Progress In Electromagnetics Research, Vol. 141, 403–414, 2013

EVOLUTION OF COS-GAUSSIAN BEAMS IN A STRONGLY NONLOCAL NONLINEAR MEDIUM

Ying Guan[1], Li-Xin Zhong[2], Khian-Hooi Chew[3], Hao Chen[4], Qiyang Wu[5], and Rui-Pin Chen[1, *]

[1]School of Sciences, Zhejiang A & F University, Lin'an, Zhejiang 311300, China

[2]School of Finance, Zhejiang University of Finance and Economics, Hangzhou 310018, China

[3]Department of Physics, University of Malaya, Kuala Lumpur 50603, Malaysia

[4]Nanjing National Laboratory of Microstructures, Department of Physics, Nanjing University, Nanjing 210093, China

[5]Department of Electrical and Computer Engineering, Ohio State University, Columbus, OH 43210, USA

Abstract—The dynamical properties of cos-Gaussian beams in strongly nonlocal nonlinear (SNN) media are theoretically investigated. Based on the moments method, the analytical expression for the root-mean-square (RMS) of the cos-Gaussian beam propagating in a SNN medium is derived. The critical powers that keep the RMS beam widths invariant during propagation in a SNN medium are discussed. The RMS beam width tends to evolve periodically when the initial power does not equal to the critical power. The analytical solution of the cos-Gaussian beams in SNN media is obtained by the technique of variable transformation. Despite the difference in beam profile symmetries and initial powers, a cos-Gaussian beam always transforms periodically into a cosh-Gaussian beam during propagation and the transformation between the two beams revives after a propagation distance.

Received 7 June 2013, Accepted 9 July 2013, Scheduled 26 July 2013

* Corresponding author: Rui-Pin Chen (chenrp123@gmail.com).

1. INTRODUCTION

The propagation properties of optical beams in nonlocal nonlinear media have attracted considerable interest in recent years. Many novel features of the nonlocal nonlinearity such as the suppression of collapse [1] and the support of vortex solitons [2, 3] have been found. In strongly nonlocal nonlinear (SNN) media, the characteristic length of the material response function is much larger than the beam width. In this case, the nonlocal nonlinear Schrödinger equation (NNLSE), which governs the propagation of an optical beam in a nonlocal nonlinear medium can be simplified to a linear equation, as suggested by A. W. Snyder and D. J. Mitchell [4]. Various solitons and breather solutions in the strongly nonlocal nonlinear media have been found [5–8]. Most of these studies focus on the shapes of optical beams invariant during propagation in linear and nonlinear media [6–13]. However, the study of the evolution of beams with a change in shape during propagation is rarely reported in literature. This is because the level of complexity can be significantly higher than that of evolution of fixed-shaped beams.

Recently, there has been growing interest in the study of Hermite-sinusoidal-Gaussian (HSG) beams as a result of the works of Casperson and Tovar [14] and Chen et al. [15]. Propagation of HSG laser beams in free space, in complex optical systems, in turbulence and in Kerr media have been studied extensively [14–20]. These studies indicate that the shape of HSG laser beams changes during propagation. However, the propagation properties of HSG laser beams in SNN media remain unexploited. As a special case of the HSG beams, the cos-Gaussian beams have many interesting applications such as optimizing the efficiency of laser amplifiers. This is because of its unique profile as a Gaussian beam that modulates with a cos function [14–17].

In the present paper, we investigate the evolution of cos-Gaussian beams in SNN media. The moments method [21–26] provides a convenient and rigorous approach to obtain the evolution of relevant parameters of beam without any assumption for the solution [27], although the beam propagates with a change in shape. Based on the method, an exact analytical description of the variation of the RMS beam width is obtained. The critical powers as a function of beam parameters require to maintain the RMS beam width invariant during propagation with a uniform wavefront are obtained. When the initial power does not equal to the critical power, the RMS beam width of the cos-Gaussian beam tends to evolve periodically during propagation in SNN media, regardless of the difference in beam profile symmetry. Using the technique of variable transformation [28], the analytical solution of cos-Gaussian beams in SNN media is derived.

