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ON THE EFFICIENCY AND GAIN OF ANTENNAS

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Abstract—The fundamental limits of the gain and efficiency of an antenna are explored. These are very important quantities for, e.g., superdirective arrays. The antenna is in this paper confined in a sphere and all of the currents are assumed to run in a material with a given conductivity. It is shown that one can find the current distribution in the sphere that optimizes the gain, given the frequency and the radius of the sphere. The results indicate the distribution of antenna elements in an antenna array in order to maximize gain, or efficiency. The analysis is based on the expansion of the electromagnetic fields in terms of vector spherical harmonics. Explicit expressions for the limits of gain and efficiency, and the corresponding current densities, are derived for different types of antennas.

1. INTRODUCTION

Small antennas suffer from physical limitations on bandwidth, gain and efficiency. The limitations are caused by the reactive electromagnetic fields in the vicinity of the antenna and the currents that create these fields. It is important to realize that the reactive electromagnetic fields and the ohmic losses are consequences of Maxwell's equations and hence inevitable. The strength of the reactive fields increases with decreasing size and increasing directivity of the antenna. Small antennas with high directivity are inefficient since they need relatively strong currents in order to radiate even a low power. The strong currents create large ohmic losses and large reactive fields in the vicinity of the antenna. This is the reason why superdirective antennas are very inefficient. The superdirective antennas are antennas with a very large directivity relative to their size, see, e.g., [1]. If one only cares about directivity at one frequency it is easy to see that one can

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create an antenna of a given size with an arbitrary large directivity. The problem arises when this antenna is to be realized. Then the strong currents that are required to create radiation make the superdirective antennas very inefficient and with a gain that is much smaller than the directivity.

This paper investigates fundamental limits of the ohmic losses in an antenna and of the gain of an antenna. The method for the investigation is based upon expansions of the electromagnetic fields in terms of spherical vector waves. A similar method was used in a classic paper by Chu [2] on the fundamental limits of the Q-value of omni-directional antennas. The results by Chu were generalized to non-axially symmetric antennas by Harrington [3]. There are a number of other papers that focus on the fundamental limits of antennas and a summary of the main results can be found in [1, 4]. There are also a number of recent results on fundamental limits of antennas with arbitrary shape, cf., [5, 6]. In [7] the spectral efficiency of a sphere is treated by a method that is related to the method in this paper.

The objective of the paper is to give measures of the efficiency and gain of antennas that can be used by antenna designers. It is possible to estimate the power efficiency of a design if one can compare it with the physical limit. If a certain power efficiency of an antenna is required the physical limits give the bound for the size of the antenna. This bound indicates the realistic size of the antenna. The results in the paper show that one can improve the efficiency and gain of superdirective antennas if one uses non-evenly distributed antenna elements with proper values of the amplitude and phase of the currents of the antenna elements. There are a number of practical problems that are not discussed in the paper. Thus mutual coupling between elements, which is an important issue, is not discussed, nor is the analysis of how to translate the current densities into distributions of antenna elements in an antenna array.

The results in the paper can help an antenna designer in at least two ways. First the limits of the efficiency and gain tell the designer how close a design is from an optimal design. The other help is that the current distributions that the optimal antennas have indicates suitable places for currents and places that are unsuitable. Currents in unsuitable positions do not contribute much to the radiation and dissipate much power.

2. PREREQUISITES

The following problem is analyzed in the paper: Consider an antenna that is circumferenced by a sphere of radius a. Outside the sphere there is vacuum and the electromagnetic fields satisfy Maxwell's equations.

The current densities are confined in the sphere and run in a metal with conductivity σ. The metal is considered to be a good conductor and hence the value of the relative permittivity is irrelevant. What are the physical limits for the efficiency and the gain of such an antenna?

The volume of the sphere is V and the angular frequency is $\omega = 2\pi f$. The time convention $e^{j\omega t}$ is adopted in the paper. The efficiency is defined as

$$\eta_{\text{eff}} = \frac{P_{\text{rad}}}{P_{\text{rad}} + P_{\text{ohm}}}, \tag{1}$$

where P_{rad} is the radiated power and P_{ohm} the power dissipated in the antenna, due to ohmic losses. The Ohm's law $\vec{J} = \sigma\vec{E}$ holds and the ohmic loss is

$$P_{\text{ohm}} = \frac{1}{2}\int_V \frac{1}{\sigma(r)}\left|\vec{J}(\vec{r})\right|^2 dv. \tag{2}$$

The far field amplitude $\vec{F}(\theta,\phi)$ of the antenna is related to the far field by

$$\vec{F}(\theta,\phi) = \lim_{kr\to\infty} \vec{E}(\vec{r})kre^{jkr}. \tag{3}$$

The radiated power is

$$P_{\text{rad}} = \frac{1}{2\eta_0 k^2}\int_0^{2\pi}\int_0^{\pi}\left|\vec{F}(\theta,\phi)\right|^2 \sin\theta d\theta d\phi. \tag{4}$$

The definition of the directivity, D, and gain, G, are

$$\begin{aligned} D &= \frac{2\pi\left|\vec{F}(\theta,\phi)\right|^2_{\text{max}}}{k^2\eta_0 P_{\text{rad}}} \\ G &= D\eta_{\text{eff}}, \end{aligned} \tag{5}$$

where max denotes the maximum w.r.t. θ and ϕ. The wave number $k = \omega\sqrt{\varepsilon_0\mu_0}$ and the wave impedance $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ refer to vacuum.

