

# Green's Dyadic, Spectral Function, Local Density of States, and Fluctuation Dissipation Theorem

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**Abstract**—The spectral functions are studied in conjunction with the dyadic Green's functions for various media. The dyadic Green's functions are found using the eigenfunction expansion method for homogeneous, inhomogeneous, periodic, lossless, lossy, and anisotropic media, guided by the Bloch-Floquet theorem. For the lossless media cases, the spectral functions can be directly related to the photon local density of states, and hence, to the electromagnetic energy density. For the lossy case, the spectral function can be related to the field correlation function. Because of these properties, one can derive properties for field correlations and the Langevin-source correlations without resorting to the fluctuation dissipation theorem. The results are corroborated by the fluctuation dissipation theorem. An expression for the local density of states for lossy, inhomogeneous, and dispersive media has also been suggested.

## 1. INTRODUCTION

The subject of thermal noise is of great interest. Thermal noise gives rise to Brownian motion, black body radiation, and noise in electrical circuit known as Langevin, Johnson, or Nyquist noise [1–4]. When a system is in thermal equilibrium with its environment, as much energy is received from the environment as is transmitted to it. The fluctuation dissipation theorem (FDT) [5–8] both in classical and quantum form has been formulated to describe this physical phenomenon. Many recent works have expounded on this phenomenon.

The recent interest in spontaneous emission, Casimir force (and the ability to measure it), has revived the interest in this area as well [9–21]. A closely related area is in near-field heat transfer where photons participate in energy transfer [22–26]. Because most electromagnetic environment is dissipative, and that quantum systems are assumed to be non-dissipative, the use of some formulas for Casimir force calculation has given rise to controversies: the most well known of which is the Lifshitz formula for Casimir force [11–13].

The subject of thermal equilibrium also emerges in quantum transport where electrons move from one reservoir to another when the two reservoirs are not in equilibrium. This gives rise to the non-equilibrium Green's function approach. Green's functions and spectral functions are often defined to describe the physics of quantum transport in micro- and nano-scopic devices [27, 28]. Even though spectral function is often used in quantum transport, its use in electromagnetics is less well known [29].

Lossless quantum systems are easier to study. The first quantization of electromagnetic fields were done in lossless systems, or weakly lossy systems [30–33]. The theory of quantum dissipation system has also been actively researched [34–36]. These systems were studied with a quantum system in equilibrium with a thermal bath, sharing some commonality with FDT. Recently, FDT has been used to motivate the quantization of electromagnetic field in lossy media [37–40].

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In this paper, first, the Green's dyadics [41, 42], and spectral functions are defined and derived in terms of modal expansions for electromagnetic system, ranging from simple homogeneous, lossless media to anisotropic, inhomogeneous, lossy media. The spectral function is related to the local density of states (LDOS) [49, 50, 53] and the density of states (DOS). Physical interpretation is given to the electromagnetic spectral functions. Since the system is in time-harmonic oscillation, the energy distribution of the harmonic oscillators can be related to the generalized Planck distribution. Finally, some equations are derived from LDOS and spectral function, which are also derivable from the FDT.

## 2. HOMOGENEOUS UNBOUNDED MEDIUM DYADIC GREEN'S FUNCTION

The dyadic Green's function for a homogeneous medium is a solution to [41, 42]

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - k^2 \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') \quad (1)$$

where  $k^2 = \omega^2 \mu_0 \epsilon_0$  and  $\omega$  is the operating frequency. A corresponding vector wave function, which is also the eigenfunction, is defined to be the solution to

$$\nabla \times \nabla \times \mathbf{F}(\boldsymbol{\kappa}, \mathbf{r}) - \kappa^2 \mathbf{F}(\boldsymbol{\kappa}, \mathbf{r}) = 0 \quad (2)$$