An interesting intensity redistribution of beams is observed during propagation in SNN media, though the RMS beam width remains constant throughout the propagation. Despite the difference in both the beam profile symmetrical ratios and the initial powers, the cos-Gaussian beam always transforms into a cosh-Gaussian beam during propagation and the transformation between the two beams revives after a propagation distance.

2. THE MOMENTS METHOD ANALYSIS

The propagation of an optical beam in a nonlocal nonlinear medium in the paraxial approximation is described by NNLSE [3]:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2ik\frac{\partial u}{\partial z}+\frac{2n_2k^2}{n_0}u\int R(|\vec{r}-\vec{r}_a|)\left|u(\vec{r}_a)\right|^2 d^2\vec{r}_a=0, \quad (1)$$

where $u(x, y, z)$ is the complex amplitude of the optical field, k the linear wave number, n_0 the linear refraction index of the medium, n_2 the nonlinear index coefficient, R the normalized symmetrical real spatial response function of the medium, and $\vec{r}$ and $\vec{r}_a$ are the two-dimensional transverse coordinate vectors. x and y are the transverse coordinates, whereas z is the longitudinal coordinate.

In SNN media, the NNLSE can be deduced to a linear model [4]

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2ik\frac{\partial u}{\partial z}-k^2\gamma^2P_0\left(x^2+y^2\right)u=0, \quad (2)$$

where γ is a material constant associated with the response function R, and $P_0=\iint |E|^2 dxdy$ is the input power. In this section, the moments method is employed to obtain the dynamics information about the cos-Gaussian beam in a SNN medium by analyzing the evolution of several integral quantities. A definition of these quantities is

$$I_1(z) = \iint_s |u|^2\, dxdy, \quad (3a)$$

$$I_{2,\xi}(z) = \iint_s \xi^2\, |u|^2\, dxdy, \quad (3b)$$

$$I_{3,\xi}(z) = \frac{i}{k}\iint_s \xi\left(u\frac{\partial u^*}{\partial \xi}-u^*\frac{\partial u}{\partial \xi}\right) dxdy, \quad (3c)$$

$$I_{4,\xi}(z) = \frac{1}{2k^2}\iint_s \left|\frac{\partial u}{\partial \xi}\right|^2 dxdy, \quad (3d)$$

where $\xi = x$ or y. These quantities are associated with the beam power, I_1, beamwidth, I_2, momentum, I_3, and Hamiltonian, I_4; and satisfy a closed set of coupled ordinary differential equations (ODEs). Thus, we have $dI_1(z)/\mathrm{d}z = 0$; $dI_{2,\xi}(z)/\mathrm{d}z = I_{3,\xi}(z)$; $dI_{3,\xi}(z)/\mathrm{d}z = 4I_{4,\xi}(z) - 2\gamma^2 P_0 I_{2,\xi}(z)$; $dI_{4,\xi}(z)/\mathrm{d}z = -\gamma^2 P_0 I_{3,\xi}(z)/2$. With the important invariant under the beam evolution, $Q = 2I_{4,\xi}I_{2,\xi} - I_{3,\xi}^2/4$, we obtain an Ermakov-Pinney (EP) equation [26] describing the dynamics of the scaled beamwidth:

$$\frac{d^2 I_{2,\xi}^{1/2}(z)}{dz^2} + \gamma^2 P_0 I_{2,\xi}^{1/2}(z) = \frac{Q}{I_{2,\xi}^{3/2}(z)}. \tag{4}$$