3. GENERAL ANTENNAS

In the region exterior to the sphere, the electric field is expanded in spherical vector waves, $\vec{u}_{\tau\kappa ml}(\vec{r})$, also referred to as partial waves. These waves satisfy Maxwell's equations and constitute a complete set of vector valued functions on a spherical surface. The details of the spherical vector waves are given in Appendix A. The expansion reads

$$\vec{E}(\vec{r}) = \sum_{l=1}^{\infty}\sum_{m=0}^{l}\sum_{\kappa=e/o}\sum_{\tau=1}^{2} a_{\tau\kappa ml}\vec{u}_{\tau\kappa ml}(\vec{r}). \tag{6}$$

The corresponding magnetic field is given by the induction law

$$\begin{aligned}\vec{H}(\vec{r}) &= \frac{j}{\omega\mu}\sum_{l=1}^{\infty}\sum_{m=0}^{l}\sum_{\kappa=e/o}\sum_{\tau=1}^{2} a_{\tau\kappa ml}\nabla\times\vec{u}_{\tau\kappa ml}(\vec{r}) \\ &= \frac{jk}{\omega\mu}\sum_{l=1}^{\infty}\sum_{m=0}^{l}\sum_{\kappa=e/o}\sum_{\tau=1}^{2} a_{\tau\kappa ml}\vec{u}_{\tau'\kappa ml}(\vec{r}),\end{aligned} \tag{7}$$

where $\tau' = 3 - \tau$. Here $\tau = 1, 2$ is the index for the two different wave types (TE and TM), $\kappa = e$ for waves that are even with respect to the azimuthal angle ϕ and $\kappa = o$ for the waves that are odd w.r.t. to ϕ. $l = 1, 2, \ldots$ is the index for the polar angle and $m = 0, \ldots, l$ the index for the azimuthal angle. For $m = 0$ only the partial waves with $\kappa = e$ are non-zero, cf. Eq. (A2). The expansion in Eq. (6) covers the fields from all possible types of time harmonic sources inside V_a.

3.1. Classification

The expansion coefficients $a_{\tau\kappa ml}$ in the expansion (6) can theoretically be altered independently of each other. Hence, each partial wave corresponds to an independent port of the antenna. The maximum number of ports, or channels, an antenna can use is then equal to the maximum number of partial waves the antenna can radiate.

In Appendix A it is shown that for small radius ka, antennas that radiate partial waves with $\tau = 1$ are inductive and antennas that radiate partial waves $\tau = 2$ are capacitive. For this reason antennas that radiate waves with $\tau = 1$ are referred to as magnetic antennas and antennas that radiate waves with $\tau = 2$ as electric antennas. Antennas that radiate both $\tau = 1$ and $\tau = 2$ waves are referred to as combined antennas.

The following classification of antennas is used in this paper:

Partial wave antenna An antenna that radiates only one partial wave $(\tau\kappa ml)$. The antenna has one port.

Magnetic multipole antenna of order l An antenna that radiates partial waves with $\tau = 1$ and index l. The maximum number of ports is $N_{l\text{port}} = 2l + 1$.

Electric multipole antenna of order l An antenna that radiates partial waves with $\tau = 2$ and index l. The maximum number of ports is $N_{l\text{port}} = 2l + 1$.

Magnetic antenna of order l_{max} An antenna that radiates partial waves with $\tau = 1$ and with $l = 1, \ldots, l_{\text{max}}$. The maximum number of ports is $N_{\text{port}} = l_{\text{max}}(l_{\text{max}} + 2)$.

Electric antenna of order $l_{\max}$ An antenna that radiates partial waves with $\tau = 2$ and with $l = 1, \ldots, l_{\max}$. The maximum number of ports is $N_{\text{port}} = l_{\max}(l_{\max} + 2)$.

Combined antenna of order $l_{\max}$ An antenna that radiates partial waves with $\tau = 1, 2$ and $l = 1, \ldots, l_{\max}$. The maximum number of ports is $N_{\text{port}} = 2l_{\max}(l_{\max} + 2)$.

4. OPTIMIZATION OF EFFICIENCY

Consider first a partial wave antenna of magnetic type, $\tau = 1$, i.e., an antenna that radiates the partial wave $\vec{u}_{1\kappa ml}$. Due to the orthogonality of the vector spherical harmonics and the expansion of the Green dyadic, Eq. (A4), and Eqs. (A7) and (A8), the current density in the sphere has to be proportional to the vector wave function $\vec{A}_{1\kappa ml}(\theta, \phi)$,

$$\vec{J}(r, \theta, \phi) = \sigma(r) f(r) \vec{A}_{1\kappa ml}(\theta, \phi). \tag{8}$$

The optimization problem is to find $f(r)$ such that the efficiency is maximized. The ohmic losses are

$$P_{\text{ohm}} = \frac{1}{2} \int_0^a \sigma(r) |f(r)|^2 r^2 dr \tag{9}$$

due to the orthonormality of the vector wave functions, cf Appendix A.

