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MINIMUM Q FOR LOSSY AND LOSSLESS ELECTRICALLY SMALL DIPOLE ANTENNAS

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Abstract—General expressions for the quality factor (Q) of antennas are minimized to obtain lower-bound formulas for the Q of electrically small, lossy or lossless, combined electric and magnetic dipole antennas confined to an arbitrarily shaped volume. The lower-bound formulas for Q are derived for dipole antennas with specified electric and magnetic dipole moments excited by both electric and magnetic surface currents as well as by electric surface currents alone. With either excitation, separate formulas are found for the dipole antennas containing only lossless or "nondispersive-conductivity" material and for the dipole antennas containing "highly dispersive lossy" material. The formulas involve the quasi-static electric and magnetic polarizabilities of the associated perfectly conducting volume of the antenna, the ratio of the powers radiated by the specified electric and magnetic dipole moments, and the efficiency of the antenna.

1. INTRODUCTION

The lower bounds on the quality factor (Q) of antennas obtained by Wheeler [1–3] and Chu [4] in the 1940s and 1950s, and in 1960 by Harrington [5], were based on circuit models for spherical and circular cylindrical wave functions. This circuit-model approach to finding the minimum Q has the advantage of reducing a complex

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Invited paper dedicated to the memory of Robert E. Collin.

problem in electromagnetic theory to the systematic investigation of a ladder network of RLC circuits [6, 7]. However, the circuit models were restricted to representing spherical and circular cylindrical modes and thus the lower bounds on Q were for spheres or circular cylinders circumscribing the antennas.

In 1964, Robert E. Collin, to whom this issue of PIER is dedicated, and S. Rothschild evaluated the Q of antennas by judiciously subtracting the infinite energy of the radiation field from the infinite energy of the total field of antennas to obtain a finite "reactive energy" of the antenna [8]. Although Collin and Rothschild also limited their method to finding the lower bounds on the Q for spherical and circular cylindrical volumes, their work provides the fundamental understanding for not only the general definitions of quality factor for any antenna [9–11] but also for the lower bounds on the Q for antennas confined to an arbitrarily shaped volume [12, 13].

The primary purpose of the present paper is to generalize the lower-bound formulas obtained in [13, 14] for a single electric-dipole or magnetic-dipole, lossless antenna to a lossy electric and magnetic dipole antenna, with specified electric and magnetic dipole moments, $\mathbf{p}$ and $\mathbf{m}$, confined to an arbitrarily shaped volume $\mathcal{V}_a$. A secondary purpose of the paper is to correct the error, found by Jonsson and Gustafsson [15], that was made in the derivation of one of the main lower-bound formulas in [13] and resulted in that formula applying exactly to only ellipsoidal volumes. (All the comparisons that were made in [13] with the sum-rule lower bounds of Gustafsson, Sohl, and Kristensson [12, 16] showed excellent agreement because the comparisons were made for different shaped ellipsoids.)†

The Q lower-bound formulas derived in this paper, like those in [13, 14], are limited to electrically small antennas with $ka \lesssim 5$, where a is the minimum circumscribing radius of the antenna volume $\mathcal{V}_a$ and k is the free-space wavenumber. Recently, Gustafsson and Nordebo [17] have obtained lower bounds on Q for larger antennas using the "convex optimization" of current distributions [18]. Also, Thal [7] has obtained lower bounds on the Q of spherical electric and magnetic dipole antennas under subsidiary conditions that maximize the gain of the antenna and increase the lower bounds by requiring extra internal tuning to maintain the proper phase between the single-port electric currents that feed the electric and magnetic dipoles. For example, Thal has shown that the lower-bound formula for the Q of an electrically small spherical Huygens source (equal-power, perpendicular, cophasal

† In the paragraph preceding Section 2.1 of [13], it was erroneously stated that the Q of an electric dipole moment perpendicular to the plane of a thin oblate spheroid approaches ∞, whereas it actually approaches $9\pi/[4(ka_s)^3]$, where a_s is the radius of the oblate spheroid.

electric and magnetic dipole moments) is twice that of the general lower bound on Q derived for equal-power electric and magnetic dipoles (the general lower bound occurring when the electric and magnetic dipole moments are 90 degrees out of phase so that their feed currents are cophasal) because extra internal tuning (in addition to the tuning necessary to make the input reactance of the antenna zero) is needed to maintain the required 90-degree phase difference between the electric currents feeding the cophasal electric and magnetic dipole moments. In the present paper, the general lower bounds on Q are determined for electric and magnetic dipole antennas, allowing for both electric and magnetic surface currents, without regard for extra internal tuning that may be required to maintain the phase differences between the given electric and magnetic dipole moments. Lower bounds on Q are also determined for electrically small, electric and magnetic dipole antennas excited by electric surface currents only. These "electric-current lower bounds" are equal to or greater than the general lower bounds obtained for electric and magnetic dipole antennas excited by electric and magnetic surface currents (magnetization being equivalent to magnetic current).

2. GENERAL EXPRESSIONS FOR QUALITY FACTOR

In [11] expressions for internal energy density were derived from Maxwell's equations and their frequency derivatives in order to determine a quality factor of antennas that is approximately equal to twice the inverse of the matched VSWR (voltage standing wave ratio) half-power fractional impedance bandwidth of antennas. In particular, the Q of a one-port, linear, passive, lossy or lossless antenna tuned at a frequency ω (so that the input reactance $X(\omega) = 0$) to resonance ($X'(\omega) > 0$) or antiresonance ($X'(\omega) < 0$) was given in [11] as

$$Q(\omega) = \frac{\omega|W(\omega)|}{P_A(\omega)} = \eta\frac{\omega|W(\omega)|}{P_R(\omega)} \tag{1}$$

where the power accepted $P_A(\omega)$ by the antenna is the power radiated by the antenna plus power lost in the antenna ($P_A = P_R + P_L = P_R/\eta$, where η is the radiation efficiency; for lossless antennas, $\eta = 1$) and the internal energy is found from

$$W(\omega) = W_e(\omega) + W_m(\omega) + W_{me}(\omega) \tag{2}$$

with

$$W_e = \frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)}\mathrm{Re}[\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}]d\mathcal{V} - \epsilon_0 r\int\limits_{4\pi}|\mathbf{F}|^2 d\Omega\right] \tag{3a}$$

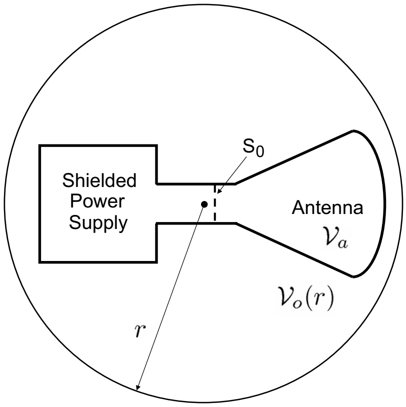

Figure 1. One-port, linear, passive antenna with feed and shielded power supply.

$$W_m = \frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)}\mathrm{Re}[\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H}]d\mathcal{V} - \epsilon_0 r\int\limits_{4\pi}|\mathbf{F}|^2 d\Omega\right] \quad (3b)$$

$$W_{me} = \frac{1}{4}\int\limits_{\mathcal{V}_a}\mathrm{Re}\left\{\mathbf{E}\cdot[\omega(\bar{\boldsymbol{\nu}}_t+\bar{\boldsymbol{\tau}}^*)]'\cdot\mathbf{H}^*\right\}d\mathcal{V}. \quad (3c)$$

Stars ($*$) denote the complex conjugate, and primes ($'$) denote differentiation with respect to the angular frequency ω. The vectors ($\mathbf{E}$, $\mathbf{D}$) and ($\mathbf{B}$, $\mathbf{H}$) are the usual time-harmonic ($e^{-i\omega t}$, $\omega > 0$) Maxwellian electric and magnetic fields related by bianisotropic constitutive parameters

$$\mathbf{D}(\mathbf{r}) = \bar{\boldsymbol{\epsilon}}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}) + \bar{\boldsymbol{\tau}}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r}) \quad (4a)$$
$$\mathbf{B}(\mathbf{r}) = \bar{\boldsymbol{\mu}}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r}) + \bar{\boldsymbol{\nu}}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}) \quad (4b)$$

where $\bar{\boldsymbol{\epsilon}}(\mathbf{r})$, $\bar{\boldsymbol{\mu}}(\mathbf{r})$, and $[\bar{\boldsymbol{\nu}}(\mathbf{r}), \bar{\boldsymbol{\tau}}(\mathbf{r})]$ are the spatially nondispersive permittivity dyadic, permeability dyadic, and magneto-electric dyadics, respectively. Like the fields, they are, in general, functions of frequency ω and position $\mathbf{r}$ within the media.[‡]

As shown in Fig. 1, $\mathcal{V}_a$ is the volume of the antenna material that lies outside the shielded power supply and the feed waveguide reference plane S_0. Any tuning elements are included in $\mathcal{V}_a$. For the purpose of

[‡] We restrict our attention to the quality factor of linear, passive antennas at isolated resonances (or antiresonances) and thus ignore increases in bandwidth that can be achieved, in principle, with overlapping multiresonances [19–21] or with nonlinear and/or active devices [22].

defining an isolated antenna volume $\mathcal{V}_a$ in free space, we can assume that an arbitrarily small shielded power supply is contained within $\mathcal{V}_a$. The surface S_a of the volume $\mathcal{V}_a$ contains the waveguide reference plane as a subsurface. The volume $\mathcal{V}_o(r)$, which includes the volume $\mathcal{V}_a$, is the entire volume outside the shielded power supply and reference plane S_0 out to a large sphere in free space of radius r that surrounds the antenna system. As $r \to \infty$, the volume $\mathcal{V}_o(r)$ becomes infinite. The solid angle integration element is $d\Omega = dS/r^2 = \sin\theta d\theta d\phi$ with (r, θ, ϕ) being the usual spherical coordinates of the position vector $\mathbf{r}$, and the complex far electric field pattern $\mathbf{F}(\theta, \phi)$ is defined by

$$\mathbf{F}(\theta, \phi) = \lim_{r\to\infty} re^{-ikr}\mathbf{E}(\mathbf{r}) \tag{5}$$

where $k = \omega/c = 2\pi/\lambda$ with c being the free-space speed of light and λ the free-space wavelength.