with  $\kappa^2$  being the eigenvalue. The eigenvalues of this system are real because the operators are Hermitian or self-adjoint [42, 43]. For Cartesian coordinates, the vector wave functions can be found easily. The general solution to (2) is of the form  $\mathbf{a}e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}$ . In an unbounded homogeneous medium,  $\boldsymbol{\kappa}$  can take on any value and is uncountably infinite in number. By the use of periodic boundary condition in a volume  $V = a \times b \times d$ ,  $\boldsymbol{\kappa}$  can be discretized and made countably infinite in number. Then

$$\begin{aligned} \boldsymbol{\kappa} &= \hat{x}\kappa_x + \hat{y}\kappa_y + \hat{z}\kappa_z, \quad \kappa_x = 2l\pi/a, \quad \kappa_y = 2m\pi/b, \quad \kappa_z = 2p\pi/d, \\ \kappa^2 &= \boldsymbol{\kappa} \cdot \boldsymbol{\kappa} = (2l\pi/a)^2 + (2m\pi/b)^2 + (2p\pi/d)^2 \end{aligned}$$

where  $(l, m, p)$  are integer triplets. To normalize the vector wave functions, then

$$\mathbf{M}(\boldsymbol{\kappa}, \mathbf{r}) = \frac{i\boldsymbol{\kappa} \times \hat{z}}{\kappa_s \sqrt{V}} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \quad (3)$$

$$\mathbf{N}(\boldsymbol{\kappa}, \mathbf{r}) = \frac{1}{\kappa} \nabla \times \mathbf{M}(\boldsymbol{\kappa}, \mathbf{r}) = i\hat{\kappa} \times \mathbf{M}(\boldsymbol{\kappa}, \mathbf{r}) \quad (4)$$

$$\mathbf{L}(\boldsymbol{\kappa}, \mathbf{r}) = \frac{i\hat{\kappa}}{\sqrt{V}} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \quad (5)$$

where  $\kappa_s = \kappa_x^2 + \kappa_y^2$ , and  $\hat{\kappa} = \boldsymbol{\kappa}/\kappa$ . In the above,  $\mathbf{M}$  is orthogonal to both  $\hat{z}$  and  $\boldsymbol{\kappa}$ ,  $\mathbf{N}$  is orthogonal to  $\mathbf{M}$  and  $\boldsymbol{\kappa}$ , while  $\mathbf{L}$  is longitudinal and parallel to  $\boldsymbol{\kappa}$ . It can be shown that  $\mathbf{M}$ ,  $\mathbf{N}$ ,  $\mathbf{L}$  above are orthonormalized, namely that

$$\int_V \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r}) \cdot \mathbf{F}_{n'}(\boldsymbol{\kappa}', \mathbf{r}) d\mathbf{r} = \delta_{ll'} \delta_{mm'} \delta_{pp'} \delta_{nn'} \quad (6)$$

where  $\mathbf{F}_n$ ,  $n = 1, 2, 3$  imply  $\mathbf{M}$ ,  $\mathbf{N}$ , or  $\mathbf{L}$ .

In the above,  $\boldsymbol{\kappa}$  is an index associated with indices  $(l, m, p)$ , and that  $\mathbf{M}$ ,  $\mathbf{N}$ ,  $\mathbf{L}$  are mutually orthonormal. Hence, assuming the completeness of these eigenfunctions, then<sup>†</sup>

$$\overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') = \sum_{\boldsymbol{\kappa}} [\mathbf{M}(\boldsymbol{\kappa}, \mathbf{r})\mathbf{a}^t(\boldsymbol{\kappa}) + \mathbf{N}(\boldsymbol{\kappa}, \mathbf{r})\mathbf{b}^t(\boldsymbol{\kappa}) + \mathbf{L}(\boldsymbol{\kappa}, \mathbf{r})\mathbf{c}^t(\boldsymbol{\kappa})] \quad (7)$$

