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FRIEDLANDER-KELLER SOLUTION FOR 3-D MAXWELL CASE

I. Andronov and D. Bouche

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1. Introduction

In the diffraction of electromagnetic waves by smooth convex obstacles the shadowed side of the body is reached by the creeping waves that then form in the far away zone some specific field of creeping rays. This problem has been studied by P.H.Pathak [1] using the method of canonical problems. This method provides a solution of the problem however it is somewhat heuristic because one has to extrapolate the particular solution for the canonical problem to the case of a general surface. An alternative method is proposed in [3]. The problem is solved by the boundary layer method. More precisely, the solution in the boundary layer in a close vicinity of the surface is matched to the Friedlander-Keller ray solution far away from the surface. However this reference deals only with the diffraction of scalar waves by a two dimensional obstacle satisfying the Dirichlet boundary condition.

In this paper we extend the method to the case of a three dimensional obstacle described by impedance boundary condition. The geometry of the problem and the systems of coordinates used in the analysis are presented in section 2. The Maxwell equation in curvilinear coordinates are given in section 3. Then the Friedlander-Keller solution is derived in section 4. It depends on arbitrary functions that are determined by matching to the solution for a creeping wave in the boundary layer presented in section 5. The details of the matching procedure are given in section 6.

2. Geometry

Let us introduce some notations. We will deal with three coordinate systems: Friedlander-Keller coordinates $(s,\ \alpha,\ l)$ and two local surface coordinate systems in the creeping point $C(s_C,\ \alpha_C,\ n_C)$ and in the point N of the normal origin $(s_N,\ \alpha_N,\ n_N)$ (see Figure)

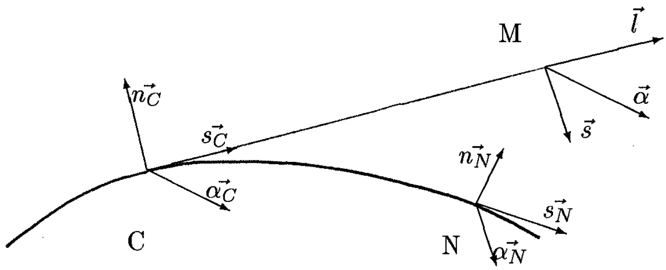

Figure. Coordinate systems.

The coordinates $(s,\ \alpha,\ l)$ will be used for constructing the asymptotics in the form of Friedlander-Keller ansatz and the coordinates $(s_N,\ \alpha_N,\ n_N)$ are the coordinates of the creeping waves asymptotics near the surface of the body. The local basis of the surface coordinates in the creeping point C is introduced as an intermediate step in the transformation from $(s,\ \alpha,\ l)$ system to the coordinates connected with the point N. Besides that the point C is involved in the Friedlander-Keller asymptotics because all the geometry characteristics are calculated in the creeping point. In the asymptotics valid near the surface the geometry characteristics are calculated in the point N The following notations are used:

- ρ radius of the surface curvature in the direction of the geodesics followed by the creeping wave propagating along the surface
- h divergence of the geodesics on the surface of the body
- σ torsion of the geodesics

The subscripts of these functions denote derivatives by surface coordinates s and α. We shall also introduce t as $l - s_C$.

The following relations between the mentioned above coordinates hold

$$s = s_C, \quad \alpha = \alpha_C, \quad l = s_C + t,$$

$$\vec{l} = \vec{s}_C, \quad \vec{s} = -\frac{t}{\rho}\vec{n}_C, \quad \vec{\alpha} = \left(1 + \frac{h_s}{h}t\right)\vec{\alpha}_C + h\sigma t\vec{n}_C,$$

$$s_N = s_C + t - \frac{1}{3\rho^2}t^3 + \frac{7}{24}\frac{\rho_s}{\rho^3}t^4, \quad \alpha_N = \alpha_C + \frac{\sigma}{3h\rho}t^3, \quad n_N = \frac{1}{2\rho}t^2 - \frac{\rho_s}{6\rho^2}t^3,$$

$$\vec{s}_N = \vec{s}_C - \frac{t}{\rho}\vec{n}_C, \quad \vec{\alpha}_N = \left(1 + \frac{h_s}{h}t\right)\vec{\alpha}_C + h\sigma t\vec{n}_C + O(t^2)\vec{s}_C,$$

$$\vec{n}_N = \vec{n}_C + \frac{t}{\rho}\vec{s}_C - \frac{\sigma t}{h}\vec{\alpha}_C \tag{1}$$

The formulae for coordinates transformation are written for small values of $t = O(k^{-1/3})$ for the zone where the matching will be done. For the coordinates α_N and n_N we neglected the terms of order $o(t^3)$. The coordinate s_N appear in the exponent multiplied by the large parameter k and we need formula accurate up to terms of order $o(t^4)$. For the vectors transformation we need lower accuracy. In the formulae we dropped the terms of order $O(t^2)$ everywhere except in the coefficient of $\vec{s}_N$ in the formula for $\vec{\alpha}_C$. Later it will be clear why we need that accuracy.

The quadratic forms matrix of the Friedlander-Keller coordinate system is

$$G = \begin{pmatrix} \frac{1}{\rho^2}t^2 & -\frac{h\sigma}{\rho}t^2 & 0 \\ -\frac{h\sigma}{\rho}t^2 & h^2\left(1 + 2\frac{h_s}{h}t + \left(\left(\frac{h_s}{h}\right)^2 + \sigma^2\right)t^2\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

We shall use the notations $G_{ss}, G_{s\alpha}$ and $G_{\alpha\alpha}$ for the elements of the matrix G.

