Progress In Electromagnetics Research, Vol. 137, 527–538, 2013

ARBITRARY LOSS FACTORS IN THE WAVE PROPAGATION BETWEEN RHM AND LHM MEDIA WITH CONSTANT IMPEDANCE THROUGHOUT THE STRUCTURE

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Abstract—We investigate the wave propagation properties in lossy structures with graded permittivity and permeability involving left-handed metamaterials. An exact analytic solution to Helmholtz' equation for a lossy case with both real and imaginary parts of permittivity and permeability profile, changing according to a hyperbolic tangent function along the direction of propagation, is obtained. It allows for different loss factors in RHM and LHM media. Thereafter, the corresponding numerical solution for the field intensity along the composite structure is obtained by means of a dispersive numerical model of lossy metamaterials that uses a transmission line matrix method based on Z-transforms. We present the expressions and graphical results for the field intensity along the composite structure and compare the analytic and numerical solutions, showing that there is an excellent agreement between them.

1. INTRODUCTION

Electromagnetic metamaterials (MM) are defined as artificial composites with electromagnetic properties not readily found in nature. A special class of MMs are the left-handed materials (LHM). The first theoretical description of LHMs was done by Veselago in 1967 [1], but only after three decades their actual practical implementations were described by Pendry [2, 3].

Received 30 January 2013, Accepted 19 February 2013, Scheduled 4 March 2013

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LHM are typically produced using subwavelength "particles" with negative effective relative permittivity and permeability as their structural units. The first proposed LHM particles were split-ring resonators (SRR) and nanowires that simultaneously furnish negative permeability and permittivity [3]. Wire media and SRRs are still widely used in the microwave domain and well understood, but a number of other "particles" like complementary split-ring resonators [4], plate pairs and cut-wire pairs [5] and double fishnets [6–8] have been studied. The first experimental confirmation of a left-handed material was published in 2001 [9]. Presently, the experimental fishnet-type LHM for the visible range of frequencies have been fabricated and described [10].

The novel and often counter-intuitive properties of the LHM, including the inverse Doppler effect, radiation tension instead of pressure and negative index of refraction (negative phase velocity) etc. [11, 12] resulted in the proposals of a number of applications. They include superlenses and hyperlenses enabling imaging far below the diffraction limit [13, 14], resonant cavities and waveguides with geometrical dimensions orders of magnitude smaller than the operating wavelength [15], as well as transformation optics [16–18], plasmonic waveguides [19] and invisibility cloaks [16–18, 21].

The use of transformation optics [16–18, 20, 21], provides an additional degree of design freedom where one is free to tailor any desired distortion of space from the point of view of wave propagation. This is basically done by mapping such distortions onto the spatial distribution of electromagnetic properties of material in the conventional Cartesian space [22]. Such artificially crafted spatially changing electromagnetic properties then allow arbitrary tailoring of the propagation of electromagnetic waves [21]. Because of the scalability of Maxwell's equations, this kind of propagation control extends in principle over any desired wavelength range, the most often targeted ranges being microwave, optical (including visible) and terahertz.

Most of the cases consider structures with abrupt interfaces towards the surrounding positive index material ("right-handed" media, RHM) and constant permittivity and permeability (and hence, the constant refractive index) within the MM structure. There is however a growing theoretical and practical interest for MMs with spatially varying permittivities and permeabilities and with gradual transitions from the RHM to LHM and vice versa, since many real-world applications would benefit from such structures. Graded permittivities and permeabilities are interesting for transformation optics and hyperlenses [23], and a class of the invisibility cloaks [16, 21] using spherically graded MM has been described [24]. Various other proposed applications of graded MM-composites include beam shaping and directing, enhancement of nonlinear effects [25], superlenses [26], etc.

The first paper dedicated to gradient refractive index LHM was published in 2005 [27]. The analytical approaches to graded MM-structures, as the one discussed in the present paper, are of special interest since they ensure fast, simple and direct determination of the field distribution and the calculation of the wave propagation parameters within such materials. Some publications include [28–34].