From Eq. (A8) and the asymptotic expressions for the Hankel functions, Eq. (A6) it follows that the current density in Eq. (8) gives rise to the far field amplitude

$$\vec{F}(\theta, \phi) = -k\omega\mu_0 \int_0^a \sigma(r) j_l(kr) f(r) r^2 dr j^{l+1} \vec{A}_{1\kappa ml}(\theta, \phi). \tag{10}$$

The corresponding radiated power is

$$\begin{aligned} P_{\text{rad}} &= \frac{1}{2k^2\eta_0} k^2\omega^2\mu_0^2 \left| \int_0^a \sigma(r) j_l(kr) f(r) r^2 dr \right]^2 \\ &= \frac{1}{2} k\omega\mu_0 \left| \int_0^a \sigma(r) j_l(kr) f(r) r^2 dr \right|^2. \end{aligned} \tag{11}$$

The efficiency is given by

$$\eta_{\text{eff}} = \left(1 + \frac{1}{k\omega\mu_0} \frac{\int_0^a \sigma(r) |f(r)|^2 r^2 dr}{\left| \int_0^a \sigma(r) j_l(kr) f(r) r^2 dr \right|^2} \right)^{-1}. \tag{12}$$

In Appendix B it is seen that the efficiency is maximum when $f(r) = j_l(kr)$ and hence the maximum efficiency for a magnetic partial wave antenna of order l is

$$\eta_{\text{eff}} = \left(1 + \frac{k}{\eta_0 \int_0^{ka} \sigma(x/k)(j_l(x))^2 x^2 dx}\right)^{-1}. \tag{13}$$

When the electric type partial wave antenna is considered the current density is

$$\vec{J}(\vec{r}) = j\sigma(r)\nabla \times \left(f(r)\vec{A}_{1\kappa ml}(\vec{r})\right). \tag{14}$$

The corresponding far field amplitude reads

$$\vec{F}(\theta,\phi) = -k\omega\mu_0 j^{l+1}\int_V \sigma(r)\vec{v}_{2\kappa ml}(\vec{r}\,')\cdot(\nabla'\times f(r')\vec{A}_{1\kappa ml}(\theta',\phi'))dv'\vec{A}_{2\kappa ml}(\theta,\phi). \tag{15}$$

The resulting efficiency is

$$\eta_{\text{eff}} = \left(1+\frac{k^{-2}\eta_0^{-1}\int_V \sigma(r)\left|\nabla\times f(r)\vec{A}_{1\kappa ml}(\theta,\phi)\right|^2 dv}{\left|\int_V \sigma(r)(\nabla\times j_l(kr)\vec{A}_{1\kappa ml}(\theta,\phi))\cdot(\nabla\times f(r)\vec{A}_{1\kappa ml}(\theta,\phi))dv\right|^2}\right)^{-1}. \tag{16}$$

The same technique that was used for the magnetic antennas is used also for the electric antennas. One can then show that $f(r)$ has to be a real function. By assuming that $f(r) = j_l(kr) + \alpha h(r)$ and finding the minimum of this function, it is seen that $\alpha = 0$. Hence the most efficient antenna of electric type has the efficiency

$$\eta_{\text{eff}} = \left(1+\frac{k\eta_0^{-1}}{\int_0^{ka}\sigma(x/k)\left(\left(j_l'(x)+\frac{1}{x}j_l(x)\right)^2+l(l+1)\left(\frac{1}{x}j_l(x)\right)^2\right)x^2dx}\right)^{-1}. \tag{17}$$

By introducing the dimensionless quantities

$$\begin{aligned} B_{1l} &= \frac{\eta_0}{k}\int_0^{ka}\sigma(x/k)(j_l(x))^2x^2dx \\ B_{2l} &= \frac{\eta_0}{k}\int_0^{ka}\sigma(x/k)\left(\left(j_l'(x)+\frac{1}{x}j_l(x)\right)^2+l(l+1)\left(\frac{1}{x}j_l(x)\right)^2\right)x^2dx, \end{aligned} \tag{18}$$

the efficiency reads

$$\eta_{\tau\text{eff}} = \frac{B_{\tau l}}{B_{\tau l}+1}, \tag{19}$$

where $\tau = 1$ for the magnetic antenna and $\tau = 2$ for the electric antenna. Notice that $f(r)$ is independent of $\sigma(r)$ but that the current density is proportional to $\sigma(r)$, cf., Eqs. (8) and (14). In the case of constant conductivity, $\sigma(r) = \sigma$, the integrals can be solved analytically

$$\begin{aligned} B_{1l} &= \frac{\eta_0 \sigma a}{2}\left((ka j_l'(ka))^2 + ka j_l(ka) j_l'(ka) + ((ka)^2 - l(l+1))(j_l(ka))^2\right) \\ B_{2l} &= \eta_0 \sigma a j_l(ka)\left(j_l(ka) + ka j_l'(ka)\right) + B_{1l}. \end{aligned} \tag{20}$$

The explicit expressions for the corresponding electric field, the far field amplitude, the radiated and the dissipated powers are given in Eqs. (C9) and(C10) in Appendix C.