The determinant of G is

$$g^2 \equiv \det G = \frac{t^2}{\rho^2}\left(h + h_s t\right)^2 \tag{3}$$

3. The Equations of Maxwell

To deal with the electromagnetic field it is convenient to introduce the vectors $\vec{E} = \varepsilon^{1/2}\mathcal{E}$ and $\vec{H} = \mu^{1/2}\mathcal{H}$ ($\mathcal{E}$ and $\mathcal{H}$ are usual electric and magnetic fields), where ε and μ are electric and magnetic permeabilities. The Maxwell equations for these vectors have the form

$$\mathrm{rot}\vec{E} = ik\vec{H}, \ \ \mathrm{rot}\vec{H} = -ik\vec{E}$$

$$k = \omega\sqrt{\varepsilon\mu}$$

The equations for the divergence are not used as they are linearly dependent on the equations for the rot.

Following [2] the covariant and contravariant components of vectors $\vec{E}$ and $\vec{H}$ are used. The correlations between these components involve the matrix G of the coordinate system

$$
\begin{aligned}
E_s &= \frac{1}{\rho^2} t^2 E^s - \frac{h\sigma}{\rho} t^2 E^\alpha, \\
E_\alpha &= -\frac{h\sigma}{\rho} t^2 E^s + h^2 \left(1 + 2\frac{h_s}{h} t + \left(\left(\frac{h_s}{h} \right)^2 + \sigma^2 \right) t^2 \right) E^\alpha, \\
E_l &= E^l
\end{aligned}
\tag{4}
$$

The Maxwell equations are symmetric with respect to changing $(H,\ E)$ into $(E,\ -H)$ and we will write out only one half of these equations

$$
\begin{aligned}
ikH^s &= \frac{1}{g} \left(\frac{\partial E_l}{\partial \alpha} - \frac{\partial E_\alpha}{\partial l} \right), \\
ikH^\alpha &= \frac{1}{g} \left(\frac{\partial E_s}{\partial l} - \frac{\partial E_l}{\partial s} \right), \\
ikH^l &= \frac{1}{g} \left(\frac{\partial E_\alpha}{\partial s} - \frac{\partial E_s}{\partial \alpha} \right)
\end{aligned}
\tag{5}
$$

The other three equations can be easily obtained by the mentioned above replacement $H \to E,\ \ E \to -H$.

4. Friedlander-Keller Solution

4.1 Friedlander-Keller ansatz

Let us use the standard Friedlander-Keller ansatz [3] for all the components of vectors $\vec{E}$ and $\vec{H}$, that is the field is searched in the form

$$
\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \exp\left(ikl + ik^{1/3}\tau \right) \sum_{j=0}^{\infty} k^{-j/3} \begin{pmatrix} \vec{E}_j \\ \vec{H}_j \end{pmatrix}
\tag{6}
$$

Here all the functions τ, E_j and H_j depend on coordinates s, α and l in such a way that their derivatives are all of order $O(1)$.

4.2. Governing recurrent system of equations

Substituting the ansatz (6) into the equations (5) to express components of $\vec{H}$ in terms of $\vec{E}$ and then using the nongiven three symmetric equations, one can get a system of equations involving only the vector $\vec{E}$. Equating terms of similar orders in $k^{1/3}$ in this system one can immediately find that the function τ does not depend on l and that the two leading coefficients in the decomposition of E^l are zeros. The dependency of τ on s and α will be found later when the matching with the creeping waves solution valid near the surface of the body will be performed. Equating the smaller order terms gives the following system of ordinary differential equations for the leading coefficients in the series (6)

$$\begin{cases} \left(\frac{\partial}{\partial l} + L_0\right) E_0^s + M_0 E_0^\alpha & = 0 \\ \left(\frac{\partial}{\partial l} + K_0\right) E_0^\alpha & = 0 \\ E_2^l + \frac{\partial \tau}{\partial \alpha} E_0^a + \frac{\partial \tau}{\partial s} E_0^s & = 0 \end{cases} \tag{7}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial l} + L_0\right) E_1^s + M_0 E_1^\alpha & = Q E_0^s \\ \left(\frac{\partial}{\partial l} + K_0\right) E_1^\alpha & = Q E_0^\alpha \end{cases} \tag{8}$$

for the first corrections. Analogously is possible to find equations for the next smaller order terms in the representation (6). These equations will have the same left-hand side operators as in (8), but the right-hand sides will be more complicated. The following functions are introduced in (7) and (8)

$$L_0 = \frac{1}{2}\frac{3h + 4h_s t}{t(h + h_s t)}, \quad M_0 = -\frac{h^2 \rho\sigma}{t(h + h_s t)}, \quad K_0 = \frac{1}{2}\frac{h + 4h_s t}{t(h + h_s t)},$$

$$Q = -\frac{i}{2}\left[\frac{\Omega^2}{(h + h_s(l - s))^2} + \frac{\tau_s^2 \rho^2}{(l - s)^2}\right], \quad \Omega = \tau_\alpha + \tau_s \rho\sigma h \tag{9}$$

4.3 Solution in the principal order

The second equation in (7) contains only E_0^α. Its solution depends on arbitrary function $C_0(s,\ \alpha)$

$$\begin{aligned} E_0^\alpha &= \frac{C_0(s,\ \alpha)}{\sqrt{l-s}J^{3/2}} \\ J &= h + h_s(l-s) \end{aligned} \tag{10}$$

Now we can solve the first equation in (7) and find the $\vec{s}$ component of vector E_0 containing one more arbitrary function D_0

$$E_0^s = \frac{1}{(l-s)^{3/2}\sqrt{J}}\left[D_0(s,\ \alpha) - \frac{h^2\rho\sigma C_0(s,\ \alpha)}{Jh_s}\right] \tag{11}$$

For the leading $\vec{l}$-component of the field we find

$$E_2^l = -\frac{1}{\sqrt{l-s}\sqrt{J}}\left\{\left[\tau_\alpha - \tau_s\frac{h^2\rho\sigma}{(l-s)h_s}\right]\frac{C_0(s,\ \alpha)}{J} + \frac{\tau_s}{l-s}D_0(s,\ \alpha)\right\} \tag{12}$$

The leading terms of the Friedlander-Keller asymptotics are found (10,11, 12). They depend on two arbitrary functions C_0 and D_0 which are defined by the incident field or by the source of electromagnetic waves. These functions and the function τ will be fixed by the matching procedure in the section **6**.