In the present paper we present an exact analytical solution of Helmholtz' equation for the wave propagation through a lossy graded MM-structure, where both the permittivity and the permeability vary according to a hyperbolic tangent function. We consider the most general case of lossy wave propagation with constant impedance throughout the entire structure, where loss factors can be chosen arbitrarily in both RHM and LHM parts. This provides the opportunity to model the significantly higher losses in LHM materials compared to those in the RHM materials.

In addition to that, a numerical model, developed in [35] by using the Transmission Line Matrix (TLM) method based on Z-transforms [36, 37] and capable to take into account the dispersive properties of metamaterials in the time-domain, is presented in some detail here and the numerical solutions compared to the analytically obtained solutions.

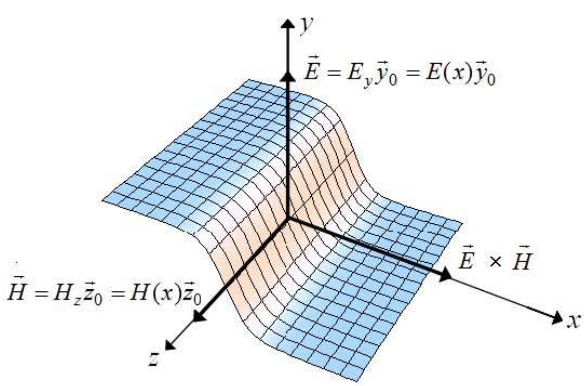

Figure 1. Propagation of a wave through a graded index structure with a hyperbolic tangent profile.

2. PROBLEM FORMULATION

We assume the time-harmonic waves with an $\exp(-i\omega t)$ dependency in isotropic materials, where the effective medium approximation is valid. The geometry of the problem is illustrated in Figure 1. The electric field is directed along the y-axis, $\vec{E}(\vec{r}) = E(x)\vec{e}_y$, whereas the magnetic field is directed along the z-axis, $\vec{H}(\vec{r}) = H(x)\vec{e}_z$. The propagation direction of the wave is along the x-axis. Since the fields depend only on the x-coordinate, the one-dimensional Helmholtz' equations have the form [30]

$$\frac{d^2E}{dx^2}-\frac{1}{\mu}\frac{d\mu}{dx}\frac{dE}{dx}+\omega^2\mu\epsilon E(x)=0,\quad \frac{d^2H}{dx^2}-\frac{1}{\epsilon}\frac{d\epsilon}{dx}\frac{dH}{dx}+\omega^2\mu\epsilon H(x)=0, \quad (1)$$

where $\epsilon = \epsilon(\omega, x)$ and $\mu = \mu(\omega, x)$ are the frequency-dependent and stratified dielectric permittivity and magnetic permeability, respectively.

3. SOLUTIONS OF THE FIELD EQUATIONS

We assume an inhomogeneous medium for which the effective permittivity and permeability vary according to following hyperbolic tangent functions

$$\mu(\omega,x)=-\mu_0\mu_R\tanh(\rho x)-i\mu_0\left[\frac{\mu_{I1}+\mu_{I2}}{2}-\frac{\mu_{I1}-\mu_{I2}}{2}\tanh(\rho x)\right], \quad (2)$$

$$\epsilon(\omega,x)=-\epsilon_0\epsilon_R(\omega)\tanh(\rho x)-i\epsilon_0\left[\frac{\epsilon_{I1}+\epsilon_{I2}}{2}-\frac{\epsilon_{I1}-\epsilon_{I2}}{2}\tanh(\rho x)\right], \quad (3)$$

where ρ is a parameter describing the steepness of the transition from the RHM material at $x < 0$ to the LHM material at $x > 0$. For passive materials, we require ϵ_{I1}, $\epsilon_{I2} > 0$ and μ_{I1}, $\mu_{I2} > 0$. A constant wave impedance throughout the structure, requires that the real and imaginary parts of the effective permittivity and permeability satisfy the condition

$$\beta(\omega) = \frac{\mu_{I1}+\mu_{I2}}{2\mu_R - i(\mu_{I1}-\mu_{I2})} = \frac{\epsilon_{I1}+\epsilon_{I2}}{2\epsilon_R - i(\epsilon_{I1}-\epsilon_{I2})}. \quad (4)$$