Using orthonormality, it follows that<sup>‡</sup>

$$\overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') = \sum_{\boldsymbol{\kappa}} \left[ \mathbf{M}(\boldsymbol{\kappa}, \mathbf{r})\mathbf{M}^\dagger(\boldsymbol{\kappa}, \mathbf{r}') + \mathbf{N}(\boldsymbol{\kappa}, \mathbf{r})\mathbf{N}^\dagger(\boldsymbol{\kappa}, \mathbf{r}') + \mathbf{L}(\boldsymbol{\kappa}, \mathbf{r})\mathbf{L}^\dagger(\boldsymbol{\kappa}, \mathbf{r}') \right] \quad (8)$$

<sup>†</sup> Usually, in the physics notation, for inner products, the transpose of the left vector is not explicit as in (6), but in linear algebraic notation, it is explicit as is used here. For outer products here, one will explicitly put the transpose of the right vector as well.

<sup>‡</sup> The completeness of these eigenfunctions can be proved along similar line as that for Fourier series. They are complete because they are eigenfunctions of a Hermitian operator.

or more concisely,

$$\bar{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') = \sum_{\boldsymbol{\kappa}, n} \mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r}') \quad (9)$$

where  $\mathbf{F}_n$ ,  $n = 1, 2, 3$  denote  $\mathbf{M}$ ,  $\mathbf{N}$ , or  $\mathbf{L}$ , respectively. If  $\hat{z}$  is assumed to be vertical, one can think of  $\mathbf{M}$  as horizontal,  $\mathbf{N}$  as vertical (with respect to  $\mathbf{M}$  and  $\boldsymbol{\kappa}$ ), and  $\mathbf{L}$  as longitudinal. By letting  $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$  in (1) as

$$\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \sum_{\boldsymbol{\kappa}, n} \mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{f}_n^t(\boldsymbol{\kappa}) \quad (10)$$

it follows that

$$\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}', k) = \sum_{\boldsymbol{\kappa}, n} \frac{\mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r}')}{\kappa_n^2 - k^2} \quad (11)$$

where  $\kappa_n^2 = \boldsymbol{\kappa} \cdot \boldsymbol{\kappa}$  when  $n = 1, 2$ , and  $\kappa_3 = 0$ . Here,  $n = 1, 2$  are used to index the transverse modes  $\mathbf{M}$  and  $\mathbf{N}$ , while  $n = 3$  indexes the longitudinal mode  $\mathbf{L}$ .

The above is very similar to the solution of a modal cavity driven by a time-harmonic source with operating frequency  $\omega$  where  $k = \omega/c$ . The  $\mathbf{M}$  and  $\mathbf{N}$  modes are dynamic modes with resonant wave number  $\kappa_n$ , while the  $\mathbf{L}$  modes are static modes with zero resonant wave number. But in this case, the modes are generated using periodic boundary condition rather than a cavity wall. In the above, (3) to (5) represent travelling waves while if cavity modes were used, they would represent standing waves.

### 3. THE DENSITY OF STATES

The density of states is the number of states (DOS) or modes per unit energy interval. For each eigenmode, there corresponds a frequency  $\omega = \kappa c$ . Since photons are described here, the relation between photon energy and wavenumber is  $\mathcal{E}_\kappa = \hbar\omega_\kappa = \hbar\kappa c$ , or  $\kappa = \mathcal{E}_\kappa/(\hbar c)$ ,  $\hbar$  is the Planck constant, and  $c$  is the velocity of light.

Since

$$\boldsymbol{\kappa} = \hat{x} \frac{2l\pi}{a} + \hat{y} \frac{2m\pi}{b} + \hat{z} \frac{2p\pi}{d} \quad (12)$$

each mode occupies a volume of  $(2\pi)^3/(abd) = 8\pi^3/V$  in  $\boldsymbol{\kappa}$ -space. The number of modes or states in a spherical shell between  $\kappa$  and  $\kappa + \Delta\kappa$  is

$$\Delta N = 2 \times 4\pi\kappa^2 \Delta\kappa / (8\pi^3/V) = \frac{1}{\pi^2} \kappa^2 \Delta\kappa V \quad (13)$$

where one assumes that there are two polarizations for every mode. Notice that if  $\Delta N$  is normalized by  $V$ , it is independent of  $V$ .

