Progress In Electromagnetics Research, Vol. 150, 13–27, 2015

Stored Electromagnetic Energy and Antenna Q

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(*Invited Paper*)

Abstract—Decomposition of the electromagnetic energy into its stored and radiated parts is instrumental in the evaluation of antenna Q and the corresponding fundamental limitations on antennas. This decomposition is not unique and there are several proposals in the literature. Here, it is shown that stored energy defined from the difference between the energy density and the far field energy equals the energy expressions proposed by Vandenbosch for many but not all cases. This also explains the observed cases with negative stored energy and suggests a possible remedy to them. The results are compared with the classical explicit expressions for spherical regions where the results only differ by the electrical size ka that is interpreted as the far-field energy in the interior of the sphere.

1. INTRODUCTION

Electrostatic energy in free space can be written as an integral of the energy density, $\epsilon_0|\mathbf{E}|^2/4$, or equivalently as an integral of the electric potential, ϕ, times the charge density, ρ, [11, 31, 34, 40]. A similar expression holds for the magnetostatic energy. The electrodynamic case is more involved. In [4], Carpenter suggests a generalization in the time domain based on $\phi\rho + \mathbf{A}\cdot\mathbf{J}$, i.e., the sum of the scalar potential times the charge density and the vector potential, $\mathbf{A}$, times the electric current density, $\mathbf{J}$, see also [9, 39]. Geyi uses a similar expression to analyze small antennas in [13]. Vandenbosch presents general integral expressions in the electric current density for the stored electric and magnetic energies in the frequency domain [41] and time domain [43, 44]. These expressions are similar to the expressions by Carpenter but include some correction terms. Analogous results were already suggested by Harrington based on differentiation of the method of moments impedance matrix [29].

Stored electromagnetic energy is used to determine lower bounds on the Q-factor for antennas [5, 8, 38, 45]. The bounds by Chu [5] and Collin & Rothschild [8] are based on mode expansions and subtraction of the power flow from the energy density. This gives analytic expressions for the lower bound on Q for small spherical antennas [5, 8]. The results are generalized to the case with electric current sheets by addition of the stored energy in the interior of the sphere [26, 38]. Stored energy for general media and its relation to the input impedance are analyzed by Yaghjian and Best [48]. The energy expressions by Vandenbosch [41] express the stored energy in the current density on the antenna structure. This is very useful for analysis of small antennas [2, 18, 21, 42] and antenna optimization [6, 18]. The expressions are verified for several antennas in [17, 30]. One problem with the expressions is that they can produce negative values of stored energy [21]. A similar relation for the differentiated antenna input impedance [23, 48] is derived in [3, 24].

In this paper, we investigate stored electric and magnetic energy expressions based on subtraction of the far-field energy density. The expressions are suitable for antenna Q and bandwidth calculations

Received 15 November 2014, Accepted 13 December 2014, Scheduled 19 December 2014
Invited paper for the Commemorative Collection on the 150-Year Anniversary of Maxwell's Equations.
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and they are closely related to the classical methods in [8, 48], see also [45], for antenna Q calculations. They are not restricted to spherical geometries and, furthermore, resembles the proposed expressions by Vandenbosch [41] and the method of moments based expressions by Harrington [29]. The results provide a new interpretation of Vandenbosch's expressions [41] and explain the observed cases with negative stored energy [21]. They also suggest a possible remedy to the negative energy and that the computed Q has an uncertainty of the order ka, where a is the radius of the smallest circumscribing sphere and k is the wavenumber. This is consistent with the use of Q for small (sub wavelength) antennas, where ka is small and Q is large [5, 8]. Analytic results for spherical structures show that the expressions in [41] for Q differ with ka from the results in [26], that is here interpreted as the far-field energy in the interior of the sphere. The results for Q are also compared with estimated values from circuit models and differentiation of the input impedance [23, 24, 48] for a dipole antenna.

The paper is organized as follows. In Section 2, the stored electric and magnetic energies defined by subtraction of far-field from the energy density are analyzed. The coordinate dependence is analyzed in Section 3. Stored energies from small structures are derived in Section 4. Analytic results for spherical geometries and numerical results for dipole antennas are presented in Section 5. The paper is concluded in Section 6. There are two appendices discussing Green's function identities Appendix A and spherical waves Appendix B.

2. STORED ELECTROMAGNETIC ENERGY

We consider time-harmonic electric and magnetic fields, $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$, respectively, with a suppressed $\mathrm{e}^{-\mathrm{i}\omega t}$ dependence, where ω denotes the angular frequency. The Maxwell equations in free space are [31]

$$\begin{cases} \nabla\times\mathbf{E} = \mathrm{i}\omega\mu_0\mathbf{H} = \mathrm{i}\eta_0 k\mathbf{H} \\ \nabla\times\mathbf{H} = -\mathrm{i}\omega\epsilon_0\mathbf{E} + \mathbf{J} = -\frac{\mathrm{i}k}{\eta_0}\mathbf{E} + \mathbf{J}, \end{cases} \tag{1}$$

where $\mathbf{J}$ denotes the current density, while ϵ_0, μ_0, and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ are the free space permittivity, permeability, and impedance, respectively. For simplicity, we interchange between the angular frequency and the free space wavenumber $k = \omega/\mathrm{c}_0$, where the speed of light $\mathrm{c}_0 = 1/\sqrt{\mu_0\epsilon_0}$. We also recall the continuity equation, $\nabla\cdot\mathbf{J} = \mathrm{i}\omega\rho$, relating the current density $\mathbf{J}$ with the charge density ρ. The discussion in this paper is restricted to electric current densities in free space, see [33] for electric and magnetic current densities and [24] for electric current densities in lossy media.

The time-harmonic electric and magnetic energy densities [31, 34, 40] are $\epsilon_0|\mathbf{E}|^2/4$ and $\mu_0|\mathbf{H}|^2/4$, respectively. The energy densities are not observable [4] and there are a few alternative suggestions in the literature [11]. The electric and magnetic energies comprise both radiated and stored energies; however, for antenna Q calculations one must extract the stored energy.

The Maxwell's Equation (1) show that the sources and fields obey the conservation of energy equation in differential form,

$$\mathrm{i}2\omega\left(\frac{\epsilon_0}{4}|\mathbf{E}|^2 - \frac{\mu_0}{4}|\mathbf{H}|^2\right) + \frac{1}{2}\mathbf{E}\cdot\mathbf{J}^* = -\frac{1}{2}\nabla\cdot\left(\mathbf{E}\times\mathbf{H}^*\right), \tag{2}$$

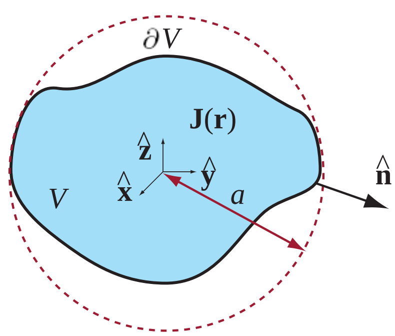

Figure 1. Illustration of the object geometry V with surface ∂V, outward normal unit vector $\hat{\mathbf{n}}$, and current density $\mathbf{J}(\mathbf{r})$. The object is circumscribed by a sphere with radius a.

where the superscript $*$ denotes complex conjugate. We consider current distributions $\mathbf{J}$ whose support is in a volume V bounded by the surface ∂V, see Fig. 1. Integrating (2) over this volume gives the real part result

$$\frac{1}{2}\operatorname{Re}\int_{\partial V}\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\cdot\hat{\mathbf{n}}(\mathbf{r})\,\mathrm{dS}=-\frac{1}{2}\operatorname{Re}\int_{V}\mathbf{E}(\mathbf{r})\cdot\mathbf{J}^*(\mathbf{r})\,\mathrm{dV},\tag{3}$$

where $\hat{\mathbf{n}}$ denotes the outward-normal unit vector of the surface ∂V. The first term in the real part expression (3) is readily identified in view of the Poynting vector [31, 40] as the time-average radiated power flow through the surface ∂V, so that (3) equates the radiated power exiting ∂V to the time average of the power generated by $\mathbf{J}$, as expected from energy conservation. Furthermore, integrating (2) over all space shows that the radiated power exiting the surface ∂V can be expressed in terms of the far field as

$$P_{\mathrm{r}}=\frac{1}{2}\operatorname{Re}\int_{\partial V}\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\cdot\hat{\mathbf{n}}(\mathbf{r})\,\mathrm{dS}=\frac{1}{2\eta_0}\int_{\Omega}|\mathbf{F}(\hat{\mathbf{r}})|^2\,\mathrm{d}\Omega,\tag{4}$$

where Ω denotes the surface of the unit sphere and the far field behaves like $\mathbf{E}(\mathbf{r})\sim\mathrm{e}^{\mathrm{i}kr}\mathbf{F}(\hat{\mathbf{r}})/r$ as $r\to\infty$, where $\mathbf{r}=r\hat{\mathbf{r}}$ and $r=|\mathbf{r}|$. Similarly, by integrating (2) over all space one obtains the imaginary part result

$$\frac{1}{4}\int_{\mathbb{R}^3}\mu_0|\mathbf{H}(\mathbf{r})|^2-\epsilon_0|\mathbf{E}(\mathbf{r})|^2\,\mathrm{dV}=\frac{1}{4\omega}\operatorname{Im}\int_{V}\mathbf{E}(\mathbf{r})\cdot\mathbf{J}^*(\mathbf{r})\,\mathrm{dV},\tag{5}$$

where we used the fact that the integral of the imaginary part of the divergence term in (2) vanishes as the integration volume approaches $\mathbb{R}^3$. The imaginary part result (5) relates the well-defined difference between the time-average electric and magnetic energies with the net reactive power delivered by $\mathbf{J}$.