It has been shown that the current density that maximizes the efficiency for a partial wave antenna of order $(\tau\kappa ml)$ is given by

$$\vec{J}(\vec{r}) = \sigma(r) a_{\tau\kappa ml} \vec{v}_{\tau\kappa ml}(\vec{r}), \tag{21}$$

where $a_{\tau\kappa ml}$ is the amplitude and where the vector wave functions $\vec{v}_{\tau\kappa ml}(\vec{r})$ are given in Appendix A.

Finally, consider a combined multipole antenna where all multipoles are independent of each other. The efficiency of this antenna is optimized when the efficiency of each multipole is optimized. Thus the radial dependence of the current density of each multipole of index l is given by $f_l \sim j_l(kr)$ in Eqs. (8) and (14). As higher order multipoles are added to an antenna, the efficiency decreases. From Eqs. (13) and (17) and Figure 1 it is seen that the efficiency is 0.5 when $B_{\tau l} = 1$. If ka is below this value it cost much power to add currents that radiate multipole fields of index l, or higher. On the other hand, if ka is above the value then the efficiency is only slightly degraded by the addition of currents that radiate multipole fields of index l. The curves in Figure 1 are valuable for an antenna designer that, e.g., intends to design an antenna with a certain number of ports.

5. GAIN

The optimal directivity of a multipole antenna of order l is $D_{\rm opt} = N_{\rm port}/2 = (2l+1)/2$, cf., [3, 8]. The corresponding optimal gain is

$$G_{\tau l} = D_{\rm opt} \eta_{\tau {\rm eff}} = \frac{2l+1}{2} \frac{B_{\tau l}}{B_{\tau l}+1}. \tag{22}$$

Notice that $G_l \to N_{\rm port}/2 = (2l+1)/2$ as $ka \to \infty$ and $G_{\tau l}$ is very close to $N_{\rm port}/2$ once ka passes the value where $B_{\tau l} = 1$.

The optimal gain of an electric or magnetic antenna of order $l_{\max}$ is somewhat harder to find. However, it turns out that the antenna with the optimal gain has a gain that is the sum of the optimal gains of the multipole antennas. Thus

$$G_\tau = \sum_{l=1}^{l_{\max}} G_{\tau l} = \sum_{l=1}^{l_{\max}} \frac{2l+1}{2}\frac{B_{\tau l}}{B_{\tau l}+1}. \tag{23}$$

The proof is given in Appendix C, cf., Eq. (C8). Also here $G \to N_{\text{port}}/2$ as $ka \to \infty$, cf., Figure 2. The efficiency of the order $l_{\max}$ antenna is

$$\eta_\tau = \frac{G_\tau}{l_{\max}(l_{\max}+2)}. \tag{24}$$

The optimal gain of a combined antenna of order $l_{\max}$ is simply $G = G_1 + G_2$.

6. EXAMPLES

The efficiency as a function of the size ka is shown in Figure 1. The maximum efficiency is close to one down to some value and then it drops very fast. The smallest efficient antenna is an electric dipole antenna. It has an efficiency close to one when the radius of the sphere that circumscribes the antenna is larger than approximately $10^{-4}\lambda$. Figure 2 shows the optimal gain as a function of the radius a when the frequency is $\sigma = 10^7\,\text{S/m}$. Also here the electric antennas are better than the magnetic antennas. In Figure 3 the optimal gain for an antenna of order $l_{\max} = 5$ is given as a function of ka for different

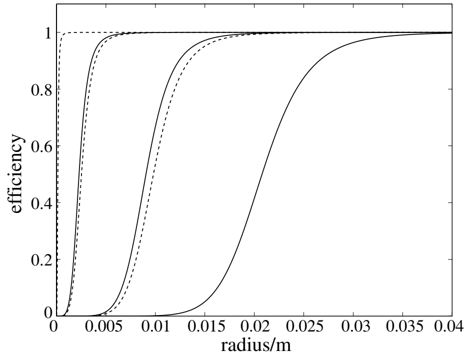

Figure 1. Efficiency for magnetic (solid line) and electric (dashed line) partial wave antenna with $l = 1$ (left curve), $l = 2$ (middle) and $l = 3$ (right) when $\sigma = 10^7\,\text{S/m}$ and $f = 1\,\text{GHz}$. Notice that the electric partial wave antenna of order l has almost the same efficiency as the magnetic partial wave antenna of order $l-1$.

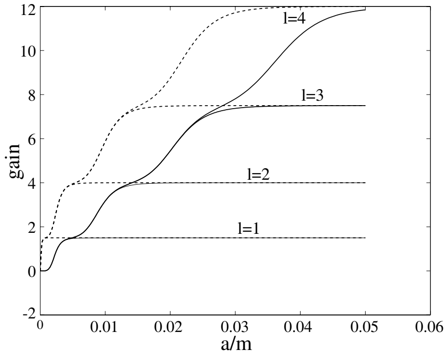

Figure 2. Optimal gain, in linear units, for an electric (dashed line) or magnetic (solid line) antenna of order $l_{\max} = 1, 2, \ldots, 4$, when $\sigma = 1 \cdot 10^7\,\mathrm{S/m}$. The frequency is $f = 1\,\mathrm{GHz}$. Asymptotically the gain approaches the maximum directivity $D_{\mathrm{opt}} = N_{\mathrm{port}}/2 = l_{\max}(l_{\max} + 2)/2$.