A succinct way to write the density of states is

$$D(E) = \sum_{\boldsymbol{\kappa}, n} \delta(E - \mathcal{E}_{\boldsymbol{\kappa}, n}) \quad (14)$$

where  $\mathcal{E}_{\boldsymbol{\kappa}, n} = \hbar\kappa c$ ,  $n = 1, 2$  represent the two polarizations, horizontal and vertical, and  $E = \hbar\omega$ . Here,  $\kappa^2 = |\boldsymbol{\kappa}|^2$  is the eigenvalue of Equation (2), which are non-zero for  $\mathbf{M}$  and  $\mathbf{N}$  modes. Since  $\boldsymbol{\kappa}$  is associated with indices  $(l, m, p)$ , there are many degeneracies in the eigenvalue  $\kappa^2$  in (2): all  $\boldsymbol{\kappa}$  vectors from the origin to the sphere defined by  $|\boldsymbol{\kappa}| = \kappa$  are degenerate. Also, in the above, one only retains states or modes with non-zero resonant wavenumbers. The  $\mathbf{L}$  modes are not counted in the above since they have zero resonant wavenumber in Equation (2).

Equation (14) is like a spectroscopy function showing spikes whenever  $E$  coincides with one of the eigenstates with energy  $\mathcal{E}_{\boldsymbol{\kappa}, n}$ . The number of states in an energy interval  $E$  and  $E + \Delta E$  is given by

$$\int_E^{E+\Delta E} D(E) dE = \Delta N, \quad (E < \mathcal{E}_{\boldsymbol{\kappa}, n} < E + \Delta E) \quad (15)$$

It is seen that Equation (14) is in fact the density of states per unit energy interval because integrating it over an energy interval gives the number of states in that interval, as shall be shown below.

In the limit when  $V \rightarrow \infty$ , one can convert the summation in Equation (14) into an integral by the substitution that  $\sum_{\boldsymbol{\kappa}} = V/(2\pi)^3 \sum \Delta\kappa_x \Delta\kappa_y \Delta\kappa_z = V/(2\pi)^3 \int d\boldsymbol{\kappa}$ . Namely,

$$D(E) = 2 \times \frac{V}{(2\pi)^3} \int d\boldsymbol{\kappa} \delta(E - \hbar\kappa c) = \frac{V}{4\pi^3} \int_0^\infty 4\pi\kappa^2 d\kappa \delta(E - \hbar\kappa c) = \frac{Vk^3}{\pi^2 E} \quad (16)$$

where it is noted again here that  $E = \hbar\kappa c = \hbar\omega$ . In the above,  $\mathcal{E}_{\boldsymbol{\kappa},n} = \hbar\kappa c$  has no angular dependence, and hence,  $\int d\boldsymbol{\kappa} = \int_0^\infty 4\pi\kappa^2 d\kappa$ . Consequently

$$\Delta N = D(E)\Delta E = \frac{Vk^2}{\pi^2} \Delta k \quad (17)$$

which is the same as Equation (13) except that one replaces  $\kappa$  in Equation (13) with  $k$  here since  $E = \hbar\omega$  is the counter. The above is self consistent with Equation (14).