As is well known [5, 8], the total energy, defined as the integral of the energy density integrated over all space, is unbounded due to the $1/r^2$ decay of the energy density in the far radiation zone. This is resolved by decomposition of the total energy into radiated and stored energy. The stored energy is, however, difficult to define and interpret. The classical approach used by Chu [5] and Collin & Rothschild [8], and subsequently by others, is based on mode expansions, and therefore restricted to canonical geometries. Spherical regions are most commonly considered but there are also some results for cylindrical [8] and spheroidal [12, 36] structures. The stored energy density is customarily defined as the difference between the total energy density and the *radiated power flow* in the radial direction [7, 8, 35], thus the stored electric energy becomes

$$W_{\mathrm{P}}^{(\mathrm{E})}=\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}}|\mathbf{E}(\mathbf{r})|^2-\eta_0\operatorname{Re}\{\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\}\cdot\hat{\mathbf{r}}\,\mathrm{dV},\tag{6}$$

where $\mathbb{R}^3_{\mathrm{r}}=\{\mathbf{r}:\lim_{r_0\to\infty}|\mathbf{r}|\leq r_0\}$ is used to indicate that the integration is over an infinite spherical volume. The classical results by Chu [5] are for spheres with vanishing interior field [8], so that the stored energy is due to the exterior field only (i.e., for the region where $r>a$ where a is the radius of the smallest sphere circumscribing the sources). The Thal bound [38] restricts the results to fields generated by electric surface currents, see also [26]. Here it is observed that there is a stored energy but no radiated energy flux in the interior of the sphere. The definition (6) is useful for spherical and cylindrical geometries [7, 8, 35]. This definition is difficult to generalize to arbitrary geometries due to its coordinate dependence that originates from the scalar multiplication with $\hat{\mathbf{r}}$. The subtraction of the radiated energy flow is equivalent to subtraction of the energy of the far field outside a circumscribing sphere, cf., (4). This suggests an alternative stored electric energy defined by subtraction of the *far-field energy*, i.e.,

$$W_{\mathrm{F}}^{(\mathrm{E})}=\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}}|\mathbf{E}(\mathbf{r})|^2-\frac{|\mathbf{F}(\hat{\mathbf{r}})|^2}{r^2}\,\mathrm{dV}=\frac{\epsilon_0}{4}\lim_{r_0\to\infty}\left(\int_{r\leq r_0}|\mathbf{E}(\mathbf{r})|^2\,\mathrm{dV}-r_0\int_{\Omega}|\mathbf{F}(\hat{\mathbf{r}})|^2\,\mathrm{d}\Omega\right),\tag{7}$$

where the integration is over the infinite sphere $\mathbb{R}^3_{\mathrm{r}}$. The subtracted far-field in the integrand can alternatively be written as a subtraction of the radius times the radiated power [10, 48].

We note that the definitions with the power flow (6) and far field (7) differ only in the interior of the smallest circumscribing sphere associated with the source support. In the interior of the smallest circumscribing sphere, which we assume next to be of radius a, this subtracted far-field energy is then

$$\frac{\epsilon_0}{4}\int_0^a\int_{\Omega}|\mathbf{F}(\hat{\mathbf{r}})|^2\,\mathrm{d}\Omega\,\mathrm{d}r=\frac{a}{2\mathrm{c}_0}P_{\mathrm{r}}.\tag{8}$$

Assuming that the contribution to the *true* stored electric energy, say $W^{(\mathrm{E})}$, due to the exterior field outside the smallest circumscribing sphere, is equal to that of $W_{\mathrm{P}}^{(\mathrm{E})}$ and $W_{\mathrm{F}}^{(\mathrm{E})}$ in (6) and (7), and that it subtracts some non-negative value less than $\epsilon_0|\mathbf{F}|^2/(4r^2)$ inside the sphere, we obtain the bound

$$W_{\mathrm{F}}^{(\mathrm{E})} \leq W^{(\mathrm{E})} \leq W_{\mathrm{F}}^{(\mathrm{E})} + \frac{a}{2\mathrm{c}_0}P_{\mathrm{r}}. \tag{9}$$

This means that the *true* stored electric energy, $W^{(\mathrm{E})}$, can be bounded from below and above by (7). The stored magnetic energy, $W_{\mathrm{F}}^{(\mathrm{M})}$, is defined analogously. The stored energy is commonly normalized with the radiated power to define Q-factors. The Q-factor is $Q = \max\{Q^{(\mathrm{E})}, Q^{(\mathrm{M})}\}$, where

$$Q^{(\mathrm{E})} = \frac{2\omega W^{(\mathrm{E})}}{P_{\mathrm{r}}} \quad \text{and} \quad Q^{(\mathrm{M})} = \frac{2\omega W^{(\mathrm{M})}}{P_{\mathrm{r}}} \tag{10}$$

and we have included a factor of 2 in the definitions of $Q^{(\mathrm{E})}$ and $Q^{(\mathrm{M})}$ to simplify the comparison with antenna Q. This translates the bound (9) into

$$\max\{0, Q_{\mathrm{F}}\} \leq Q \leq Q_{\mathrm{F}} + ka, \tag{11}$$

where we have added that Q is non-negative.

We show that the stored energy with the subtracted far field (7) is similar to the energy defined by Vandenbosch in [41] for the free space case. For simplicity we express the energy using the scalar potential ϕ and the vector potential $\mathbf{A}$ in the Lorentz gauge [31, 34, 40], so that $(\nabla^2+k^2)\phi(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon_0$ and $(\nabla^2 + k^2)\mathbf{A}(\mathbf{r}) = -\mu_0\mathbf{J}(\mathbf{r})$ and therefore

$$\phi(\mathbf{r}) = \epsilon_0^{-1}(G * \rho)(\mathbf{r}) = \frac{1}{\epsilon_0}\int_V G(\mathbf{r}-\mathbf{r}_1)\rho(\mathbf{r}_1)\,\mathrm{dV}_1 \tag{12}$$

and

$$\mathbf{A}(\mathbf{r}) = \mu_0(G * \mathbf{J})(\mathbf{r}) = \mu_0\int_V G(\mathbf{r}-\mathbf{r}_1)\mathbf{J}(\mathbf{r}_1)\,\mathrm{dV}_1, \tag{13}$$

where $*$ denotes convolution and G is the outgoing Green's function i.e., $G(\mathbf{r}) = \mathrm{e}^{\mathrm{i}kr}/(4\pi r)$ and $r = |\mathbf{r}|$. The vector and scalar potentials are related by $\nabla\cdot\mathbf{A} = \mathrm{i}k\phi/\mathrm{c}_0$ and the electric and magnetic fields are given by [31]

$$\mathbf{E} = \mathrm{i}\omega\mathbf{A} - \nabla\phi \quad \text{and} \quad \mathbf{H} = \mu_0^{-1}\nabla\times\mathbf{A}. \tag{14}$$

We also use the corresponding far-field potentials defined by

$$\phi_\infty(\hat{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}_1)\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{r}_1}\,\mathrm{dV}_1 \quad \text{and} \quad \mathbf{A}_\infty(\hat{\mathbf{r}}) = \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}_1)\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{r}_1}\,\mathrm{dV}_1 \tag{15}$$

giving the electric far-field

$$\mathbf{F}(\hat{\mathbf{r}}) = \mathrm{i}\omega\mathbf{A}_\infty(\hat{\mathbf{r}}) - \hat{\mathbf{r}}\mathrm{i}k\phi_\infty(\hat{\mathbf{r}}). \tag{16}$$

Using that the far-field is orthogonal to $\hat{\mathbf{r}}$, i.e., $\hat{\mathbf{r}}\cdot\mathbf{F} = 0$, the far-field radiation pattern obeys

$$|\mathbf{F}(\hat{\mathbf{r}})|^2 = \omega^2|\mathbf{A}_\infty(\hat{\mathbf{r}})|^2 - k^2|\phi_\infty(\hat{\mathbf{r}})|^2. \tag{17}$$