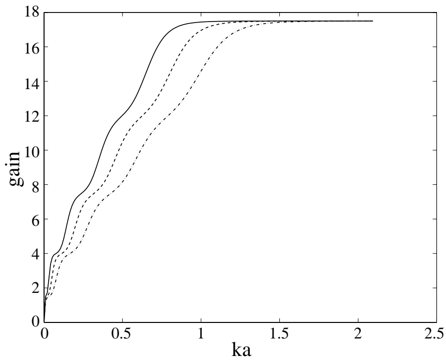

Figure 3. Optimal gain for an electric antenna of order $l_{\max} = 5$, when $\sigma = 1 \cdot 10^7\,\mathrm{S/m}$ as a function of ka. The frequency is $f = 10\,\mathrm{GHz}$ (dash-dot line), $f = 1\,\mathrm{GHz}$ (dashed line) and $f = 100\,\mathrm{MHz}$ (solid line). Notice that the maximum gain does not scale with frequency if the conductivity is kept constant.

frequencies. This is to show that for a given conductivity the maximum gain does not scale with frequency. The reason is that the efficiency is frequency dependent for a given ka, as can be seen from Eqs. (18) and (19).

In Figure 4 the far field patterns are shown for three electric antennas. All three antennas are transmitting at the frequency 2 GHz. The conductivity is $2 \cdot 10^7\,\mathrm{S/m}$ and the radius of the antennas are,

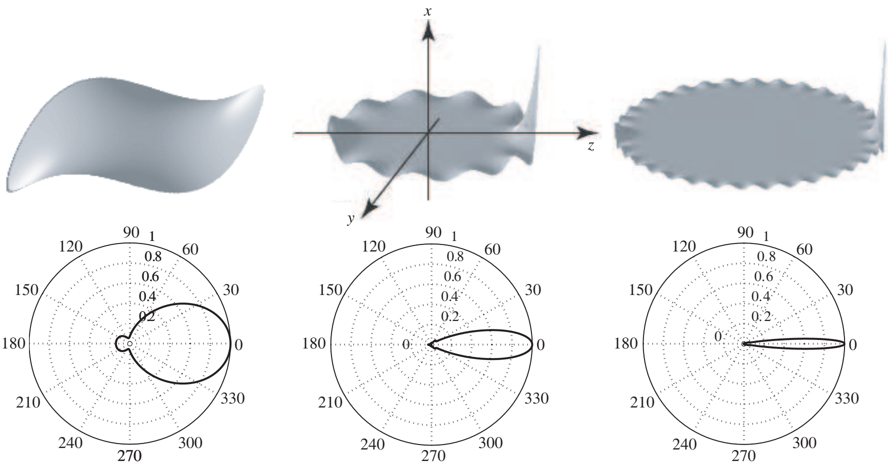

Figure 4. The current distributions in the plane $x = 0$ of electric antennas with maximum gain that radiate in the positive z-direction and their corresponding far field patterns. The conductivity is $2 \cdot 10^7$ S/m and the frequency is 2 GHz for all three antennas. The radius are from left to right, 1 cm, 10 cm and 50 cm. The far field is polarized in the $\hat{\vec{y}}$ direction and thus the current distribution at $x = 0$ is also directed in the $\hat{\vec{y}}$ direction.

from left to right, 1 cm, 10 cm and 50 cm. The figures on the top depict the current density $\vec{J}(\vec{r}, t)$ at a fixed time t in a cross section at x =0 of the antenna. The antennas are designed for maximum gain with its maximum radiation in the positive z-direction and with polarization in the direction $\hat{\vec{y}}$. This means that in the plane $x = 0$ the current density is directed in the y-direction. The current densities follow from the theory. There are some things to be learned from the distribution of the current. The first thing is that if one varies the time t the pattern is seen to move as a wave in the positive z-direction. The antenna resembles a traveling wave antenna, which is reasonable since a traveling wave antenna makes sure that the waves generated at different positions in the antenna are in phase in the forward direction. The next thing to notice is that as the radius of the antenna increases the currents are squeezed towards the surface of the antenna. In particular the currents of the higher order multipoles are very strong and are concentrated close to the surface. In fact the current distribution resembles the current distribution of a parabolic, or spherical, reflector antenna.

7. CONCLUDING REMARKS

The currents that give the most optimal antennas in this paper were chosen without physical restrictions. This is necessary in order to obtain the optimal current densities. Needless to say, it is very hard to realize the optimal designs since the currents that can be created inside a spherical volume suffer from inductive and capacitive couplings that are hard to tamper with, e.g., the skin effect. Nevertheless, the physical limits of antennas give the antenna designer indications on the achievable efficiency, gain and bandwidth for an antenna of a certain size and frequency. The limits also serve as measures of the quality of a design. If the values of efficiency, gain and bandwidth are far from the physical limits, it might be worthwhile to modify the design of an antenna. This paper gives no rules of thumb on what can be considered as far from the physical limits, that is left to the designers to explore. It is quite straightforward to write a computer program that illustrates the current densities in Eq. (C9) in two-dimensional graphs. From such graphs a designer can get ideas on how to construct an antenna with high gain. It is seen that an antenna that is large compared to the wavelength should have its currents close to the surface of the sphere in order to maximize the gain whereas an antenna that is small compared to the wavelength should have its currents distributed over the entire volume. The amplitude and phase of these currents can be obtained from a graph of the optimal current density.

