Progress In Electromagnetics Research, *PIER 15, 221–252, 1997*

THE GREEN'S FUNCTION PERTURBATION METHOD FOR SOLUTION OF ELECTROMAGNETIC SCATTERING PROBLEMS

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1. Introduction

In this paper we will introduce a new approximate method for solving large scale two-dimensional electromagnetic boundary value problems over perfect electric conductors *PEC*. Specifically, we will expand the Green's function in the kernel of the electric field integral equation *EFIE* [3] as a perturbation series with respect to the arclength variable s. The series is restricted by demanding that the first term is Toeplitz [15] (*Toeplitz* operators are also known as cyclic or convolution operators) with respect to s. The approximation is achieved by using the series form of the Green's function, truncated after the Toeplitz term, in the *EFIE*. The reasoning behind such a scheme lies in the computational efficiency of inverting Toeplitz operators. In particular, Toeplitz operators can be inverted, using the fast Fourier transform FFT [6] thus reducing an $\mathcal{O}(N^3)$ operation, where N is the number of discretizations along the scatterer, to an $\mathcal{O}(N \log N)$ operation. There is also a saving in core storage from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$.

In section 1, beginning with the Electric Field Integral Equation we show how this can be expressed as a perturbation series where the first term is a convolution integral. In section 2 we examine the conditions for convergence of the series and obtain a bound for the neglected perturbation terms when using the GFPM. In section 3 we consider how best to choose the integral transformation and show how this may be done to minimize the error in an average sense. In section 4 we show that the method is reciprocal and in section 5 the GFPM is compared to the Kirchhoff and height perturbation methods. Finally, some numerical examples are presented along with expreimental data for UHF propagation over irregular terrain.

We would like to mention another different method due to Urusovskii [17] who used a convolution to calculate scattering from periodic surfaces. He considered sinusoids with very small maximum slope such that the surface normal at every point could be approximated by the normal to the mean plane. In essence the surface was projected onto the mean plane but the boundary condition was not.

In an earlier paper [9] presented at the ICAP'95 conference, the authors outlined the the Green's Function Perturbation Method applied to UHF propagation over undulating terrain. The purpose of this paper is to describe the Green's Function Perturbation Method for electromagnetic scattering in greater detail.

In this chapter the acronym PEC always denotes that the scatterer is a pefect electrical conductor. In the main we deal with open surfaces. Also, since in reality, the surface current will often be negligible outside a certain finite region, in there cares we can use a function defined over a finite domain to represent the surface without loss of accuracy.

2. Integral Transformation

Referring to the geometry in Figure (1) the *EFIE* for a *PEC* is

$$E^s(\boldsymbol{\rho}) = -\frac{k\eta}{4}\int_S J(\boldsymbol{\rho}')H_0^{(2)}\left(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|\right)\, dp' \tag{1}$$

where $J(\boldsymbol{\rho}')$ is the induced current on the *PEC* surface, $H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)$ is a Hankel function of the second kind of order zero and E^s is the scattered electric field.

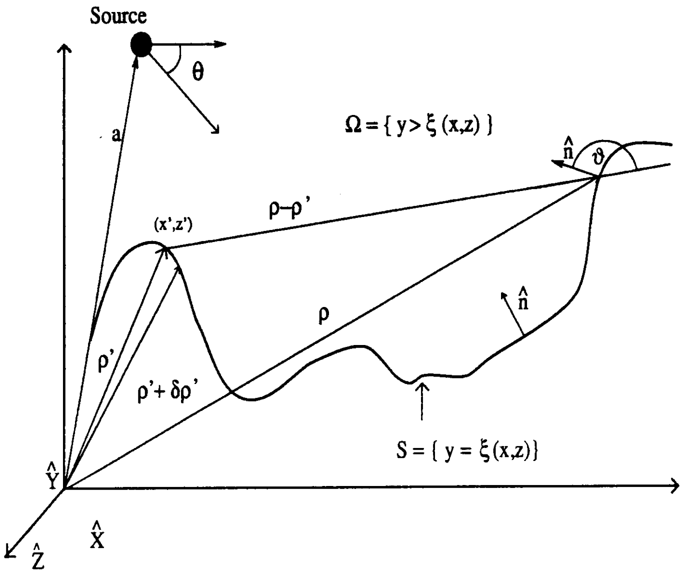

Figure 1. Scattering Geometry

The Green's function for the *EFIE* is the Hankel function

$$H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) \tag{2}$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are vectors representing points of integration and points of contribution, respectively, in the integrand of (1) and $k = \beta - j\alpha$ is the propagation constant. The propagating medium is chosen to be slightly absorptive so that the solution $E^s(\rho)$ is of order $e^{-\alpha/\boldsymbol{\rho}}$ as $\boldsymbol{\rho} \to \infty$ (i.e., the radiation condition at infinity is guaranteed).

In Cartesian coordinates the surface S is represented in terms of the single variable x as

$$z = \xi(x) \tag{3}$$

which is assumed to be continuous, bounded and differentiable, and we define the vector $\boldsymbol{\rho}$ which traces the surface S as

$$\boldsymbol{\rho} = x\,\hat{\boldsymbol{x}} + \xi(x)\,\hat{\boldsymbol{z}} \tag{4}$$

The distance between any two points, $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$, on the surface is given by

$$|\boldsymbol{\rho} - \boldsymbol{\rho}'| = \sqrt{(x-x')^2 + (\xi(x) - \xi(x'))^2} \tag{5}$$

The independent variable x can be changed into a function of the variable of integration; the arclength s, i.e., $x \to f(s)$. This may be expressed as

$$s = \int_0^x \sqrt{1 + \left(\frac{d\xi(x)}{dx}\right)^2}\, dx \tag{6}$$

Equations of the form (6) have closed form solutions only for certain classes of functions ξ but all can be solved numerically. The actual values for s are not of critical importance for this analysis.

We continue by introducing a function $g(x)$. This function is chosen by some means to best approximate the Euclidean distance function $\mathbf{d}(s,s')$ such that

$$g(s-s') \approx \sqrt{(f(s) - f(s'))^2 + (\xi(f(s)) - \xi(f(s')))^2} = \mathbf{d}\left(s,s'\right) \tag{7}$$

where we have replaced x by $f(s)$. It is clear that $\mathbf{d}(s,s')$ is a function of position and that $g(s-s')$ is a function of arclength between two points ρ and ρ' described by s and s'. The properties of $g(x)$ are chosen so as to match those of $\mathbf{d}(s,s')$. Explicitly they are as follows;

- The distance between two points in the plane is real so $g(x)$ will be real.
- The distance between two points is greater than or equal to zero so $g(x)$ will be positive.
- The distance from A to B is the same as from B to A so $g(x)$ will be even.
- The distance function defined along any continuous differentiable surface is differentiable so $g(x)$ will be differentiable.