When the condition (4) is satisfied, we have

$$\begin{aligned}\mu(\omega,x) &= -\mu_0\frac{\mu_{I1}+\mu_{I2}}{2\beta}(\tanh(\rho x)+i\beta),\\ \epsilon(\omega,x) &= -\epsilon_0\frac{\epsilon_{I1}+\epsilon_{I2}}{2\beta}(\tanh(\rho x)+i\beta),\end{aligned} \quad (5)$$

Note that the wave impedance $Z = Z_0 Z(\omega) = \sqrt{\mu(\omega, x)/\epsilon(\omega, x)}$ is constant throughout the entire structure and there is no reflection on the graded interface between the two materials. The two differential Equation (1) have the exact solutions

$$E(x) = E_0 e^{-\kappa\beta x} \left[2\cosh(\rho x)\right]^{i\frac{\kappa}{\rho}}, \ H(x) = H_0 e^{-\kappa\beta x} \left[2\cosh(\rho x)\right]^{i\frac{\kappa}{\rho}}, \quad (6)$$

where E_0 and H_0 are the amplitudes of the electric and magnetic fields at the boundary $x = 0$, and

$$\kappa = k + i\alpha = \frac{\omega}{c}\sqrt{\mu_R \epsilon_R} + i\frac{\omega}{2c}\sqrt{\frac{\epsilon_R}{\mu_R}}(\mu_{I2} - \mu_{I1}), \quad (7)$$

We note that in the absence of losses, the results (6) are reduced to the results in references [30] and [34] as special cases. Specifically in [34] we consider a solution for the oblique incidence on the graded interface with a less general profile function which is also used in [30]. Choosing the normal incidence with incident angle $\theta = 0$ in [34] we obtain the lossy solutions with uniform loss factors in the entire space. The empirical data suggests however that the loss factors in RHM and LHM media are generally very different. Thus in the present paper, unlike the analysis in [34], we introduce a more general graded interface model such that the loss factors in the RHM and LHM media can be chosen different from each other. This point will be further highlighted below. The one-dimensional field distribution based on the results (6) is shown in Figure 2. The field amplitudes are related by $E_0 = Z_0 Z(\omega) H_0$. The exact solutions (6) are valid for arbitrary steepness ρ of the graded index interface and arbitrary losses. In the RHM material, we obtain for $x \to -\infty$

$$\begin{aligned} &E(x,t) \sim E_0 e^{-\gamma_1 x}\cos(\omega t - kx), \ H(x,t) \sim H_0 e^{-\gamma_1 x}\cos(\omega t - kx), \\ &\quad \gamma_1 = \frac{\omega}{c}\sqrt{\frac{\epsilon_R}{\mu_R}}\mu_{I1} \end{aligned} \quad (8)$$

In the LHM material, we obtain for $x \to +\infty$ that

$$\begin{aligned} &E(x,t) \sim E_0 e^{-\gamma_2 x}\cos[\omega t - (-k)x], \\ &H(x,t) \sim H e^{-\gamma_2 x}\cos[\omega t - (-k)x], \ \gamma_2 = \frac{\omega}{c}\sqrt{\frac{\epsilon_R}{\mu_R}}\mu_{I2} \end{aligned} \quad (9)$$

For $x \to -\infty$, it follows (8) that the wave in the RHM with the wavevector $\vec{k}_{\text{RHM}} = +k\vec{e}_x$ propagates in the $+x$-direction. For $x \to +\infty$, it follows (9) that the wave in the LHM with wavevector $\vec{k}_{\text{LHM}} = -k\vec{e}_x$ propagates in the $-x$-direction. The energy flux (the Poynting vector) is still in the $+x$-direction in both media as expected.