4.4 First corrections

Now we pass to the system of equations (8). This system have the same differential operator as in (7) in the left-hand side. And the second equation as previously is an ordinary differential equation for the $\vec{\alpha}$-coefficient E_1^α. So we shall first solve this second equation and then find the function E_1^s from the first equation of the system (8). The terms E_1^α and E_1^s are the sum of some particular solutions of inhomogeneous differential equations and general solutions of the homogeneous ones. As previously these terms contain arbitrary functions C_1 and D_1 of arguments α and s. The solutions are

$$E_1^\alpha = \frac{1}{\sqrt{l-s}J^{3/2}}\left\{C_1(s,\ \alpha) + \frac{i}{2}C_0(s,\ \alpha)\left(\frac{\tau_s^2\rho^2}{t} + \frac{\Omega^2}{h_sJ}\right)\right\}, \tag{13}$$

$$E_1^s = \frac{1}{(l-s)^{3/2}\sqrt{J}} \left\{ D_1(s,\ \alpha) - \frac{h^2\rho\sigma C_1(s,\ \alpha)}{Jh_s} \right.$$
$$\left. + \frac{i}{2}\left[D_0(s,\ \alpha) - \frac{h^2\rho\sigma C_0(s,\ \alpha)}{Jh_s}\right]\left(\frac{\tau_s^2\rho^2}{t} + \frac{\Omega^2}{h_s J}\right)\right\} \tag{14}$$

Here subscripts s and α denote differentiation by s and α respectively, the function Ω is given by the last formula (9).

Note that the corrections have stronger singularities on the surface of the body when $t = 0$ than the leading terms. In the next order terms of the asymptotics (6), which can be found analogously to the previous derivations, the singularities will increase.

The formulae (10–14) are valid for the case when $h_s \neq 0$. The case of constant surface divergence ($h_s = 0$) should be considered specially. The final formulae nevertheless will be presented in the form which can be used for the case $h_s = 0$, too. This is possible due to some successful cancellation of terms which infinitely increase when h_s tends to zero.

5. Creeping Waves

5.1 Ansatz and recurrent formulae

Following [2] we represent the creeping waves in the boundary layer in the form of ansatz

$$\vec{E} = \exp\left(iks + ik^{1/3}\varphi(s,\ \alpha)\right)\sum_{j=0}^{\infty}\vec{E}_j(s,\ \alpha,\ k^{2/3}n)k^{-j/3} \tag{15}$$

and analogous formula for the vector $\vec{H}$. Substituting these expressions into the Maxwell equations and equating terms of similar orders in $k^{1/3}$ similarly to [4] one can get the system of relations between the coefficients of (15) and nonpresented decomposition for $\vec{H}$. To find the principal order coefficients one needs the equations of orders k, $k^{2/3}$, $k^{1/3}$ and k^0. The equations of orders k and $k^{2/3}$ permit to express the $\vec{s}$ and $\vec{n}$ components of the fields $\vec{E}$ and $\vec{H}$ via its binormal $\vec{\alpha}$

components

$$\begin{array}{ll} H_0^s = 0, & E_0^s = 0, \\ E_0^n = -hH_0^\alpha, & H_0^n = hE_0^\alpha, \\ H_1^s = ih\frac{\partial E_0^\alpha}{\partial \nu_1}, & E_1^s = -ih\frac{\partial H_0^\alpha}{\partial \nu_1}, \\ E_1^n = -hH_1^\alpha, & H_1^n = hE_1^\alpha, \\ \dots & \dots \end{array}$$

All the following derivations up to the final formulae of this section will be performed for the binormal components. The equations in the order $k^{1/3}$ lead to the following formula for the attenuation factor φ

$$\varphi(s,\ \alpha) = 2^{-1/3} \int^s \rho^{-2/3} \xi ds \tag{16}$$

where ξ is the solution of one of the equations involving the Airy functions w_1 (the Fock notations [3] are used)

$$w_1'(\xi) + i(k\rho/2)^{1/3} Z w_1(\xi) = 0 \tag{17}$$

or

$$w_1'(\xi) + i(k\rho/2)^{1/3} Z^{-1} w_1(\xi) = 0 \tag{18}$$

In these equations Z is the impedance of the boundary and it is assumed to be of appropriate order in k so that the two terms in (17) or (18) be of similar orders. That provides the uniform property of the solution with respect to the impedance that is the formulas be valid both for the Dirichlet and Neumann conditions.

The imaginary part of ξ is responsible for the attenuation of the waves. So, only the solution with smallest imaginary part of ξ is significant, from the practical point of view. The behavior of ξ with respect to the impedance Z is examined in [5], [6].

If we take for ξ the solution of the equation (17) the leading E_α component is zero and we call this solution magnetic α-polarized creeping wave. In the opposite case, when ξ is the solution of (18) the α component of the magnetic field is zero and the field is α-polarized electric creeping wave. Generally saying, the electromagnetic field near the surface of the body is a sum of these two types of creeping waves. For simplicity we assume that the magnetic creeping wave is absent.

From the same $k^{1/3}$ order equations one can find also the dependency of the field on the normal coordinate in the leading term

$$E_0^\alpha = A_0(s,\ \alpha) w_1(\xi - \nu) \tag{19}$$

where we introduced new normal coordinate $\nu = k^{2/3}\rho^{-1/3}2^{1/3}n$.