With some straightforward algebraic manipulation we may write the Hankel function as

$$H_0^{(2)}\left(kg(s-s')\,[1+T]^{\frac{1}{2}}\right), \tag{8}$$

where

$$|\boldsymbol{\rho} - \boldsymbol{\rho}'| = g\left(s - s'\right)\sqrt{1+T} \tag{9}$$

and

$$T = T(s, s') = -1 + \left[\frac{f(s) - f(s')}{g(s - s')}\right]^2 + \left[\frac{\xi\left(f(s)\right) - \xi(f(s'))}{g(s - s')}\right]^2 \tag{10}$$

We proceed by introducing the restriction that

$$|T| < 1 \tag{11}$$

This presents a further limitation on the set of permissible functions $g(x)$. Equation (11) will be satisfied when $g(s-s')$ is greater than half of $\mathbf{d}(s, s')$ and allows us to remove the square root dependency in the argument of the Hankel function by expanding the term $(1+T)^{1/2}$ in (8) into a binomial series giving

$$H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) = H_0^{(2)}\left(kg(s - s') + kg(s - s')B\right), \tag{12}$$

where

$$B = B\left(s, s'\right) = \sum_{m=1}^{\infty} \binom{1/2}{m} T^m = \sqrt{1+T} - 1 \tag{13}$$

Equation (12) can now be expanded as a Taylor series about $kg(s-s')$ giving

$$\begin{aligned} H_0^{(2)}\left(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|\right) = & H_0^{(2)}\left(kg(s - s')\right) + \\ & \sum_{n=1}^{\infty} \frac{[kg(s - s')B]^n}{n!} H_0^{(2)^{(n)}}(kg(s - s')) \end{aligned} \tag{14}$$

where

$$H_0^{(2)^{(n)}}(x) = \frac{d^n}{dx^n} H_0^{(2)}(x) \tag{15}$$

Replacing the Hankel function in (1) we have

$$\begin{aligned} E^i(s') = & \frac{k\eta}{4} \int_S J(s') H_0^{(2)}\left(kg(s - s')\right) ds' + \\ & \frac{k\eta}{4} \int_S J(s') \sum_{n=1}^{\infty} \frac{[kg(s - s')B]^n}{n!} H_0^{(2)^{(n)}}\left(kg(s - s')\right) ds' \end{aligned} \tag{16}$$

We have thus created a series formulation for the *EFIE*. We choose to take the first term and use it as an approximation. We denote the approximate current density so calculated by $J_c = J_c(f(s'), \xi(f(s')))$. It is also clear that the first term approximation is a convolution in the scalar variable s. This allows us to calculate J_c by means of a Fourier transform technique which is computationally $\mathcal{O}(N \log N)$ [6].

$$J_c = \mathcal{F}^{-1}\left[\frac{\mathcal{F}\left[E^i(s')\right]}{\mathcal{F}\left[H_0^{(2)}(kg(s'))\right]}\right] \tag{17}$$

where $\mathcal{F}$ denotes Fourier transform and $\mathcal{F}^{-1}$ inverse Fourier transform. Any method for calculating the approximate current distribution on the surface of a *PEC* by equation (17) will henceforth be referred to as the Green's function perturbation method *GFPM*. It is clear that equation (17) defines a class of solution methods defined by the function g.

3. Condition for Convergence

To investigate the region of validity of the approximation J_c to the Helmholtz integral consider the iterative solution to the perturbation problem described in equation (16) where the first term (the zeroth order solution) is given by the *GFPM*. For the *GFPM* to represent a sufficiently accurate solution to equation (16) requires the contributions to J, the exact current, from the the remaining series terms to be negligible.

We shall, for this purpose, exploit the more compact notation of linear operators, defining:

$$\mathcal{L}J = (\mathcal{A} + \mathcal{C})\, J = E^i \tag{18}$$

where the approximation operator is

$$\mathcal{A} = \frac{k\eta}{4}\int_{s'} H_0^{(2)}(kg(s-s'))ds' \tag{19}$$

and the correction operator is

$$\mathcal{C} = \frac{k\eta}{4}\int_{s'}\sum_{n=1}^{\infty}\frac{(kg(s-s')B(s,s'))^n}{n!}\,H_0^{(2)^{(n)}}(kg(s-s'))ds' \tag{20}$$

The operators $\mathcal{A}$ and $\mathcal{C}$ are bounded (by the fact that $\mathcal{L}$ is) and are continuous. A solution J to the integral equation exists because $\mathcal{L}$ is complex valued and non-singular [10]. The system described in equation (18) describes a Fredholm [8] integral equation of the first kind. We can easily transform it to a Fredholm integral equation of the second kind by multiplying across by the inverse of the operator $\mathcal{A}$. We can show this explicitly by multiplying (18) by $\mathcal{A}^{-1}$ to give

$$\left(I + \mathcal{A}^{-1}\mathcal{C}\right) J = \mathcal{A}^{-1} E^i \tag{21}$$

which is a Fredholm equation of the second kind. The unknown J can thus be found if we can find the operator $\left(\mathcal{I} + \mathcal{A}^{-1}\mathcal{C}\right)^{-1}$. If $\|\mathcal{A}^{-1}\mathcal{C}\| < 1$, where $\|\cdot\|$ is any operator norm, this operator can be expanded, using the operator form of the Binomial series, in a power series as,

$$\mathcal{I} - \mathcal{A}^{-1}\mathcal{C} + \left(\mathcal{A}^{-1}\mathcal{C}\right)^2 - \ldots + \left(\mathcal{A}^{-1}\mathcal{C}\right)^n - \ldots \tag{22}$$

which is absolutely convergent [1]. An iterative scheme solving this for unknown J is

$$J_{n+1} = -\mathcal{A}^{-1}\mathcal{C} \;\; J_n + \mathcal{A}^{-1} \; E^i \tag{23}$$

The series (23), is called the Neumann or Born series. The condition for convergence is method driven and, therefore, the fact that a solution exists does not guarantee convergence. In terms of the individual operator norms we have the condition for convergence

$$\frac{\|\mathcal{C}\|}{\|\mathcal{A}\|} < 1 \tag{24}$$

In equation (24) $\|\mathcal{A}\|$, which is fixed by a choice of g, is bounded above by $\|\mathcal{L}\|$ and is only zero when $\mathcal{L} = 0$. Thus, to determine the functions which need to be minimized in order to get a sufficiently small contribution from perturbation terms beyond the *GFPM* term we will bound the term $\mathcal{C}J_c$ from equation (24).

Consider the integral $\mathcal{C}J_c$ defined by

$$\mathcal{C}J_c = -\frac{k\eta}{4}\left(I_1 + I_2\right) \tag{25}$$

[1] A series $\sum_n a_n$ is *absolutely convergent* if the series $\sum_n \|a_n\|$ is convergent.

with

$$I_1 = \int_{kg(s-s')\leq 1} J_c \sum_{n=1}^{\infty} \frac{(kg(s-s')B(s,s'))^n}{n!} H_0^{(2)^{(n)}}(kg(s-s'))\, ds' \tag{26}$$

representing contributions in the region local to an integration point, i.e., $kg \leq 1$ and

$$I_2 = \int_{kg(s-s')>1} J_c \sum_{n=1}^{\infty} \frac{(kg(s-s')B(s,s'))^n}{n!} H_0^{(2)^{(n)}}(kg(s-s'))\, ds' \tag{27}$$

representing regions away from integration points. We wish to find a bound for $\mathcal{C}J_c$ and thus indicate the terms that need to be minimized for a valid approximation to be obtained. It can be shown (Appendix 8) that

$$\left|\frac{d^n}{dx^n} H_0^{(2)}(x)\right| \leq \begin{cases} \left|\sqrt{\frac{2}{\pi x}} e^{-jx}\right| & , x > 1 \\ \frac{4}{\pi\gamma}|x^{-1}| & , x \leq 1 \end{cases} \tag{28}$$

where γ is Euler's constant [4].

3.1. A Bound for I_2

We will consider I_2 first. To begin we note that J_c can be bounded by its least upper bound *lub* [2]

$$J_c^m = lub\,|J_c| \tag{29}$$

the two-dimensional function $B(s,s')$ can be bounded by a cyclic function $B_m(s-s')$ defined by

$$B_m(s-s') = \max_s [B(s, s-s')] \tag{30}$$

which is symmetric about each integration point s, and that the exponential function is defined by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{31}$$

[2] A number x is a *lub* of A if; x is an upper bound of A; y is an upper bound of A and $x \leq y \forall y$.