For the following simple scalar constitutive relations

$$\mathbf{D} = (\epsilon_r + i\epsilon_i)\mathbf{E}, \quad \mathbf{B} = (\mu_r + i\mu_i)\mathbf{H}, \quad (\bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\tau}} = 0) \tag{6}$$

the magnetic, electric, and magneto-electric internal energies in (3a)–(3c) reduce to

$$W_e = \frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)} (\omega\epsilon_r)'|\mathbf{E}|^2 d\mathcal{V} - \epsilon_0 r \int\limits_{4\pi} |\mathbf{F}|^2 d\Omega\right] \tag{7a}$$

$$W_m = \frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)} (\omega\mu_r)'|\mathbf{H}|^2 d\mathcal{V} - \epsilon_0 r \int\limits_{4\pi} |\mathbf{F}|^2 d\Omega\right] \tag{7b}$$

$$W_{me} = 0. \tag{7c}$$

We showed in [11] that the Q in (1), which depends on the definition of the internal energy in (3) or (7), was approximately equal to twice the inverse of the matched VSWR half-power fractional bandwidth (FBW_{hp}), that is

$$Q(\omega) = \eta\frac{\omega|W(\omega)|}{P_R(\omega)} \approx \frac{2}{FBW_{hp}(\omega)} \tag{8}$$

for a sufficiently isolated resonance or antiresonance with $Q \gg 1$ ($Q \gtrsim 2$ often suffices), except when the antenna is dominated by lossy dispersive materials; see, for example, [11, Fig. 19]. However, it is noted that hypothetical materials with conductivities ($\sigma_e \geq 0$, $\sigma_m \geq 0$) independent of frequency such that $\epsilon_i(\omega) = \epsilon_{ei}(\omega) + \sigma_e/\omega$ and $\mu_i(\omega) = \mu_{mi}(\omega) + \sigma_m/\omega$, *where* $\epsilon_{ei}(\omega)$ *and* $\mu_{mi}(\omega)$ *are equal to or* *greater than zero for all frequencies as well as equal to zero in a frequency window (band) about the* ω *of interest*, should not be included in the exceptions because these nondispersive conductivities do not affect the internal energy [23]. They merely change the efficiency η in (8) to maintain the high accuracy of the inverse relationship in (8) between bandwidth and Q. This is further corroborated in the improved formulas (11) where a frequency independent σ_e or σ_m does not contribute to the Q-energy because $(\omega\epsilon_i)' = \sigma_e' = 0$ and $(\omega\mu_i)' = \sigma_m' = 0$. We shall refer to antennas containing material characterized by a scalar permittivity $\epsilon(\omega) = \epsilon_r(\omega)+i\epsilon_{ei}(\omega)+i\sigma_e/\omega$ and a scalar permeability $\mu(\omega) = \mu_r(\omega)+i\mu_{mi}(\omega)+i\sigma_m/\omega$ with constant σ_e and σ_m for all frequencies as *nondispersive-conductivity* antennas (or antennas containing *nondispersive-conductivity* material). All other antennas or antenna material will be referred to as *highly dispersive lossy* antennas or antenna material. The μ_r, μ_i, ϵ_r, ϵ_i (σ_e, σ_m, ϵ_{ei}, μ_{mi}) can all be functions of the position vector $\mathbf{r}$.

For antennas containing materials with highly dispersive conductivities, it was found in [24] that a quality-factor energy[§] or simply Q-energy $|W^Q(\omega)|$, given in the following formulas, proves to be a replacement for $|W(\omega)|$ that produces a Q that maintains the accuracy of the relationship between Q and matched VSWR half-power fractional bandwidth in (8)

$$Q(\omega) = \eta\frac{\omega|W^Q(\omega)|}{P_R(\omega)} \tag{9}$$

§ The term "quality-factor energy" or simply "Q-energy" was introduced in [24] as an alternative to the term "internal energy" to describe the generalized formulas applied to highly dispersive lossy media because these formulas involve dispersive dissipative energy as well as stored energy per unit volume. Regardless of the terminology, the purpose is to define energy densities, which when integrated, will produce a total Q-energy that determines with reasonable accuracy the inverse-bandwidth Q of antennas including those that contain highly dispersive lossy materials. The quality-factor energy densities defined here are not circuit-model dependent [25, 26] but depend only on the macroscopic constitutive parameters and fields of the antenna media (and thus are useful for antenna design). The Q-energy differs considerably from both the equivalent-circuit energy of Tretyakov [25, 26] and the energy obtained by Vorobyev [27] in determining electromagnetic wave velocities in lossy dispersive material, but only slightly from the magnetic electrodynamic energy of Boardman and Marinov [28] in the frequency range where their magnetic energy is positive.

with

$$W^Q(\omega)=\frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)}\left\{\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}+\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H}\right.\right.$$
$$\left.\left.+\left[\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\tau}})'\cdot\mathbf{H}+\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\nu}})'\cdot\mathbf{E}\right]\right\}d\mathcal{V}-2\epsilon_0 r\int\limits_{4\pi}|\mathbf{F}|^2 d\Omega\right] \quad (10)$$

instead of (2)–(3),[‖] and $W^Q(\omega)=W_e^Q(\omega)+W_m^Q(\omega)+W_{me}^Q(\omega)$ with

$$W_e^Q = \frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)}(\omega\epsilon)'|\mathbf{E}|^2 d\mathcal{V}-\epsilon_0 r\int\limits_{4\pi}|\mathbf{F}|^2 d\Omega\right] \quad (11a)$$

$$W_m^Q = \frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)}(\omega\mu)'|\mathbf{H}|^2 d\mathcal{V}-\epsilon_0 r\int\limits_{4\pi}|\mathbf{F}|^2 d\Omega\right] \quad (11b)$$

$$W_{me}^Q = 0 \quad (11c)$$

instead of (7). We note that for highly dispersive lossy material, the values of the real and imaginary parts of $(\omega\epsilon)'$ and $(\omega\mu)'$ can be less than ϵ_0 and μ_0, respectively, and even less than or equal to zero.

If the medium is lossless ($\eta = 1$) *in a frequency window* about ω, not only does $\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}_t^*$, $\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}_t^*$, and $\bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\tau}}_t^*$, but also $\bar{\boldsymbol{\epsilon}}' = \bar{\boldsymbol{\epsilon}}_t'^*$, $\bar{\boldsymbol{\mu}}' = \bar{\boldsymbol{\mu}}_t'^*$, and $\bar{\boldsymbol{\nu}}' = \bar{\boldsymbol{\tau}}_t'^*$ (subscript t denoting the transpose). Then (10) and (11) reduce to (2)–(3) and (7), respectively. This can be proven by showing that the imaginary parts of $\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}$, $\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H}$, and $\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\tau}})'\cdot\mathbf{H}+\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\nu}})'\cdot\mathbf{E}$ are zero. In addition, we have [14, Eqs. (44) and (46)]

$$\mathrm{Re}\left\{\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H}+\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}+\mathbf{E}\cdot[(\omega(\bar{\boldsymbol{\nu}}_t+\bar{\boldsymbol{\tau}}^*)]'\cdot\mathbf{H}^*\right\}$$
$$\geq\mu_0|\mathbf{H}|^2+\epsilon_0|\mathbf{E}|^2 \quad (12a)$$

$$\mathrm{Re}\left[\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}\right]=\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}\geq\epsilon_0|\mathbf{E}|^2 \quad (12b)$$

$$\mathrm{Re}\left[\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H}\right]=\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H}\geq\mu_0|\mathbf{H}|^2 \quad (12c)$$

[‖] The $[\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\tau}})'\cdot\mathbf{H}+\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\nu}})'\cdot\mathbf{E}]$ term in (10) was mistakenly written as $|\mathbf{E}\cdot[(\omega(\bar{\boldsymbol{\nu}}_t+\bar{\boldsymbol{\tau}}^*)]'\cdot\mathbf{H}^*|$ in [24, Eqs. (13) and (17a)]. Also, the absolute value signs in [24, Eqs. (13) and (17a)], as well as in [14, Eqs. (59)–(60)], should have been placed outside the integrals as in (9)–(11).

in a lossless frequency window about ω. In lossless frequency windows, we also have the inequalities [14, Eqs. (39) and (42)]¶

$$(\omega\epsilon_r)' - \epsilon_0 \geq \frac{\omega\epsilon_r'}{2} \geq 0 \tag{14a}$$

$$(\omega\mu_r)' - \mu_0 \geq \frac{\omega\mu_r'}{2} \geq 0 \tag{14b}$$

for scalar permittivities and permeabilities.+

3. SIMPLIFICATIONS FOR ELECTRICALLY SMALL ANTENNAS

The expressions given in (1) or (9) do not define Q uniquely [11]. In particular, these expressions for Q depend on the position chosen for the origin of the coordinate system in which the integration of the fields are performed unless $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2 d\Omega = 0$ (for example, if the far-field pattern $|\mathbf{F}|$ is symmetric about the origin — as in the case of a single spherical multipole). Also, the Q of an antenna as defined in (1) or (9) can be increased with "surplus" capacitors and inductors without changing the input impedance and thus without changing the bandwidth of the antenna [32, p. 176]. Both of these ambiguities in the definition of Q can be dealt with effectively, if necessary, by the procedure given in [11, Section 4.6]. However, for electrically small antennas ($ka \lesssim 5$, a being the minimum circumscribing radius) where the quasi-static fields dominate the stored energy or Q-energy, the

¶ The inequalities in (14) are equivalent to those derived by Landau and Lifshitz [29, p. 287] from the first Kramers-Kronig causality relations

$$\epsilon_r(\omega) - \epsilon_0 = \frac{2}{\pi}\int_0^\infty \frac{\nu\epsilon_i(\nu)}{\nu^2-\omega^2}d\nu, \quad \mu_r(\omega) - \mu_0 = \frac{2}{\pi}\int_0^\infty \frac{\nu\mu_i(\nu)}{\nu^2-\omega^2}d\nu \tag{13}$$

given that $\epsilon_i(\nu)$ and $\mu_i(\nu)$ are zero in a frequency window about ω (and using the passivity conditions, $\epsilon_i(\nu) \geq 0$ and $\mu_i(\nu) \geq 0$). For hypothetical material with nondispersive conductivity ($\epsilon(\nu) = \epsilon_r(\nu) + i\epsilon_{ei}(\nu) + i\sigma_e/\nu$ and $\mu(\nu) = \mu_r(\nu) + i\mu_{mi}(\nu) + i\sigma_m/\nu$ with σ_e and σ_m independent of frequency, $\epsilon_{ei}(\nu) \geq 0$ and $\mu_{mi}(\nu) \geq 0$, and $\epsilon_{ei}(\nu) = 0$ and $\mu_{mi}(\nu) = 0$ in a frequency window about ω), one can subtract and add $\sigma_e/(\nu^2-\omega^2)$ in the first integrand of (13) and $\sigma_m/(\nu^2-\omega^2)$ in the second integrand of (13), then use the result that $⨍_0^\infty 1/(\nu^2-\omega^2)d\nu = 0$ to obtain integrands that are zero in the frequency band about ω so that the inequalities in (14) also hold for nondispersive-conductivity material.
+ For diamagnetic $\bar{\boldsymbol{\mu}}$ produced by induced, microscopic, Amperian magnetic dipole moments rather than by the alignment of initially randomly oriented permanent microscopic magnetic dipoles, the $\mu_0|\mathbf{H}|^2$ terms on the right-hand sides of (12a) and (12c) change to $|\mathbf{B}|^2/\mu_0$, and the μ_0 in (14b) changes to μ_r^2/μ_0 [30]. It is possible for (14b) to remain valid for lossless diamagnetic material if μ_0 is changed to $\mu_\infty = \mu(\omega = \infty)$ such that $\mu_r(\omega \to 0) \geq \mu_\infty$. However, this possibility is unrealistic for spatially nondispersive continua because the magnetic dipole moment of a given molecule or inclusion excited by a free-space incident field usually goes to zero as $\omega \to \infty$ and thus $\mu_\infty = \mu_0$ [31].

origin dependence of the expressions for Q is virtually eliminated by choosing the origin within the antenna volume so that the equation for the Q-energy in (10) reduces to

$$W^Q(\omega) \approx \frac{1}{4}\int\limits_{\mathcal{V}_\infty} \Big\{\mathbf{E}^* \cdot (\omega\bar{\boldsymbol{\epsilon}})' \cdot \mathbf{E} + \mathbf{H}^* \cdot (\omega\bar{\boldsymbol{\mu}})' \cdot \mathbf{H} + \big[\mathbf{E}^* \cdot (\omega\bar{\boldsymbol{\tau}})' \cdot \mathbf{H} + \mathbf{H}^* \cdot (\omega\bar{\boldsymbol{\nu}})' \cdot \mathbf{E}\big]\Big\} d\mathcal{V} \qquad (15)$$

and in (11) to

$$W_e^Q \approx \frac{1}{4}\int\limits_{\mathcal{V}_\infty} (\omega\epsilon)'|\mathbf{E}|^2 d\mathcal{V} \qquad (16a)$$

$$W_m^Q \approx \frac{1}{4}\int\limits_{\mathcal{V}_\infty} (\omega\mu)'|\mathbf{H}|^2 d\mathcal{V} \qquad (16b)$$

$$W_{me}^Q = 0 \qquad (16c)$$

where $\mathcal{V}_\infty$ denotes all space and the fields in (15)–(16) are the quasi-static fields of the electrically small antenna. Also, for the sake of obtaining lower bounds on Q, surplus capacitors and inductors are avoided. Neglecting the far-field terms in (15)–(16) for a spherical volume $\mathcal{V}_a$ with the origin at the center of the sphere amounts to neglecting terms in Q of order ka [11, Eqs. (C4)–(C5)].