For cos-Gaussian beams, the initial field is of the form [9, 10]

$$u\,(x, y, z = 0) = A_0 \exp\left(-\frac{x^2 + y^2}{w_0^2}\right) \cos(\beta_x x) \cos(\beta_y y), \tag{5}$$

where A_0 is the amplitude, w_0 is the beam width associated with the Gaussian part, β_x and β_y are the parameters associated with the cos part. Then, the invariant is given by

$$Q = I_1^2 S_{2,\xi} S_{3,\xi}/\left(k^2 S_{1,\xi}^2\right), \tag{6}$$

with

$$S_{1,\xi} = \sqrt{2\pi}w_0\left[1 + \exp\left(-\beta_\xi^2 w_0^2/2\right)\right]/4, \tag{7a}$$

$$S_{2,\xi} = \sqrt{2\pi}w_0^3\left[1 + \left(1 - \beta_\xi^2 w_0^2\right)\exp\left(-\beta_\xi^2 w_0^2/2\right)\right]/16, \tag{7b}$$

$$S_{3,\xi} = \sqrt{2\pi}\left[1 + \beta_\xi^2 w_0^2 + \exp\left(-\beta_\xi^2 w_0^2/2\right)\right]/4w_0. \tag{7c}$$

The solution of Eq. (4) for cos-Gaussian beams as the initial field is [21]

$$I_{2,\xi}(z) = I_{2,\xi}(z = 0)\cos^2(z/z_p) + I_1 S_{3,\xi} z_p^2 \sin^2(z/z_p)/\left(k^2 S_{1,\xi}\right), \tag{8}$$

where $z_p = (\gamma^2 P_0)^{-1/2}$. Equation (8) describes the variation of the scaled beamwidth in any transversal direction (e.g., x or y) in a SNN medium. Since the RMS beam width is $W_\xi^2 = I_{2,\xi}/I_1$, we can obtain the evolution dynamics of the RMS beam width in any transversal direction by rewriting Eq. (8) as:

$$W_\xi^2(z) = S_{2,\xi}\cos^2(z/z_p)/S_{1,\xi} + S_{3,\xi} Z_p^2 \sin^2(z/z_p)/\left(k^2 S_{1,\xi}\right). \tag{9a}$$

If $d^2 I_{2,\xi}^{1/2}(z)/dz^2 = 0$, the critical power for an invariant RMS beam width in any transversal direction during propagation, $P_{c,\xi}$, can be obtained from Eq. (5):

$$P_{c,\xi} = S_{3,\xi}/\left(\gamma^2 k^2 S_{2,\xi}\right). \tag{9b}$$

In the above discussion, we consider only the variation of transversal direction of beams in one-dimensional (1D) case. In order to study the evolution of cos-Gaussian beams in two-dimensional (2D) case, we need to obtain the 2D total RMS beam width evolution and the 2D critical power for invariant total RMS beam width as follows

$$W^2(z) = W_x^2(z) + W_y^2(z) = \left(\frac{S_{2,x}}{S_{1,x}} + \frac{S_{2,y}}{S_{1,y}}\right)\cos^2(z/z_p) + \left(\frac{S_{3,x}}{S_{1,x}} + \frac{S_{3,y}}{S_{1,y}}\right)\frac{z_p^2 \sin^2(z/z_p)}{k^2}, \tag{10a}$$

$$P_{cr} = (S_{3,x}S_{1,y} + S_{3,y}S_{1,x}) / \left[\gamma^2 k^2 \left(S_{2,x}S_{1,y} + S_{2,y}S_{1,x}\right)\right]. \tag{10b}$$

Note here that if $\beta_x = \beta_y = 0$, Eq. (10) reduces to a Gaussian beam $W_G(z)$, which can be expressed as

$$W_G^2(z) = w_0^2 \cos^2(z/z_p)/2 + 2z_p^2 \sin^2(z/z_p)/\left(k^2 w_0^2\right), \tag{11a}$$

$$P_{cr}^G = 4/\left(\gamma^2 k^2 w_0^4\right). \tag{11b}$$

Figure 1 illustrates the critical powers of cos-Gaussian beams that is normalized with respect to the critical power of a Gaussian beam. It is seen that P_{cr}/P_{cr}^G increases with increasing the beam parameters β_x/w_0 and β_y/w_0, as expected.

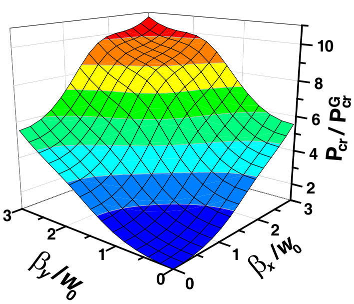

Figure 1. The critical powers of cos-Gaussian beam for different parameters β_x and β_y.