The electric energy density is proportional to

$$|\mathbf{E}|^2 = \omega^2|\mathbf{A}|^2 - 2\,\mathrm{Re}\{\mathrm{i}\omega\mathbf{A}\cdot\nabla\phi^*\} + |\nabla\phi|^2 = \omega^2|\mathbf{A}|^2 - 2k^2|\phi|^2 + |\nabla\phi|^2 - 2\,\mathrm{Re}\left\{\mathrm{i}\omega\nabla\cdot(\phi^*\mathbf{A})\right\}, \tag{18}$$

where we used $\nabla\cdot(\phi^*\mathbf{A}) = \phi^*\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\phi^* = \mathrm{i}k|\phi|^2/\mathrm{c}_0 + \mathbf{A}\cdot\nabla\phi^*$. We integrate this result over a large sphere to get the far-field type stored electric energy (7) expressed in the potentials

$$\begin{aligned} W_{\mathrm{F}}^{(\mathrm{E})} &= \frac{\epsilon_0}{4}\int_{\mathbb{R}_{\mathrm{r}}^3} |\mathbf{E}(\mathbf{r})|^2 - \frac{|\mathbf{F}(\hat{\mathbf{r}})|^2}{r^2}\,\mathrm{dV} \\ &= \frac{\epsilon_0}{4}\int_{\mathbb{R}_{\mathrm{r}}^3} |\nabla\phi(\mathbf{r})|^2 - k^2|\phi(\mathbf{r})|^2 + \omega^2\left(|\mathbf{A}(\mathbf{r})|^2 - \frac{|\mathbf{A}_\infty(\hat{\mathbf{r}})|^2}{r^2}\right) - k^2\left(|\phi(\mathbf{r})|^2 - \frac{|\phi_\infty(\hat{\mathbf{r}})|^2}{r^2}\right)\mathrm{dV}, \end{aligned} \tag{19}$$

where we applied the divergence theorem to the integration of the last term in (18), obtaining via the discussion in (16) and (17) that $\int_\Omega \mathrm{Im}\{\phi^*(r\hat{\mathbf{r}})A_{\mathrm{r}}(r\hat{\mathbf{r}})\}r^2\,\mathrm{d}\Omega \to 0$ as the radius $r\to\infty$ in $\mathbb{R}_{\mathrm{r}}^3$, see (16).

Use the energy identity for the Helmholtz equation, $|\nabla\phi|^2 - k^2|\phi|^2 = \epsilon_0^{-1}\operatorname{Re}\{\phi\rho^*\} + \nabla\cdot\operatorname{Re}\{\phi^*\nabla\phi\}$, and that $\phi^*\nabla\phi \to \mathrm{i}k\hat{\mathbf{r}}|\phi|^2$ for large enough r, to rewrite the first two terms in (19) as

$$\int_{\mathbb{R}^3_{\mathrm{r}}} |\nabla\phi(\mathbf{r})|^2 - k^2|\phi(\mathbf{r})|^2 \,\mathrm{dV} = \epsilon_0^{-1}\operatorname{Re}\int_V \phi(\mathbf{r})\rho^*(\mathbf{r})\,\mathrm{dV} = \int_V\int_V \rho(\mathbf{r}_1)\frac{\cos(kr_{12})}{4\pi\epsilon_0^2 r_{12}}\rho^*(\mathbf{r}_2)\,\mathrm{dV}_1\,\mathrm{dV}_2, \tag{20}$$

where we also used that the surface term vanishes. The Green's function identity, see Appendix A

$$\int_{\mathbb{R}^3_{\mathrm{r}}} G(\mathbf{r}-\mathbf{r}_1)G^*(\mathbf{r}-\mathbf{r}_2) - \frac{\mathrm{e}^{-\mathrm{i}k(\mathbf{r}_1-\mathbf{r}_2)\cdot\hat{\mathbf{r}}}}{16\pi^2 r^2}\,\mathrm{dV} = -\frac{\sin(kr_{12})}{8\pi k} + \mathrm{i}\frac{r_1^2 - r_2^2}{8\pi r_{12}}\,\mathrm{j}_1(kr_{12}), \tag{21}$$

where $\mathrm{j}_1(z) = (\sin(z) - z\cos(z))/z^2$ is a spherical Bessel function [40], is used to rewrite the two remaining terms in (19) as

$$\begin{aligned}\int_{\mathbb{R}^3_{\mathrm{r}}} |G*\mathbf{J}|^2 - \frac{|\int_V \mathrm{e}^{-\mathrm{i}k\mathbf{r}_1\cdot\hat{\mathbf{r}}}\mathbf{J}(\mathbf{r}_1)\,\mathrm{dV}_1|^2}{16\pi^2 r^2}\,\mathrm{dV} &= -\int_V\int_V \mathbf{J}(\mathbf{r}_1)\cdot\frac{\sin(k|\mathbf{r}_1-\mathbf{r}_2|)}{8\pi k}\mathbf{J}^*(\mathbf{r}_2)\,\mathrm{dV}_1\,\mathrm{dV}_2 \\ &\quad + \mathrm{i}\int_V\int_V \mathbf{J}(\mathbf{r}_1)\cdot\frac{r_1^2-r_2^2}{8\pi r_{12}}\,\mathrm{j}_1(kr_{12})\mathbf{J}^*(\mathbf{r}_2)\,\mathrm{dV}_1\,\mathrm{dV}_2\end{aligned} \tag{22}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3_{\mathrm{r}}} |G*\rho|^2 - \frac{|\int_V \mathrm{e}^{-\mathrm{i}k\mathbf{r}_1\cdot\hat{\mathbf{r}}}\rho(\mathbf{r}_1)\,\mathrm{dV}_1|^2}{16\pi^2 r^2}\,\mathrm{dV} &= -\int_V\int_V \rho(\mathbf{r}_1)\frac{\sin(k|\mathbf{r}_1-\mathbf{r}_2|)}{8\pi k}\rho^*(\mathbf{r}_2)\,\mathrm{dV}_1\,\mathrm{dV}_2 \\ &\quad + \mathrm{i}\int_V\int_V \rho(\mathbf{r}_1)\frac{r_1^2-r_2^2}{8\pi r_{12}}\,\mathrm{j}_1(kr_{12})\rho^*(\mathbf{r}_2)\,\mathrm{dV}_1\,\mathrm{dV}_2\,.\end{aligned} \tag{23}$$

We note that the first terms in the right-hand side of (22) and (23) only depend on the distance $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ and are hence coordinate independent, whereas the last terms depend on the coordinate system due to the factor $r_1^2 - r_2^2 = (\mathbf{r}_1 + \mathbf{r}_2)\cdot(\mathbf{r}_1 - \mathbf{r}_2)$. The coordinate dependence originates in the division with r^2 and the explicit evaluation of the integral in (21) over large spherical volumes $\mathbb{R}^3_{\mathrm{r}}$, see Appendix A.

Collecting the terms in (20), (22), and (23), we get a quadratic form in the current density $\mathbf{J}$ for the far-field type stored electric energy (19) as

$$W_{\mathrm{F}}^{(\mathrm{E})} = W_{\mathrm{F}_0}^{(\mathrm{E})} + W_{\mathrm{F}_1} + W_{\mathrm{F}_2}, \tag{24}$$

where $W_{\mathrm{F}_0}^{(\mathrm{E})} + W_{\mathrm{F}_1}$ is the coordinate independent part

$$\begin{aligned}W_{\mathrm{F}_0}^{(\mathrm{E})} + W_{\mathrm{F}_1} &= \frac{\eta_0}{4\omega}\int_V\int_V \nabla_1\cdot\mathbf{J}(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}^*(\mathbf{r}_2)\frac{\cos(kr_{12})}{4\pi k r_{12}} \\ &\quad - \left(k^2\mathbf{J}(\mathbf{r}_1)\cdot\mathbf{J}(\mathbf{r}_2)^* - \nabla_1\cdot\mathbf{J}(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}^*(\mathbf{r}_2)\right)\frac{\sin(kr_{12})}{8\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2\end{aligned} \tag{25}$$

and $W_{\mathrm{F}_0}^{(\mathrm{E})}$ and W_{F_1} contain the cos and sin parts, respectively. The coordinate dependent part is

$$W_{\mathrm{F}_2} = \frac{\eta_0}{4\omega}\int_V\int_V \operatorname{Im}\left\{k^2\mathbf{J}(\mathbf{r}_1)\cdot\mathbf{J}^*(\mathbf{r}_2) - \nabla_1\cdot\mathbf{J}(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}^*(\mathbf{r}_2)\right\}\frac{r_1^2-r_2^2}{8\pi r_{12}}k\,\mathrm{j}_1(kr_{12})\,\mathrm{dV}_1\,\mathrm{dV}_2\,. \tag{26}$$

The coordinate independent part $W_{\mathrm{F}_0}^{(\mathrm{E})} + W_{\mathrm{F}_1}$ is identical to the energy by Vandenbosch in [41] for free space and hence presents a clear interpretation of the energy [41] in terms of (7). We also see that the definition (7) partly explains the peculiar effects of negative stored energies [21] and suggests a remedy to it in (11). The coordinate dependent part W_{F_2} is more involved. A similar coordinate dependent term is observed in [48]. Obviously the actual stored energy, as any physical quantity, should be independent of the coordinate system. First, we observe that $W_{\mathrm{F}_2} = 0$ for any current density that has a constant phase. This includes the fields originating from single spherical modes on spherical surfaces and hence most cases in [5, 8, 26, 38]. It also includes currents in the form of single characteristic modes [2]. We also get the coordinate independent part by taking the average of the stored energy from $\mathbf{J}$ and $\mathbf{J}^*$. The term W_{F_2} is further analyzed in Sections 3 and 4.