The equations in the next order permit to calculate the dependency of the factor A_0 on arc-length s

$$A_0(s,\ \alpha) = A_0(0,\ \alpha)\left(\frac{h(s,\ \alpha)}{h(0,\ \alpha)}\right)^{-3/2}\left(\frac{\rho(s,\ \alpha)}{\rho(0,\ \alpha)}\right)^{-1/6}\left(\frac{d(s,\ \alpha)}{d(0,\ \alpha)}\right)^{-1/2}$$
$$d = \left(w_1'(\xi)\right)^2 - \xi w_1(\xi)^2 \tag{20}$$

The function $A_0(0,\ \alpha)$ characterizes the excitation of creeping rays following different geodesics. From the equations of the same order k^0 we get the differential equation for the first correction E_1^α

$$\frac{\partial^2 E_1^\alpha}{\partial \nu^2} + (\xi - \nu)E_1^\alpha = \mathcal{F} \tag{21}$$

and leading term H_1^α for the α component in (15)

$$\frac{\partial^2 H_1^\alpha}{\partial \nu^2} + (\xi - \nu)H_1^\alpha = \mathcal{G} \tag{22}$$

with the right-hand sides $\mathcal{F}, \mathcal{G}$ being result of some operators on E_0^α. The equations (21, 22) are supplemented with the boundary conditions which are also inhomogeneous in this order. It is not difficult to transfer these right-hand sides in the boundary conditions to the equations (21 , 22). That will bring only to some changes in the right-hand sides. The solutions of the equations (21, 22) are the sums of particular solutions E_{1p}^α, H_{1p}^α of the inhomogeneous equations and arbitrary solutions E_{1a}^α, H_{1a}^α of the homogeneous Airy equation. The solution H_{1a}^α is fixed immediately by the boundary condition because we have chosen ξ from (18), but not from (17). The solution E_{1a}^α satisfies the homogeneous boundary condition automatically and it remains arbitrary till the next order equations are used.

The further computations follows exactly the scheme presented in [4] where only the principal terms were calculated. The intermediate formulae become rather cumbersome and are not presented here.

5.2 Asymptotics of the electric transversally polarized creeping wave

The final results in two orders in $k^{1/3}$ are presented in this paragraph. To make formulae more compact we dropped the common for all the formulae multiplier $A_0(0,\ \alpha)$. We introduced some notations

$$S = \left(\frac{h(s,\ \alpha)}{h(0,\ \alpha)}\right)^{-3/2} \left(\frac{\rho(s,\ \alpha)}{\rho(0,\ \alpha)}\right)^{-1/6} \left(\frac{d(s,\ \alpha)}{d(0,\ \alpha)}\right)^{-1/2},$$

$$m = \left(\frac{k\rho}{2}\right)^{1/3} \frac{1}{Z}, \quad r = \left(\frac{\rho}{2}\right)^{2/3},$$

$$f = \frac{\partial}{\partial \alpha} \left(2^{-1/3} \int^{s} \rho^{-2/3} \xi ds \right)$$

For ψ see [2] formula (14). All the characteristics of the geometry and boundary impedance are calculated in the point N of the normal origin. The subscripts $_s$ denote the derivative by arc-length s in the surface geodesic system of coordinates.

We write out here only the components of electric vector $\vec{E}$

$$\begin{aligned}
E_0^{\alpha} &= S w_1(\xi - \nu), \\
E_1^{\alpha} &= iSr\frac{\xi_s}{\xi + m^2} w_1'(\xi - \nu) + S\left[\psi + i\xi_s r\nu + \frac{i}{6}\frac{\rho_s}{\rho} r\nu^2\right] w_1(\xi - \nu), \\
E_0^{n} &= 0, \\
E_1^{n} &= -\, 2irSh\sigma w_1'(\xi - \nu) \\
&\qquad -\, 2rS\frac{h\sigma\xi + hm^2\sigma Z^2 + 2^{-2/3}\rho^{-1/3} f}{m(Z^2 - 1)} w_1(\xi - \nu), \\
E_0^{s} &= 0, \\
E_1^{s} &= 0, \\
E_2^{s} &= -Sfw_1(\xi - \nu)cr
\end{aligned} \tag{23}$$

6. Matching Procedure

6.1 Asymptotic decomposition of the creeping waves in the intermediate zone

The matching of the asymptotics (10 – 13) and (14) can be performed in the point M in the region where the coordinate $t = O(k^{-\delta})$, $\frac{1}{4} < \delta < \frac{1}{3}$. The asymptotics (23) of the creeping waves is written in the surface geodesic coordinates. That is the geometry characteristics $h,\ \rho,\ \sigma$, their derivatives and other parameters of the diffraction problem are calculated in the point N of the origin of the normal to M. In the Friedlander-Keller asymptotic expansion the evolute coordinates $(s,\ \alpha,\ l)$ are used and the characteristics of the surface are calculated in the creeping point C. Thus is necessary first to rewrite the asymptotics in one coordinate system. We shall use the evolute coordinate system. The parameters of the problem involved in the asymptotics (23) can be decomposed in Taylor series by $\triangle s = s_N - s_C$ and $\triangle \alpha = \alpha_N - \alpha_C$. Then these differences $\triangle s$ and $\triangle \alpha$ with the help of formulae (1) are expressed in terms of t. Then one should take into account that in the creeping waves asymptotics the basis for the vectors decomposition is $\vec{\alpha_N},\ \vec{n_N},\ \vec{s_N}$. So for the matching it is necessary to redecompose the electric vector, that is to find its $\vec{\alpha},\ \vec{s}$ and $\vec{l}$ components. These transformations are carried with the help of formulae (1) describing the decomposition of the vectors $\vec{\alpha_N},\ \vec{n_N},\ \vec{s_N}$ by vectors $\vec{\alpha},\ \vec{s}$ and $\vec{l}$. The component E^α is $k^{2/3}$ times larger than the component E^s and not to lose accuracy we need to know the corresponding coefficient in the vectors transformation up to terms of order $O(t^2)$

$$\vec{\alpha}_N = \vec{\alpha},\ \ \vec{s}_N = \vec{l} + \vec{s},\ \ \vec{n}_N = -\frac{\rho}{t}\vec{s} + \frac{t}{\rho}\vec{l} - \frac{\sigma t}{h}\vec{\alpha}$$