APPENDIX A. VECTOR WAVES AND GREEN DYADIC

The definition of spherical vector waves can be found in different textbooks, e.g., [3, 9]. In this paper they are defined using vector spherical harmonics, cf., [10]

$$\begin{aligned}
\vec{A}_{1\kappa ml}(\theta,\phi) &= \frac{1}{\sqrt{l(l+1)}}\nabla\times(\vec{r}\,Y_{\kappa ml}(\theta,\phi))\\
\vec{A}_{2\kappa ml}(\theta,\phi) &= \frac{1}{\sqrt{l(l+1)}}r\nabla Y_{\kappa ml}(\theta,\phi)\\
\vec{A}_{3\kappa ml}(\theta,\phi) &= \hat{r}Y_{ml}(\theta,\phi).
\end{aligned}\tag{A1}$$

The following definition of the spherical harmonics is used:

$$Y_{\kappa ml}(\theta,\phi)=\sqrt{\frac{\varepsilon_m}{2\pi}}\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix},\tag{A2}$$

where $\varepsilon_m = 2 - \delta_{m0}$ and κ, m, l take the values

$$\kappa = \begin{pmatrix} e \\ o \end{pmatrix}, \quad m = 0,\, 1,\, 2,\, \ldots,\, l, \quad l = 0,\, 1, \ldots \tag{A3}$$

In the current application the index l will never take the value 0, since there are no monopole antennas. The vector spherical harmonics constitute an orthogonal set of vector functions on the unit sphere

$$\int_\Omega \vec{A}_{\tau n}(\theta,\phi) \cdot \vec{A}_{\tau' n'}(\theta,\phi) d\Omega = \delta_{\tau\tau'}\delta_{nn'}, \tag{A4}$$

where the integration is over the unit sphere and where $n = \kappa ml$. The outgoing divergence-free spherical vector waves are defined by

$$\begin{cases} \vec{u}_{1n}(\vec{r}\,) = h_l(kr)\vec{A}_{1n}(\theta,\phi) \\ \vec{u}_{2n}(\vec{r}\,) = \dfrac{1}{k}\nabla \times \Big(h_l(kr)\vec{A}_{1n}(\theta,\phi)\Big) = h_l'(kr)\vec{A}_{2n}(\theta,\phi) \\ \qquad\qquad + \dfrac{1}{kr}h_l(kr)\left(\vec{A}_{2n}(\theta,\phi) + \sqrt{l(l+1)}\vec{A}_{3n}(\theta,\phi)\right), \end{cases} \tag{A5}$$

where $h_l(kr) = h_l^{(2)}(kr)$ is the spherical Hankel function of the second kind. The asymptotic behavior in the far zone of the spherical Hankel functions is

$$h_l^{(2)}(kr) \to j^{l+1}\frac{e^{-jkr}}{kr} \quad \text{when} \quad |k|r \to \infty. \tag{A6}$$

The regular wave function $\vec{v}_{\tau n}(\vec{r}\,)$ are obtained by replacing the spherical Hankel functions $h_l(kr)$ with the corresponding spherical Bessel functions $j_l(kr)$.

The Green dyadic is given by

$$\vec{G}(\vec{r},\vec{r}\,') = -j\sum_n \vec{v}_n(\vec{r}_<)\vec{u}_n(\vec{r}_>). \tag{A7}$$

where $\vec{v}_n(\vec{r}\,)$ is the regular wave function. Here $\vec{r}_<$ is $\vec{r}$ if $r < r'$ and $\vec{r}\,'$ if $r' < r$ and vice versa for $\vec{r}_>$. In a homogenous space with current density $\vec{J}$ the electric field is given by

$$\vec{E}(\vec{r}\,) = -j\omega\mu_0 k \int\limits_V \vec{G} \cdot \vec{J} dv. \tag{A8}$$

The complex power radiated by an antenna is proportional to the radiation impedance of the antenna. An antenna enclosed in a region $r < a$ and with an electric field $\vec{E}(\vec{r}\,) = \vec{u}_1(\vec{r}\,)$ for $r > a$ has the magnetic field $\vec{H} = j\eta_0^{-1}\vec{u}_2(\vec{r}\,)$. By using the expressions for the wave functions it is seen that radiated complex power through a sphere of radius a is

$$\frac{1}{2}\int\left(\vec{E}\times\vec{H}^{*}\right)\cdot\hat{\vec{r}}dS=\frac{1}{2k^{2}\eta_{0}}+j\frac{(l+1)((2l-1)!!)^{2}}{2k^{2}\eta_{0}(ka)^{2l+1}}+j\mathcal{O}\left((ka)^{-2l-3}\right). \quad \text{(A9)}$$

Hence the imaginary part of the radiation impedance is positive for small ka and the antenna is inductive. For an antenna with electric field $\vec{E}(\vec{r}) = \vec{u}_2(\vec{r})$ the complex radiated power is the complex conjugate of the right hand side of Eq. (A9) and the radiation impedance is then capacitive for small ka.