For the stored magnetic energy we can use $|\mathbf{B}|^2 = |\nabla\times\mathbf{A}|^2$ or simpler the energy identity (5), to directly get the difference

$$\int_{\mathbb{R}^3_{\mathrm{r}}} \mu_0|\mathbf{H}(\mathbf{r})|^2 - \epsilon_0|\mathbf{E}(\mathbf{r})|^2 \,\mathrm{dV} = \mathrm{Re}\int_V \mathbf{A}(\mathbf{r})\cdot\mathbf{J}^*(\mathbf{r}) - \phi(\mathbf{r})\rho^*(\mathbf{r})\,\mathrm{dV}, \tag{27}$$

where we used $\mathbf{E}\cdot\mathbf{J}^* = \mathrm{i}\omega\mathbf{A}\cdot\mathbf{J}^* - \nabla\cdot(\phi\mathbf{J}^*) - \mathrm{i}\omega\phi\rho^*$. This gives the far-field type stored magnetic energy $W_{\mathrm{F}}^{(\mathrm{M})} = W_{\mathrm{F}_0}^{(\mathrm{M})} + W_{\mathrm{F}_1} + W_{\mathrm{F}_2}$, where the coordinate independent part

$$\begin{aligned} W_{\mathrm{F}_0}^{(\mathrm{M})} + W_{\mathrm{F}_1} = \frac{\eta_0}{4\omega}\int_V\int_V & k^2\mathbf{J}(\mathbf{r}_1)\cdot\mathbf{J}^*(\mathbf{r}_2)\frac{\cos(kr_{12})}{4\pi k r_{12}} \\ & -\big(k^2\mathbf{J}(\mathbf{r}_1)\cdot\mathbf{J}(\mathbf{r}_2)^* - \nabla_1\cdot\mathbf{J}(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}^*(\mathbf{r}_2)\big)\frac{\sin(kr_{12})}{8\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2 \end{aligned} \tag{28}$$

is expressed as a quadratic form in $\mathbf{J}$, see also [41]. We also have the radiated power [14, 41]

$$P_{\mathrm{r}} = \frac{\eta_0}{2}\int_V\int_V \big(k^2\mathbf{J}(\mathbf{r}_1)\cdot\mathbf{J}^*(\mathbf{r}_2) - \nabla_1\cdot\mathbf{J}(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}^*(\mathbf{r}_2)\big)\frac{\sin(kr_{12})}{4\pi k r_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2\,. \tag{29}$$

It is illustrative to rewrite the coordinate independent far-field stored energy in the potentials:

$$W_{\mathrm{F}_0}^{(\mathrm{E})} = \frac{1}{4}\mathrm{Re}\int_V \rho^*(\mathbf{r})\phi(\mathbf{r})\,\mathrm{dV} \quad\text{and}\quad W_{\mathrm{F}_0}^{(\mathrm{M})} = \frac{1}{4}\mathrm{Re}\int_V \mathbf{J}^*(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})\,\mathrm{dV}\,. \tag{30}$$

We note that the sum of the first terms, $W_{\mathrm{F}_0}^{(\mathrm{E})} + W_{\mathrm{F}_0}^{(\mathrm{M})}$, corresponds to a frequency-domain version of the energy expression by Carpenter [4], see also [9, 39]. Moreover, they reduce to well-known electrostatic and magnetostatic expressions in the low-frequency limit [31].

We follow standard notation in the method of moments (MoM) and use the operator $\mathcal{L}$ as the integral operator associated with the electric field integral equation (EFIE) [32]. Here, the operator is generalized to volumes and defined as

$$\langle\mathbf{J},\mathcal{L}\mathbf{J}\rangle = \langle\mathbf{J},(\mathcal{L}_{\mathrm{m}}-\mathcal{L}_{\mathrm{e}})\mathbf{J}\rangle = \mathrm{i}\int_V\int_V \Big(k\mathbf{J}^*(\mathbf{r}_1)\cdot\mathbf{J}(\mathbf{r}_2) - \frac{1}{k}\nabla_1\cdot\mathbf{J}^*(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}(\mathbf{r}_2)\Big)\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2\,. \tag{31}$$

The total stored energy is related to the frequency derivative of $\mathcal{L}$, i.e.,

$$W_{\mathrm{F}_0}^{(\mathrm{E})} + W_{\mathrm{F}_0}^{(\mathrm{M})} + 2W_{\mathrm{F}_1} = \frac{\eta_0}{4\omega}\mathrm{Im}\left\langle\mathbf{J},k\frac{\partial\mathcal{L}}{\partial k}\mathbf{J}\right\rangle, \tag{32}$$

where

$$\begin{aligned} \left\langle\mathbf{J},k\frac{\partial\mathcal{L}}{\partial k}\mathbf{J}\right\rangle &= \mathrm{i}\int_V\int_V \left(k\mathbf{J}^*(\mathbf{r}_1)\cdot\mathbf{J}(\mathbf{r}_2) + \frac{1}{k}\nabla_1\cdot\mathbf{J}^*(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}(\mathbf{r}_2)\right)\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2 \\ & -k\int_V\int_V \left(k\mathbf{J}^*(\mathbf{r}_1)\cdot\mathbf{J}(\mathbf{r}_2) - \frac{1}{k}\nabla_1\cdot\mathbf{J}^*(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}(\mathbf{r}_2)\right)\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2\,. \end{aligned} \tag{33}$$

The difference between the stored magnetic and electric energies is

$$W_{\mathrm{F}_0}^{(\mathrm{M})} - W_{\mathrm{F}_0}^{(\mathrm{E})} = \frac{\eta_0}{4\omega}\mathrm{Im}\langle\mathbf{J},\mathcal{L}\mathbf{J}\rangle = \frac{\eta_0}{4\omega}\int_V\int_V (k^2\mathbf{J}^*(\mathbf{r}_1)\cdot\mathbf{J}(\mathbf{r}_2) - \nabla_1\cdot\mathbf{J}^*(\mathbf{r}_1)\nabla_2\cdot\mathbf{J}(\mathbf{r}_2))\frac{\cos(kr_{12})}{4\pi k r_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 \tag{34}$$

that expresses the stored electric and magnetic energies as

$$W_{\mathrm{F}_0}^{(\mathrm{E})} + W_{\mathrm{F}_1} = \frac{\eta_0}{8\omega}\mathrm{Im}\left\langle\mathbf{J},\left(k\frac{\partial\mathcal{L}}{\partial k} - \mathcal{L}\right)\mathbf{J}\right\rangle \quad\text{and}\quad W_{\mathrm{F}_0}^{(\mathrm{M})} + W_{\mathrm{F}_1} = \frac{\eta_0}{8\omega}\mathrm{Im}\left\langle\mathbf{J},\left(k\frac{\partial\mathcal{L}}{\partial k} + \mathcal{L}\right)\mathbf{J}\right\rangle, \tag{35}$$

respectively. These relations are similar to the method of moments expressions in [24, 29]. The corresponding total radiated power is $P_{\mathrm{r}} = \frac{-\eta_0}{2}\mathrm{Re}\langle\mathbf{J},\mathcal{L}\mathbf{J}\rangle$.

3. COORDINATE DEPENDENT TERM

The stored electric (24) and magnetic energies defined by subtraction of the far-field energy (7) contain the potentially coordinate dependent part W_{F_2} defined in (26). Assume that $W_{\mathrm{F}_2} = W_{\mathrm{F}_2,\mathbf{0}}$ for one coordinate system. Consider a shift of the coordinate system $\mathbf{r} \to \mathbf{d} + \mathbf{r}$ and use that $r_1^2 - r_2^2 \to r_1^2 - r_2^2 + 2\mathbf{d} \cdot (\mathbf{r}_1 - \mathbf{r}_2)$. This expresses the coordinate dependent term as

$$W_{\mathrm{F}_2,\mathbf{d}} = W_{\mathrm{F}_2,\mathbf{0}} + k\mathbf{d} \cdot \mathbf{W}, \tag{36}$$

where $\mathbf{W} = \mathbf{W}_\rho + \mathbf{W}_\mathrm{J}$ and

$$\begin{aligned}\mathbf{W}_\rho &= \frac{\mathrm{i}}{2\epsilon_0} \int_V \int_V \rho(\mathbf{r}_1)\nabla_1 \frac{\sin(kr_{12})}{8\pi kr_{12}} \rho^*(\mathbf{r}_2)\,\mathrm{dV}_1\,\mathrm{dV}_2 \\ &= \frac{k\epsilon_0}{4} \int_\Omega \hat{\mathbf{r}} \left| \int_V \frac{\rho(\mathbf{r}_1)\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{r}_1}}{4\pi\epsilon_0}\,\mathrm{dV}_1 \right|^2 \mathrm{d}\Omega = \frac{k\epsilon_0}{4} \int_\Omega |\phi_\infty(\hat{\mathbf{r}})|^2 \hat{\mathbf{r}}\,\mathrm{d}\Omega \end{aligned} \tag{37}$$