Identities (29) and (30) allow us to bound $|I_2|$ by

$$|I_2| \leq \int_{kg(s-s')>1} J_c^m \sum_{n=1}^{\infty} \frac{(kg(s-s')B(s,s'))^n}{n!} \sqrt{\frac{2}{\pi kg(s-s')}} e^{-\jmath\beta g(s-s')} e^{-\alpha g(s-s')}\, ds' \tag{32}$$

In equation (32) $\exp(-\jmath\beta g(s-s'))\,/\,[kg(s-s')]^{-1}$ can be replaced by one because, by definition, $kg(s-s')$ is greater than one and $\exp(-\jmath\beta g(s-s'))$ is always bounded by one. Now, using the definition of e, the exponential function, in equation (32) gives

$$|I_2| \leq J_c^m \sqrt{\frac{2}{\pi}} \int_{kg(s-s')>1} [e^{\beta g(s-s')B_m(s-s')} - 1] e^{-\alpha g(s-s')} ds' \tag{33}$$

It is clear that the right hand side of equation (33) is bounded for the following two cases

$$\beta B_m(s-s') - \alpha \leq 0 \tag{34}$$

and

$$|\beta g(s-s')B_m(s-s')| < 1 \tag{35}$$

We note that for $B_m(s-s') = 0$ the bound is trivially zero. Equation (35) tells us that $|B_m(s-s')|$ must be sufficiently small everywhere, while equation (34) only requires that $B_m(s-s')$ be negative everywhere if it is to be valid for all attenuation factors α. From (13) we have the limits for $|B_m(s-s')|$ as

$$-1 < |B_m(s-s')| < \sqrt{2} - 1 \tag{36}$$

We will first look at the case described by equation $\beta B_m(s-s') - \alpha \leq 0$. Extending the bounds of integration to plus and minus infinity for the function (33), we have

$$|I_2| \leq 2kJ_c^m \sqrt{\frac{2}{\pi}} \int_{g(s-s')=0}^{g(s-s')=\infty} [e^{\beta g(s-s')B_m(s-s')} - 1] e^{-\alpha g(s-s')} ds' \tag{37}$$

where the constant 2 appears for the symmetry in the integral. Equation (37) is a generalized Fourier integral [5] and can be solved asymptotically [18] (using integration by parts). This gives as an upper bound for $|I_2|$

$$|I_2| \leq 2J_c^m\sqrt{\frac{2}{\pi}}\left[\frac{1}{\frac{d}{ds'}[(\beta B_m(s-s')-\alpha)g(s-s')]_{|_{s'=0}}}+\right.$$
$$\left.+\frac{1}{\frac{d}{ds'}[\alpha g(s-s')]_{|_{s'=0}}}\right] \tag{38}$$

Performing the same steps with the case described by $|\beta g(s-s')B_m(s-s')| < 1$ we find the same bound for $|I_2|$. The bound in (38) suggests that $\frac{d}{ds'}\ [\beta B_m(s-s')g(s-s')]$ should be minimized for fast method convergence of the perturbative scheme and validity of the one term solution.

3.2. A Bound for I_1

Returning to I_1 we now bound this integral.

$$|I_1| = \int_{kg(s-s')\leq 1} J_c \sum_{n=1}^{\infty} \frac{(kg(s-s')B(s,s'))^n}{n!}\, H_0^{(2)}(kg(s-s'))\, ds' \tag{39}$$

Again, using the asymptotic bound for the Hankel function (28) (this time for small argument) and the bound J_c^m for J_c we have

$$\begin{aligned}
&|I_1| \\
&\quad \leq J_c^m \frac{4}{\pi\gamma} \int_{kg(s-s')\leq 1} \sum_{n=1}^{\infty} \frac{(kg(s-s')B(s,s'))^n}{n!} \left(\frac{1}{kg(s-s')}\right)^n n!\, ds' \\
&\quad \leq J_c^m \frac{4}{\pi\gamma} \int_{kg(s-s')\leq 1} \frac{1}{1-B_M} - 1\, ds'
\end{aligned} \tag{40}$$

where we have defined

$$B_M = \max_{s,s'} B(s,s') \tag{41}$$

Thus an upper bound for I_1 is

$$J_c^m \frac{4}{\pi\gamma}\left[\frac{1}{1-B_M} - 1\right] \int_{kg(s-s')\leq 1} ds' = J_c^m \frac{4}{\pi\gamma}\left[\frac{1}{1-B_M} - 1\right] 2\Delta s \tag{42}$$

where Δs is a constant dependent on k and g. In order to minimize $|I_1|$ we require B_M to be as small as possible in the domain $kg(s-s') \leq 1$.

3.3. A Bound for the Neglected Perturbation Terms when using the *GFPM*

Using the upper bounds (42) for I_1 and (38) for I_2 we can bound the neglected perturbations terms $\mathcal{C}J_c = -k\eta/4\,(I_1+I_2)$, when using the *GFPM*, by

$$\begin{aligned}\|\mathcal{C}J_c\| \leq & -\frac{k\eta}{4}2\left[J_c^m\frac{4}{\pi\gamma}\left[\frac{1}{1-B_M}-1\right]\Delta s \; + \right.\\ & J_c^m\sqrt{\frac{2}{\pi}}\left[\frac{1}{\frac{d}{ds'}\left[(\beta B_m(s-s')-\alpha)g(s-s')\right]_{|_{s'=0}}}+\right.\\ & \left.\left.+\frac{1}{\frac{d}{ds'}\left[\alpha g(s-s')\right]_{|_{s'=0}}}\right]\right]\end{aligned} \tag{43}$$

We notice immediately that as the attenuation factor α reduces we need to decrease the value of $B(s,s')$ i.e., restrict the set of surfaces for which the one term approximation is valid. We have shown that convergence of the method to the true current J is dependent on the value $\|\mathcal{C}J\|$ and that this value must be less than one. Thus the perturbative solution is absolutely and uniformly convergent for B_M and $\frac{d}{ds'}[\beta B_m(s-s')g(s-s')]$ sufficiently small.

4. Choosing $g(x)$

We have shown that for the *GFPM* to produce valid results requires

$$B(s,s') \ll 1 \qquad\qquad , kg(s-s') \leq 1 \tag{44}$$

and

$$\frac{d}{ds'}[\beta B_m(s-s')g(s-s')] \ll 1 \qquad\qquad , kg(s-s') > 1 \tag{45}$$

Both cases (equation (44) and (45)) are optimally satisfied when B is zero. We will now show the two cases for which this optimal solution exists. Explicitly, B is written as

$$B = \left(\left[\frac{f(s) - f(s')}{g(s - s')} \right]^2 + \left[\frac{\xi\,(f(s)) - \xi(f(s'))}{g(s - s')} \right]^2 \right)^{\frac{1}{2}} - 1 \tag{46}$$

or, after extracting g

$$B = g(s - s')^{-1} \left[\left(f\,(s) - f\,(s')\right)^2 + \left(\xi\,(f\,(s)) - \xi\,(f\,(s'))\right)^2 \right]^{1/2} - 1. \tag{47}$$

Thus B is zero for surfaces where

$$\left[\left(f\,(s) - f\,(s')\right)^2 + \left(\xi\,(f\,(s)) - \xi\,(f\,(s'))\right)^2 \right]^{1/2} \tag{48}$$

can be expressed as a function of $s - s'$ only. There are two such surfaces; the straight line for which equation (48) is $s - s'$ and the circle for which equation (48) is $2r\sin((s - s')/2r)$, where r is the circles radius.