In Figure 2, we show the evolution of the RMS beam widths of a cos-Gaussian beam with the beam parameters $\beta_x = 4/w_0$ and $\beta_y = 5/w_0$ for different initial powers. The relations between the 2D critical power and any 1D critical power of the beam can be obtained from Eqs. (9b) and (10b) as $P_{c,x} = 0.83P_{cr}$; $P_{c,y} = 1.3P_{cr}$. Figure 2(a) illustrates the evolution of the total (solid line), x-component (dashed line), y-component (dotted line) cos-Gaussian RMS beam width for a particular case with the initial power equals to the 2D critical power, i.e., $P_{in} = P_{cr}$. It is interesting to see that each transversal direction of the RMS beam width of the beam propagates in a periodic

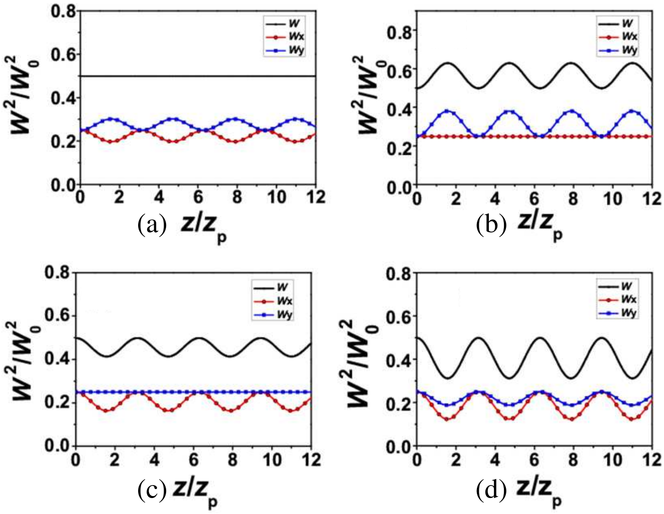

Figure 2. Evolution of the total (solid line), x direction (dashed line), y direction (dotted line) cos-Gaussian RMS beam width with the beam parameters $\beta_x = 4/w_0$ and $\beta_y = 5/w_0$ for different initial powers: (a) $P_{in} = P_{cr}$; (b) $P_{in} = P_{c,x}$; (c) $P_{in} = P_{c,y}$; (d) $P_{in} = 1.6P_{cr}$

manner, whereas the total RMS beam width retains a constant value during propagation. Figures 2(b) and 2(c) show the evolution of beam widths with the initial powers equal to the critical powers for x-direction and y-direction, respectively. The evolution of the beam widths with the initial powers equal to 1.6 times the 2D critical power, $P_{in} = 1.6P_{cr}$ is depicted in Figure 2(d). From Figure 2, it is clear that the corresponding RMS beam width of the cos-Gaussian beam remains invariant during propagation, if the initial power equals to the corresponding critical power. If the initial power is smaller than the critical power, the beam width inclines to oscillate periodically and the beam width always larger than the initial beam width, and vice versa. If $P_{in} < P_{cr}$, the RMS beam width at $z/z_p = 0$ increases. On the other hand, the RMS beam width at $z/z_p = 0$ decreases if $P_{in} > P_{cr}$.

The Eq. (2) that is reduced from the nonlinear Schrodinger equation for a SNN medium is analogous to the linear harmonic oscillator or the propagation of light in a parabolic refractive index medium. But it still describes the high nonlinear optics phenomenon. Firstly, although the evolution of an optical field in a SNN medium can be described by the linear Schrodinger equation, the evolution of the beam profiles and the corresponding RMS beam widths of the beam are dependent on the input powers. The sizes of beam profiles and the corresponding RMS beam width of the beam are different during propagation with different input powers as shown in Figure 2. Secondly, the underlying physics of SNN medium is different from linear optics, such as the nonlocality in laser-induced thermal nonlinearities is thermal diffusion, and the nonlocal response in photorefractive materials is induced by the photogeneration of carriers. If take a Gaussian distribution as an initial field, the beam preserves its Gaussian shape but its RMS beam width "breathes" sinusoidally as it travels in a straight path. When the initial power equals the critical power $P_{in} = P_{cr}$, beam diffraction is balanced by beam-induced refraction and then form a soliton [4].