and we used (A5), the identity

$$\nabla_1 \frac{\sin(kr_{12})}{4\pi kr_{12}} = -\mathrm{i}k \lim_{r\to\infty} \int_{|\mathbf{r}|=r} \hat{\mathbf{r}} G(\mathbf{r}-\mathbf{r}_1) G^*(\mathbf{r}-\mathbf{r}_2)\,\mathrm{dS} = \frac{-\mathrm{i}k}{16\pi^2} \int_\Omega \hat{\mathbf{r}} \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{r}}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\,\mathrm{d}\Omega, \tag{38}$$

and the far-field potential (15). Similarly, the current part is

$$\mathbf{W}_\mathrm{J} = -\frac{\mathrm{i}\mu_0}{2} \int_V \int_V \mathbf{J}(\mathbf{r}_1) \cdot \mathbf{J}^*(\mathbf{r}_2) \nabla_1 \frac{\sin(kr_{12})}{8\pi kr_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 = -\frac{k}{4\mu_0} \int_\Omega |\mathbf{A}_\infty(\hat{\mathbf{r}})|^2\, \hat{\mathbf{r}}\,\mathrm{d}\Omega. \tag{39}$$

Using the far-field identity (17) simplifies $\mathbf{W}$ to

$$\mathbf{W} = -\frac{\epsilon_0}{4k} \int_\Omega |\mathbf{F}(\hat{\mathbf{r}})|^2\, \hat{\mathbf{r}}\,\mathrm{d}\Omega. \tag{40}$$

The corresponding Q factor is hence shifted as

$$\varDelta Q_{\mathrm{F}_2} = \frac{-k\mathbf{d} \cdot \int_\Omega \hat{\mathbf{r}}\, |\mathbf{F}(\hat{\mathbf{r}})|^2\,\mathrm{d}\Omega}{2\int_\Omega |\mathbf{F}(\hat{\mathbf{r}})|^2\,\mathrm{d}\Omega}, \tag{41}$$

where we see that $|\varDelta Q_{\mathrm{F}_2}| \leq ka$ for all coordinate shifts within the smallest circumscribing sphere, see Fig. 1. We note that this term is similar to the coordinate dependence observed in [48].

4. SMALL STRUCTURES

Evaluation of the stored energy for antenna Q is most interesting for small structures, where Q is large, e.g., $Q \geq 10$, and can be used to quantify the bandwidth of antennas [5, 21, 23, 42, 48]. The low-frequency expansion of the stored energy are presented in [13, 21, 41, 42]. Here, we base it on the low-frequency expansion $\mathbf{J} = \mathbf{J}^{(0)} + k\mathbf{J}^{(1)} + \mathcal{O}(k^2)$ as $k \to 0$, where $\nabla \cdot \mathbf{J}^{(0)} = 0$ and the static terms $\mathbf{J}^{(0)}$ and $\rho_0 = -\mathrm{i}\nabla \cdot \mathbf{J}^{(1)}/\mathrm{c}_0$ have a constant phase. For the corresponding asymptotic expansions of the Q-factor components in (24), we note that the coordinate dependent part vanishes if $\mathbf{J}$ and $\rho(\mathbf{r})$ have constant phase. This gives

$$\begin{aligned}\mathrm{Im}\{\rho(\mathbf{r}_1)\rho^*(\mathbf{r}_2)\} &= \mathrm{Im}\{(\rho_0(\mathbf{r}_1) + k\rho_1(\mathbf{r}_1))(\rho_0^*(\mathbf{r}_2) + k\rho_1^*(\mathbf{r}_2))\} + \mathcal{O}\left(k^2\right) \\ &= k\,\mathrm{Im}\left\{\rho_0\left(\mathbf{r}_1\right)\rho_1^*\left(\mathbf{r}_2\right) + \rho_1\left(\mathbf{r}_1\right)\rho_0^*\left(\mathbf{r}_2\right)\right\} + \mathcal{O}\left(k^2\right)\end{aligned} \tag{42}$$

as $k \to 0$ and similarly for $\mathbf{J}$. The different parts of the stored energy (24) contribute to the Q-factor asymptotically

$$Q_{\mathrm{F}_0}^{(\mathrm{E,M})} \sim \frac{1}{(ka)^3}, \quad Q_{\mathrm{F}_1} \sim \frac{1}{ka}, \quad \text{and} \quad Q_{\mathrm{F}_2} \sim ka \tag{43}$$

as $ka \to 0$, where a is the radius of smallest circumscribing sphere, and the coordinate system is centered inside the sphere.

We can compare the expansion (43) with the Chu [5] and Thal [38] bounds

$$Q_{\text{Chu}} = \frac{1}{(ka)^3} + \frac{1}{ka} \quad \text{and} \quad Q_{\text{Thal}} = \frac{3}{2(ka)^3}, \tag{44}$$

respectively, where it is seen that Q_{Chu} has components that are of the same order as $Q_{\text{F}_0}^{(\text{E,M})}$ and Q_{F_1} and hence that these terms are essential to produce reliable results. This is also the conclusion from (46) that shows that the Q-factors differ by ka for spherical regions. The lower bound on small antennas is inversely proportional to the polarizability [16, 19–21, 33, 42, 46, 47].

The coordinate dependent part Q_{F_2} is negligible for small structures and is of the same order as the difference between the far-field (7) and power (6) type as seen by the bound (11). We also note that the importance of Q diminishes as Q approaches unity. This also restricts the interest of the results to small antennas.

5. EXAMPLES

To interpret the different proposals for stored electromagnetic energy, we consider analytic and numerical examples. The first analytic example illustrates the relation between the stored energy defined by subtraction of the power flow (6), used in [5, 8, 35], and the far-field power (7) similar to [41] for spherical modes and shows that their Q factors differ by ka. The second example considers a dipole antenna and compares the Q factors from differentiation [23, 48] with the stored energy determined from the current density (25) [41], see also [17] for additional examples.

5.1. Numerical Example for Spherical Shells

The two formulations (6) and (7) for the stored energy can be compared for electric surface currents on spherical shells. This is the case analyzed by Thal [38] and Hansen & Collin [26], see also [27] for the case with electric and magnetic surface currents. We expand the surface current on a sphere with radius a in vector spherical harmonics $\mathbf{Y}_{\tau\sigma lm}$, see Appendix B.3, to compute the electric and magnetic Q factors (7)

$$Q_{\tau l,\text{F}}^{(\text{E})}(\kappa)=\frac{2\omega W_{\text{F}}^{(\text{E})}(\kappa)}{P_{\text{r}}(\kappa)}=-\frac{\left(\kappa\,\mathrm{R}_{\tau l}^{(1)}(\kappa)\,\mathrm{R}_{\tau l}^{(2)}(\kappa)\right)'}{2(\mathrm{R}_{\tau l}^{(1)}(\kappa))^2} \quad \text{and} \quad Q_{\tau l,\text{F}}^{(\text{M})}=\frac{2\omega W_{\text{F}}^{(\text{M})}(\kappa)}{P_{\text{r}}(\kappa)}=Q_{\tau l,\text{F}}^{(\text{E})}(\kappa)-\frac{\mathrm{R}_{\tau l}^{(2)}(\kappa)}{\mathrm{R}_{\tau l}^{(1)}(\kappa)}, \tag{45}$$

respectively. Here, $\tau = 1$ for the TE and $\tau = 2$ for the TM cases and l is the order of the spherical mode. We note that the expressions for the TE and TM are written in identical forms by using the radial functions $R_{\tau l}^{(p)}$ (B8), see also [25]. The corresponding power flow stored energy (6) is

$$Q_{\tau 1,\text{P}}^{(\text{E,M})}(\kappa) = \frac{2\omega W_{\text{P}}^{(\text{E,M})}(\kappa)}{P_{\text{r}}(\kappa)} = \kappa + Q_{\tau 1,\text{F}}^{(\text{E,M})}(\kappa), \tag{46}$$

where $Q_{\tau 1,\text{F}}^{(\text{E,M})}$ denotes the electric and magnetic far-field type Q factors in (45), see [22]. The difference $\kappa = ka$ is consistent with the interpretation of a standing wave in the interior of the sphere, cf., (11). Moreover, the expressions (45) unifies the TE and TM cases and offer an alternative to the expressions in [26].

The electric and magnetic Q-factors are depicted in Fig. 2 for $l = 1$, 2. The relative differences are negligible for small ka where Q is large. For larger ka, where Q can be small, the relative difference is significant although the absolute difference is exactly ka. We also note that the Q factors oscillate and can be significant even for large ka. This is mainly due to small values of $\mathrm{R}_{\tau l}^{(1)}(ka)$ that can be interpreted as a negligible radiated power. Moreover, the Q-factors related to the far-field type stored energy (7) is negative in some frequency bands. The corresponding Q-factors related to (6) are always non-negative. Moreover, it is observed that $Q_{11,\text{P}}^{(\text{M})} \geq Q_{11,\text{P}}^{(\text{E})}$ for low ka but there are frequency intervals with $Q_{11,\text{P}}^{(\text{M})} < Q_{11,\text{P}}^{(\text{E})}$ for larger ka.

