $NGD$ -bandwidth product value at the center frequency,  $t_g \cdot \Delta\omega = 3.77$ .

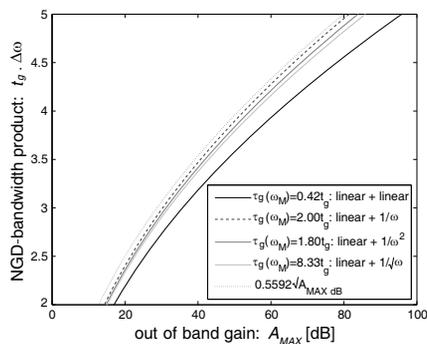
The amplitude responses corresponding to Fig. 10 cases are derived following a similar procedure as in Section 5.1, and they are shown in Fig. 11. Since all cases have the same  $NGD$ -bandwidth product,  $t_g \cdot \Delta\omega = 3.77$ , the case with the smallest  $A_{MAX}$  value will have the largest  $NGD/A_{MAX}$  ratio (optimum case).



**Figure 10.** Piece-wise linear and reciprocal  $(1/\omega^k)$  decay (a) phase responses and corresponding (b) group-delay-bandwidth products for  $k = 1/2$ ,  $k = 1$ ,  $k = 2$ , and  $k \rightarrow \infty$  (linear decay). Each case shown corresponds to the maximized  $NGD/A_{MAX}$  ratio, and it has  $t_g \cdot \Delta\omega = 3.77$ .



**Figure 11.** Magnitude responses for piece-wise linear and reciprocal ( $1/\omega^k$ ) phase decays with  $k = 1/2$ ,  $k = 1$ ,  $k = 2$ , and  $k \rightarrow \infty$  (linear decay). Each case shown corresponds to the maximized NGD/ $A_{MAX}$  ratio, and it has  $t_g \cdot \Delta\omega = 3.77$ .



**Figure 12.** Center frequency NGD-bandwidth product as a function of out-of-band gain  $A_{MAX}$ , for piece-wise linear and reciprocal ( $1/\omega^k$ ) phase decays with  $k = 1/2$ ,  $k = 1$ ,  $k = 2$ , and  $k \rightarrow \infty$  (linear decay).

As evident from Fig. 11, the reciprocal decay case ( $1/\omega$ ) has the smallest out-of-band gain, and it is therefore optimal. This is confirmed in Fig. 12, where NGD-bandwidth versus out-of-band gain curves are produced in a similar manner as in Section 5.1. The asymptotic limit for the optimum case is determined as

$$t_g \cdot \Delta\omega \leq 0.5592 \sqrt{A_{MAX\text{-dB}}}. \tag{19}$$

The asymptotic limit given by (19) does not contradict limits given by (18), (8) and (2). Changing the order of phase decay to a higher ( $1/\omega^2$ ), or a lower ( $1/\omega^{1/2}$ ) value from the optimum case ( $1/\omega$ ) produces asymptotic limits smaller than (19). All cases shown in Fig. 11 have their maximum out-of-band values,  $A_{MAX}$ , larger than their corresponding out-of-band values at high frequencies,  $A_{dB}(\infty)$ . Therefore, amplitude characteristics are not monotonic functions of frequency, unlike the medium presented in Section 3. Decay slopes at the phase reversal points,  $\tau_g(\omega_M^+)$ , can be adjusted for the  $k = 1$  and  $k = 1/2$  cases, to make their  $A_{MAX}$  coincide with  $A_{dB}(\infty)$ . However, this would result in lowering the NGD/ $A_{MAX}$  ratio. For example, the limit for the optimum case ( $k = 1$ ) given by (19) would decrease by 2.25%. Since the primary objective of this paper is deriving the upper asymptotic limit, we will retain the limit given by expression (19).

A similar derivation as the one presented in the last two sections was performed for several other types of phase responses, such as a 2nd

order approximation around the center frequency, with a continuous phase derivative (group delay) over the entire frequency domain. However, the upper limit given by expression (19) is still higher than any of the ones obtained for these cases. The same is expected for any higher order phase response around the center frequency.