For the components that gives

$$E^\alpha = E^{\alpha_N} - \frac{\sigma(l-s)}{h} E^{n_N} \tag{24}$$

$$E^s = E^{s_N} - \frac{\rho}{l-s} E^{n_N} \tag{25}$$

$$E^l = E^{s_N} + \frac{l-s}{\rho} E^{n_N} \tag{26}$$

In the intermediate region of matching the value of $t = l - s$ is small and due to this we neglect the second term in (24), which is of order $O(k^{-2/3})$, and the first term in (25), which is also of order $O(k^{-2/3})$. The asymptotics (23) gives for E^n component only higher order term, and according to (25) the E^s component is calculated only in the principal order. The both terms in (26) become of order $O(k^{-2/3})$, and we find the E^l component again only in the principal order $O(k^{-2/3})$. This order coincides with the Friedlander-Keller asymptotic decomposition.

Then one can note that in the intermediate zone the argument of the Airy function and its derivative is large negative and the asymptotic decompositions [7]

$$w_1(-\gamma) = \gamma^{-1/4} \exp\left(i\frac{2}{3}\gamma^{3/2} + i\frac{\pi}{4}\right)\left(1 + O(\frac{1}{\gamma^{3/2}})\right) \tag{27}$$

$$w_1'(-\gamma) = \gamma^{1/4} \exp\left(i\frac{2}{3}\gamma^{3/2} - i\frac{\pi}{4}\right)\left(1 + O(\frac{1}{\gamma^{3/2}})\right) \tag{28}$$

We present the main steps of our derivations. In the formulae below the functions ρ, φ and ξ are calculated in points N and C. In the first case we write $\rho(N)$, $\varphi(N)$, $\xi(N)$ in the second we drop the arguments.

$$\nu = k^{2/3}2^{1/3}\rho(N)\left(\frac{t^2}{2\rho} - \frac{\rho_s t^3}{6\rho^2}\right) = 2^{-2/3}\rho^{-4/3}t_1^2 - \frac{k^{-1/3}}{3}2^{1/3}\rho^{-4/3}\frac{\rho_s}{\rho}t_1^3,$$

$$\xi(N) = \xi + k^{-1/3}\xi_s t_1,$$

The exponent argument in the ansatz of creeping waves is decomposed into the following expression

$$\begin{aligned} iks(N) + ik^{1/3}\varphi(N) =& ikl + ik^{1/3}\varphi + i2^{-1/3}\rho^{-2/3}\xi t_1 - \frac{i}{3\rho^2}t_1^3 \\ &+ \frac{7i}{24}k^{-1/3}\frac{\rho_s}{\rho^3}t_1^4 + \frac{i}{2}k^{-1/3}2^{-1/3}\rho^{-2/3}\xi\left(\frac{\xi_s}{\xi} - \frac{2}{3}\frac{\rho_s}{\rho}\right)t_1^2, \end{aligned}$$

Here we neglected the terms of order $O(k^{-2/3})$ and terms containing

t_1^2. The phase function arising from the asymptotics (27, 28) gives

$$\frac{i\pi}{4}+\frac{i}{3}\frac{t_1^3}{\rho^2}-i2^{-1/3}\xi\rho^{-2/3}t_1+\frac{i}{4}2^{1/3}\rho^{2/3}\xi^2t_1^{-1}$$
$$+ik^{-1/3}\Big\{-\frac{1}{3}\frac{\rho_s}{\rho}t_1^4-2^{-1/3}\rho^{-2/3}\xi_st_1^2+\frac{1}{3}2^{-1/3}\rho^{-2/3}\xi\frac{\rho_s}{\rho}t_1^2$$
$$+2^{-2/3}\rho^{2/3}\xi\xi_s+2^{-2/3}\rho^{2/3}\xi^2\left(\frac{1}{6}\frac{\rho_s}{\rho}+\frac{\xi_s}{\xi}\right)\Big\}$$

Finally for the exponent in the creeping waves asymptotics one gets

$$\begin{aligned}\mathbf{\Psi}_{cr}=&ks+k^{2/3}t_1+k^{1/3}\varphi+\frac{\pi}{4}+\frac{1}{4}2^{1/3}\rho^{2/3}\xi^2t_1^{-1}\\&+k^{-1/3}\left\{-\frac{1}{24}\frac{\rho_s}{\rho^3}t_1^4-2^{-4/3}\rho^{-2/3}\xi_st_1^2+2^{-2/3}\rho^{2/3}\xi^2\left(\frac{\xi_s}{\xi}+\frac{1}{6}\frac{\rho_s}{\rho}\right)\right\}\end{aligned}\tag{29}$$

The factor $S(N)$ is decomposed as

$$S-k^{-1/3}S\left(\frac{3}{2}\frac{h_s}{h}+\frac{1}{6}\frac{\rho_s}{\rho}+\frac{1}{2}\frac{\xi_s}{m^2+\xi}\right)t_1$$