These simplified equations can be rewritten in terms of integrations over the volume $\mathcal{V}_a$ of the material of the antenna and the free space outside the antenna, $\mathcal{V}_{out} = \mathcal{V}_\infty - \mathcal{V}_a$; for example, (15) can be rewritten as

$$W^Q(\omega) \approx \frac{1}{4}\int\limits_{\mathcal{V}_a} \Big\{\mathbf{E}^* \cdot (\omega\bar{\boldsymbol{\epsilon}})' \cdot \mathbf{E} + \mathbf{H}^* \cdot (\omega\bar{\boldsymbol{\mu}})' \cdot \mathbf{H} + \big[\mathbf{E}^* \cdot (\omega\bar{\boldsymbol{\tau}})' \cdot \mathbf{H} + \mathbf{H}^* \cdot (\omega\bar{\boldsymbol{\nu}})' \cdot \mathbf{E}\big]\Big\} d\mathcal{V} + \frac{1}{4}\int\limits_{\mathcal{V}_{out}} \left(\epsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2\right) d\mathcal{V}. \qquad (17)$$

For electrically small dipole antennas, the quasi-static electric and magnetic fields, $\mathbf{E}_e$ and $\mathbf{H}_m$, produced by the sources of the electric and magnetic dipole moments, respectively, dominate the fields outside the antenna volume $\mathcal{V}_a$, so that (17) becomes

$$W^Q(\omega) \approx \frac{1}{4}\int\limits_{\mathcal{V}_a}\Big\{\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}+\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H}+\big[\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\tau}})'\cdot\mathbf{H} +\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\nu}})'\cdot\mathbf{E}\big]\Big\}d\mathcal{V}+\frac{1}{4}\int\limits_{\mathcal{V}_{out}}\left(\epsilon_0|\mathbf{E}_e|^2+\mu_0|\mathbf{H}_m|^2\right)d\mathcal{V} \quad (18)$$

and the quality factor from (9) is

$$Q(\omega)\approx\frac{\eta\omega}{4(P_e+P_m)}\left|\int\limits_{\mathcal{V}_a}\Big\{\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}+\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H} +\big[\mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\tau}})'\cdot\mathbf{H}+\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\nu}})'\cdot\mathbf{E}\big]\Big\}d\mathcal{V}+\int\limits_{\mathcal{V}_{out}}\left(\epsilon_0|\mathbf{E}_e|^2+\mu_0|\mathbf{H}_m|^2\right)d\mathcal{V}\right| \quad (19)$$

where P_e and P_m are the powers radiated by the electric and magnetic dipole moments, $\mathbf{p}$ and $\mathbf{m}$, respectively.

Since the antenna is assumed to be tuned at the frequency ω, that is, its input reactance is zero, we also have the relationship [11, Eq. (53)]

$$X(\omega)=\frac{\omega}{|I(\omega)|^2}\mathrm{Re}\int\limits_{\mathcal{V}_\infty}\left[\mathbf{H}\cdot\bar{\boldsymbol{\mu}}^*\cdot\mathbf{H}^*-\mathbf{E}\cdot\bar{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}^*+\mathbf{H}\cdot(\bar{\boldsymbol{\nu}}^*-\bar{\boldsymbol{\tau}}_t)\cdot\mathbf{E}^*\right]d\mathcal{V}=0 \quad (20a)$$

or

$$\mathrm{Re}\int\limits_{\mathcal{V}_a}\left[\mathbf{H}\cdot\bar{\boldsymbol{\mu}}^*\cdot\mathbf{H}^*-\mathbf{E}\cdot\bar{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}^*+\mathbf{H}\cdot(\bar{\boldsymbol{\nu}}^*-\bar{\boldsymbol{\tau}}_t)\cdot\mathbf{E}^*\right]d\mathcal{V} \approx \int\limits_{\mathcal{V}_{out}}\left(\epsilon_0|\mathbf{E}_e|^2-\mu_0|\mathbf{H}_m|^2\right)d\mathcal{V} \quad (20b)$$

where $I(\omega)$ is the input current to the antenna. The approximations in (19) and (20b) become increasingly more accurate with decreasing electrical size of the antenna. A comparison of (20) and (3) shows that the input reactance is not necessarily proportional to the differences of electric and magnetic energies (and differences between magneto-electric energies) unless the constitutive parameters are temporally nondispersive.

4. EXPRESSIONS FOR MINIMIZING THE QUALITY FACTOR

Imagine that we have found the minimum Q in (19) for a given $\mathcal{V}_a$ and given values of the efficiency η and the ratio of the powers radiated by the given electric and magnetic dipole moments, $\mathbf{p}$ and $\mathbf{m}$. One could then replace the volume of this minimum-Q antenna with equivalent electric and magnetic surface currents on the surface S_a of $\mathcal{V}_a$ to keep the fields outside $\mathcal{V}_a$ the same while reducing the fields inside $\mathcal{V}_a$ to zero. Thus, this thought experiment using the "extinction theorem" appears to indicate that the minimum-Q antenna has zero fields inside $\mathcal{V}_a$. However, all the fields inside $\mathcal{V}_a$ cannot be zero, because if the fields inside $\mathcal{V}_a$ are zero, the zero-reactance condition in (20b) cannot be satisfied (unless the right-hand side of (20b) is equal to zero).

4.1. Highly Dispersive Lossy Antennas

Despite the foregoing considerations, *suppose the fields inside $\mathcal{V}_a$ are first made to equal zero by means of equivalent electric and magnetic surface currents*, and that the right-hand side of (20b) is greater than zero. Then a tuning inductor with a core of scalar permeability $\mu(\omega) = \mu_r(\omega) + i\mu_i(\omega)$ could be added to $\mathcal{V}_a$ such that

$$\int\limits_{\mathcal{V}_a} \mu_r |\mathbf{H}|^2 d\mathcal{V} = \int\limits_{\mathcal{V}_{out}} \left(\epsilon_0 |\mathbf{E}_e|^2 - \mu_0 |\mathbf{H}_m|^2\right) d\mathcal{V}. \tag{21}$$

Furthermore, assume a $\mu(\omega)$ satisfying $(\omega\mu)' = 0$ so that the integral over $\mathcal{V}_a$ in (19) is zero; that is

$$\int\limits_{\mathcal{V}_a} (\omega\mu)' |\mathbf{H}|^2 d\mathcal{V} = 0. \tag{22}$$

A permeability with a nondispersive magnetic conductivity would disallow (22) because of the inequality in (14b) along with the footnotes associated with (14). Nonetheless, it is conceivable that using magnetic material with highly dispersive lossy permeability, the integral over $\mathcal{V}_a$ in (22) can be made zero (at the single frequency of interest ω)* while maintaining the tuned condition in (21). Similarly, if the right-hand side of (20b) is less than zero, a dielectric material with a highly dispersive lossy permittivity could presumably tune the antenna (at the single frequency of interest ω) while increasing the Q-energy negligibly.

* For example, the Lorentzian permeability or permittivity function, $f(\omega) = B\{1+2b^2/[1-(\omega/\omega_0)^2 - 2ib(\omega/\omega_0)]\}$ has $(\omega f)' = 0$ at $\omega = \omega_0$.

In either case, one obtains the following expression for the minimum value of Q in (19) for a highly dispersive lossy electric and magnetic dipole antenna

$$Q_{lb}^{hdl}(\omega) = \eta\omega \mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_{out}} \left(\epsilon_0|\mathbf{E}_e|^2 + \mu_0|\mathbf{H}_m|^2\right) d\mathcal{V}}{4(P_e + P_m)}\right] \tag{23}$$

minimized for a fixed η and power ratio P_m/P_e of the electric and magnetic dipole moments, $\mathbf{p}$ and $\mathbf{m}$ (on $\mathcal{V}_a$), which are assumed given to within a constant factor. The subscript "*lb*" stands for "lower bound" and the superscript "*hdl*" stands for "highly dispersive lossy" constitutive parameters. The minimization problem in (23) is equivalent to minimizing the Q of a dipole antenna with zero fields inside the volume $\mathcal{V}_a$ but without the requirement of tuning the antenna.

The minimization of the quotient in (23) involving the integral over the quasi-static fields is done assuming that the shape of $\mathcal{V}_a$ is given along with the radiation efficiency η and power ratio P_m/P_e for the electric and magnetic dipole moments, $\mathbf{p}$ and $\mathbf{m}$, specified to within a constant factor (since $\mathbf{p}$ and $\mathbf{m}$ are proportional to the voltage or current applied to the antenna). There is an infinite set of different electric and magnetic surface-current distributions on the surface S_a of $\mathcal{V}_a$ that will produce zero fields inside $\mathcal{V}_a$ and radiate predominantly electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$. Each surface-current distribution will produce approximately the same dipolar fields a wavelength or so outside the sphere that circumscribes the given volume $\mathcal{V}_a$. However, between the surface S_a of $\mathcal{V}_a$ and a wavelength or so outside the surface of the circumscribing sphere, different surface-current distributions can produce very different fields and thus very different values of the integral in (23) and very different values of the quotient in the square brackets of (23). The Q lower-bound is given by the surface-current distribution that produces the minimum value of this quotient. This minimum value will be determined in Section 5.

4.2. Lossless and Nondispersive-Conductivity Dipole Antennas

For lossless or nondispersive-conductivity antennas with conductivities ($\sigma_e \geq 0, \sigma_m \geq 0$) independent of frequency such that $\epsilon_i(\omega) = \epsilon_{ei}(\omega) + \sigma_e/\omega$ and $\mu_i(\omega) = \mu_{mi}(\omega) + \sigma_m/\omega$, where $\epsilon_{ei}(\omega)$ and $\mu_{mi}(\omega)$ are equal to or greater than zero for all frequencies as well as equal to zero in a frequency window (band) about the ω of interest, the inequalities in (14) do not allow the integral in (22) to equal zero. In fact, we shall now show that for antennas with nondispersive-conductivity materials (which include lossless materials as a subset)

$$(\omega\mu_r)' \geq \mu_r \tag{24a}$$
$$(\omega\epsilon_r)' \geq \epsilon_r \tag{24b}$$

in the above stated frequency window. Because of (21) and the corresponding equation with ϵ_r for a tuning capacitor, the values of μ_r and ϵ_r must be positive. To prove (24a), assume the contradiction; that is, $\mu_r > (\omega\mu_r)'$ in the above stated frequency window. From this inequality, we find $\omega\mu_r' < 0$, which violates the second inequality in (14b) and, thus, (24a) holds; similarly for (24b).