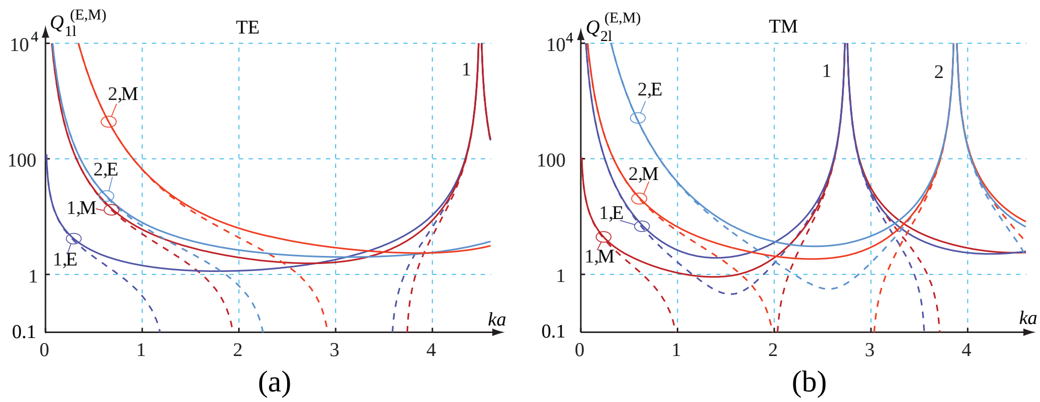

Figure 2. Electric and magnetic Q factors for electrical surface currents $\mathbf{J}(\mathbf{r}) = J_0\,\mathbf{Y}_{\tau\sigma ml}(\hat{\mathbf{r}})\delta(r-a)$ for $l=1,2$. Power (solid curves) and far-field (dashed curves) stored energies. They differ by ka (46). (a) TE ($\tau = 1$) modes. (b) TM ($\tau = 2$) modes.

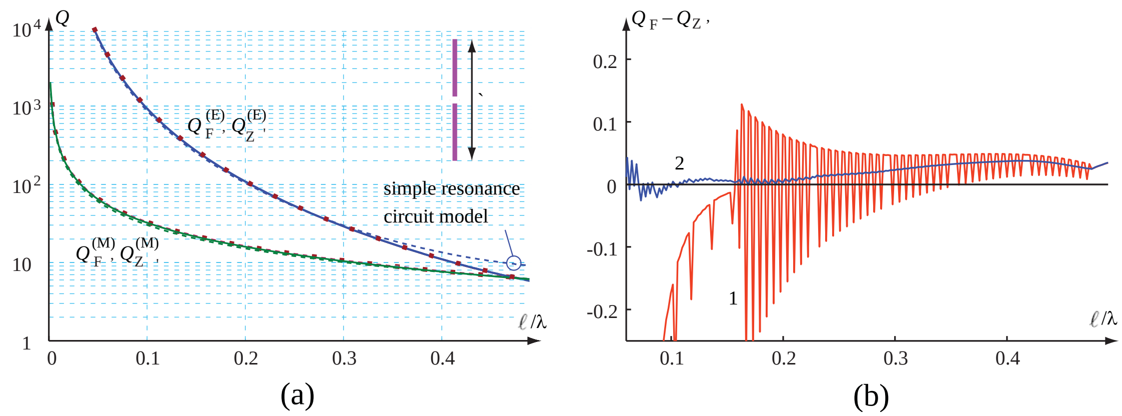

Figure 3. Illustration of the Q factor for a center feed strip dipole with length ℓ and width $\ell/100$. The Q factors are determined from the stored energies (25) and (28) and from differentiation of the input impedance [23, 24, 48]. (a) Electric and magnetic Q-factors from (25), (28), the circuit model (dashed curves), and differentiation of the input impedance $Q_{\mathrm{Z}'}$. (b) Difference between the computed Q-factors $Q_{\mathrm{F}} - Q_{\mathrm{Z}'}$, where $Q_{\mathrm{Z}'}$ is computed from a difference scheme and analytic differentiation of a high order rational approximation in 1 and 2, respectively.

5.2. Strip Dipole

Consider a center fed strip dipole with length ℓ and width $\ell/100$ modeled as a perfect electric conductor (PEC). The Q-factors (10) determined from the integral expressions $Q_{\mathrm{F}}^{(\mathrm{E})}$ in (25) and $Q_{\mathrm{F}}^{(\mathrm{M})}$ in (28), the simple resonance circuit model [15], and differentiation of the input impedance [23, 48] are compared in Fig. 3(a). The circuit model is based on the circuit representations of the lowest order spherical modes [37] with the lumped elements determined with the approach in [15]. The Q-factors from the simple resonance circuit model approximates the integral expression very well for $\ell < 0.3\lambda$ but starts to differ for shorter wavelengths where the circuit model is less accurate, see Fig. 3(a). The difference Q_{F}-$Q_{\mathrm{Z}'}$ is also depicted in Fig. 3(b). We see that the difference is negligible for the considered wavelengths. Curve (1) shows $Q_{\mathrm{Z}'}$ computed with a finite difference scheme. The curve is sensitive to noise and the used discretization. The noise is suppressed by approximating the input impedance with a high order polynomial and performing analytic differentiation as seen by curve (2).

6. CONCLUSIONS

The analyzed expression (7) for the stored energy defined by subtraction of the far-field energy density from the energy density is mainly motivated by the formulation of Collin & Rothschild [8], Fante [10], McLean [35], Yaghjian & Best [48] and the expressions by Vandenbosch in [41]. We show that the stored energy (7) is identical to the energy in [41] for many currents. However, some current densities have an additional coordinate dependent term. This term is very small for small antennas but it can contribute for larger structures, see also [48]. Here, it is also important to realize that the classical definition [8] with the subtracted power flow (6) is inherently coordinate dependent. The identification of the energy expressions in [41] with (7) offers a simple interpretation of the observed cases with a negative stored energy [21]. The analysis also suggests that the resulting Q factor has an uncertainty of the order ka. This is consistent with the use of the results for small (sub wavelength) antennas [18, 21], where ka is small and the Q-factor is large.

The energy expressions proposed by Vandenbosch in [41] are well suited for optimization formulations as they are simple quadratic forms of the current density. The quadratic form is very practical as it allows for various optimization formulations such as Lagrangian [21] and convex optimization [18] and has already led to many new antenna results. Their resemblance of the electric field integral equation (EFIE) makes the numerical implementation very simple. The new formulation also produce simplified analytic expressions (46) the unifies the TE and TM cases for the Q factor of current densities on spherical shells [26, 27]. These results for the spherical modes and the antennas are very interesting for the interpretation of the Q factor, Q_{F}, defined by subtraction of the far field (7). The analytic results for the spherical modes show that the subtracted far-field inside the sphere defines a stored energy that can be negative, see also [21], and a Q_{F} that differ from the classical definition (6) by the electrical size ka, i.e., $Q_{\mathrm{P}} = Q_{\mathrm{F}} + ka$, where Q_{P} denotes the Q defined using (6).

ACKNOWLEDGMENT

This work was supported by the Swedish Research Council (VR), the Crafoord Foundation, the Swedish Governmental Agency for Innovation Systems (VINNOVA) within the VINN Excellence Center CHASE, and the Swedish Foundation for Strategic Research (SSF).

Prof. Edwin Marengo at Northeastern University, MA, USA is also acknowledged for valuable discussions and his hospitality during a visit of MG to Boston, August 2012.

APPENDIX A. GREEN'S FUNCTION IDENTITIES

Multiply the Helmholtz Green's function for G_1: $(\nabla^2 + k^2)G_1 = -\delta(\mathbf{r} - \mathbf{r}_1)$ with G_2^*, and similarly for G_2^*. Adding the results together with a standard vector calculus identity gives $2(\nabla G_1 \cdot \nabla G_2^* - k^2 G_1 G_2^*) = G_1\delta_2 + G_2^*\delta_1 + \nabla^2(G_1 G_2^*)$, where $\delta_n = \delta(\mathbf{r} - \mathbf{r}_n)$ with $n = 1, 2$ denotes the Dirac delta distribution. Integration yields the identity [41]

$$\int_{\mathbb{R}^3_{\mathrm{r}}} \nabla G(\mathbf{r} - \mathbf{r}_1) \cdot \nabla G^*(\mathbf{r} - \mathbf{r}_2) - k^2 G(\mathbf{r} - \mathbf{r}_1)\, G^*(\mathbf{r} - \mathbf{r}_2)\, \mathrm{dV} = \frac{\cos(k|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{A1}$$

where we used Gauss's theorem together with the observation that

$$\nabla(G_1 G_2^*) \to \frac{-\hat{\mathbf{r}}\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot(\mathbf{r}_2 - \mathbf{r}_1)}}{8\pi^2 r^3} \tag{A2}$$

for large enough radius.