4.1. Choosing $g(x)$ to Minimize the Error in an Average Sense

In general, it is not possible to pick a function g such that B is zero. Consider the situation where the minimization criterion, given by equation (45), is bounded by a two-dimensional error function $\varepsilon(s, s')$, where s is the arclength parameterization variable. Equationally this is

$$\frac{d}{ds'} \left[\frac{\mathbf{d}(s, s')}{g(s - s')} - 1 \right] g(s - s') < \varepsilon(s, s'). \tag{49}$$

The function $\varepsilon(s, s')$ represents a surface and we would like this function to be a minimum in some sense. To continue we introduce the definition for the average of a function over a finite domain

$$\langle f \rangle_s = \frac{1}{L} \int_s f\, ds, \tag{50}$$

with L equal to the scalar length of the domain in terms of the variable s. To make $\varepsilon(s,s')$ small in an average sense we apply equation (50) over the variables s and s' to get

$$\int_s \left[\int_{s'} \frac{d}{ds'}\left[\mathbf{d}(s,s') - g(s-s')\right]\, ds'\right] ds < \int_s \int_{s'} \varepsilon(s,s')\, ds'\, ds \quad (51)$$

which reduces to

$$\int_s \mathbf{d}(s,s')\, ds - \int_s g(s-s')\, ds < \int_s \int_{s'} \varepsilon(s,s') \quad (52)$$

Thus we can minimize the error in an average sense, i.e., set $\int_s \int_{s'} \varepsilon(s,s') = 0$, if

$$g(s') = \left\langle \mathbf{d}(s, s-s') \right\rangle_s \quad (53)$$

We note that $g(x) = g(-x)$ by definition, thus $g(s-s')$ is well defined everywhere.

5. Reciprocity

Reciprocity is a necessary but not sufficient condition for any solution to Maxwell's equations. This is evident in the fact that non-consistent numerical solutions may exhibit reciprocity. In its most simple form the reciprocity theorem tells us that the measured response of a system to a source is unchanged when the source and measurer are interchanged. It is sufficient to show that either the impedance or admittance operator of a linear system is symmetric to show reciprocity. In the *GFPM* we have chosen $g(x)$ such that the approximation operator (impedance) (18) is symmetric, as is its inverse, the admittance operator. Thus the approximate solution system obeys reciprocity as does its error (by definition of the total system being reciprocal). It follows that any necessary criterion for the validation of the *GFPM* should also satisfy reciprocity. The first of our criteria (equation (30)) is satisfied by symmetry in $g(x)$ and noticing that

$$B_m(s-s') = \max_s B(s, s-s') = \max_{s'} B(s', s'-s) = B_m(s'-s) \quad (54)$$

so that

$$\frac{d}{ds'} B_m(s-s') \equiv \frac{d}{ds} B_m(s'-s) \quad (55)$$

The second (41) is trivially true because B_M is a constant.

6. Analytic Comparison of the *GFPM* and the Kirchhoff and Height Perturbation Method

In Meecham's paper [13] on the validity of the Kirchhoff method the scalar Helmholtz equation is expressed in terms of the Kirchhoff term and a series of integrals. Meecham continues by investigating the neglected terms of the series and shows that for the Kirchhoff term to converge to the exact solution the first integral I in the neglected series must be as small as possible. A bound for this term is given by,

$$|I| \leq M' \left| \frac{d\xi(x)^m}{dx} \right| + M'' \frac{1}{|k| R_m} \tag{56}$$

where M' and M'' are constants independent of the surface, R_m is the minimum radius of curvature along the surface and $|d\xi(x)^m/dx|$ is the maximum slope of the surface. The conditions for convergence are then given by

$$|I| \approx 0 \tag{57}$$

$$\left| \frac{d\xi(x)^m}{dx} \right| \ll 1 \tag{58}$$

$$|k|\, R_m \gg 1 \tag{59}$$

Using these conditions we will show that if these are true then the *GFPM* will also be valid.

Consider the function B, i.e., the function that needs to be minimized for validity of the *GFPM* given by

$$B(s,s') = \left(\left[\frac{f(s) - f(s')}{g(s-s')} \right]^2 \left\{ 1 + \frac{[\xi(f(s)) - \xi(f(s'))]^2}{[f(s) - f(s')]^2} \right\} \right)^{1/2} - 1 \tag{60}$$

If we consider condition (58) then B, which is zero for exact solutions, is approximated by,

$$B(s,s') \approx \left[\frac{f(s) - f(s')}{g(s-s')} \right] - 1 \tag{61}$$

and

$$g(s-s') = f(s) - f(s') \tag{62}$$

gives the required result.

Using the radius of curvature R given by

$$R = \frac{\left[1 + \left(\frac{d\xi}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2\xi}{dx^2}\right|} \tag{63}$$

condition (59) is satisfied when

$$\begin{gathered} \left|\frac{d^2\xi}{dx^2}\right|^{2/3} - 1 \ll \left(\frac{d\xi}{dx}\right)^2 \\ \left|\frac{d^2\xi}{dx^2}\right| \ll 2^{3/2} \end{gathered} \tag{64}$$

But if the change in ξ with respect to x is small everywhere, as demanded by condition (58), then the rate of change must also be small and so equation (62) gives one suitable *GFPM* for when Kirchhoff's method is valid.

The conditions of validity for the height perturbation method [14] are given by

$$\left|\max_x \frac{d\xi(x)}{dx}\right| \ll 1 \tag{65}$$

$$k \max_x |\xi(x)| \ll 1 \tag{66}$$

We consider that all perturbations such that they are about the mean plane $z = 0$. We may write $B(s,s')$ as

$$B(s,s') = \left(\left[\frac{f(s) - f(s')}{g(s-s')}\right]^2 \left\{1 + \frac{[\xi(f(s)) - \xi(f(s'))]^2}{[f(s) - f(s')]^2}\right\}\right)^{1/2} - 1 \tag{67}$$

Using condition (66) gives

$$B(s,s') \approx \left[\frac{f(s) - f(s')}{g(s-s')}\right] - 1 \tag{68}$$

and the required g function is

$$g(s - s') = f(s) - f(s') \tag{69}$$

Condition (66) gives the same result.

7. Numerical Examples

As an example of the method and some of the functions used we consider here a cubic surface of the form

$$\xi(x) = A(x - r)(x + r)x \tag{70}$$

where $-r, r, 0$ are the roots of $\xi(x)$ and A is a scaling variable. Let us examine the case where $r = 20$ and $A = 1/2000$ in the domain $x \in [-20\,20]$ as illustrated in Figure (2). All dimensions are in metres. It is apparent that as an approximation the height perturbation method [14] is unsuitable as the necessary requirements are

$$\beta\,|\xi(x)| \ll 1 \tag{71}$$

$$|\nabla\xi(x)| \ll 1 \tag{72}$$

The maximum slope for the example is approximately $11.3°$ and the maximum absolute height $1.5m$ thus β must be much less than $2/3$; λ much greater than $9.3m$. Kirchhoff theory overcomes the difficulty in restrictive wavenumber. The requirements for this method are [13]

$$\left|\max_x \frac{d\xi(x)}{dx}\right| \ll 1 \tag{73}$$

$$|\beta| \min_x \rho \gg 1 \tag{74}$$

where ρ is the radius of curvature. Equation (73) is satisfied by the inequality $0.2 \ll 1.0$ and equation (74) is satisfied if the inequality $\beta \times 31.051 \gg 1$ is true. In addition to the above a criterion must be satisfied for grazing incident of the impinging radiation. In particular this is [16] $2\beta|\rho|sin^3\theta \gg 1$, where θ is the grazing angle.