The full amplitude characteristic expression corresponding to the optimum linear and reciprocal ( $1/\omega$ ) phase response is obtained from (11), and given by

$$A_{\text{dB}}(\omega) = \frac{10(t_g \Delta\omega)}{\pi \ln 10} \left[ \frac{2\omega_M}{\omega_C} \ln(2) + \frac{|\omega|}{\omega_C} \ln \left( \frac{\omega_M + |\omega|}{|\omega_M - |\omega||} \right) + \frac{\omega_M^2}{\omega_C} \frac{\ln(2|\omega_M - |\omega||/\omega_M)}{2|\omega| - \omega_M} - \frac{\omega_M^2}{\omega_C} \frac{\ln(2(\omega_M + |\omega|)/\omega_M)}{2|\omega| + \omega_M} \right], \quad (20)$$

where the values chosen for the maximum NGD-bandwidth product,  $t_g \cdot \Delta\omega$ , and the linear to reciprocal phase transition frequency,  $\omega_M$ , are related such that  $A_{\text{dB}}(\omega_C) = 10 \log(2) \approx 3$  dB condition is satisfied. In the example depicted in Figs. 10(a) and 11, the chosen value of  $t_g \cdot \Delta\omega = 3.77$  requires a value of  $\omega_M \approx 2.606\omega_C$  for the linear and reciprocal phase response medium, to satisfy the 3 dB bandwidth condition at  $\omega_C$ .

## 6. DISCUSSION

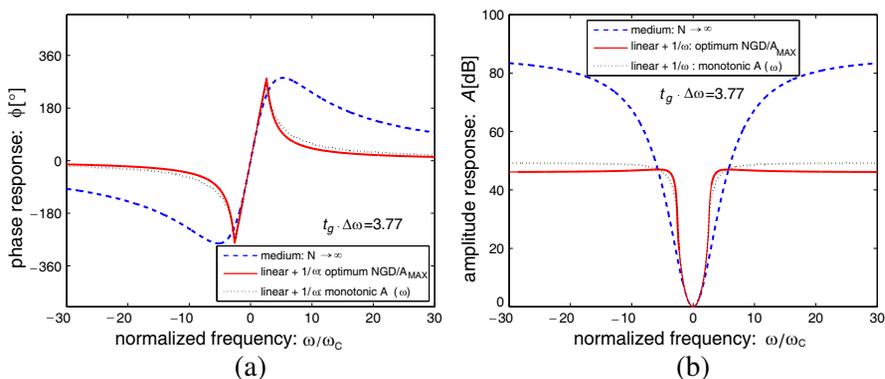
The asymptotic limit for the distributed high-pass filter based NGD medium presented in Section 3 is compared with the optimally engineered medium from Section 5.2 corresponding to the asymptotic limit given by expression (19). The comparison is carried out for both amplitude and phase frequency responses, for a selected case. In this comparison, an infinitely distributed high-pass filter-based medium exhibiting a NGD-bandwidth product of  $t_g \cdot \Delta\omega = 3.77$  is chosen, corresponding to a maximum out-of-band gain  $A_{\text{MAX}} = 86$  dB. This is compared with a piece-wise linear and reciprocal ( $1/\omega$ ) decay phase response, which is chosen to have the same NGD-bandwidth product and with  $\tau_g(\omega_M^+) = 2t_g$ , yielding a maximum NGD/ $A_{\text{MAX}}$  ratio as before. The amplitude characteristic for this case was obtained by Kramers-Kronig relations as before.

The phase and amplitude response plots are shown in Figs. 13(a) and 13(b), respectively. We can see that for the same NGD-bandwidth product chosen for these cases, the piece-wise linear and reciprocal decay phase response medium has a much smaller maximum out-of-band gain of  $A_{\text{MAX}} = 47$  dB. If the parameters of this medium type from Section 5.2 are slightly adjusted to yield a monotonic

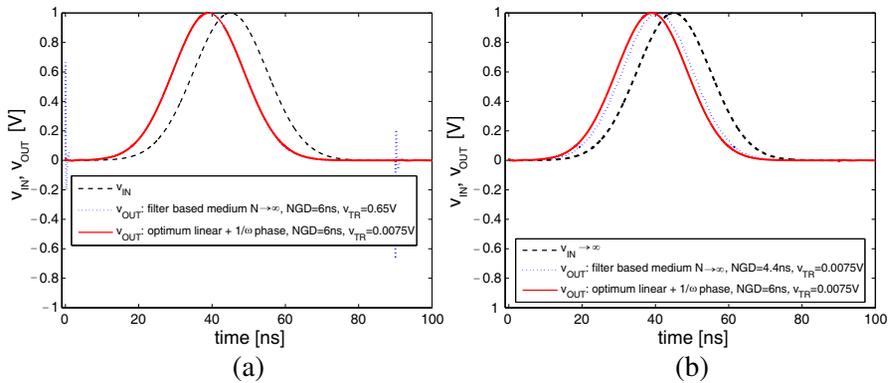
amplitude characteristic, a slightly larger out-of-band gain value of  $A_{MAX} = 49$  dB results. This case is also shown in Figs. 13(a) and 13(b).