The k-derivative of the Helmholtz Green's equation for G_1 is $(\nabla^2 + k^2)\partial_k G_1 + 2kG_1 = 0$. Similarly to the derivation of (A1) we multiply with G_2^*, and repeat the procedure with the k-derivative of G_2^*. Adding the result and applying vector calculus identities to move ∇^2 away from the k-derivative results in the identity

$$4kG_1G_2^* = \delta_2\partial_k G_1 + \delta_1\partial_k G_2^* - \nabla\cdot\mathbf{q}, \tag{A3}$$

where

$$\begin{aligned}\hat{\mathbf{r}}\cdot\mathbf{q} &= \hat{\mathbf{r}}\cdot\left(G_1\nabla\partial_k G_2 - (\partial_k G_2^*)\nabla G_1 + G_2^*\nabla\partial_k G_1 - (\partial_k G_1)\nabla G_2^*\right)\\ &\to \frac{-k}{8\pi^2 r}\left[2+\frac{1}{r}\left(\hat{\mathbf{r}}\cdot(\mathbf{r}_1+\mathbf{r}_2)+\mathrm{i}\left(|\hat{\mathbf{r}}\times\mathbf{r}_1|^2-|\hat{\mathbf{r}}\times\mathbf{r}_2|^2\right)\right)+\mathcal{O}\left(\frac{1}{r^2}\right)\right]\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{r}}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\end{aligned}\tag{A4}$$

for large enough radius. Collecting term of decay rate r^{-1} on the left-hand side and the remaining terms on the right-hand side. Integration over a large sphere, together with Gauss's theorem and elementary integrals result in

$$\begin{aligned}\int_{\mathbb{R}^3_{\mathrm{r}}} G(\mathbf{r}-\mathbf{r}_1)G^*(\mathbf{r}-\mathbf{r}_2)-\frac{\mathrm{e}^{-\mathrm{i}k(\mathbf{r}_1-\mathbf{r}_2)\cdot\hat{\mathbf{r}}}}{16\pi^2r^2}\,\mathrm{dV} &= -\frac{\sin(kr_{12})}{8\pi k}+\mathrm{i}\frac{r_1^2-r_2^2}{8\pi k^2 r_{12}^3}\left(\sin(kr_{12})-kr_{12}\cos(kr_{12})\right)\\ &= -\frac{\sin(kr_{12})}{8\pi k}+\mathrm{i}\frac{(\mathbf{r}_1+\mathbf{r}_2)\cdot(\mathbf{r}_1-\mathbf{r}_2)}{8\pi r_{12}}\,\mathrm{j}_1(kr_{12})\\ &= -\frac{\sin(kr_{12})}{8\pi k}-\mathrm{i}\frac{(\mathbf{r}_1+\mathbf{r}_2)}{k}\cdot\nabla_1\frac{\sin(kr_{12})}{8\pi k r_{12}}.\end{aligned}\tag{A5}$$

Here $\mathrm{j}_1(z)=(\sin(z)-z\cos(z))/z^2$ and $r_{12}=|\mathbf{r}_1-\mathbf{r}_2|$. Note that (A5) generalizes the result in [41] to the case $\mathbf{r}_1+\mathbf{r}_2\neq\mathbf{0}$ and shows that the integral depends of the coordinate system. The result also shows that it is necessary to specify how the integration over $\mathbb{R}^3$ is performed, i.e., here as the limit $\mathbb{R}^3_{\mathrm{r}}=\{\mathbf{r}:\lim_{r_0\to\infty}|\mathbf{r}|<r_0\}$.

APPENDIX B. ELECTRIC SURFACE CURRENTS ON A SPHERE

The two formulations (6) and (7) for the stored energy can be compared for electric surface currents on spherical shells. This is the case analyzed by Thal [38] and Hansen & Collin [26], see also [27] for the case with electric and magnetic surface currents. We expand the surface current on a sphere with radius a in vector spherical harmonics $\mathbf{Y}$, see (B9). For simplicity, consider the surface current $\mathbf{J}(\mathbf{r})=J_0\,\mathbf{Y}_{\tau\sigma ml}(\hat{\mathbf{r}})\delta(r-a)$ that induces the electric and magnetic fields

$$\mathbf{E}(\mathbf{r})=\mathrm{i}\eta_0\tilde{J}_0\frac{\mathbf{u}^{(p)}_{\tau\sigma ml}(k\mathbf{r})}{\mathrm{R}^{(p)}_{\tau l}(ka)}\quad\text{and}\quad\mathbf{H}(\mathbf{r})=\tilde{J}_0\frac{\mathbf{u}^{(p)}_{\bar{\tau}\sigma ml}(k\mathbf{r})}{\mathrm{R}^{(p)}_{\tau l}(ka)},\tag{B1}$$

where $p=1$ for $r<a$ and $p=3$ for $r>a$, $\mathbf{u}^{(p)}_{\tau\sigma ml}$ is the spherical vector waves, and $\mathrm{R}^{(p)}_{\tau l}$ the radial functions in Hansen [25], defined in (B8). We note that the derivatives of $\mathrm{R}^{(p)}_{\tau l}(\kappa)$ are easily expressed in $z^{(p)}$, see (B8). Here, $\tau=1$ is transverse electric (TE) and $\tau=2$ transverse magnetic (TM) waves. Moreover, the dual index $\bar{\tau}$ is $\bar{\tau}=2$ if $\tau=1$ and $\bar{\tau}=1$ if $\tau=2$. The current in (B1) is rescaled as $\tilde{J}_0=J_0\,\mathrm{R}^{(1)}_{\tau l}(ka)\,\mathrm{R}^{(3)}_{\tau l}(ka)$ and below we let J_0 be real valued to simplify the notation. We also note that the coordinate dependent term (26) vanishes for single spherical modes.

B.1. Far-field Type Stored Energy for the TE Case W_{F}

We start with the transverse electric (TE) case $\tau=1$, i.e., $\mathbf{J}(\mathbf{r})=\mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}})\delta(r-a)$ that is divergence free, $\nabla\cdot\mathbf{J}=0$. The integrals in (24) are evaluated analytically by expanding the Green's functions (B10) in spherical modes (B9). Using $\nabla\cdot\mathbf{Y}_{1\sigma ml}=0$, we get $\langle\mathbf{J},\mathcal{L}_{\mathrm{e}}\mathbf{J}\rangle=0$ for (31) and hence the first part of the stored electric energy $W^{(\mathrm{E})}_{\mathrm{F}_0}=0$. The expansion of the full Green's dyadic, $\mathbf{G}=G\mathbf{I}$, (B11) gives

$$\begin{aligned}\frac{1}{\mathrm{i}kJ_0^2}\langle\mathbf{J},\mathcal{L}_{\mathrm{m}}\mathbf{J}\rangle &= \int_V\int_V \mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}}_1)\delta(r_1-a)\cdot\mathbf{G}(\mathbf{r}_1-\mathbf{r}_2)\cdot\mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}}_2)\delta(r_2-a)\,\mathrm{dV}_1\,\mathrm{dV}_2\\ &= a^4\int_\Omega\int_\Omega \mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}}_1)\cdot\mathbf{G}(\mathbf{r}_1-\mathbf{r}_2)\cdot\mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}}_2)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2=\mathrm{i}a^4k\,\mathrm{R}^{(3)}_{1l}(\kappa)\,\mathrm{R}^{(1)}_{1l}(\kappa)\end{aligned}\tag{B2}$$

for the terms in (31) to get the first part of the stored magnetic energy as $4\omega\eta_0^{-1}W^{(\mathrm{M})}_{\mathrm{F}_0}=-a^2\kappa^2J_0^2\,\mathrm{R}^{(2)}_{1l}\,\mathrm{R}^{(1)}_{1l}$. The radiated power follow from $2\eta_0^{-1}P_{\mathrm{r}}=-\mathrm{Re}\langle\mathbf{J},\mathcal{L}_{\mathrm{m}}\mathbf{J}\rangle=a^2\kappa^2J_0^2(\mathrm{R}^{(1)}_{1l})^2$. The corresponding expansion of the frequency derivative of the Green's function (B11) is used for the terms related to (31)

$$\frac{-2}{\mathrm{i}k^2a^4J_0^2}\langle\mathbf{J},\mathcal{L}_{\mathrm{em}}\mathbf{J}\rangle=\int_\Omega\int_\Omega\mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}}_1)\cdot\frac{\partial\mathbf{G}(\mathbf{r}_1-\mathbf{r}_2)}{\partial k}\cdot\mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}}_2)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2=\mathrm{i}\frac{\partial}{\partial\kappa}\left(\kappa\,\mathrm{R}_{1l}^{(3)}(\kappa)\,\mathrm{R}_{1l}^{(1)}(\kappa)\right)$$
$$=\mathrm{i}\left(\kappa\,\mathrm{R}_{1l}^{(3)}(\kappa)\,\mathrm{R}_{1l}^{(1)}(\kappa)\right)'=\mathrm{i}\left(\mathrm{R}_{1l}^{(3)}\,\mathrm{R}_{1l}^{(1)}+\kappa\,\mathrm{R}_{1l}^{(3)\,\prime}\,\mathrm{R}_{1l}^{(1)}+\kappa\,\mathrm{R}_{1l}^{(3)}\,\mathrm{R}_{1l}^{(1)\,\prime}\right),\tag{B3}$$

where $'$ denotes differentiation with respect to κ, giving $4\omega\eta_0^{-1}W_{\mathrm{F}_1}=-\frac{a^2\kappa^2}{2}J_0^2(\kappa\,\mathrm{R}_{1l}^{(2)}\,\mathrm{R}_{1l}^{(1)})'$.