For the Green's function perturbation method to be valid we expect $\beta\frac{d}{ds'}g(s - s')B(s, s')$ and $B(s, s')|_{\beta g(s-s')<1}$ to be small. The following figures show these functions for three different choices of g. The choices are:

1. is calculated exactly in the average sense as in equation (53). By exactly we mean that average is calculated over all N match points. This is an $\mathcal{O}(N^2)$ operation and thus dominates the computation time. We shall call this scheme the N point average method.
2. g is calculated in the average sense by polynomial curve fitting to a set of M points, where M is much less than N. The computational cost of this method is $\mathcal{O}(M^2)$. M is chosen such that $M^2 \leq N \log N$, and the overall computational cost of calculating the current is $\mathcal{O}(N \log N)$. We shall call this scheme the polynomial average fit method. The order of the polynomial is left open.
3. g is chosen such that $g(s - s') = g(s_n - s'_m) = |n - m|\Delta s$. We shall refer to this scheme as the position index method.

The three g functions are shown, normalized by the exact average value so that we can see a difference, in Figure (3). The polynomial average fit is third order.

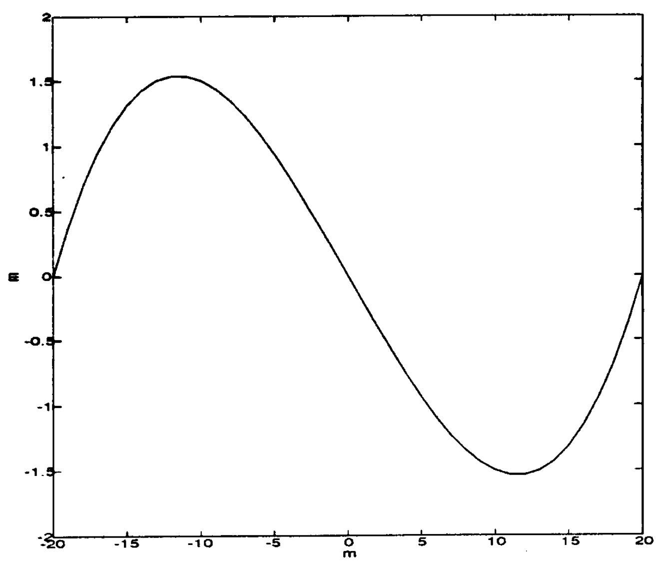

Figure 2. Cubic surface. $\xi(x) = 1/2000(x-20)(x+20)x$**. The** x **and** y **ases are scaled in meters.**

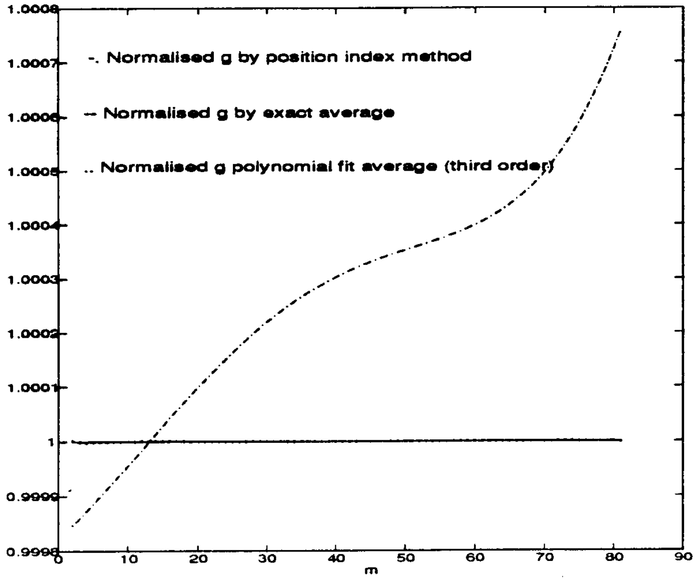

Figure 3. Normalized distance functions g. The normalization is by the N point average g function.

Figures (4) and (5) show the functions $\frac{d}{ds'}g(s-s')B(s,s')$ and $B(s,s')_{|_{\beta g(s-s')<1}}$ for case (1). Similarly Figures (6) and (7) show the same functions for case (2) and Figures (8) and (9) for case (3). To avoid any ambiguity the curves drawn in the figures are representative of matrices, defined by $\frac{d}{ds'}g(s-s')B(s,s')$ and $B(s,s')_{|_{\beta g(s-s')<1}}$, skewed such that the leftmost position in the figure refers to the diagonal elements of the matrix. The matrices are symmetric about the diagonal and the x-axis refers to positions to the right of the diagonal. Each curve represents one line of the matrix. The individual curves are not labelled as it is the general structure of the matrix which is considered important.

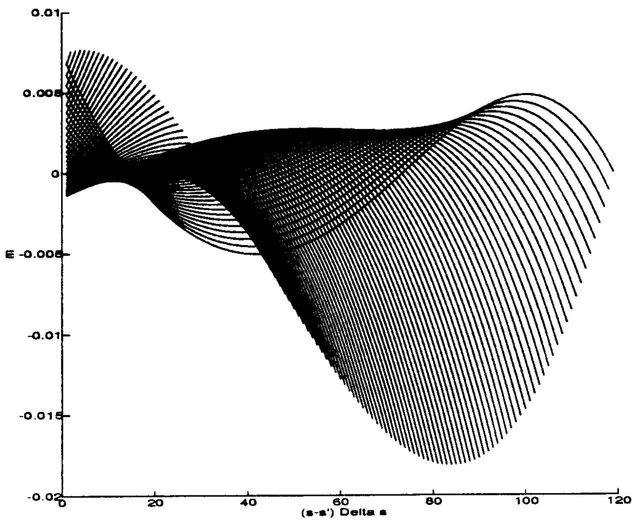

Figure 4. $\frac{d}{ds'}g(s-s')B(s,s-s')$. **The g function is by the N point average method.**

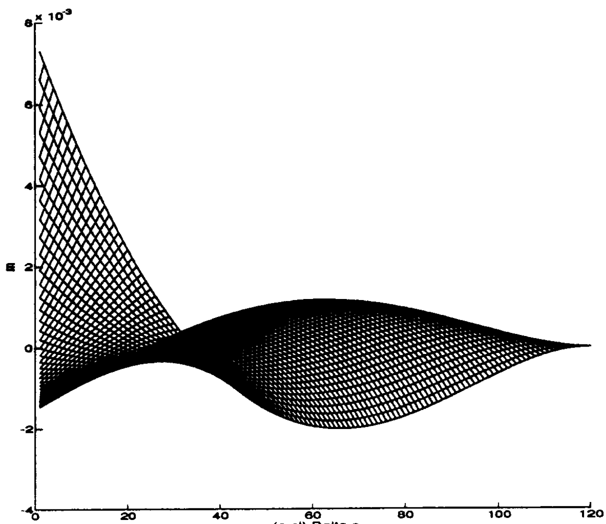

Figure 5. $B(s,s-s')$. **The g function is by the N point average method.**

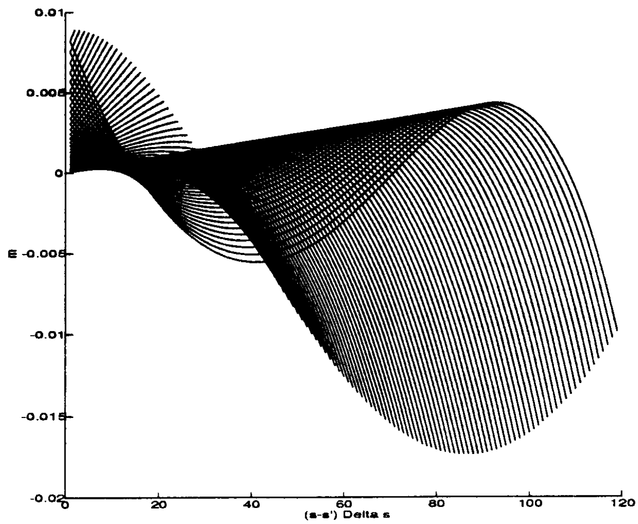

Figure 6. $\frac{d}{ds'}g(s-s')B(s,s-s')$**. The** g **function is by the polynomial average fit method (third order).**