Combining the inequality in (24a) with the equations in (21) and (22) yields for the tuning inductor

$$\int\limits_{\mathcal{V}_a}(\omega\mu)'|\mathbf{H}|^2 d\mathcal{V} \geq \int\limits_{\mathcal{V}_a}\mu_r|\mathbf{H}|^2 d\mathcal{V} = \int\limits_{\mathcal{V}_{out}}\left(\epsilon_0|\mathbf{E}_e|^2 - \mu_0|\mathbf{H}_m|^2\right) d\mathcal{V} \geq 0. \tag{25}$$

Consequently, the minimum value of Q in (19) for a nondispersive-conductivity electric and magnetic dipole antenna with $\int_{\mathcal{V}_{out}}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V} \geq \int_{\mathcal{V}_{out}}\mu_0|\mathbf{H}_m|^2 d\mathcal{V}$ is

$$Q^{nc}_{e,lb}(\omega) = \frac{\eta\omega}{2(1+P_m/P_e)}\mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_{out}}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V}}{P_e}\right] \tag{26a}$$

where the superscript "nc" stands for "nondispersive conductivity" (which includes the lossless case) and the subscript "e" denotes that the electric energy dominates.

Likewise, if the magnetic energy dominates, that is, $\int_{\mathcal{V}_{out}}\mu_0|\mathbf{H}_m|^2 d\mathcal{V} \geq \int_{\mathcal{V}_{out}}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V}$, then

$$Q^{nc}_{m,lb}(\omega) = \frac{\eta\omega}{2(1+P_e/P_m)}\mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_{out}}\mu_0|\mathbf{H}_m|^2 d\mathcal{V}}{P_m}\right]. \tag{26b}$$

The equations in (26) can also be rewritten in terms of $\mathrm{Max}[W_e^Q, W_m^Q]$ by noting that tuning the antenna amounts to replacing $\int_{\mathcal{V}_{out}}(\epsilon_0|\mathbf{E}_e|^2 + \mu_0|\mathbf{H}_e|^2)d\mathcal{V}/4$ with $2\mathrm{Max}[W_e^Q, W_m^Q]$. The criterion

$$\int\limits_{\mathcal{V}_{out}}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V} \geq \int\limits_{\mathcal{V}_{out}}\mu_0|\mathbf{H}_m|^2 d\mathcal{V} \quad \text{or} \quad \int\limits_{\mathcal{V}_{out}}\mu_0|\mathbf{H}_m|^2 d\mathcal{V} \geq \int\limits_{\mathcal{V}_{out}}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V}$$

does not necessarily imply that $P_e \geq P_m$ ($|\mathbf{p}|^2/\epsilon_0 \geq \mu_0|\mathbf{m}|^2$) or $P_m \geq P_e$ ($\mu_0|\mathbf{m}|^2 \geq |\mathbf{p}|^2/\epsilon_0$), respectively, except for the special case of a spherical volume $\mathcal{V}_a$.

5. DETERMINATION OF THE MINIMUM Q FOR DIPOLE ANTENNAS

In this section, we determine the minimum values of the dipolar expressions for quality factor in (23) and (26), beginning with the nondispersive-conductivity expressions in (26).

5.1. Minimization for the Quasi-static Electric Field of the Electric Dipole

The quasi-static electric-dipole electric field $\mathbf{E}_e$ of an electrically small electric and magnetic dipole antenna dominates the electric field both inside and outside the surface S_a of the antenna $\mathcal{V}_a$. Since the surface S_a contains the equivalent electric and magnetic currents that reduce the fields to zero inside $\mathcal{V}_a$, one can divide $\mathbf{E}_e$ into separate contributions, $\mathbf{E}_{e1}$ and $\mathbf{E}_{e2}$, respectively, from electric and magnetic surface currents on S_a; that is,

$$\mathbf{E}_e(\mathbf{r}) = \mathbf{E}_{e1}(\mathbf{r}) + \mathbf{E}_{e2}(\mathbf{r}) \tag{27}$$

such that $\mathbf{E}_e(\mathbf{r}) \approx 0$ for $\mathbf{r} \in \mathcal{V}_a$ with the approximation becoming more accurate as ka gets smaller. In the far field, both $\mathbf{E}_{e1}$ and $\mathbf{E}_{e2}$ represent electric dipoles with dipole moments that can be designated as

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2. \tag{28}$$

The total power radiated by these electric dipoles is given by [33, p. 437]

$$P_e = \frac{\omega k^3}{12\pi\epsilon_0}|\mathbf{p}|^2 = \frac{\omega k^3}{12\pi\epsilon_0}|\mathbf{p}_1 + \mathbf{p}_2|^2. \tag{29}$$

We also have from (27) that

$$\begin{aligned}\int\limits_{\mathcal{V}_{out}} \epsilon_0|\mathbf{E}_e|^2 d\mathcal{V} &= \epsilon_0 \int\limits_{\mathcal{V}_{out}} \left(|\mathbf{E}_{e1}|^2 + |\mathbf{E}_{e2}|^2 + 2\mathrm{Re}\left[\mathbf{E}_{e1} \cdot \mathbf{E}_{e2}^*\right]\right) d\mathcal{V} \\ &= \epsilon_0 \int\limits_{\mathcal{V}_{\infty}} \left(|\mathbf{E}_{e1}|^2 + |\mathbf{E}_{e2}|^2 + 2\mathrm{Re}\left[\mathbf{E}_{e1} \cdot \mathbf{E}_{e2}^*\right]\right) d\mathcal{V}. \end{aligned} \tag{30}$$

The second equality in (30) holds because $\mathbf{E}_e \approx 0$ inside $\mathcal{V}_a$. Substitution from (29) and (30) into (26a) gives

$$Q^{nc}_{e,lb}(\omega)=\frac{6\pi\epsilon_0^2\eta}{k^3(1+P_m/P_e)}\mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_\infty}\left(|\mathbf{E}_{e1}|^2+|\mathbf{E}_{e2}|^2+2\mathrm{Re}\left[\mathbf{E}_{e1}\cdot\mathbf{E}^*_{e2}\right]\right)d\mathcal{V}}{|\mathbf{p}_1+\mathbf{p}_2|^2}\right] \tag{31}$$

as an expanded minimization expression for the nondispersive-conductivity, electric-energy dominated ($W_e^Q \geq W_m^Q$) lower bound on Q for dipole antennas with electric and magnetic dipole moments, $\mathbf{p}$ and $\mathbf{m}$, radiating powers in the ratio P_m/P_e.

In order to minimize the ratio in (31), we ask what quasi-static fields incident upon the electrically small volume $\mathcal{V}_a$ will produce a scattered quasi-static electric field outside $\mathcal{V}_a$ with the least energy (represented by the integrals in (30) and (31) for a given electric dipole moment $\mathbf{p}$). The fields $\mathbf{E}_{e1}$ and $\mathbf{E}_{e2}$ are produced, respectively, by electric and magnetic surface currents on S_a and thus they can be induced, respectively, by an electric field $\mathbf{E}_{01}$ incident on a perfect electric conductor (PEC) filling the volume $\mathcal{V}_a$ and an electric field $\mathbf{E}_{02}$ incident on a perfect magnetic conductor (PMC) filling the volume $\mathcal{V}_a$. The total field is zero inside a PEC or a PMC. Therefore, it follows that the combined scattered field inside $\mathcal{V}_a$ is $-(\mathbf{E}_{01}+\mathbf{E}_{02})$, which must equal zero because inside $\mathcal{V}_a$ we have $\mathbf{E}_{e1} = -\mathbf{E}_{01}$, $\mathbf{E}_{e2} = -\mathbf{E}_{02}$, and $\mathbf{E}_e = \mathbf{E}_{e1} + \mathbf{E}_{e2} = 0$; that is, $\mathbf{E}_{01} + \mathbf{E}_{02} = 0$.

Divide either of these two incident electric fields within $\mathcal{V}_a$ into the sum of a uniform electric field, which is not a function of $\mathbf{r}$ in $\mathcal{V}_a$, and a nonuniform electric field whose spatial average over $\mathcal{V}_a$ is zero. The uniform electric field applied to the PEC or PMC in $\mathcal{V}_a$ will induce predominantly electric-dipole fields a fraction of a wavelength (typically $\lambda/(2\pi)$) outside the sphere that circumscribes $\mathcal{V}_a$, and reactive fields between the surface S_a of $\mathcal{V}_a$ and a fraction of a wavelength outside the circumscribing sphere. The nonuniform electric field applied to the PEC or PMC will induce predominantly quadrupole and higher-order multipole fields a fraction of a wavelength outside the circumscribing sphere, and reactive fields between S_a and a fraction of a wavelength outside the circumscribing sphere, but with a ratio of the energy stored in the reactive fields to the power radiated by the electric dipole that increases faster with decreasing electrical size of $\mathcal{V}_a$ than the same ratio for the fields induced by the uniform electric field [4]. In other words, for $ka \ll 1$, a nonuniform incident electric field will produce scattered fields that add significantly to the reactive energy while contributing negligible power to the radiated electric-dipole fields. Therefore, for $ka \ll 1$, *the minimum possible Q for an* *antenna confined to $\mathcal{V}_a$ will be obtained with spatially uniform incident electric fields*

$$\mathbf{E}_0 = \mathbf{E}_{01} = -\mathbf{E}_{02}. \tag{32}$$

The electric dipole moments, $\mathbf{p}_1$ and $\mathbf{p}_2$, are determined by the uniform electric field $\mathbf{E}_0$ under the conditions in (32).

The spatially uniform quasi-static electric field $\mathbf{E}_0$ must satisfy Maxwell's equations; in particular

$$\nabla \times \mathbf{E}_0 \;=\; 0 = i\omega\mathbf{B}_0 - \mathbf{J}_{m0} \tag{33a}$$

$$\nabla \times \mathbf{B}_0 \;=\; -i\omega\mu_0\epsilon_0\mathbf{E}_0 \Rightarrow \mathbf{B}_0 = i\omega\mu_0\epsilon_0\mathbf{r} \times \mathbf{E}_0/2. \tag{33b}$$

The hypothetical magnetic volume current density $\mathbf{J}_{m0}$ is required in the first Maxwell equation of (33a) to enable $\mathbf{B}_0(\mathbf{r})$ to satisfy the second Maxwell equation in (33b).

The quasi-static electric field $\mathbf{E}_{e1}$ induced by the PEC is produced by electric charge-current and thus satisfies the quasi-static equations

$$\nabla\times\mathbf{E}_{e1} = i\omega\mathbf{B}_{e1} = O\left(\omega^2\right) \Rightarrow \mathbf{E}_{e1} = -\nabla\psi + O\left(\omega^2\right) \tag{34a}$$

$$\nabla\cdot\mathbf{E}_{e1} = \sigma\delta(n-n_s)/\epsilon_0 \Rightarrow \nabla^2\psi = -\sigma\delta(n-n_s)/\epsilon_0 + O\left(\omega^2\right) \tag{34b}$$

where σ is the induced surface electric charge density on the surface of the PEC and ψ is the electric scalar potential. The variable n in the argument of the delta function is the normal coordinate for the family of surfaces parallel to S_a such that $n = n_s$ defines the surface S_a. The $O(\omega^2)$ term (which is of order ω^2 rather than ω because $\mathbf{B}_{e1}$ for an electric dipole approaches zero as ω [33, p. 436]) becomes negligible as $\omega \to 0$ or, alternatively, as $ka \to 0$. Then with the help of Green's first identity and (34), we have

$$\begin{aligned}
&\int\limits_{\mathcal{V}_\infty} |\mathbf{E}_{e1}|^2 d\mathcal{V}\\
&= \int\limits_{\mathcal{V}_\infty} |\nabla\psi|^2 d\mathcal{V} + O\left[(ka)^2\right] = -\int\limits_{\mathcal{V}_\infty} \psi^*\nabla^2\psi d\mathcal{V} + O\left[(ka)^2\right]\\
&= \frac{1}{\epsilon_0}\int\limits_{S_a} \psi^*\sigma dS + O\left[(ka)^2\right] = -\frac{1}{\epsilon_0}\int\limits_{S_a} \psi_0^*\sigma dS + O\left[(ka)^2\right]\\
&= \frac{1}{\epsilon_0}\mathbf{E}_0^*\cdot\int\limits_{S_a} \sigma\mathbf{r} dS + O\left[(ka)^2\right] = \frac{1}{\epsilon_0}\mathbf{E}_0^*\cdot\mathbf{p}_1 + O\left[(ka)^2\right] \overset{ka\to 0}{\to} \frac{1}{\epsilon_0}\mathbf{E}_0^*\cdot\mathbf{p}_1
\end{aligned} \tag{35}$$

where use has been made of the total electric charge on the PEC being zero ($\int_{S_a} \sigma dS = 0$) and the total electric scalar potential being constant on S_a, namely $\psi(\mathbf{r}) + \psi_0(\mathbf{r}) = \text{constant}$, $\mathbf{r} \in S_a$ with $\psi_0(\mathbf{r}) = -\mathbf{E}_0 \cdot \mathbf{r}$. (The surface integrals at an infinite radius in Green's first identity vanish because ψ decays as $1/r$ and thus $\psi^* \nabla \psi$ decays as $1/r^3$ as $r \to \infty$.)