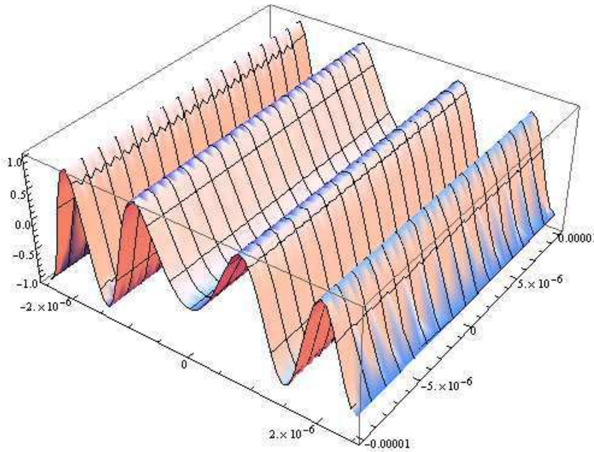

Figure 2. 1D field distribution based on the Equation (6).

4. NUMERICAL MODEL OF METAMATERIALS

A dispersive TLM Z-transform model of the lossy MTM-composite, described in [35], is used here to verify the analytical solution for gradient index metamaterials with arbitrary loss factor in RHM and LHM presented in Sections 2 and 3. This model follows the notation used in [36, 37] to describe various types of conventional linear time-dependent materials with the purpose that it can be easily incorporated into the algorithm of so-called Z-transform-based TLM method given in [36]. TLM Z-transform model of the lossy MTM-composite is based on Drude dispersive model as it allows to characterize MTM-composite response in much wider frequency range than for example, Lorentz dispersion model. Hovewer, it could be easily adapted to describe any higher-order material responses. In this paper, the Drude model describing the frequency dependence of electric and magnetic conductivities is used

$$\sigma_e(\omega) = \frac{\sigma_{e0}}{1+i\omega\tau_e} = \frac{\omega_{pe}^2\tau_e\epsilon_0}{1+i\omega\tau_e},\ \sigma_m(\omega) = \frac{\sigma_{m0}}{1+i\omega\tau_m} = \frac{\omega_{pm}^2\tau_m\mu_0}{1+i\omega\tau_m},\quad (10)$$

where $\omega_{pe,pm}$, $\tau_{e,m}$ and $\sigma_{e0,m0}$ are the electric and magnetic plasma frequencies and the corresponding collision times and static conductivities, respectively. For LHM which is matched to free-space, the static electric and magnetic conductivities are related by $\sigma_{m0} = Z_0^2\sigma_{e0}$ where Z_0 is the wave impedance of free-space. As an alternative, Drude model describing the frequency dependence of permittivity and permeability (i.e., electric and magnetic susceptibilities) of MTM

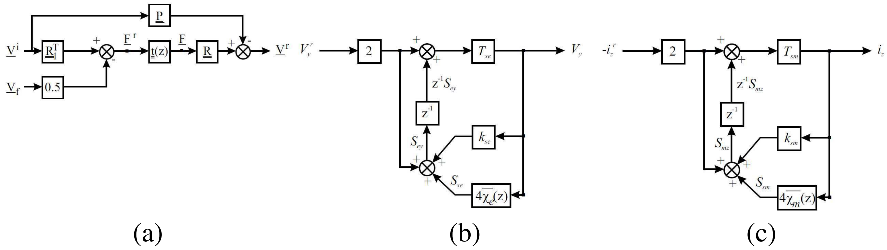

Figure 3. (a) Signal flow diagram of the general algorithm of TLM method based on Z-transform [36]. (b) Dispersive TLM Z-transform model for metamaterials — calculation of E_y in the time-domain. (c) Dispersive TLM Z-transform model for metamaterials — calculation of H_z in the time-domain.