3. ANALYTICAL SOLUTIONS OF COS-GAUSSIAN BEAMS IN STRONGLY NONLOCAL NONLINEAR MEDIA

By the technique of variable transformation, the Eq. (2) can be deduced to the paraxial diffraction equation. The beams propagate in SNNM can be transferred to their counterparts in free space. The relationship between the solution in free space and its counterpart in a SNN medium has been given by D. Lu et al. [28] as

$$u(x,y,z) = (-1)^a \left(1+\tan^2\alpha\right) \exp\left[-\frac{ik\left(x^2+y^2\right)\tan\alpha}{2z_p}\right] E\left[\left(1+\tan^2\alpha\right)x, \left(1+\tan^2\alpha\right)y, z_p\tan\alpha\right], \tag{12}$$

where $a = \{\alpha - \arctan[\tan(\alpha)]\}/\pi$ and $\alpha = z/z_p$, $z_p = \left(\gamma^2 P_0\right)^{-1/2}$. $E(x,\ y,\ z)$ is the solution in free space. The propagation of a cos-Gaussian beam in free space can be expressed in terms of the Huygens-Fresnel diffraction integral as

$$E(x,y,z) = \frac{i}{\lambda z}\iint u(x_0,y_0,z=0) \exp\left[-\frac{ik(x-x_0)^2+ik(y-y_0)^2}{2z}\right] dx_0dy_0. \tag{13}$$

Substituting Eq. (5) into Eq. (13), we obtain the analytical solution in free space as

$$E(x,y,z) = \frac{i\pi w_0^2 A_0}{i\pi w_0^2+\lambda z}\exp\left[-\frac{\lambda z w_0^2\left(\beta_x^2+\beta_y^2\right)}{4i\pi w_0^2+4\lambda z}\right]\exp\left[-\frac{i\pi(x^2+y^2)}{i\pi w_0^2+\lambda z}\right] \times\cos\left(\frac{i\pi w_0^2\beta_x x}{i\pi w_0^2+\lambda z}\right)\cos\left(\frac{i\pi w_0^2\beta_y y}{i\pi w_0^2+\lambda z}\right). \tag{14}$$

Using Eq. (14), the analytical solution of the cos-Gaussian beam in a SNN medium becomes

$$\begin{aligned} u(x,y,z) \; = \; & \frac{(-1)^a i\pi N_w A_0 \left(1+\tan^2\alpha\right)}{i\pi N_w + 1} \exp\left[-\frac{ik(x^2+y^2)\tan\alpha}{2z_p}\right] \\ & \exp\left[-\frac{w_0^2(\beta_x^2+\beta_y^2)}{4i\pi N_w + 4}\right] \times \exp\left[-\frac{i\pi(1+\tan^2\alpha)(x^2+y^2)}{i\pi w_0^2 + \lambda z_p \tan\alpha}\right] \\ & \cos\left(\frac{i\pi N_w \beta_x x}{i\pi N_w + 1}\right) \cos\left(\frac{i\pi N_w \beta_y y}{i\pi N_w + 1}\right), \end{aligned} \tag{15}$$