Expressing the electric dipole moment $\mathbf{p}_1$ in terms of the the realvalued, symmetric [34] electric polarizability dyadic $\bar{\boldsymbol{\alpha}}_e$ of a PEC volume $\mathcal{V}_a$ in a uniform electric field $\mathbf{E}_{01} = \mathbf{E}_0$, namely

$$\mathbf{p}_1 = \epsilon_0 \bar{\boldsymbol{\alpha}}_e \cdot \mathbf{E}_0 \Rightarrow \mathbf{E}_0 = \bar{\boldsymbol{\alpha}}_e^{-1} \cdot \mathbf{p}_1 / \epsilon_0 \tag{36}$$

recasts (35) into the form

$$\int\limits_{\mathcal{V}_\infty} |\mathbf{E}_{e1}|^2 d\mathcal{V} = \mathbf{p}_1 \cdot \bar{\boldsymbol{\alpha}}_e^{-1} \cdot \mathbf{p}_1^* / \epsilon_0^2. \tag{37}$$

The electric polarizability of a PEC volume $\mathcal{V}_a$ is equal to the magnetic polarizability of a PMC volume $\mathcal{V}_a$. Note that for a diagonal electric-PEC (magnetic-PMC) polarizability dyadic [$\bar{\boldsymbol{\alpha}}_e = \sum_{j=1}^{3} \alpha_{ej} \hat{\mathbf{x}}_j \hat{\mathbf{x}}_j$, $\hat{\mathbf{x}}_j = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$], Eq. (37) implies that the scalar polarizabilities are equal to or greater than zero ($\alpha_{ej} \geq 0$, $j = 1, 2, 3$).

The quasi-static electric field $\mathbf{E}_{e2}$ induced by the PMC is produced by magnetic charge-current and thus satisfies the quasi-static equations

$$\begin{aligned} \nabla \times \mathbf{E}_{e2} &= i\omega \mathbf{B}_{e2} - \mathbf{K}_m \delta(n - n_s) \\ &= -\mathbf{K}_m \delta(n - n_s) + O\left[(ka)^2\right] \end{aligned} \tag{38a}$$

$$\begin{aligned} \mathbf{E}_{e2} &= -\frac{\nabla \times \mathbf{B}_{e2}}{(i\omega\mu_0\epsilon_0)} \Rightarrow \nabla \times \nabla \times \mathbf{B}_{e2} \\ &= i\omega\mu_0\epsilon_0 \mathbf{K}_m \delta(n - n_s) + O\left[(ka)^3\right] \end{aligned} \tag{38b}$$

where $\mathbf{K}_m$ is the induced surface magnetic current density on the surface of the PMC. With the help of the vector analogue of Green's first identity and (38), we have

$$\begin{aligned}
\int\limits_{\mathcal{V}_\infty} |\mathbf{E}_{e2}|^2 d\mathcal{V} &= \frac{1}{(\omega\mu_0\epsilon_0)^2}\int\limits_{\mathcal{V}_\infty} |\nabla\times\mathbf{B}_{e2}|^2 d\mathcal{V}\\
&= \frac{1}{(\omega\mu_0\epsilon_0)^2}\int\limits_{\mathcal{V}_\infty} \mathbf{B}_{e2}^*\cdot\nabla\times\nabla\times\mathbf{B}_{e2} d\mathcal{V}\\
&= \frac{i}{\omega\mu_0\epsilon_0}\int\limits_{S_a} \mathbf{B}_{e2}^*\cdot\mathbf{K}_m dS + O\left[(ka)^2\right]\\
&= -\frac{i}{\omega\mu_0\epsilon_0}\int\limits_{S_a} \mathbf{B}_{02}^*\cdot\mathbf{K}_m dS + O\left[(ka)^2\right]\\
&= \mathbf{E}_0^*\cdot\frac{1}{2}\int\limits_{S_a} \mathbf{K}_m\times\mathbf{r} dS + O\left[(ka)^2\right]\\
&= \frac{1}{\epsilon_0}\mathbf{E}_0^*\cdot\mathbf{p}_2 + O\left[(ka)^2\right] \overset{ka\to 0}{\to} \frac{1}{\epsilon_0}\mathbf{E}_0^*\cdot\mathbf{p}_2
\end{aligned} \tag{39}$$

where use has been made of the total tangential B-field being zero on S_a, namely $[\mathbf{B}_{e2}(\mathbf{r}) + \mathbf{B}_{02}(\mathbf{r})]_{\tan} = 0$, $\mathbf{r} \in S_a$ with $\mathbf{B}_{02}(\mathbf{r}) = -\mathbf{B}_0 = -i\omega\mu_0\epsilon_0\mathbf{r}\times\mathbf{E}_0/2$. (The surface integrals at an infinite radius in the vector Green's first identity vanish because $\mathbf{B}_{e2}$ decays as $1/r^2$ and thus $\mathbf{B}_{e2}^*\cdot\nabla\times\mathbf{B}_{e2}$ decays as $1/r^5$ as $r\to\infty$.)

Expressing the electric dipole moment $\mathbf{p}_2$ in terms of the realvalued, symmetric [34] electric polarizability dyadic $\bar{\boldsymbol{\alpha}}_m$ of a PMC volume $\mathcal{V}_a$ in a uniform quasi-static electric field $\mathbf{E}_{02} = -\mathbf{E}_0$, namely

$$\mathbf{p}_2 = -\epsilon_0\bar{\boldsymbol{\alpha}}_m\cdot\mathbf{E}_0 \Rightarrow \mathbf{E}_0 = -\bar{\boldsymbol{\alpha}}_m^{-1}\cdot\mathbf{p}_2/\epsilon_0 \tag{40}$$

recasts (39) into the form

$$\int\limits_{\mathcal{V}_\infty} |\mathbf{E}_{e2}|^2 d\mathcal{V} = -\mathbf{p}_2\cdot\bar{\boldsymbol{\alpha}}_m^{-1}\cdot\mathbf{p}_2^*/\epsilon_0^2. \tag{41}$$

The electric polarizability of a PMC volume $\mathcal{V}_a$ is equal to the magnetic polarizability of a PEC volume $\mathcal{V}_a$ and, thus, the traditional subscript "m" on $\bar{\boldsymbol{\alpha}}_m$. Note that for a diagonal electric-PMC (magnetic-PEC) polarizability dyadic $[\bar{\boldsymbol{\alpha}}_m = \sum_{j=1}^{3}\alpha_{mj}\hat{\mathbf{x}}_j\hat{\mathbf{x}}_j$, $\hat{\mathbf{x}}_j = (\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})]$, Eq. (41) implies that the scalar polarizabilities are equal to or less than zero ($\alpha_{mj}\le 0$, $j = 1, 2, 3$).

Before substituting the volume integrals from (37) and (41) into (31), we shall show that the third term in the volume integral of (31) is zero.

$$\begin{aligned}\int\limits_{\mathcal{V}_\infty} \mathbf{E}_{e1}\cdot\mathbf{E}_{e2}^* d\mathcal{V} &= \frac{-1}{i\omega\mu_0\epsilon_0}\int\limits_{\mathcal{V}_\infty} \nabla\psi\cdot\nabla\times\mathbf{B}_{e2}^* d\mathcal{V} + O\left[(ka)^2\right]\\ &= \frac{-1}{i\omega\mu_0\epsilon_0}\int\limits_{\mathcal{V}_\infty} \nabla\cdot(\mathbf{B}_{e2}^*\times\nabla\psi) d\mathcal{V} + O\left[(ka)^2\right]\\ &= \frac{-1}{i\omega\mu_0\epsilon_0}\int\limits_{S_\infty} \hat{\mathbf{n}}\cdot(\mathbf{B}_{e2}^*\times\nabla\psi) dS + O\left[(ka)^2\right]. \quad (42)\end{aligned}$$

Since the quasi-static fields $\mathbf{B}_{e2}$ and $\nabla\psi$ both decay as $1/r^2$, the surface integral over S_∞ in (42) is zero and we are left with

$$\int\limits_{\mathcal{V}_\infty} \mathbf{E}_{e1}\cdot\mathbf{E}_{e2}^* d\mathcal{V} = O\left[(ka)^2\right] \overset{ka\to 0}{\to} 0. \quad (43)$$

The results in (37), (41), and (43) allow (31) to be re-expressed as

$$\begin{aligned}Q_{e,lb}^{nc}(\omega) &= \frac{6\pi\eta}{k^3(1+P_m/P_e)}\left[\frac{\mathbf{p}_1\cdot\bar{\boldsymbol{\alpha}}_e^{-1}\cdot\mathbf{p}_1^* - \mathbf{p}_2\cdot\bar{\boldsymbol{\alpha}}_m^{-1}\cdot\mathbf{p}_2^*}{|\mathbf{p}_1+\mathbf{p}_2|^2}\right]\\ &= \frac{6\pi\eta}{k^3(1+P_m/P_e)}\left[\frac{\mathbf{E}_0\cdot(\bar{\boldsymbol{\alpha}}_e-\bar{\boldsymbol{\alpha}}_m)\cdot\mathbf{E}_0^*}{|(\bar{\boldsymbol{\alpha}}_e-\bar{\boldsymbol{\alpha}}_m)\cdot\mathbf{E}_0|^2}\right] \quad (44)\end{aligned}$$

for the nondispersive-conductivity, electric-energy dominated ($W_e^Q \geq W_m^Q$) lower bound on Q for dipole antennas with $\mathbf{E}_0$ determined by the specified $\mathbf{p}$. Reciprocity implies that $\bar{\boldsymbol{\alpha}}_e$ and $\bar{\boldsymbol{\alpha}}_m$ are real symmetric dyadics [34] and thus the xyz coordinate system of $\mathcal{V}_a$ can be oriented to make $\bar{\boldsymbol{\alpha}}_e$ or $\bar{\boldsymbol{\alpha}}_m$ a diagonal dyadic with three principal directions. If the symmetry of $\mathcal{V}_a$ is such that the three principal directions for $\bar{\boldsymbol{\alpha}}_e$ and $\bar{\boldsymbol{\alpha}}_m$ are the same, and $\mathbf{E}_0$ (and thus $\mathbf{p}_1$ and $\mathbf{p}_2$) are in one of these principal directions, then (44) reduces to

$$Q_{e,lb}^{nc}(\omega) = \frac{6\pi\eta}{k^3(1+P_m/P_e)}\left[\frac{1}{\alpha_e-\alpha_m}\right] \quad (45)$$

where, as we proved above, $\alpha_e \geq 0$ and $\alpha_m \leq 0$. Since $\mathbf{p}$ and $\mathbf{m}$ are given to within a constant factor, the ratio P_m/P_e is fixed and, thus, the ratio W_e^Q/W_m^Q is also fixed (and assumed ≥ 1 in (44)–(45)).