composite can be used but both models give identical results as shown in [35]. In addition, using the relations

$$\epsilon(\omega) = \epsilon_0 \left(1 - i\frac{\sigma_e(\omega)}{\omega\epsilon_0}\right), \quad \mu(\omega) = \mu_0 \left(1 - i\frac{\sigma_m(\omega)}{\omega\mu_0}\right), \tag{11}$$

it is possible to determine permittivity and permeability from known electric and magnetic conductivities and vice versa. In [36] the Drude model was also used but only to describe an electric conductivity of an unmagnetized plasma with collisions. Also, instead of exponential Z-transform employed in [36] to transfer frequency dependence of considered material properties to the Z-domain, the bilinear Z-transform is adopted in [35] to develop TLM Z-transform model of the lossy MTM as the bilinear discretisation provides a much more accurate scheme.

The TLM scattering process incorporating the dispersive model of the lossy MTM composite, can be represented, for considered field components, with the flow graph shown in Figure 3.

Further details of the inclusion into the 3D TLM method of the Z-transform models, developed for various types of conventional time-dependent materials, as well as the expressions for vectors and matrices from Figure 3(a) are given in [36]. The complete description of each block in Figure 3 can also be found in [35].

5. GRAPHICAL PRESENTATION AND DISCUSSION OF THE RESULTS

A comparison between the exact analytical solutions for the electric field $E(x)$, given by Equation (6) with the corresponding numerical

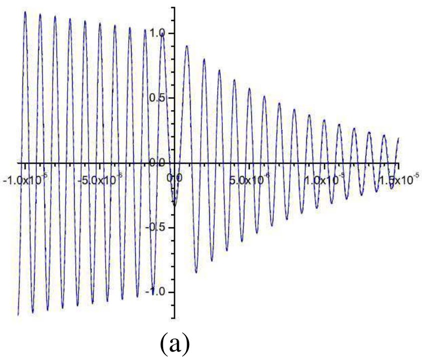

Figure 4. Numerical results (blue dash line) vs analytical results (black solid line) for electric field $E(x)$ as a function of x, with $E_0 = 1$, $k = 2\pi/(10^{-6}\,\mathrm{m})$, $\rho = 1/(10^{-6}\,\mathrm{m})$, $\kappa\beta = 2\pi/(10^{-4}\,\mathrm{m})$ as well as (a) $\alpha = 0.75\kappa\beta$ and (b) $\alpha = 0.50\kappa\beta$.

results obtained by means of the dispersive TLM Z-transform model, for two different values of the numerical parameters are presented in Figure 4. From Figure 4, we see that the numerical and analytical results show an excellent agreement with each other. Furthermore, we see that there is no reflection at the interface between RHM and LHM, as expected, since the impedance is constant throughout the entire space. However we see that the loss factors γ_1 and γ_2 in RHM and LHM respectively are different.

6. CONCLUSION

We present an exact analytic solution to lossy Helmholtz' equations with a hyperbolic tangent profile function along the direction of propagation. The analytic expressions and graphical results for the field intensities along the MM-composite structure are presented. The model allows for arbitrary loss factors and arbitrary temporal dispersion.

Furthermore, the enhanced Z-transform based TLM method which enables the direct time-domain modelling of lossy metamaterials is presented. The accuracy and stability of the model is demonstrated for the gradient transitions of effective electromagnetic parameters for which analytical solutions are presented in this paper. The model can be easily adapted to describe Lorentz or higher-order material responses and is generally usable to arbitrary gradient metamaterial profiles.

Finally, we note that the numerical benchmark of the exact analytical results obtained in the present paper can be done very accurately using other numerical methods, like for example the finite-difference time-domain (FDTD) method. However for the present purposes we have chosen the TLM-method, being an enhanced and modified version of the finite difference method written using the engineering notation. In our previous publications, we have used other numerical techniques like for example the finite element method in [30]. The main purpose of the comparisons of our exact analytical results with the corresponding numerical results is the exact benchmark of the numerical methods and confirmation of the validity of the exact analytical results.

ACKNOWLEDGMENT

The work of T.A. and N.D. was funded by the Serbian Ministry of Science and Technology within the project TR-32024.

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