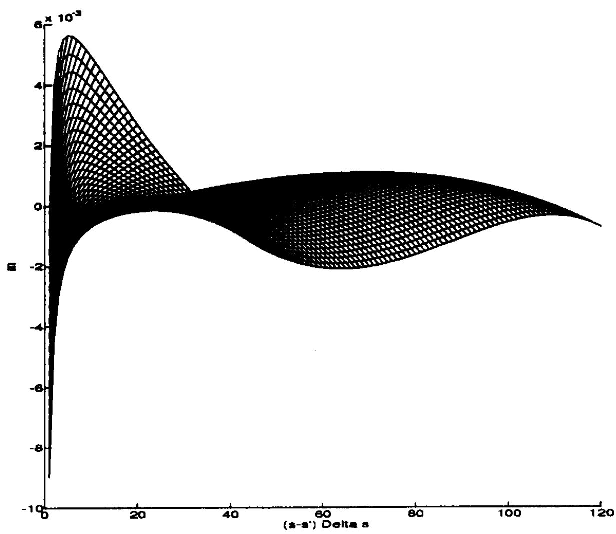

Figure 7. $B(s,s-s')$**. The** g **function is by the polynomial average fit method (third order).**

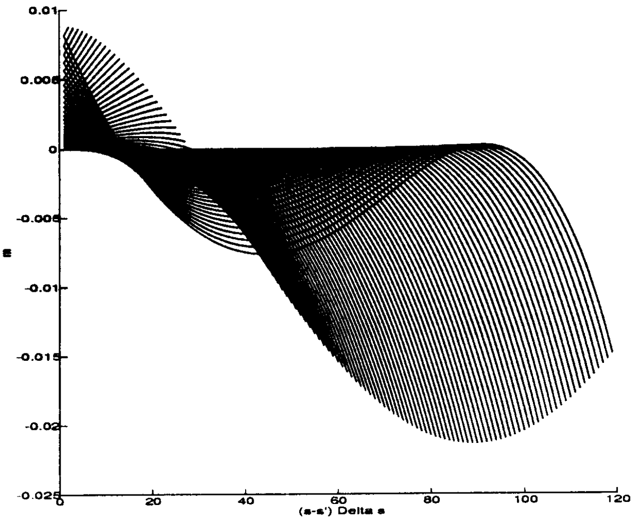

Figure 8. $\frac{d}{ds'}g(s-s')B(s,s-s')$**. The** g **function is by the position index method.**

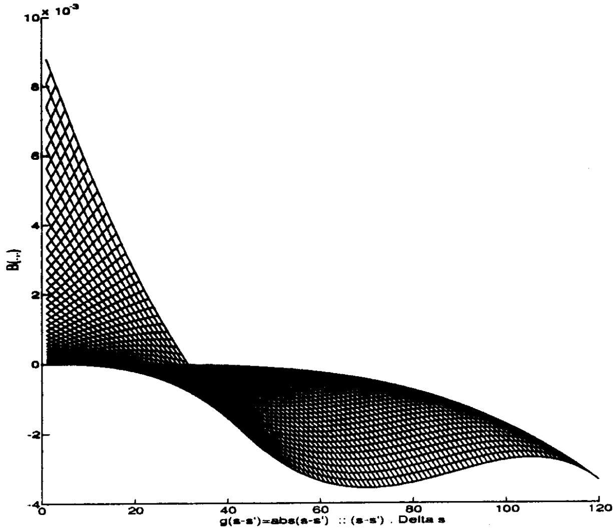

Figure 9. $B(s,s-s')$**. The** g **function is by the position index method.**

The limiting value of $g(s-s')B(s,s')$ as $|s-s'| \to 0$ can be seen for all three cases in Figures (10), (11) and (12) thus ensuring a small error in the local region about any integration point in all cases.

Estimations for the current density induced on the surface of the cubic scatterer for the stated choices of g are shown in Figure (13) for the case of a TM line source at $(-12, 12)$ and $\beta = 8$.

As with all methods, validation is best demonstrated against exact solutions and other well known methods. Results with $g(s-s')$, in the *GFPM*, satisfying average error minimization by third order polynomial fit are shown against an exact solution by the method of moments (discretizations every $\lambda/7$ wavelengths and point matching) and the Kirchhoff approximation with a TM line source at $(-12, 12)$ and $\beta = 8$ are shown in Figure (13).

Figure (16) has the source moved to $(-40, 8)$ which includes grazing incidence and $\beta = 10$. It includes current estimations for g functions of the following types; N point average method, polynomial fit average method and index position method.

In Figure (14) the polynomial average fit current is compared against an exact moment method solution and a physical optics solution.

The total field is calculated at $2.4m$ above the surface. Readings for the N point average method, the polynomial average method and the index position methods are compared against the physical optics method and an exact moment method and are shown in Figure (15). The mean error and standard deviations from exact solutions are tabled in (1).

Figure (17) has the source moved to $(-4000, 12)$ and $\beta = 8$ so that diffraction is a significant feature. The plot indicates the total field $2.4m$ above the surface.

The total field from the current calculated by the Kirchhoff method does not include shadowing [7, 11] but the result is similar due to the grazing incidence of the source field. Table (2) indicates the mean error and standard deviation in relative dB for Kirchhoff and the *GFPM* in Figure (17).

It is clear that the Green's function perturbation method is the most suitable approximation in all cases.

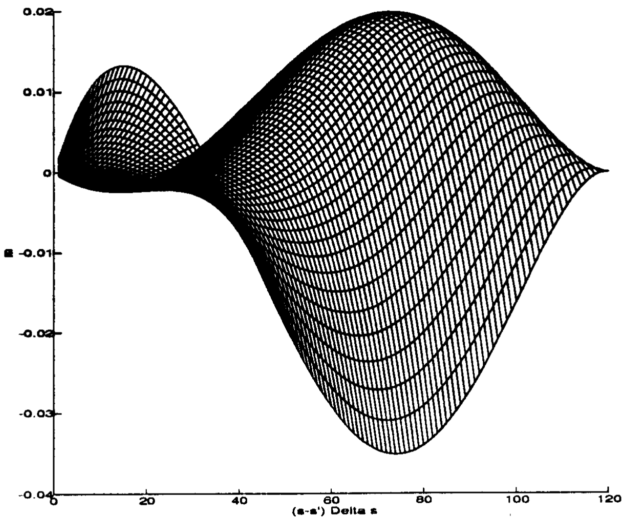

Figure 10. $g(s-s')B(s,s-s')$. **The** g **function is by the** N **point average method.**