The exponent of small terms can be decomposed in series and combined with the amplitudes. We shall perform that decomposition in two steps. First we decompose $\exp(O(k^{-1/3}))$ and find that t_1 is present in powers not higher than $1/2$. That means that we need not take into account the terms of order $O(t_1^{-3})$ in the exponent. On the second step we decompose $\exp(O(t_1^{-1}))$. Substituting these formulae into the asymptotics (23) one can find the principal terms of the electric creeping wave decomposition in the intermediate zone

$$\begin{aligned}\vec{E}=Se^{i\frac{\pi}{4}}\exp\left\{ikl+ik^{1/3}\varphi\right\}2^{1/6}\rho^{1/3}t_1^{-1/2}&\\ \left\{\left[e_0^\alpha+k^{-1/3}e_1^\alpha\right]\vec{\alpha}+e_0^s\vec{s}+k^{-2/3}e_2^l\vec{l}\right\}&\end{aligned}\tag{30}$$

where

$$e_0^\alpha=1+\frac{i}{4}2^{1/3}\rho^{2/3}\xi^2t_1^{-1}+O(t_1^{-2}),\tag{31}$$
$$e_1^\alpha=-\frac{3}{2}\frac{h_s}{h}t_1+\psi+i2^{-2/3}\rho^{2/3}\xi^2\left(\frac{\xi_s}{\xi}+\frac{1}{6}\frac{\rho_s}{\rho}\right)$$

$$
\begin{aligned}
&- \frac{3i}{8} 2^{1/3} \rho^{2/3} \xi^2 \frac{h_s}{h} + O(t_1^{-1}), && (32)\\
e_0^s =& h\rho\sigma + \Big(\frac{i}{4} 2^{1/3} \rho^{5/3} h\sigma\xi^2 \\
&+ 2^{-2/3} \rho^{5/3} \frac{h\sigma\xi + hm^2\sigma Z^2 + 2^{-2/3}\rho^{-1/3} f}{m(Z^2-1)} \Big) t_1^{-1} + O(t_1^{-2}), && (33)\\
e_2^l =& - h\sigma t_1 - \frac{i}{4} 2^{1/3} \rho^{2/3} h\sigma\xi^2 \\
&- 2^{-2/3} \rho^{5/3} \frac{h\sigma\xi + hm^2\sigma Z^2 + 2^{-2/3}\rho^{-1/3} f}{m(Z^2-1)} - f + O(t_1^{-1}) && (34)
\end{aligned}
$$

6.2 Asymptotic decomposition of the Friedlander-Keller solution in the intermediate zone

The creeping wave asymptotics (23) permitted to find two terms in the decomposition of E^α and only one term in the decompositions of E^s and E^l. So in the Friedlander-Keller asymptotics in the intermediate zone we need to know the leading terms for all the components and first corrections only for the component E^α.

In the intermediate zone the function J in the asymptotics (10 – 14) can be decomposed in Taylor series by small t which we present as $k^{-1/3}t_1$. We use only two terms of the decomposition

$$J = h + 2h_s k^{-1/3} t_1 + \dots$$

The Friedlander-Keller asymptotics takes the form

$$
\begin{aligned}
E^\alpha =& k^{1/6} \frac{C_0(s,\ \alpha)}{h^{3/2}} \left\{ t_1^{-1/2} + \frac{i}{2} \rho^2 \tau_s^2 t_1^{-3/2} \right\} \\
&+ k^{-1/6} \frac{1}{h^{3/2}} \Big\{ C_1(s,\ \alpha) t_1^{-1/2} \\
&-C_0(s,\ \alpha) \left[\frac{3}{2} \frac{h_s}{h} t_1^{1/2} + \frac{i}{2} \left(-\frac{3}{2} \rho^2 \tau_s^2 \frac{h_s}{h} + \frac{\Omega^2}{hh_s} \right) t_1^{-1/2} \right] \Big\}, \qquad (35)
\end{aligned}
$$

The decomposition of the other components becomes more simple if we introduce the functions

$$B_j(s,\ \alpha) = D_j(s,\ \alpha) - \frac{h\rho\sigma}{h_s} C_j(s,\ \alpha) \qquad (36)$$

and express D_j via B_j and C_j .

$$
\begin{aligned}
E^s =& k^{1/2}\frac{1}{t_1^{3/2}\sqrt{h}}B_0(s,\ \alpha)\left(1+\frac{i}{2}\frac{\tau_s^2\rho^2}{t_1}\right) \\
&+ k^{1/6}\frac{1}{t_1^{3/2}\sqrt{h}}\{B_1(s,\ \alpha) \\
&+\rho\sigma C_0(s,\ \alpha)\left(t_1+\frac{3i}{4}\tau_s^2\rho^2\right)+\frac{1}{2}\frac{i\Omega^2-h_s^2t_1}{hh_s}B_0(s,\ \alpha)\Bigg\} \qquad (37) \\
E^l =& -k^{-1/6}\frac{B_0(s,\ \alpha)\tau_s}{t_1^{3/2}\sqrt{h}} \\
&+ k^{-1/2}\frac{1}{\sqrt{t_1}h^{3/2}}\left(C_0(s,\ \alpha)\Omega-\frac{\tau_s h_s}{2}B_0(s,\ \alpha)\right), \qquad (38)
\end{aligned}
$$

The exponential factor in the intermediate zone is

$$
\mathbf{\Psi}_{fk} = iks + ik^{2/3}t_1 + ik^{1/3}\tau \qquad (39)
$$

Now we have all the formulae to do the matching and to find the functions $C_0,\ D_0,\ C_1,\ D_1$.

6.3 Matching of the phase function

Let us match first the phase functions of the two asymptotic decompositions. Comparing the phase functions in the Friedlander-Keller ansatz (39) and in the formula (29) one can find that the terms of orders k and $k^{2/3}$ coincide automatically. The terms of order $k^{1/3}$ determine the expression for the function τ

$$
\tau = 2^{-1/3}\int^{s_C}\frac{\xi ds}{\rho^{2/3}} \qquad (40)
$$

Note that τ does not depend on l , but only on the coordinates of the creeping point C .

6.4 Matching of the amplitudes

The terms of order $O(1)$ in the phase function and the leading terms in the amplitude permit to find the unknown functions $C_0(s,\ \alpha)$ and $D_0(s,\ \alpha)$ in (10, 1). We shall use the α and s components of the electric vectors for the matching.