APPENDIX B. A PROOF

It is here proven that $f(r) = j_l(kr)$ in Eq. (12). The efficiency is given by

$$\eta_{\text{eff}} = \left(1+\frac{1}{k\omega\mu_0}\frac{\int_0^a \sigma(r)|f(r)|^2 r^2 dr}{\left|\int_0^a \sigma(r) j_l(kr) f(r) r^2 dr\right|^2}\right)^{-1}. \quad \text{(B1)}$$

The aim is to find a complex value function $f(r)$ such that the quotient

$$\frac{\int_0^a \sigma(r)|f(r)|^2 r^2 dr}{\left|\int_0^a \sigma(r) j_l(kr) f(r) r^2 dr\right|^2} \quad \text{(B2)}$$

is minimized. First write $f(r) = g(r)e^{j\phi(r)}$, where $g(r)$ and $\phi(r)$ are a real valued functions that are allowed to be discontinuous, such that $j_l(kr)g(r)$ is positive for all r and try to find $g(r)$ and $\phi(r)$ such that

$$\frac{\int_0^a \sigma(r)|g(r)|^2 r^2 dr}{\left|\int_0^a \sigma(r) j_l(kr) g(r) e^{j\phi(r)} r^2 dr\right|^2} \quad \text{(B3)}$$

is minimized. It is seen that

$$\left|\int_0^a \sigma(r) j_l(kr) g(r) e^{j\phi(r)} r^2 dr\right|^2 \leq \left(\int_0^a \sigma(r) j_l(kr) g(r) r^2 dr\right)^2. \quad \text{(B4)}$$

Thus $f(r)$ can be chosen to be a real function. Write $f(r,\alpha) = j_l(kr) + \alpha b(r)$ where $b(r)$ is a an arbitrary real valued function and α is a parameter, and form

$$F(\alpha) = \frac{\int_0^a \sigma(r)|f(r,\alpha)|^2 r^2 dr}{\left|\int_0^a \sigma(r) j_l(kr) f(r,\alpha) r^2 dr\right|^2}. \quad \text{(B5)}$$

Since $f(r,\alpha) = j(r) + \alpha b(r)$ it follows that $F(\alpha)$ is a fraction of two polynomials of order two. The derivative is simple enough such that one can solve the equation $F'(\alpha) = 0$ and see that $\alpha = 0$. Thus the efficiency is maximized when $f(r) = j_l(kr)$.

APPENDIX C. OPTIMAL GAIN OF AN ELECTRIC OR MAGNETIC ANTENNA OF ORDER $L_{\max}$

For a given far field amplitude the most efficient current distribution for each partial wave is the same as for a multipole antenna of order l. Thus the current densities read

$$\vec{J}(\vec{r}) = \sigma(r) \sum_{\tau=1}^{2} \sum_{n} \gamma_{\tau n} j^{-l+\tau} \vec{v}_{\tau n}(\vec{r}), \tag{C1}$$

where n is the multi-index $n = \kappa ml$. $\gamma_{\tau n}$ are the so far unknown amplitudes of the currents, and the factor $j^{-l-\tau}$ has been introduced for convenience. The corresponding far field amplitude, the radiated power, and the dissipated power read

$$\begin{aligned} \vec{F}(\theta,\phi) &= \sum_{\tau=1}^{2} \sum_{n} \gamma_{\tau n} B_{\tau l} \vec{A}_{\tau n}(\theta,\phi) \\ P_{\text{rad}} &= \frac{1}{2\eta_0 k^2} \sum_{\tau=1}^{2} \sum_{n} (\gamma_{\tau n} B_{\tau l})^2 \\ P_{\text{ohm}} &= \frac{1}{2\eta_0 k^2} \sum_{\tau=1}^{2} \sum_{n} \gamma_{\tau n}^2 B_{\tau l}. \end{aligned} \tag{C2}$$

The gain is given by

$$G = \frac{2\pi |\vec{F}(\theta,\phi)|^2_{\max}}{k^2 \eta_0 (P_{\text{rad}} + P_{\text{ohm}})}. \tag{C3}$$

That results in the following expression

$$G = \frac{4\pi \left| \sum_{\tau=1}^{2} \sum_n \gamma_{\tau n} B_{\tau l} \vec{A}_{\tau n}(\theta,\phi) \right|^2_{\max}}{\sum_{\tau=1}^{2} \sum_n \gamma_{\tau n}^2 \left(B_{\tau l}^2 + B_{\tau l} \right)}, \tag{C4}$$

where max is with respect to θ and ϕ and where $B_{\tau l}$ is given by Eq. (18). At this stage one can use the same technique as in [3, 8] to find the maximum gain. Let the direction of maximum gain be $\hat{\vec{z}}$, i.e., $\theta = 0$. The maximum gain is independent of which polarization is chosen on the electric far field and one may let the polarization be in the $\hat{\vec{x}}$ direction. Then

$$G = \frac{4\pi \left(\sum_{\tau=1}^{2} \sum_n \gamma_{\tau n} B_{\tau l} \left| \hat{\vec{x}} \cdot \vec{A}_{\tau n}(0,\phi) \right| \right)^2_{\max}}{\sum_{\tau=1}^{2} \sum_n \gamma_{\tau n}^2 \left(B_{\tau l}^2 + B_{\tau l} \right)}, \tag{C5}$$

where $|\hat{\vec{x}} \cdot \vec{A}_{\tau n}(0, \phi)|$ are seen to be independent of ϕ and given by

$$\begin{aligned} \left|\hat{\vec{x}} \cdot \vec{A}_{1n}(0, \phi)\right| &= \delta_{m1}\delta_{\kappa o}\sqrt{\frac{2l+1}{8\pi}} \\ \left|\hat{\vec{x}} \cdot \vec{A}_{2n}(0, \phi)\right| &= \delta_{m1}\delta_{\kappa e}\sqrt{\frac{2l+1}{8\pi}}. \end{aligned} \tag{C6}$$