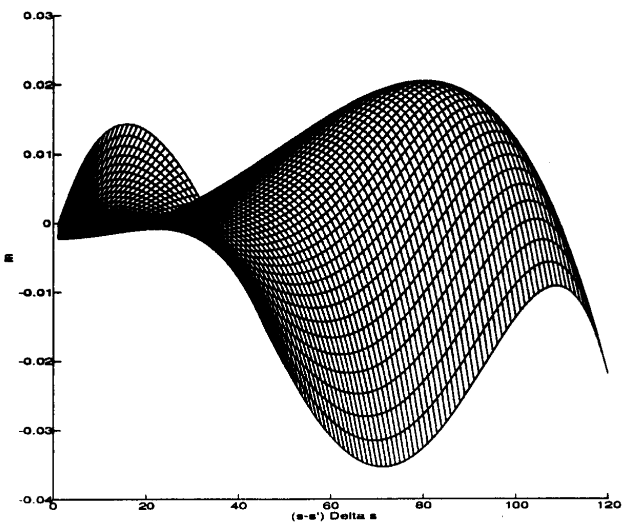

Figure 11. $g(s-s')B(s,s-s')$. **The** g **function is by the polynomial average fit method with a polynomial of order three.**

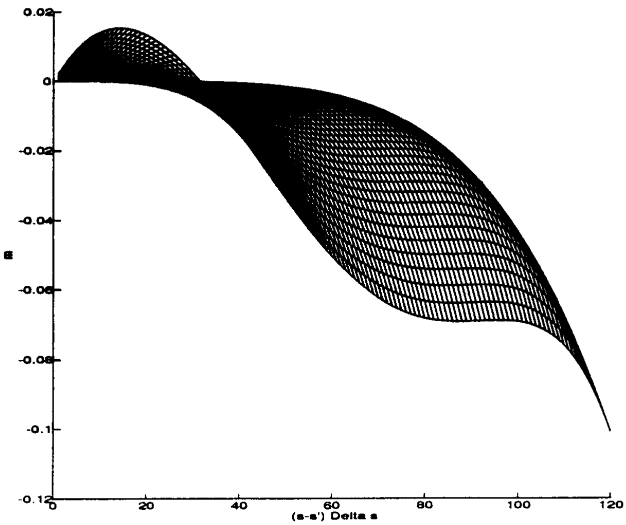

Figure 12. $g(s - s')B(s, s - s')$. **The** g **function is by the position index method.**

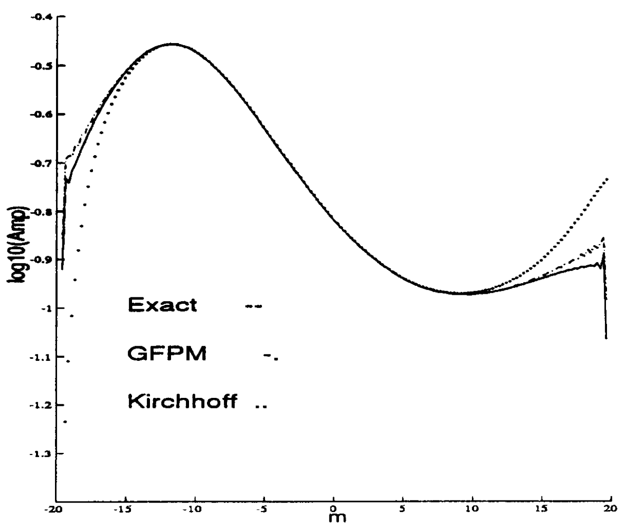

Figure 13. Induced current on surface illustrated in Figure (2) with a line source positioned at (12, 12) with frequency $382MHz(\beta = 8)$.

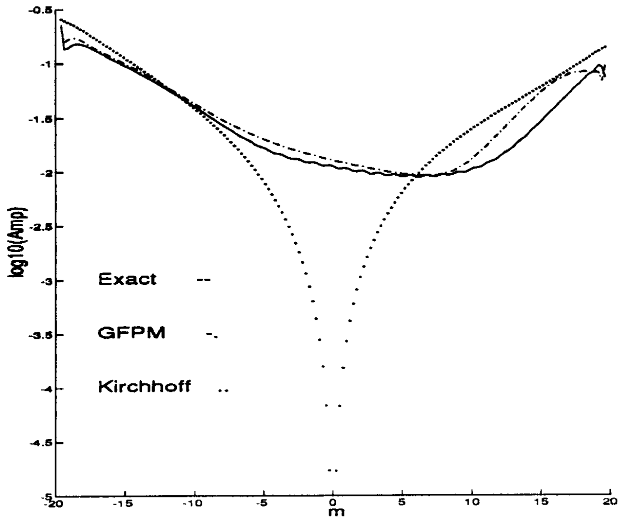

Figure 14. Comparison of currents calculated by exact moment method, physical options and *GFPM* using polynomical average fit scheme for a line source at (-40, 8) with frequency $477.5MHz(\beta = 10)$.

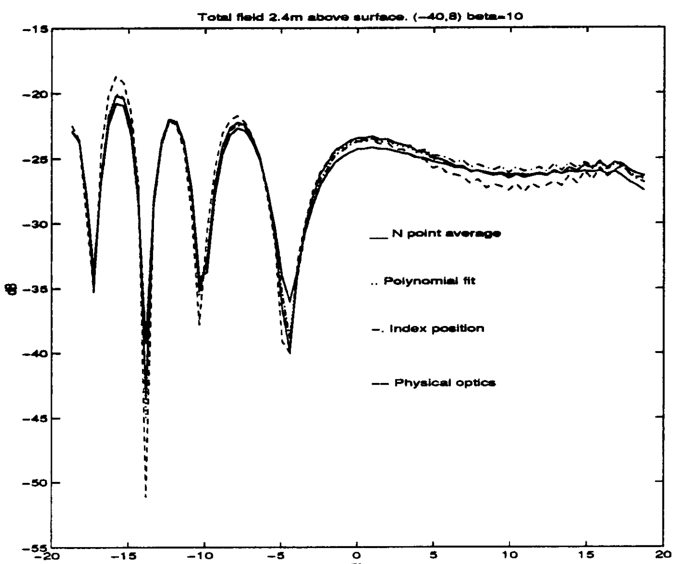

Figure 15. Total fields $2.4m$ above surface calculated from currents calculated by exact moment method, physical optics method and polynomial average fit scheme of the *GFPM* from Figure (14).

Relative errors in GFPM and Kirchhoff theory in dB		
	Mean error	Standard deviation
Physical optics	1.1301	1.5501
GFPM Polynomial average fit	0.5709	0.7701
GFPM N point average	0.5947	0.6941
GFPM Index position	0.5449	0.5101

Table 1. Error quantifications for the solutions of approximate methods shown in Figure (15).