Comparing the formula (31) with the leading term of the Friedlander-Keller asymptotics in the intermediate zone (35) it is not difficult to find C_0

$$C_0 = 2^{1/6}k^{-1/6}\rho^{1/3}h^{3/2}Se^{\frac{i}{4}\pi} \tag{41}$$

Here we matched the coefficients of the terms $t_1^{-1/2}$ and $t_1^{-3/2}$ in the principal in $k^{1/3}$ order. We did not consider the terms of higher negative powers of t_1. To add these terms one should find the higher order corrections in the Friedlander-Keller solution. These corrections will have terms with higher singularities on the surface, that is with higher negative powers of t_1, and in the intermediate zone they all will participate in the leading in k term of the decomposition. One can check that the result (41) will not change in this case.

To find the function C_1 one should match the terms of order $O(k^{-1/3})$ (32) with terms of order $O(k^{-1/6})$ in (35). The principal terms of order $O(t_1^{1/2})$ coincide due to (41) and terms of order $O(t_1^{-1/2})$ permit to find the expression for C_1

$$C_1 = i2^{1/6}k^{-1/6}\rho^{1/3}h^{3/2}Se^{\frac{i}{4}\pi}\left[2^{-2/3}\rho^{2/3}\xi^2\left(\frac{\xi_s}{\xi}+\frac{1}{6}\frac{\rho_s}{\rho}\right)-\frac{\Omega^2}{2hh_s}-i\psi\right] \tag{42}$$

Analysis of the orders of E^s components in the Friedlander-Keller and creeping waves solutions shows that

$$B_0 \equiv 0 \tag{43}$$

The principal terms of order $O(t_1^{-1/2})$ coincides in the two formulas automatically due to (41). Comparing the terms of order $O(t_1^{-3/2})$ one can find for B_1 the following formula

$$B_1 = 2^{1/6}k^{-1/6}\rho^2h^{3/2}Se^{\frac{i}{4}\pi}$$
$$\left[-\frac{i}{4}2^{-2/3}\sigma\xi^2 + 2^{-2/3}\frac{h\sigma\xi + hm^2\sigma Z^2 + 2^{-2/3}\rho^{-1/3}f}{hm(Z^2-1)}\right] \tag{44}$$

For the E^l component we have only the principal term containing C_0 and D_0. In the intermediate zone where the matching is performed we can get only the principal term which is proportional to B_0 and due to (43) disappears.

7. Final Formulae

Substituting the expressions for C_0, C_1, B_0 and B_1 $(41-44)$ and performing evident derivations one can find that in the principal order the asymptotics of the electric vector have the form

$$\begin{aligned}\vec{E} = \ & 2^{1/6}k^{-1/6}\rho^{1/3}h^{3/2}S\exp\left(ikl+ik^{1/3}\tau+\frac{i}{4}\pi\right)(l-s)^{-1/2}J^{-3/2}\times \\ & \times\left\{\left[1+k^{-1/3}\left(i2^{-2/3}\rho^{2/3}\xi^2\left(\frac{1}{2}\frac{1}{l-s}+\frac{\xi_s}{\xi}+\frac{1}{6}\frac{\rho_s}{\rho}\right)+\psi-\right.\right.\right. \\ & \left.\left.-\frac{i\Omega^2}{2h}\frac{t}{j}\right)+O(k^{-2/3})\right]\vec{\alpha}+h\rho\sigma\left[1+k^{-1/3}\left(i2^{-2/3}\rho^{2/3}\xi^2\right.\right. \\ & \left(\frac{1}{2}\frac{1}{l-s}-\frac{1}{4h}\frac{J}{l-s}+\frac{\xi_s}{\xi}+\frac{1}{6}\frac{\rho_s}{\rho}\right)+\psi-i\frac{\Omega^2}{2h}\frac{l-s}{J}2^{-2/3}\rho^{2/3} \\ & \left.\left.\frac{\xi+m^2Z^2+2^{-2/3}\rho^{-1/3}h^{-1}\sigma^{-1}f}{hm(Z^2-1)}\frac{J}{l-s}\right)+O(k^{-2/3})\right]\vec{s} \\ & \left.+\,k^{-2/3}\left[-\Omega+O(k^{-1/3})\right]\vec{l}\right\}\end{aligned} \tag{45}$$

This result is also valid when $h_s = 0$ and correlates with the asymptotics given in [8].

It is more convenient to represent the formula (45) giving the asymptotics of Friedlander-Keller field using the basis of vectors in the creeping point α_C, n_C and s_C

$$
\begin{aligned}
\vec{E} = {} & 2^{1/6}k^{-1/6}\rho^{1/3}h^{1/2}S\exp\Big\{ikl + ik^{1/3}\tau + \\
& +\frac{i}{4}\pi + k^{-1/3}\left(\psi - i\frac{\Omega^2}{2h}\frac{l-s}{J}\right)\Big\}(l-s)^{-1/2}J^{-1/2}\times \\
& \times\Big\{\Big[\,1 + k^{-1/3}\left(i2^{-2/3}\rho^{2/3}\xi^2\left(\frac{1}{2}\frac{1}{l-s} + \frac{\xi_s}{\xi} + \frac{1}{6}\frac{\rho_s}{\rho}\right)\right) \\
& +O(k^{-2/3})\Big]\,\vec{\alpha}_C + \ h\sigma k^{-1/3}2^{-2/3}\rho^{2/3}\Big[\frac{i}{4}\xi^2 - \\
& -\frac{\xi + m^2Z^2 + 2^{-2/3}\rho^{-1/3}h^{-1}\sigma^{-1}f}{hm(Z^2-1)} + O(k^{-1/3})\Big]\,\vec{n}_C \\
& + \ k^{-2/3}\left[-\frac{\Omega}{J} + O(k^{-1/3})\right]\vec{s}_C\Big\}
\end{aligned}
\tag{46}
$$