where $N_w = w_0^2/\lambda z_p \tan\alpha$. In order to further illustrate how the cos-Gaussian beam evolves in a SNN medium, we examine the intensity distribution of a cos-Gaussian beam with initial power $P_{in} = P_{cr}$ and $P_{in} = 1.6P_{cr}$ at different propagation distances, as shown in Figure 3. Numerical calculations are carried out by using the analytical formula Eq. (15). It is interesting to see that an obvious redistribution of beam intensity occurs during propagation, though the RMS beam width remains constant that predicted by the moments method. Despite the difference in symmetries and initial powers, the cos-Gaussian beam transforms periodically into a cosh-Gaussian beam during propagation in the SNN medium. In addition, the transformation between the two beams revives after certain propagation distance, as expected from Eq. (15). The beam profiles are qualitatively the same during the evolution of the beams with initial powers $P_{in} = P_{cr}$ and $P_{in} = 1.6P_{cr}$ as shown in Figures 3(a) and (b). However, the intensity distributions of the beam for the case $P_{in} = 1.6P_{cr}$ are more focused to the center of the beam compared with that for the case $P_{in} = P_{cr}$ except the same initial profiles as shown in Figures 3(a) and (b), i.e., the corresponding RMS beam width of the beam with $P_{in} = 1.6P_{cr}$ are always less than that for the case $P_{in} = P_{cr}$ except the positions $z/z_p = n\pi$ (n is an integer). It can also be confirmed for the evolution of RMS beam width of the beam with $P_{in} = 1.6P_{cr}$ and $P_{in} = P_{cr}$ in Figures 2(a) and 2(d). In the other hand, although the beam transforms periodically into a cosh-Gaussian beam during propagation, the corresponding RMS beam width of the beam remains constant when $P_{in} = P_{cr}$ as shown in Figure 3(a). It should also be noted that although the beam profiles evolve periodically during propagation, the RMS beam widths (e.g., W, W_x, W_y) remain constant or oscillate periodically, as predicted by the moments method.

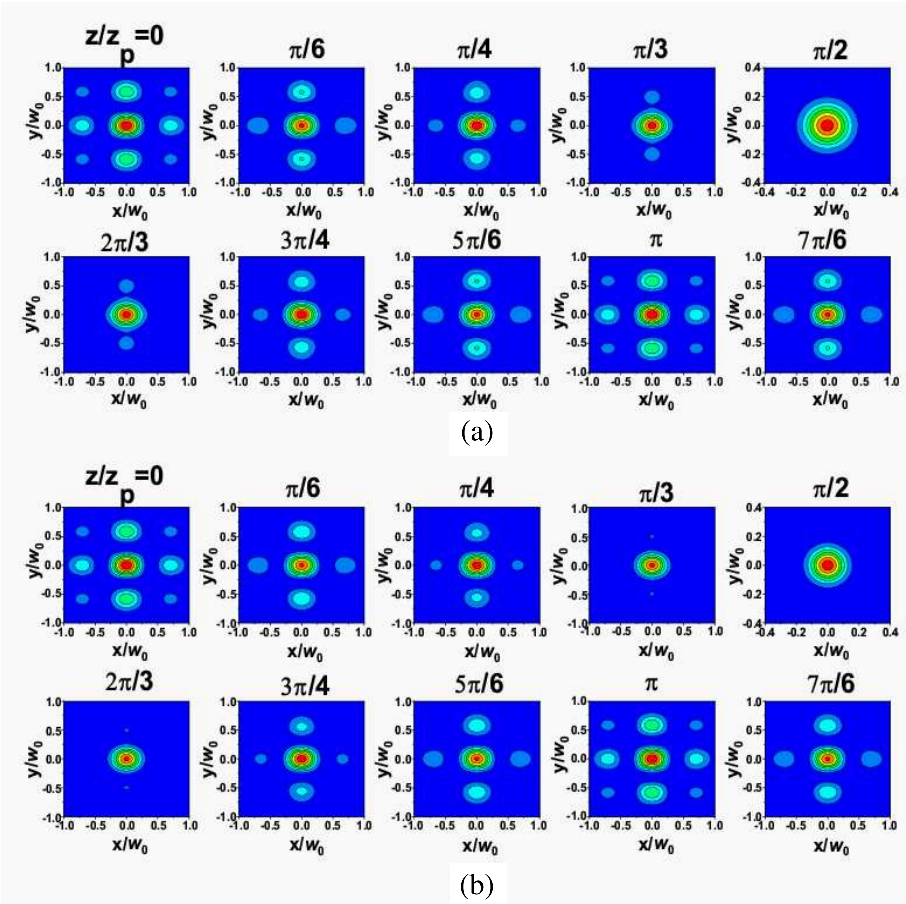

Figure 3. The intensity distribution of cos-Gaussian beams with the beam parameters $\beta_x = 4/w_0$ and $\beta_y = 5/w_0$ at several different propagation distances for initial powers: (a) $P_{in} = P_{cr}$ and (b) $P_{in} = 1.6P_{cr}$.