The scalar quasi-static electric-PEC (magnetic-PMC) and magnetic-PEC (electric-PMC) polarizabilities can be determined analytically and numerically, for example, for ellipsoids [35] and for regular polyhedra [36]. For a sphere, $\alpha_e = 3V$, $\alpha_m = -1.5V$ and (45) gives

$$Q_{e,lb}^{nc,sph}(\omega) = \frac{4\pi\eta}{3k^3V(1+P_m/P_e)} = \frac{\eta}{(ka)^3(1+P_m/P_e)} \tag{46}$$

the generalization of Chu's lower bound for combined lossy (nondispersive conductivities) electric and magnetic dipoles with electric energy dominating. For $P_m = 0$ and radiation efficiency $\eta = 1$, (46) reduces to the original Chu lower bound for electrically small electric dipoles. The lower bound for an electric-dipole antenna confined to a sphere is less than that of any other volume $\mathcal{V}_a$ circumscribed by the sphere because of the extra stored energy between $\mathcal{V}_a$ and its circumscribing sphere. (For $ka = 0.5$, the $1/(ka)$ term in the more accurate Chu lower bound given by $1/(ka)^3 + 1/(ka)$ [4, 37] adds an amount equal to 25% of the $1/(ka)^3$ term. This percentage decreases rapidly with decreasing ka.)

The polarizabilities of a PEC ellipsoid for the direction of the dipole moment $\mathbf{p}$ parallel to one of its principal axes can be found from [35] as

$$\alpha_e = \mathcal{V}_a/N_0 \tag{47a}$$

$$\frac{1}{\alpha_m} = \frac{1}{\alpha_e} - \frac{1}{\mathcal{V}_a} \tag{47b}$$

where N_0 is the depolarization factor for a given principal-axis direction and $\mathcal{V}_a = 4\pi a_s b_s c_s/3$ with a_s, b_s, and c_s equal to the lengths of the principal semi-axes. Consequently, for an ellipsoid, the lower bound in (45) can be written as

$$Q_{e,lb}^{nc,elp}(\omega) = \frac{6\pi\eta}{k^3(1+P_m/P_e)\alpha_e}\left(1 - \frac{\mathcal{V}_a}{\alpha_e}\right). \tag{48}$$

The expression in (48) for the Q lower bound of an electrically small ellipsoid holds only approximately for volume shapes other than ellipsoids. However, in our original article [13] on the subject of lower bounds for arbitrarily shaped electrically small antennas, we obtained (48) as a generally valid result. This mistake was discovered by Jonsson and Gustafsson in their work with determining stored energies [15] from optimized current integrals [18]. The derivation in [13] proceeded along the lines of the present derivation of (44)–(45) but with the implicit assumption that the magnetic surface current, which nulls the fields inside the volume $\mathcal{V}_a$, produce essentially the same dipole-field distribution outside $\mathcal{V}_a$ as the electric surface current. This assumption holds perfectly for ellipsoids but only approximately for other shapes.

The minimum value, with respect to different directions of $\mathbf{E}_0$ (or, equivalently, different directions of $\mathbf{p}$), of the quotient in the square brackets of the last line of (44) is given in terms of the maximum eigenvalue (call it α_{em}) of the positive-semidefinite matrix $(\bar{\boldsymbol{\alpha}}_e - \bar{\boldsymbol{\alpha}}_m)$, namely

$$Q^{nc}_{e,\mathrm{Min}}(\omega) = \frac{6\pi\eta}{k^3(1+P_m/P_e)\alpha_{em}} \tag{49}$$

for $\mathbf{E}_0$ equal to the corresponding eigenvector $\mathbf{E}_0^{em}$, where $\alpha_{em} \geq (\alpha_e - \alpha_m)$. Of course, choosing $\mathbf{E}_0 = \mathbf{E}_0^{em}$ determines the direction of $\mathbf{p}$, so that the minimum Q in (49) cannot be obtained for an arbitrary $\mathbf{p}$.

5.2. Minimization for the Quasi-static Magnetic Field of the Magnetic Dipole

The magnetic-dipole magnetic field $\mathbf{H}_m$ of an electrically small electric and magnetic dipole antenna dominates the magnetic field both inside and outside the surface S_a of the antenna $\mathcal{V}_a$. Since the surface S_a contains the equivalent magnetic and electric currents that reduce the fields to zero inside $\mathcal{V}_a$, one can divide $\mathbf{H}_m$ into separate contributions, $\mathbf{H}_{m1}$ and $\mathbf{H}_{m2}$, respectively, from magnetic and electric surface currents on S_a (similarly to what we did for the electric-dipole electric fields in the previous subsection) in order to evaluate the minimum $Q^{nc}_{m,lb}(\omega)$ in (26b). Since the steps are entirely analogous to those for evaluating $Q^{nc}_{e,lb}(\omega)$ in Subsection 5.1, we shall present only the final results, namely

$$\begin{aligned} Q^{nc}_{m,lb}(\omega) &= \frac{6\pi\eta}{k^3(1+P_e/P_m)}\left[\frac{\mathbf{m}_1\cdot\bar{\boldsymbol{\alpha}}_e^{-1}\cdot\mathbf{m}_1^* - \mathbf{m}_2\cdot\bar{\boldsymbol{\alpha}}_m^{-1}\cdot\mathbf{m}_2^*}{|\mathbf{m}_1+\mathbf{m}_2|^2}\right] \\ &= \frac{6\pi\eta}{k^3(1+P_e/P_m)}\left[\frac{\mathbf{H}_0\cdot(\bar{\boldsymbol{\alpha}}_e-\bar{\boldsymbol{\alpha}}_m)\cdot\mathbf{H}_0^*}{|(\bar{\boldsymbol{\alpha}}_e-\bar{\boldsymbol{\alpha}}_m)\cdot\mathbf{H}_0|^2}\right] \end{aligned} \tag{50}$$

for the nondispersive-conductivity, magnetic-energy dominated ($W_m^Q \geq W_e^Q$) lower bound on Q for dipole antennas (with $\mathbf{H}_0$ determined by the specified $\mathbf{m}$), where $\mathbf{m}_1 = \bar{\boldsymbol{\alpha}}_e \cdot \mathbf{H}_0$ and $\mathbf{m}_2 = -\bar{\boldsymbol{\alpha}}_m \cdot \mathbf{H}_0$ are the magnetic dipole moments of the magnetic surface current (induced by the uniform magnetic field $\mathbf{H}_0$ on a PMC in $\mathcal{V}_a$) and electric surface current (induced by the uniform magnetic field $-\mathbf{H}_0$ on a PEC in $\mathcal{V}_a$) producing $\mathbf{H}_{m1}$ and $\mathbf{H}_{m2}$. The total power radiated by the magnetic dipoles is given analogously to (29) as [33, p. 438]

$$P_m = \frac{\omega k^3\mu_0}{12\pi}|\mathbf{m}|^2 = \frac{\omega k^3\mu_0}{12\pi}|\mathbf{m}_1+\mathbf{m}_2|^2. \tag{51}$$

If the symmetry of $\mathcal{V}_a$ is such that the three principal directions for $\bar{\boldsymbol{\alpha}}_e$ and $\bar{\boldsymbol{\alpha}}_m$ are the same, and $\mathbf{H}_0$ (and thus $\mathbf{m}_1$ and $\mathbf{m}_2$) are in one of these principal directions, then (50) reduces to

$$Q^{nc}_{m,lb}(\omega) = \frac{6\pi\eta}{k^3(1+P_e/P_m)}\left[\frac{1}{\alpha_e-\alpha_m}\right]. \tag{52}$$

Since $\mathbf{p}$ and $\mathbf{m}$ are given to within a constant factor, the ratio P_e/P_m is fixed and, thus, the ratio W^Q_m/W^Q_e is also fixed (and assumed ≥ 1 in (50) and (52)).

For a sphere, $\alpha_e = 3V$, $\alpha_m = -1.5V$ and (52) gives

$$Q^{nc,sph}_{m,lb}(\omega) = \frac{4\pi\eta}{3k^3V(1+P_e/P_m)} = \frac{\eta}{(ka)^3(1+P_e/P_m)}. \tag{53}$$

The only difference between $Q^{nc}_{m,lb}$ in (50)–(52), which applies if the magnetic energy dominates, and $Q^{nc}_{e,lb}$ in (44)–(45), which applies if the electric energy dominates, is the interchange of P_m and P_e. The lower bounds in (46) and (53) for the sphere with $\eta = 1$ were given previously in [14, Eq. (73)]. When $P_e = P_m$, the spherical antenna radiates equal power in the electric and magnetic dipole fields and both $Q^{nc,sph}_{e,lb}$ and $Q^{nc,sph}_{m,lb}$ become equal to one half the minimum Q of a single electric or magnetic dipole because the two spherical dipoles form a self-tuned antenna requiring no external tuning capacitor or inductor. (As mentioned in the Introduction, for cophasal electric and magnetic dipole moments forming a Huygens source, Thal [7] has shown that extra internal tuning, which adds to the Q-energy, is required to feed these Huygens-source electric and magnetic dipole moments with a common electric current.)

The minimum value, with respect to different directions of $\mathbf{H}_0$ (or, equivalently, different directions of $\mathbf{m}$), of the quotient in the square brackets of the last line of (50) is given in terms of the maximum eigenvalue (α_{em}) of the positive-semidefinite matrix ($\bar{\boldsymbol{\alpha}}_e - \bar{\boldsymbol{\alpha}}_m$), namely

$$Q^{nc}_{m,\mathrm{Min}}(\omega) = \frac{6\pi\eta}{k^3(1+P_e/P_m)\alpha_{em}} \tag{54}$$

for $\mathbf{H}_0$ equal to the corresponding eigenvector $\mathbf{H}^{em}_0$, where $\alpha_{em} \geq (\alpha_e - \alpha_m)$. Of course, choosing $\mathbf{H}_0 = \mathbf{H}^{em}_0$ determines the direction of $\mathbf{m}$, so that the minimum Q in (54) cannot be obtained for an arbitrary $\mathbf{m}$.