For the perfectly conducting body ($Z = 0$) the formula simplifies

$$
\begin{aligned}
\vec{E} = {} & 2^{1/6}k^{-1/6}\rho^{1/3}h^{1/2}S\exp\Big\{ikl + ik^{1/3}\tau + \frac{i}{4}\pi + \\
& +k^{-1/3}\left(\psi - i\frac{\Omega^2}{2h}\frac{l-s}{J}\right)\Big\}(l-s)^{-1/2}J^{-1/2} \\
& \times\Big\{\left[1 + k^{-1/3}\left(i2^{-2/3}\rho^{2/3}\xi^2\left(\frac{1}{2}\frac{1}{l-s} + \frac{1}{6}\frac{\rho_s}{\rho}\right)\right) + O(k^{-2/3})\right]\vec{\alpha}_C \\
& + \ h\sigma k^{-1/3}2^{-2/3}\rho^{2/3}\left[\frac{i}{4}\xi^2 + O(k^{-1/3})\right]\vec{n}_C \\
& + \ k^{-2/3}\left[-\frac{\Omega}{J} + O(k^{-1/3})\right]\vec{s}_C\Big\}
\end{aligned}
\tag{47}
$$

The formulae (46, 47) give the field of electrical transversally polarized creeping wave. For the magnetic transversally polarized creeping wave the formulae can be derived by substituting (45) into the Maxwell equations to express the magnetic field components and then by replacing Z by $1/Z$. That gives

$$\vec{E} = 2^{1/6}k^{-1/6}\rho^{1/3}h^{1/2}S\exp\Big\{ikl + ik^{1/3}\tau + \frac{i}{4}\pi$$
$$+ k^{-1/3}\left(\psi - i\frac{\Omega^2}{2h}\frac{l-s}{J}\right)\Big\}(l-s)^{-1/2}J^{-1/2}$$
$$\times\Bigg\{ k^{-1/3}\Bigg[2^{-2/3}\rho^{2/3}h\sigma\xi^2\Big(-\frac{i}{4h}+$$
$$+ \frac{\xi + m^2Z^2 + 2^{-2/3}\rho^{-1/3}h^{-1}\sigma^{-1}f}{hmZ^2(1-Z^2)}\Big) + O(k^{-1/3})\Bigg]\vec{\alpha}_C \tag{48}$$
$$+ \Bigg[1 + k^{-1/3}i2^{-2/3}\rho^{2/3}\xi^2\left(\frac{1}{2}\frac{1}{l-s} + \frac{\xi_s}{\xi} + \frac{1}{6}\frac{\rho_s}{\rho}\right)$$
$$+ O(k^{-2/3})\Bigg]\vec{n}_C + k^{-2/3}\left[\frac{\rho\tau_s}{l-s} + O(k^{-1/3})\right]\vec{s}_C\Bigg\}$$

Again for the case of perfect conductor the formula simplifies

$$\vec{E} = 2^{1/6}k^{-1/6}\rho^{1/3}h^{1/2}S\exp\Bigg\{ikl + ik^{1/3}\tau + \frac{i}{4}\pi +$$
$$+k^{-1/3}\left(\psi - i\frac{\Omega^2}{2h}\frac{l-s}{J}\right)\Bigg\}(l-s)^{-1/2}J^{-1/2}\times$$
$$\times\Bigg\{ k^{-1/3}\left[-\frac{i}{4}2^{-2/3}\rho^{2/3}\sigma\xi^2 + O(k^{-1/3})\right]\vec{\alpha}_C$$
$$+ \left[1 + k^{-1/3}i2^{-2/3}\rho^{2/3}\xi^2\left(\frac{1}{2}\frac{1}{l-s} + \frac{1}{6}\frac{\rho_s}{\rho}\right) + O(k^{-2/3})\right]\vec{n}_C$$
$$+ k^{-2/3}\left[\frac{\rho\tau_s}{l-s} + O(k^{-1/3})\right]\vec{s}_C\Bigg\} \tag{49}$$

The term $(l-s)^{-1/2}J^{-1/2}$ has natural geometrical interpretation, it gives geometrical divergence of the free space part of the creeping ray. As expected for the electric creeping ray the main part of the electric field is directed along the vector $\vec{\alpha}_C$ in the creeping point. There are two corrections of order $k^{-1/3}$, one is for $\vec{\alpha}_C$ component and the other is for $\vec{n}_C$ component. There is also a phase correction of the same order. Part of this correction (ψ) is due to the creeping wave propagation along the surface of the body up to the creeping point C.

The other part takes into account the propagation of the creeping ray in free space. This part is proportional to the factor Ω^2 which is zero for the case of a cylinder and sphere.

For the perfectly conducting case the $\vec{\alpha}_C$ component includes the term $\rho_s/(6\rho)$ proportional to the derivative of the radius of curvature in the direction of the creeping ray. The other correction in this component which turns out to be proportional to $1/(l-s)$ namely to the inverse of the length of the space part of the creeping ray rapidly decreases far away from the creeping point. The correction of the component along $\vec{n}_C$ is proportional to the product of the radius of curvature and the torsion of the creeping ray.

The field directed along the ray ($\vec{s_C}$ component) is of order $k^{-2/3}$, moreover it decreases with the distance from the surface.

8. Conclusion

In this paper we have computed the field carried by a creeping ray propagating on an arbitrary impedance convex surface both in the boundary layer in the close vicinity of the surface and in the ray zone outside this boundary layer. We give formulas both for electric and magnetic transversaly polarized creeping waves. The main order term of the solution presented in this paper is well known, it is just the standard result for the field carried by a creeping ray [1], [8]. The second order terms of our solution permit to examine the effect of torsion, variation of the curvature and impedance on the electromagnetic field.

The application of these formulas to real problems ask for the computation of the excitation coefficients of the creeping waves. For example, for the problem of excitation of creeping waves by a source located on the surface these coefficients are given in [9]. If the source is far away from the surface these coefficients can be found in [1].

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