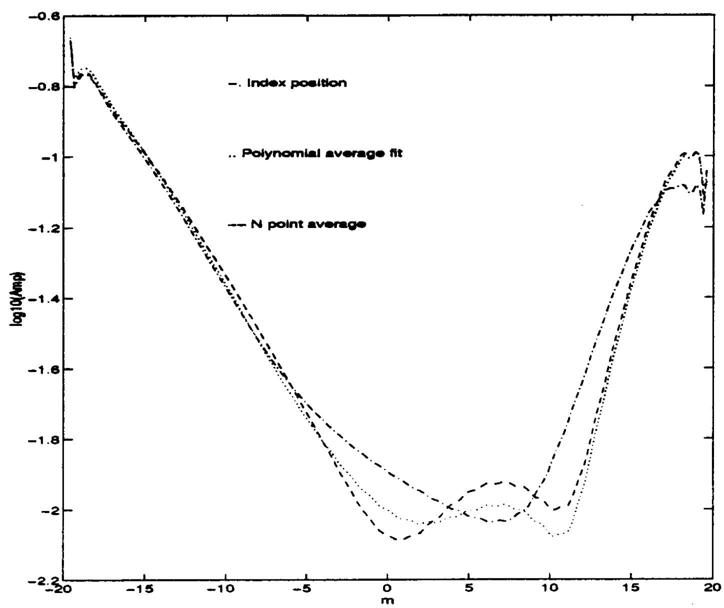

Figure 16. Induced current on surface with source at (-40, 8) and $\beta = 10$**.**

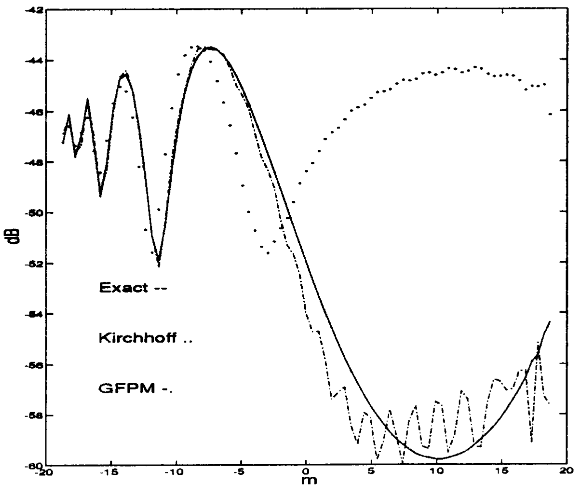

Figure 17. Total field $2.4m$ above surface with source at (-4000, 12) and $\beta = 8$.

Relative errors in *GFPM* and Kirchhoff theory in dB		
	Mean Error	Standard Deviation
Kirchhoff	6.8841	5.8139
GFPM	0.8724	0.8607

Table 2. Error quantifications for the solutions of approximate methods shown in Figure (17).

7.1. Application to Undulating Terrain

The method has been applied to terrain height data for a number of regions in Denmark [2]. The method is used in a general way, using the function $g(s-s') = |s-s'|$, as described above with the exception that where the method does not satisfy its validity criteria the incident field is set to zero. This allows for a free space decay of the current in these regions without adding any complexity to the method. This preserves the properties of reciprocity and self-consistency. The data calculated using the Green's function perturbation method *GFPM* are compared with a fast exact moment method solution of the electric field integral equation using a natural basis (described by Moroney and Cullen in [19]) and field measurements which were provided with the terrain data. The data for Figures (18) and (19) were calculated at a frequency of $1900MHz$. The data for Figures (20) and (21) were calculated at a frequency of $435MHz$. For both the above data sets the radiating source was positioned $10.4m$ above the leftmost terrain element and the field intensity was calculated at $2.4m$ above the terrain.

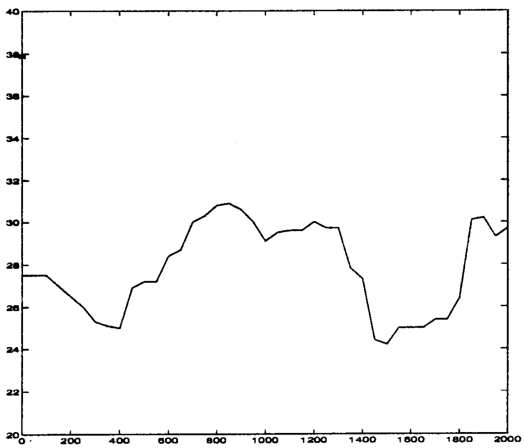

Figure 18. Jerslev terrain profile.

Figure 19. Jerslev field intensity.

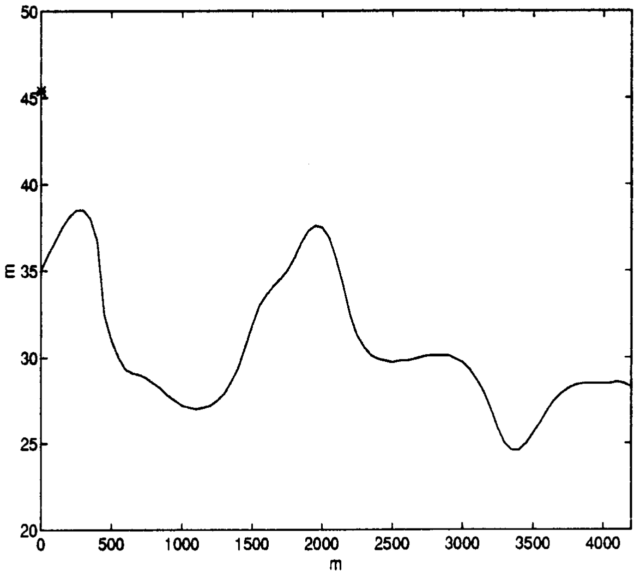

Figure 20. Hjorring terrain profile.

Figure 21. Hjorring field intensity.

8. Conclusion

In this paper we have indicated a new perturbation method based on the Green's function for solving the Helmholtz integral. We have shown that the method is absolutely and uniformly convergent and specified criteria which should be minimized for fast method convergence. We have shown that the method is more suitable for a wide class of surfaces than other approximate methods, namely the height perturbation method and the Kirchhoff approximation. We have described a computationally simple method for picking the function g, on which the method hinges and shown that the solution and its validating measures obey reciprocity.

We have elucidated a simple example to show the steps of the method and compared it with the above approximate methods. We have also compared it with another integral equation approximation over natural terrain profiles at different frequencies and found the *GFPM* a justifiable candidate.

The *GFPM* has also been applied to sinusoidally rough surfaces [12], where the $\mathcal{O}(N^2)$ N point average method of picking the g function was used, with positive results.

Finally one should note that if we define E^i by

$$E^i = \breve{E}\exp(-\jmath k_b x) \tag{75}$$

where $\breve{E}$ is the complex envelope of E^i with respect to the monochromatic function $\exp(-\jmath k_b x)$ then provided E^i and $H_0^{(2)}(k|g|)$ are bandpass with centre wavenumber k_b and bandwidth b less than $k_b/2$ we have

$$\breve{E} = 1/2\breve{J} * \breve{H_0^{(2)}(k|g|)} \tag{76}$$

where $*$ defines convolution. The complex envelopes can often (for example where forward propagation dominates) be sampled at a significantly lower rate than the complex fields.

Acknowledgements

The authors wish to express their gratitude to Prof. Jorgen Bach Andersen of the University of Aalborg for his kind permission to use the data cited in [2].

Appendix

A bound for $\frac{d^n}{dx^n}H_0^{(2)}(x)$. It is necessary for our calculations to be able to bound the n^{th} derivative of the zeroth order Hankel function of the second kind with respect to its entire argument. The n^{th} derivative can be represented [1] as

$$\begin{aligned}\frac{d^n}{dx^n}H_0^{(2)}(x) =& \frac{1}{2^n}\left[H_{-n}^{(2)}(x) - \binom{n}{1}H_{2-n}^{(2)}(x)\right.\\ &\left. + \binom{n}{2}H_{4-n}^{(2)}(x) - \ldots + (-1)^n H_n^{(2)}(x)\right]\end{aligned} \tag{77}$$

An upper bound for this term can be found by taking the absolute value of all terms on the right hand side of the above formula.

$$\left|\frac{d^n}{dx^n}H_0^{(2)}(x)\right| \leq \frac{1}{2^n}\left|H_1^{(2)}(x)\right|\sum_{k=1}^{n}\binom{n}{k}|(-1)^n| \tag{78}$$

$$= \frac{1}{2^n}\left|H_1^{(2)}(x)\right| 2^n \tag{79}$$

$$= \left|H_1^{(2)}(x)\right| \tag{80}$$

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