The optimum frequency domain NGD/ $A_{MAX}$  ratio of the piece-wise linear and reciprocal decay phase response medium translates into the optimum NGD to transient amplitude ratio in the time domain, as shown in Figs. 14(a) and 14(b). For the same Gaussian pulse chosen, an optimally engineered medium exhibits a smaller transient amplitude for a given NGD (Fig. 14(a)), or for a given transient amplitude it exhibits a larger NGD (Fig. 14(b)), compared to an infinitely distributed high-pass filter-based medium.

A large NGD-bandwidth product requires a large out-of-band gain. In practice, for gain-compensated active media, achieving very large gain with realistic amplifiers while maintaining stability is an issue. If the individual-stage amplifiers have non-ideal directivity and non-ideal matching, a cascaded circuit can become unstable. Further, active device non-linear effects in the out-of-band high-gain region could cause distortion. This part of the signal spectrum also corresponds to the transients. Signals with their spectral energy limited to a very narrow-band region around the center frequency would not be affected by issues related to large out-of-band gain. However, as shown in Fig. 2, any practical signal with a defined turn-on/off time has some of its spectral energy in the out-of band region and therefore would result in large transients due to a large out-of-band gain of the circuit. In fact, if the out-of-band gain is large enough, the output noise floor can increase to a point where it is on the order of



**Figure 13.** (a) Phase and (b) amplitude response characteristics of a distributed high-pass filter-based medium, and a piece-wise linear and reciprocal decay phase response medium with NGD-bandwidth product  $t_g \cdot \Delta\omega = 3.77$ .



**Figure 14.** Gaussian waveform ( $\sigma_t = 10$  ns, turn-on/off times at 45 ns away from the peak) time-domain responses of a distributed high-pass filter-based medium, and a piece-wise linear and reciprocal decay phase response medium with (a) parameters from Fig. 13 yielding different transient amplitudes but the same  $NGD = 6$  ns for both media, and (b) parameters yielding different  $NGD$  values but the same transient amplitude  $v_{TR} = 0.0075$  V for both media.

magnitude, or larger, than the signal.

Note that the limits derived in this paper apply to both active and passive media, where for the passive media the out-of-band gain corresponds to center frequency attenuation [19]. The difficulties encountered with active devices do not apply for passive media, even though transients and noise issues still do.

## 7. CONCLUSION

In this paper, an asymptotic  $NGD$ -bandwidth limit as a function of maximum out-of-band gain was derived for an optimally engineered linear causal medium. Several examples of phase response characteristics of linear causal media were examined and their corresponding amplitude responses were derived using Kramers-Kronig relations. The optimal phase response function was identified, comprised of a linear part around the center frequency and reciprocal decay at high frequencies. The corresponding causal amplitude response was analytically obtained in each case.

It was shown that there is a trade-off between the maximum  $NGD$  value within a specified bandwidth on one hand, and the undesired maximum out-of-band gain in the amplitude response on the other. Moreover, an upper limit for  $NGD$ -bandwidth-product was shown to

be an asymptotic square root function of a logarithm of the maximum out-of-band gain. Alternatively, we can say that the out-of-band gain, and therefore the transient response amplitude of finite-duration signals as well, increase exponentially with the square of the achieved NGD. The obtained asymptotic NGD limit was shown to apply to a Lorentzian dielectric model medium, as well as to a distributed gain-compensated NGD medium comprised of ideally matched high-pass filters. A practical distributed NGD filter-based medium may exhibit non-ideal matching between stages which would reduce its NGD-bandwidth product further, however, the derived limit would still apply.

For the wide range of classes of linear causal media examined in this paper, it was shown that the NGD-bandwidth product as a function of the maximum out-of-band gain has an upper limit given by expression (19).

## ACKNOWLEDGMENT

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