5.3. Minimization for the Highly Dispersive Lossy Antennas

Although it is conjectural that practical highly dispersive lossy materials could be found to realize the highly dispersive lossy lower bound on Q given in (23), for the sake of completeness, we can use the results in (44)–(45) and (50)–(52) to immediately evaluate (23). First rewrite (23) as

$$Q_{lb}^{hdl}(\omega) = \frac{\eta\omega P_e}{4(P_e+P_m)}\mathrm{Min}\left[\frac{\int_{\mathcal{V}_{out}} \epsilon_0|\mathbf{E}_e|^2 d\mathcal{V}}{P_e}\right] + \frac{\eta\omega P_m}{4(P_e+P_m)}\mathrm{Min}\left[\frac{\int_{\mathcal{V}_{out}} \mu_0|\mathbf{H}_m|^2 d\mathcal{V}}{P_m}\right] \tag{55}$$

where the minimum of the sum of the positive integrals in (23) is equal to the sum of the minimum of the integrals because, to within the approximation of electrically small antennas, the value of the electric field of the magnetic dipole, and the value of the magnetic field of the electric dipole is negligible. Then we can use the foregoing results for each of these minimizations to obtain

$$\begin{aligned} Q_{lb}^{hdl}(\omega) &= \frac{3\pi\eta P_e/P_m}{k^3(1+P_e/P_m)}\left[\frac{\mathbf{p}_1\cdot\bar{\boldsymbol{\alpha}}_e^{-1}\cdot\mathbf{p}_1^* - \mathbf{p}_2\cdot\bar{\boldsymbol{\alpha}}_m^{-1}\cdot\mathbf{p}_2^*}{|\mathbf{p}_1+\mathbf{p}_2|^2}\right] \\ &\quad + \frac{3\pi\eta P_m/P_e}{k^3(1+P_m/P_e)}\left[\frac{\mathbf{m}_1\cdot\bar{\boldsymbol{\alpha}}_e^{-1}\cdot\mathbf{m}_1 - \mathbf{m}_2\cdot\bar{\boldsymbol{\alpha}}_m^{-1}\cdot\mathbf{m}_2^*}{|\mathbf{m}_1+\mathbf{m}_2|^2}\right] \\ &= \frac{3\pi\eta P_e/P_m}{k^3(1+P_e/P_m)}\left[\frac{\mathbf{E}_0\cdot(\bar{\boldsymbol{\alpha}}_e-\bar{\boldsymbol{\alpha}}_m)\cdot\mathbf{E}_0^*}{|(\bar{\boldsymbol{\alpha}}_e-\bar{\boldsymbol{\alpha}}_m)\cdot\mathbf{E}_0|^2}\right] \\ &\quad + \frac{3\pi\eta P_m/P_e}{k^3(1+P_m/P_e)}\left[\frac{\mathbf{H}_0\cdot(\bar{\boldsymbol{\alpha}}_e-\bar{\boldsymbol{\alpha}}_m)\cdot\mathbf{H}_0^*}{|(\bar{\boldsymbol{\alpha}}_e-\bar{\boldsymbol{\alpha}}_m)\cdot\mathbf{H}_0|^2}\right] \end{aligned} \tag{56}$$

for the highly dispersive lossy lower bound on Q of dipole antennas with $\mathbf{E}_0$ and $\mathbf{H}_0$ determined by the specified $\mathbf{p}$ and $\mathbf{m}$, respectively. If the symmetry of $\mathcal{V}_a$ is such that the principal directions of $\bar{\boldsymbol{\alpha}}_e$ and $\bar{\boldsymbol{\alpha}}_m$ are the same, and the applied uniform fields $\mathbf{E}_0$ and $\mathbf{H}_0$ (and thus $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{m}_1$, and $\mathbf{m}_2$) are in one of these principal directions, then (56) reduces to

$$Q_{lb}^{hdl}(\omega) = \frac{3\pi\eta}{k^3}\left[\frac{1}{\alpha_e-\alpha_m}\right] \tag{57}$$

which makes sense because it is equal to the least possible value of $Q_{e,lb}^{nc}$ in (45) or $Q_{m,lb}^{nc}$ in (52) obtained when $P_e = P_m$, which for a sphere is $Q_{lb}^{hdl,sph} = \eta/[2(ka)^3]$, the value of (57) obtained for a sphere in [14, Eq. (70)]. However, (57) holds for any ratio P_e/P_m and thus applies to a single electric or magnetic dipole. The factor of two reduction in the Chu lower bound for a single dipole brings to mind the approximate factor of two increase in half-power bandwidth for electrically small antennas using Bode-Fano matching networks [19, 20] ($\pi/\ln(s+1/s-1) = 1.95 \approx 2$ for half-power voltage standing wave ratio $s = 5.828$). In principle, the Bode-Fano networks create a lossless multi-resonance antenna rather than a single resonance antenna with highly dispersive lossy material required by the lower bound in (57).

The minimum value of (56), with respect to different directions of $\mathbf{E}_0$ and $\mathbf{H}_0$ (or, equivalently, different directions of $\mathbf{p}$ and $\mathbf{m}$), of the quotients in the square brackets of the last two lines of (56) is given in terms of the maximum eigenvalue (α_{em}) of the positive-semidefinite matrix ($\bar{\boldsymbol{\alpha}}_e - \bar{\boldsymbol{\alpha}}_m$), namely

$$Q^{hdl}_{\text{Min}}(\omega) = \frac{3\pi\eta}{k^3\alpha_{em}} \tag{58}$$

for $\mathbf{E}_0$ and $\mathbf{H}_0$ equal to the corresponding eigenvectors $\mathbf{E}_0^{em}$ and $\mathbf{H}_0^{em}$, where $\alpha_{em} \geq (\alpha_e - \alpha_m)$. Of course, choosing $\mathbf{E}_0 = \mathbf{E}_0^{em}$ and $\mathbf{H}_0 = \mathbf{H}_0^{em}$ determines the direction of $\mathbf{p}$ and $\mathbf{m}$, so that the minimum Q in (58) cannot be obtained for an arbitrary $\mathbf{p}$ and $\mathbf{m}$.

6. MINIMUM Q WITH ELECTRIC SURFACE CURRENTS ONLY

The lower bounds on the quality factor given in Section 5 for electric and magnetic dipole antennas confined to an electrically small volume $\mathcal{V}_a$ were derived assuming the possibility of magnetic as well as electric surface currents on the surface S_a of the volume $\mathcal{V}_a$. Since magnetic charge does not exist per se, magnetic surface currents would have to be produced as thin layers of magnetization in natural magnetic material or in metamaterials synthesized from small Amperian current loops (or possibly slots in electrically conducting surfaces). Although it has been shown that high-permeability magnetic material can, in principle, be used to approach the lower bounds for spherical electric dipoles [38] as well as magnetic dipoles [39], low-loss magnetic material may be difficult to obtain for frequencies above a few MHz (and magnetic material with high enough loss to approximate a PMC may also be difficult to obtain). Therefore, in this section, the lower-bound expressions in Section 5 will be modified to obtain the lower bounds on the quality factor for electric and magnetic dipole antennas confined to an arbitrarily shaped free-space (except for the tuning inductor or capacitor) volume $\mathcal{V}_a$ with applied electric surface currents alone on the surface S_a of $\mathcal{V}_a$. Thal has determined the lower bounds of lossless electrically small spherical antennas allowing only global applied electric surface currents in free space [6, 40].

6.1. Highly Dispersive Lossy, Electric-Current-Only Lower Bound

With the volume inside $\mathcal{V}_a$ consisting of free space, the lower bound on quality factor using a tuning inductor or capacitor filled with highly dispersive lossy material that does not contribute to the Q-energy (as discussed in Section 4.1) can be written immediately from (19) as

$$Q_{lb}^{hdl,ec}(\omega) = \eta\omega \mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_\infty}\left(\epsilon_0|\mathbf{E}_e|^2 + \mu_0|\mathbf{H}_m|^2\right)d\mathcal{V}}{4(P_e+P_m)}\right] \tag{59}$$

minimized for a given $\mathcal{V}_a$, η, and power ratio P_m/P_e of the given electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$, where the additional "*ec*" superscripts indicate that now the minimization is restricted to using electric surface current only. Note that (59) is identical to (23) except that the integration of the quasi-static fields in (59) is over all space ($\mathcal{V}_\infty = \mathcal{V}_a + \mathcal{V}_{out}$) because the restriction to electric surface currents only implies nonzero fields inside $\mathcal{V}_a$. Rewriting (59) as

$$Q_{lb}^{hdl,ec}(\omega) = \frac{\eta\omega P_e}{4(P_e+P_m)}\mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_\infty}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V}}{P_e}\right] + \frac{\eta\omega P_m}{4(P_e+P_m)}\mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_\infty}\mu_0|\mathbf{H}_m|^2 d\mathcal{V}}{P_m}\right] \tag{60}$$

we obtain instead of (56)

$$\begin{aligned} Q_{lb}^{hdl,ec}(\omega) &= \frac{3\pi\eta P_e/P_m}{k^3(1+P_e/P_m)}\left[\frac{\mathbf{p}\cdot\bar{\boldsymbol{\alpha}}_e^{-1}\cdot\mathbf{p}^*}{|\mathbf{p}|^2}\right] \\ &\quad - \frac{3\pi\eta P_m/P_e}{k^3(1+P_m/P_e)}\left[\frac{\mathbf{m}\cdot\bar{\boldsymbol{\alpha}}_m^{-1}\cdot\mathbf{m}^*}{|\mathbf{m}|^2}\right] \\ &= \frac{3\pi\eta P_e/P_m}{k^3(1+P_e/P_m)}\left[\frac{\mathbf{E}_0\cdot\bar{\boldsymbol{\alpha}}_e\cdot\mathbf{E}_0^*}{|\bar{\boldsymbol{\alpha}}_e\cdot\mathbf{E}_0|^2}\right] \\ &\quad - \frac{3\pi\eta P_m/P_e}{k^3(1+P_m/P_e)}\left[\frac{\mathbf{H}_0\cdot\bar{\boldsymbol{\alpha}}_m\cdot\mathbf{H}_0^*}{|\bar{\boldsymbol{\alpha}}_m\cdot\mathbf{H}_0|^2}\right] \end{aligned} \tag{61}$$

because without magnetic surface currents $\mathbf{p}_2 = 0$ and $\mathbf{m}_1 = 0$, so that $\mathbf{p}_1 = \mathbf{p}$ and $\mathbf{m}_2 = \mathbf{m}$. If the symmetry of $\mathcal{V}_a$ allows a principal direction of $\bar{\boldsymbol{\alpha}}_e$ and $\bar{\boldsymbol{\alpha}}_m$ to be the same, and $\mathbf{p}$ and $\mathbf{m}$ are in this direction, then (61) reduces to

$$Q_{lb}^{hdl,ec}(\omega) = \frac{3\pi\eta}{k^3(P_e + P_m)} \left(\frac{P_e}{\alpha_e} + \frac{P_m}{|\alpha_m|} \right). \tag{62}$$

(Recall that $\alpha_e \geq 0$ and $\alpha_m \leq 0$.) If $|\alpha_m/\alpha_e| < 1$, as for a sphere, then the least lower bound (with respect to different values of P_m/P_e) in (62) occurs when $P_m = 0$, that is for the electric dipole. For a spherical electric dipole ($P_m = 0$), this highly dispersive lossy, electric-current-only, least lower bound is $3\eta/[4(ka)^3]$, half the Thal [6] lower bound (times η) for an electric dipole. The highly dispersive lossy, electric-current-only, lower bound for the spherical magnetic dipole ($P_e = 0$) is $3\eta/[2(ka)^3]$, again half the Thal [6] lower bound (times η) for a magnetic dipole.

The minimum value of (61), with respect to different directions of $\mathbf{E}_0$ and $\mathbf{H}_0$ (or, equivalently, different directions of $\mathbf{p}$ and $\mathbf{m}$), of the quotients in the square brackets of the last line of (61) is given in terms of the maximum eigenvalues (α_{ee} and $-\alpha_{mm}$) of the positive-semidefinite matrices ($\bar{\boldsymbol{\alpha}}_e$ and $-\bar{\boldsymbol{\alpha}}_m$), namely

$$Q_{\rm Min}^{hdl,ec}(\omega) = \frac{3\pi\eta}{k^3(P_e + P_m)} \left(\frac{P_e}{\alpha_{ee}} + \frac{P_m}{|\alpha_{mm}|} \right) \tag{63}$$

for $\mathbf{E}_0$ and $\mathbf{H}_0$ equal to the corresponding eigenvectors $\mathbf{E}_0^{ee}$ and $\mathbf{H}_0^{mm}$, where $\alpha_{ee} \geq \alpha_e$ and $|\alpha_{mm}| \geq |\alpha_m|$. Of course, choosing $\mathbf{E}_0 = \mathbf{E}_0^{ee}$ and $\mathbf{H}_0 = \mathbf{H}_0^{mm}$ determines the directions of $\mathbf{p}$ and $\mathbf{m}$, so that the minimum Q in (63) cannot be obtained for an arbitrary $\mathbf{p}$ and $\mathbf{m}$.