That means that only $m = 1$ terms are non-zero in the sum.

The extreme value of G is when $\frac{\partial G}{\partial \gamma_{\tau l}} = 0$ for all l. That leads to the relations

$$\gamma_{\tau l} = \sqrt{\frac{2l+1}{3}}\frac{B_{11}+1}{B_{\tau l}+1}\gamma_{11} = \sqrt{\frac{2l+1}{3}}\frac{B_{21}+1}{B_{\tau l}+1}\gamma_{21}, \tag{C7}$$

where γ_{11} and γ_{21} are arbitrary constants, and the gain

$$G = \sum_{\tau=1}^{2}\sum_{l=1}^{l_{\max}} \frac{2l+1}{2}\frac{B_{\tau l}}{B_{\tau l}+1}. \tag{C8}$$

If only electric or magnetic antennas are used, the sum in τ is omitted. The same optimal value is obtained for a polarization in the $\hat{\vec{y}}$ direction. Hence any combination of $\hat{\vec{x}}$ and $\hat{\vec{y}}$ polarization, including elliptical polarization, give the same optimal value.

The optimal current density is

$$\vec{J}(r,\theta,\phi) = \sigma(r)\sum_{\tau=1}^{2}\sum_{l=1}^{l_{\max}} j^{-l+\tau}\gamma_{\tau l}(\vec{v}_{1o1l}(r,\theta,\phi)\delta_{\tau 1} + \vec{v}_{2e1l}(r,\theta,\phi)\delta_{\tau 2}). \tag{C9}$$

The regular vector waves $\vec{v}_{\tau\kappa ml}(\vec{r})$ are given in Appendix A. These current densities result in a far field that is maximum in the direction $\theta = 0$ and with the electric field polarized in the x-direction. The corresponding far field amplitude and the electric field are given by

$$\begin{aligned} \vec{F}(\theta,\phi) &= \sum_{\tau=1}^{2}\sum_{l=1}^{l_{\max}} \gamma_{\tau l}\left(\vec{A}_{1o1l}(r,\theta,\phi)\delta_{\tau 1} + \vec{A}_{2e1l}(r,\theta,\phi)\delta_{\tau 2}\right) \\ \vec{E}(r,\theta,\phi) &= \sum_{\tau=1}^{2}\sum_{l=1}^{l_{\max}} j^{-l+\tau+2}\gamma_{\tau l}(\vec{u}_{1o1l}(r,\theta,\phi)\delta_{\tau 1} + \vec{u}_{2e1l}(r,\theta,\phi)\delta_{\tau 2}). \end{aligned} \tag{C10}$$

The expressions in Eqs. (C9) and (C10) are valid for the general conductivity case $\sigma(r)$.

REFERENCES

1. Hansen, R. C., "Fundamental limitations in antennas," *Proc. IEEE*, Vol. 69, No. 2, 170–182, 1981.
2. Chu, L. J., "Physical limitations of omni-directional antennas," *Appl. Phys.*, Vol. 19, 1163–1175, 1948.
3. Harrington, R. F., *Time Harmonic Electromagnetic Fields*, McGraw-Hill, New York, 1961.
4. Collin, R. E., "Minimum Q of small antennas," *Journal of Electromagnetic Waves and Applications*, Vol. 12, No. 10, 1369–1393, 1998.
5. Sohl, C. and M. Gustafsson, "A priori estimates on the partial realized gain of ultra-wideband (UWB) antennas," *Quart. J. Mech. Appl. Math.*, Vol. 61, No. 3, 415–430, 2008.
6. Gustafsson, M., C. Sohl, and G. Kristensson, "Illustrations of new physical bounds on linearly polarized antennas," *IEEE Trans. on Antennas and Propagat.*, Vol. 57, No. 5, 1319–1327, May 2009.
7. Gustafsson, M. and S. Nordebo, "On the spectral efficiency of a sphere," *Progress In Electromagnetics Research*, Vol. 67, 275–296, 2007.
8. Karlsson, A., "Physical limitations of antennas in a lossy medium," *IEEE Trans. on Antennas and Propagat.*, Vol. 52, 2027–2033, 2004.
9. Hansen, J. E., *Spherical Near-Field Antenna Measurements*, IEE Electromagnetic Waves Series, No. 26, Peter Peregrinus Ltd., Stevenage, UK, 1988, ISBN: 0-86341-110-X.
10. Arfken, G., *Mathematical Methods for Physicists*, 3rd Edition, Academic Press, Orlando, 1985.