4. CONCLUSION

In this work, we have studied the propagation of cos-Gaussian beams into a SNN medium. Based on the moments method, the evolution of a cos-Gaussian RMS beam width is analytically derived. The critical powers require to maintain the RMS beam widths constant for different β_x and β_y are obtained. The RMS beam width varies periodically during propagation when the input power does not equal to the critical power. The analytical solution of the cos-Gaussian beam in a SNN medium is obtained by the technique of variable transformation. Interesting redistribution of intensity is found when the beam propagates into a SNN medium, though the RMS beam width remains a constant value. The cos-Gaussian beam transforms into a cosh-Gaussian beam when the beam propagates into a SNN medium, regardless the differences both the beam profile symmetries and initial powers. Moreover, the transformation between the cos-Gaussian beam and the cosh-Gaussian beam revives after certain propagation distance. Our analytical results agree reasonable well with the prediction of the moments method.

ACKNOWLEDGMENT

This work is partially supported by the Key Project of the Education Commission of Zhejiang Province of China No. Z201120128 and the National Basic Research Program and National Natural Science Foundation of China (61178062 and 60990322). This work was also supported by the Humanities and Social Sciences Fund sponsored by Ministry of Education of China (Grant No. 10YJAZH137), Natural Science Foundation of Zhejiang Province (Grant No. Y6110687), Social Science Foundation of Zhejiang Province (Grant No. 10CGGL14YB).

REFERENCES

1. Bang, O., W. Krolikowski, J. Wyller, and J. J. Rasmussen, "Collapse arrest and soliton stabilization in nonlocal nonlinear media," *Phys. Rev. E*, Vol. 66, 046619, 2002.
2. Yakimenko, A. I., V. M. Lashkin, and O. O. Prikhodko, "Dynamics of two-dimensional coherent structures in nonlocal nonlinear media," *Phys. Rev. E*, Vol. 73, 066605, 2006.
3. Buccoliero, D., A. S. Desyatnikov, W. Krolikowski, and Y. S. Kivshar, "Spiraling multivortex solitons in nonlocal nonlinear media," *Opt. Lett.*, Vol. 33, No. 2, 198–200, 2008.
4. Snyder, A. W. and D. J. Mitchell, "Accessible solitons," *Science*, Vol. 276, No. 5318, 1538–1541, 1997.
5. Buccoliero, D., A. S. Desyatnikov, W. Krolikowski, and Y. S. Kivshar, "Laguerre and hermite soliton clusters in nonlocal nonlinear media," *Phys. Rev. Lett.*, Vol. 98, No. 5, 053901, 2007.
6. Lopez-Aguayo, S., A. S. Desyatnikov, and Y. S. Kivshar, "Azimuthons in nonlocal nonlinear media," *Opt. Express*, Vol. 14, No. 17, 7903, 2006.
7. Królikowski, W. and O. Bang, "Solitons in nonlocal nonlinear

media: Exact solutions," *Phys. Rev. E*, Vol. 63, No. 1, 016610, 2000.