Collecting the terms gives the electric and magnetic Q-factors in (45). We note that $\mathrm{R}_{1l}^{(1)}=\mathrm{j}_l$ and $\mathrm{R}_{1l}^{(2)}=\mathrm{n}_l$ can be used to rewrite the Q-factors, however the form with the radial functions simplifies the comparison with the TM case below. The differentiated terms are easily evaluated using (B3) and (B8).

B.2. Far-field Type Stored Energy for the TM Case W_F

The transverse magnetic (TM) case is given by $\tau=2$ and generated by the current density $\mathbf{J}(\mathbf{r})=J_0\,\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}})\delta(r-a)$ that has the divergence $\nabla\cdot\mathbf{Y}_{2\sigma ml}=-\sqrt{l(l+1)}\,\mathrm{Y}_{\sigma ml}/r$. With the expansion of the Green's function (B10) we get the part related to the charge density (31)

$$\frac{-\mathrm{i}k}{a^4J_0^2}\langle\mathbf{J},\mathcal{L}_\mathrm{e}\mathbf{J}\rangle=\int_\Omega\int_\Omega\nabla_1\cdot\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}_1)G(\mathbf{r}_1-\mathbf{r}_2)\nabla_2\cdot\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}_2)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2=\frac{\mathrm{i}kl(l+1)}{a^2}\,\mathrm{j}_l(\kappa)\mathrm{h}_l^{(1)}(\kappa)\tag{B4}$$

and the full Green's Dyadic expansion (B11) gives

$$\frac{1}{\mathrm{i}ka^4J_0^2}\langle\mathbf{J},\mathcal{L}_\mathrm{m}\mathbf{J}\rangle=\int_\Omega\int_\Omega\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}_1)\cdot\mathbf{G}(\mathbf{r}_1-\mathbf{r}_2)\cdot\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}_2)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2$$
$$=\mathrm{i}k\left(\mathrm{R}_{2l}^{(1)}(\kappa)\,\mathrm{R}_{2l}^{(3)}(\kappa)+l(l+1)\frac{\mathrm{h}_l^{(1)}(\kappa)\,\mathrm{j}_l(\kappa)}{\kappa^2}\right)\tag{B5}$$

for the part related to the current density (31). The expansions of the frequency derivatives of the Green's function (B10) and Green's dyadic (B11) give

$$\mathrm{Re}\int_\Omega\int_\Omega\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}_1)\cdot\frac{\partial\mathbf{G}(\mathbf{r}_1-\mathbf{r}_2)}{\partial k}\cdot\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}_2)-\nabla_1\cdot\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}_1)\frac{\partial G(\mathbf{r}_1-\mathbf{r}_2)}{k^2\partial k}\nabla_2\cdot\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}_2)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2$$
$$=2l(l+1)\mathrm{n}_l(\kappa)\,\mathrm{j}_1(\kappa)-\kappa^2\left(\kappa\,\mathrm{R}_{2l}^{(1)}(\kappa)\,\mathrm{R}_{2l}^{(2)}(\kappa)\right)'\tag{B6}$$

for the part related to (31).

Collecting the terms gives that the normalized radiated power is $2\eta_0^{-1}P_\mathrm{r}/J_0^2=\mathrm{Re}\langle\mathbf{J},(\mathcal{L}_\mathrm{e}-\mathcal{L}_\mathrm{m})\mathbf{J}\rangle/J_0^2=a^3\kappa(\mathrm{R}_{2l}^{(1)})^2$. The electric and magnetic Q factors are finally determined to (45). We note that the expressions for the TE and TM cases are written in identical forms by using the radial functions (B8).

B.3. Spherical Waves

The radiated electromagnetic field is expanded in spherical vector waves or modes [25]:

$$\begin{cases}\mathbf{u}_{1\sigma ml}^{(p)}(k\mathbf{r})=\mathrm{R}_{1l}^{(p)}(kr)\,\mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}})\\ \mathbf{u}_{2\sigma ml}^{(p)}(k\mathbf{r})=\mathrm{R}_{21}^{(p)}(kr)\,\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}})+\sqrt{l(l+1)}z_l^{(p)}(kr)/(kr)\,\mathrm{Y}_{\sigma ml}(\hat{\mathbf{r}})\hat{\mathbf{r}}\\ \mathbf{u}_{3\sigma ml}^{(p)}(k\mathbf{r})=z_l^{(p)\prime}(kr)\,\mathrm{Y}_{\sigma ml}(\hat{\mathbf{r}})\hat{\mathbf{r}}+\sqrt{l(l+1)}z_l^{(p)}(kr)/(kr)\,\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}})\end{cases}\tag{B7}$$

where $\mathbf{r}$ is the spatial coordinate, $\hat{\mathbf{r}}=\mathbf{r}/r$, $r=|\mathbf{r}|$ and k the wavenumber. The radial functions $\mathrm{R}_l^{(p)}(kr)$ of order l and their derivatives are:

$$\mathrm{R}_{\tau l}^{(p)}(\kappa)=\begin{cases}z_l^{(p)}(\kappa) & \tau=1\\ \dfrac{1}{\kappa}\dfrac{\partial(\kappa z_l^{(p)}(\kappa))}{\partial\kappa} & \tau=2\end{cases}\quad\text{and}\quad\frac{\partial\,\mathrm{R}_{\tau l}^{(p)}}{\partial\kappa}=\begin{cases}\dfrac{\partial}{\partial\kappa}z_l^{(p)} & \tau=1\\ \dfrac{-\,\mathrm{R}_{\tau l}^{(p)}}{\kappa}+\dfrac{l(l+1)-\kappa^2}{\kappa^2}z_l^{(p)} & \tau=2.\end{cases}\tag{B8}$$

For regular waves ($p = 1$) $z_l^{(1)} = \mathrm{j}_l$ is a spherical Bessel function, irregular waves ($p = 2$) $z_l^{(2)} = \mathrm{n}_l$ is a spherical Neumann function, and outgoing waves ($p = 3$) $z_l^{(3)} = \mathrm{h}_l^{(1)}$ is an outgoing spherical Hankel function. The indices are $\sigma = \{\mathrm{e}, \mathrm{o}\}$, $m = 0, \ldots, l$, $l = 1, \ldots$, see [1, 28]. In addition, $\mathbf{Y}_{\tau\sigma ml}(\hat{\mathbf{r}})$ denotes the vector spherical harmonics defined as

$$\mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\mathbf{r}\, \mathrm{Y}_{\sigma ml}(\hat{\mathbf{r}})) \tag{B9}$$

and $\mathbf{Y}_{2\sigma ml}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} \times \mathbf{Y}_{1\sigma ml}(\hat{\mathbf{r}})$ where $\mathrm{Y}_{\sigma ml}$ denotes the ordinary spherical harmonics [1]. Here, we follow [1, 28] and use $\cos m\phi$ and $\sin m\phi$ as basis functions in the azimuthal coordinate. The modes labeled by $\tau = 1$ are TE modes (or magnetic 2^l-poles) while those labeled by $\tau = 2$ correspond to TM modes (or electric 2^l-poles). The Green functions are expanded in spherical waves to analyze spherical geometries. The scalar Green's function has the expansion [1]

$$G(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}_1 - \mathbf{r}_2|}}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} = \mathrm{i}k \sum_{\sigma ml} \mathrm{j}_l(kr_<) \mathrm{h}_l^{(1)}(kr_>)\, \mathrm{Y}_{\sigma ml}(\hat{\mathbf{r}}_1)\, \mathrm{Y}_{\sigma ml}(\hat{\mathbf{r}}_2), \tag{B10}$$

where $r_< = \min\{|\mathbf{r}_1|, |\mathbf{r}_2|\}$ and $r_> = \max\{|\mathbf{r}_1|, |\mathbf{r}_2|\}$. In addition, the full Green's dyadic, $\mathbf{G} = \mathbf{I}G$, can be expanded as [1]

$$\mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2) = \mathrm{i}k \sum_{\tau\sigma ml} \mathbf{u}_{\tau\sigma ml}^{(1)}(k\mathbf{r}_<)\, \mathbf{u}_{\tau\sigma ml}^{(3)}(k\mathbf{r}_>), \tag{B11}$$

where $\tau = 1, 2, 3$. We also use the frequency derivatives of the Green's function and the Green's dyadic expansions.

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