6.2. Lossless and Nondispersive-Conductivity, Electric-Current-Only Lower Bounds

Next we consider the more conventional case of a tuning inductor or capacitor filled with lossless or nondispersive-conductivity material that adds to the Q-energy of the dipole antenna but still assuming that the dipolar fields are produced by electric surface current only in free space. In that case, it is required that the energy in the tuning element be equal to

$$\frac{1}{4} \int\limits_{\mathcal{V}_\infty} \left| \epsilon_0 |\mathbf{E}_e|^2 - \mu_0 |\mathbf{H}_m|^2 \right| d\mathcal{V}. \tag{64}$$

Therefore, if

$$\int\limits_{\mathcal{V}_\infty} \left(\epsilon_0 |\mathbf{E}_e|^2 - \mu_0 |\mathbf{H}_m|^2 \right) d\mathcal{V} \geq 0 \tag{65}$$

that is, the electric energy dominates, then the lower-bound expressions in (59)–(61) are replaced by

$$
\begin{aligned}
&Q_{e,lb}^{nc,ec}(\omega)\\
&= \eta\omega \mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_\infty}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V}}{2(P_e+P_m)}\right] = \frac{\eta\omega P_e}{2(P_e+P_m)}\mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_\infty}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V}}{P_e}\right]\\
&= \frac{6\pi\eta P_e/P_m}{k^3(1+P_e/P_m)}\left[\frac{\mathbf{p}\cdot\bar{\boldsymbol{\alpha}}_e^{-1}\cdot\mathbf{p}^*}{|\mathbf{p}|^2}\right] = \frac{6\pi\eta P_e/P_m}{k^3(1+P_e/P_m)}\left[\frac{\mathbf{E}_0\cdot\bar{\boldsymbol{\alpha}}_e\cdot\mathbf{E}_0^*}{|\bar{\boldsymbol{\alpha}}_e\cdot\mathbf{E}_0|^2}\right]
\end{aligned}
\tag{66}
$$

for the lossless or nondispersive-conductivity, electric-current-only, electric-energy dominated ($W_e^Q \geq W_m^Q$) dipole antenna with $\mathbf{E}_0$ determined by the specified $\mathbf{p}$. If $\mathbf{p}$ is in one of the principal directions of $\bar{\boldsymbol{\alpha}}_e$, then (66) reduces to

$$
Q_{e,lb}^{nc,ec}(\omega) = \frac{6\pi\eta P_e/P_m}{k^3(1+P_e/P_m)\alpha_e} \tag{67}
$$

where, as we proved above, $\alpha_e \geq 0$ and $\alpha_m \leq 0$. From (29) and (37), it follows that $\int_{\mathcal{V}_\infty}\epsilon_0|\mathbf{E}_e|^2 d\mathcal{V} = 12\pi P_e/(\omega k^3\alpha_e)$ and, similarly, if $\mathbf{m}$ is in a principal direction, $\int_{\mathcal{V}_\infty}\mu_0|\mathbf{H}_m|^2 d\mathcal{V} = 12\pi P_m/(\omega k^3|\alpha_m|)$, so that the inequality in (65) can be rewritten as

$$
P_e \geq \frac{\alpha_e}{|\alpha_m|}P_m. \tag{68}
$$

For a sphere, $\alpha_e = 3V$ and (67) gives

$$
Q_{e,lb}^{sph,ec}(\omega) = \frac{3\eta P_e/P_m}{2(ka)^3(1+P_e/P_m)} \tag{69}
$$

the generalization of Thal's lower bound [6] for combined lossy (nondispersive conductivities), electric-current-only electric and magnetic dipoles under the inequality in (68), which for a sphere is simply $P_e \geq 2P_m$.

The minimum value of (66), with respect to different directions of $\mathbf{E}_0$ (or, equivalently, different directions of $\mathbf{p}$), of the quotients in the square brackets of the last line of (66) is given in terms of the maximum eigenvalue (α_{ee}) of the positive-semidefinite matrix $\bar{\alpha}_e$, namely

$$
Q_{e,\mathrm{Min}}^{nc,ec}(\omega) = \frac{6\pi\eta P_e/P_m}{k^3(1+P_e/P_m)\alpha_{ee}} \tag{70}
$$

for $\mathbf{E}_0$ equal to the corresponding eigenvector $\mathbf{E}_0^{ee}$, where $\alpha_{ee} \geq \alpha_e$. Of course, choosing $\mathbf{E}_0 = \mathbf{E}_0^{ee}$ determines the direction of $\mathbf{p}$, so that the minimum Q in (70) cannot be obtained for an arbitrary $\mathbf{p}$.

If instead of (65), the fields obey the inequality for magnetic energy dominating, namely

$$\int\limits_{\mathcal{V}_\infty} \left(\mu_0|\mathbf{H}_m|^2 - \epsilon_0|\mathbf{E}_e|^2\right) d\mathcal{V} \geq 0 \tag{71}$$

then the electric-energy lower-bound expression in (66) is replaced by the corresponding magnetic-energy one, namely

$$\begin{aligned}
Q_{m,lb}^{nc,ec}(\omega) &= \eta\omega \mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_\infty} \mu_0|\mathbf{H}_m|^2 d\mathcal{V}}{2(P_e+P_m)}\right] \\
&= \frac{\eta\omega P_m}{2(P_e+P_m)}\mathrm{Min}\left[\frac{\int\limits_{\mathcal{V}_\infty} \mu_0|\mathbf{H}_m|^2 d\mathcal{V}}{P_m}\right] \\
&= \frac{-6\pi\eta P_m/P_e}{k^3(1+P_m/P_e)}\left[\frac{\mathbf{m}\cdot\bar{\boldsymbol{\alpha}}_m^{-1}\cdot\mathbf{m}^*}{|\mathbf{m}|^2}\right] \\
&= \frac{-6\pi\eta P_m/P_e}{k^3(1+P_m/P_e)}\left[\frac{\mathbf{H}_0\cdot\bar{\boldsymbol{\alpha}}_m\cdot\mathbf{H}_0^*}{|\bar{\boldsymbol{\alpha}}_m\cdot\mathbf{H}_0|^2}\right]
\end{aligned} \tag{72}$$

for the lossless or nondispersive-conductivity, electric-current-only, magnetic-energy dominated ($W_m^Q \geq W_e^Q$) dipole antenna with $\mathbf{H}_0$ determined by the specified $\mathbf{m}$. If $\mathbf{m}$ is in one of the principal directions of $\bar{\boldsymbol{\alpha}}_m$, then (72) and (71) reduce to

$$Q_{m,lb}^{nc,ec}(\omega) = \frac{6\pi\eta P_m/P_e}{k^3(1+P_m/P_e)|\alpha_m|} \tag{73}$$

and if $\mathbf{p}$ is in a principal direction

$$P_m \geq \frac{|\alpha_m|}{\alpha_e} P_e. \tag{74}$$

Note that for $P_m = |\alpha_m|P_e/\alpha_e$, the electric and magnetic energies are equal, the antenna is self-tuned, and the quality factors in (67) and (73) are equal. For a sphere, $\alpha_m = -1.5V$ and (73) gives

$$Q_{m,lb}^{sph,ec}(\omega) = \frac{3\eta P_m/P_e}{(ka)^3(1+P_m/P_e)} \tag{75}$$

the generalization of Thal's lower bound [6] for combined lossy (nondispersive conductivities), electric-current-only electric and magnetic dipoles under the inequality in (74), which for a sphere is simply $P_m \geq P_e/2$.

For a self-tuned, free-space, electric-current, spherical dipole antenna, $P_e = 2P_m$ and both (69) and (75) give the lower-bound quality factor

$$Q_{lb}^{sph,ec}(\omega) = \frac{\eta}{(ka)^3} \tag{76}$$

which agrees with the self-tuned, free-space, electric-current, spherical dipole lower bound determined in [23] by direct integration.

The minimum value of (72), with respect to different directions of $\mathbf{H}_0$ (or, equivalently, different directions of $\mathbf{m}$), of the quotients in the square brackets of the last line of (72) is given in terms of the maximum eigenvalue $(-\alpha_{mm})$ of the positive-semidefinite matrix $-\bar{\boldsymbol{\alpha}}_m$, namely

$$Q_{m,\mathrm{Min}}^{nc,ec}(\omega) = \frac{6\pi\eta P_m/P_e}{k^3(1+P_m/P_e)|\alpha_{mm}|} \tag{77}$$

for $\mathbf{H}_0$ equal to the corresponding eigenvector $\mathbf{H}_0^{mm}$, where $|\alpha_{mm}| \geq |\alpha_m|$. Of course, choosing $\mathbf{H}_0 = \mathbf{H}_0^{mm}$ determines the direction of $\mathbf{m}$, so that the minimum Q in (77) cannot be obtained for an arbitrary $\mathbf{m}$.

7. CONCLUSION

Beginning with the general expressions for the quality factor of lossy or lossless antennas derived in [11, 24], simplifications of these expressions are derived for electrically small antennas and, in particular, for electrically small dipole antennas confined to an arbitrarily shaped volume $\mathcal{V}_a$. These latter expressions are recast in a form convenient for the minimization of the quality factor of highly dispersive lossy dipole antennas as well as lossless or nondispersive-conductivity dipole antennas. The antennas are allowed to have arbitrary combinations of electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$. The minimizations are accomplished through the use of scalar and vector Green's identities applied to the quasi-static fields of the dipole antennas. Convenient formulas for the lower bounds on the quality factor are given in terms of the quasi-static electric and magnetic perfectly conducting polarizabilities of the volume $\mathcal{V}_a$, the ratios of the powers radiated by the electric and magnetic dipoles, and the efficiency of the antenna. Expressions are also found for the lower bounds further minimized with respect to varying the directions of $\mathbf{p}$ and $\mathbf{m}$ relative to the volume $\mathcal{V}_a$.

The lower-bound formulas for quality factor are first derived assuming the possibility of both electric and magnetic surface currents (or effective magnetic surface currents in the form of magnetization) on the surface S_a of the volume $\mathcal{V}_a$ of the antenna. These general lower bounds are found for nondispersive-conductivity antennas in the formulas (44)–(45) if the electric energy of the dipoles dominates and in the formulas (50), (52) if the magnetic energy of the dipoles dominates. For highly dispersive lossy antennas, the general formulas for the Q lower bounds are given in (56)–(57), which, in principle, can have a smaller value than the lossless or nondispersive-conductivity lower bounds on Q.

Lastly, analogous formulas for lower bounds on quality factor are derived allowing only for electric surface currents in free space on the surface S_a of the volume $\mathcal{V}_a$ of the antenna. These electric-current-only lower bounds on Q are equal to or larger than the general lower bounds on Q and are given for electric-energy-dominated and magnetic-energy-dominated, nondispersive-conductivity dipole antennas in (66)–(67) and (72)–(73), respectively. For highly dispersive lossy, electric-current-only antennas, the formulas for the lower bounds on Q are given in (61)–(62).

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