8. Lopez-Aguayo, S. and J. C. Gutiérrez-Vega, "Elliptically modulated self-trapped singular beams in nonlocal nonlinear media: ellipticons," *Opt. Express*, Vol. 15, No. 26, 18326–18338, 2007.
9. Biswas, A., "Temporal-soliton solution of the complex Ginzburg-Landau equation with power law nonlinearity," *Progress In Electromagnetics Research*, Vol. 96, 1–7, 2009.
10. Mitatha, S., "Dark soliton behaviors within the nonlinear micro and nanoring resonators and applications," *Progress In Electromagnetics Research*, Vol. 99, 383–404, 2009.
11. Chen, R. P. and C. H. R. Ooi, "Evolution and collapse of a Lorentz beam in Kerr medium," *Progress In Electromagnetics Research*, Vol. 121, 39–52, 2011.
12. Gharakhili, F. G., M. Shahabadi, and M. Hakkak, "Bright and dark soliton generation in a left-handed nonlinear transmission line with series nonlinear capacitors," *Progress In Electromagnetics Research*, Vol. 96, 237–249, 2009.
13. Khalique, C. M. and A. Biswas, "Optical solitons with parabolic and dual-power nonlinearity via lie symmetry analysis," *Journal of Electromagnetic Waves and Applications*, Vol. 23, No. 7, 963–973, 2009.
14. Casperson, L. W. and A. A. Tovar, "Hermite-sinusoidal-Gaussian beams in complex optical systems," *J. Opt. Soc. Am. A*, Vol. 15, No. 4, 954–961, 1998.
15. Chen, R. P., Y. Ni, and X. X. Chu, "Propagation of a cos-Gaussian beam in a Kerr medium," *Opt. & Laser Tech.*, Vol. 43, No. 3, 483–487, 2011.
16. Tovar, A. A. and L. W. Casperson, "Production and propagation of Hermite-sinusoidal-Gaussian laser beams," *J. Opt. Soc. Am. A*, Vol. 15, No. 9, 2425–2432, 1998.
17. Chen, R. P., H. P. Zheng, and X. X. Chu, "Propagation properties of a sinh-Gaussian beam in a Kerr medium," *Appl. Phys. B*, Vol. 102, 695–698 2011.
18. Chu, X. X., "Propagation of a cosh-Gaussian beam through an optical system in turbulent atmosphere," *Opt. Express*, Vol. 15, No. 26, 17613–17618, 2007.
19. Eyyuboğlu, H. T. and Y. Baykal, "Reciprocity of cos-Gaussian and cosh-Gaussian laser beams in turbulent atmosphere," *Opt. Express*, Vol. 12, 4659–4674, 2004.

20. Chen, R. P. and C. Q. Dai, "The effect of Kerr nonlinearity on a cosh-Gaussian beam," *Laser in Engineering*, Vol. 19, 255–264, 2009.
21. Pérez-Garćia V. M., P. Torres, and G. D. Montesinos, "The method of moments for nonlinear schrödinger equations: Theory and applications," *SIAM J. Appl. Math.*, Vol. 67, No. 4, 990–1015, 2007.
22. Garćia-Ripoll, J. J., V. M. Pérez-Garćia, and P. Torres, "Extended parametric resonances in nonlinear schrödinger systems," *Phys. Rew. Lett.*, Vol. 83, No. 9, 1715, 1999.
23. Vlasov, S. N., V. A. Petrishchev, and V. I. Talanov, "Averaged description of wave beams in linear and nonlinear media (the method of moments)," *Radio. Quan. Electron.*, Vol. 14, No. 9, 1062–1070, 1971.
24. Chen, R. P., K. H. Chew, and S. He, "Dynamic control of collapse in a vortex airy beam," *Scientific Reports*, Vol. 3, 1406, 2013.
25. Chen, R. P., C. F. Yin, X. X. Chu, and H. Wang, "Effect of Kerr nonlinearity on an airy beam," *Phys. Rev. A*, Vol. 82, No. 4, 043832, 2010.
26. Ermakov, V. P., "Second-order differential equations: Conditions of complete integrability," *Appl. Anal. Discrete Math.*, Vol. 2, 123–145, 2008.
27. Chen, R. P. and H. T. Wang, "Propagation of Laguerre-Gaussian beams in cubic-quintic nonlinear media by variational approach," *Opt. & Laser Tech.*, Vol. 42, 1318–1322, 2010.
28. Lu, D. Q., W. Hu, and Q. Guo, "The relation between optical beam propagation in free space and in strongly nonlocal nonlinear media," *Eur. Phys. Lett.*, Vol. 86, 44004, 2009.