#### 4. SPECTRAL FUNCTION AND LOCAL DENSITY OF STATES

The photon (electromagnetics) local density of states (LDOS) is defined as

$$D(\mathbf{r}, E) = \sum_{\boldsymbol{\kappa}, n} |\mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r})|^2 \delta(E - \mathcal{E}_{\boldsymbol{\kappa}, n}) \quad (18)$$

where  $n = 1, 2$  represent the two polarizations, and  $\boldsymbol{\kappa}$  indicates the plane wave direction of the eigenfunction since they are traveling plane waves. The above has the property that

$$D(E) = \int_V d\mathbf{r} D(\mathbf{r}, E) \quad (19)$$

since

$$\int_V |\mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r})|^2 d\mathbf{r} = 1 \quad (20)$$

The above implies that the LDOS, when integrated over  $\mathbf{r}$ , yields DOS. Therefore, LDOS has the unit of states per unit energy per unit volume. It also associates different probabilities of finding a photon in different locations denote by  $\mathbf{r}$ .

The LDOS can be related to the dyadic Green's function as shall be shown next. To this end, a spectral function is defined as

$$\overline{\mathbf{A}}(\mathbf{r}, \mathbf{r}') = i \left[ \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - \overline{\mathbf{G}}^\dagger(\mathbf{r}', \mathbf{r}) \right] \quad (21)$$

The second Green's operator is the adjoint of the first one. The Green's dyadic is an integral operator acting on a functional Hilbert space made of 3 vectors. The adjoint of this Green's dyadic operator is defined to satisfy the following relationship

$$\int d\mathbf{r} \mathbf{f}_2^\dagger(\mathbf{r}) \cdot \int d\mathbf{r}' \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{f}_1(\mathbf{r}') = \int d\mathbf{r}' \int d\mathbf{r} \left[ \overline{\mathbf{G}}^a(\mathbf{r}', \mathbf{r}) \cdot \mathbf{f}_2(\mathbf{r}) \right]^\dagger \cdot \mathbf{f}_1(\mathbf{r}') \quad (22)$$

In this manner, the transpose of a scalar integral operator is such that  $g^t(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r})$ . This is similar to that for matrices, where matrix elements of transpose matrices are related by  $[\overline{\mathbf{A}}^t]_{ij} = [\overline{\mathbf{A}}]_{ji}$ . Hence, one swaps the argument of the second Green's dyadic in Equation (21) to make it the adjoint of the first Green's dyadic operator. To elaborate, from the above, one obtains

$$\overline{\mathbf{G}}^a(\mathbf{r}', \mathbf{r}) = \overline{\mathbf{G}}^\dagger(\mathbf{r}, \mathbf{r}'), \quad \text{or} \quad \overline{\mathbf{G}}^a(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{G}}^\dagger(\mathbf{r}', \mathbf{r}) \quad (23)$$

In this manner, the spectral function so defined above is a self-adjoint operator.

It can be shown that the above represents a Hermitian system. For a reciprocal Green's function, it further has the property that  $\overline{\mathbf{G}}^t(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{G}}(\mathbf{r}', \mathbf{r})$ . Hence, the spectral function can be written as

$$\overline{\mathbf{A}}(\mathbf{r}, \mathbf{r}') = i \left[ \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - \overline{\mathbf{G}}^*(\mathbf{r}, \mathbf{r}') \right] = -2\Im m \left[ \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right] \quad (24)$$

From Equation (11), it can be shown that

$$\overline{\mathbf{G}}^*(\mathbf{r}, \mathbf{r}') = \sum_{\boldsymbol{\kappa}, n} \frac{\mathbf{F}_n^*(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^t(\boldsymbol{\kappa}, \mathbf{r}')}{\kappa_n^2 - k^{*2}} = \sum_{\boldsymbol{\kappa}, n} \frac{\mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r}')}{\kappa_n^2 - k^{*2}} \quad (25)$$

where  $\kappa_n^2 = \boldsymbol{\kappa} \cdot \boldsymbol{\kappa}$ , and the last equality is obtained by letting  $\boldsymbol{\kappa} \rightarrow -\boldsymbol{\kappa}$  and one notices that  $\mathbf{F}_n(-\boldsymbol{\kappa}, \mathbf{r}) = \mathbf{F}_n^*(\boldsymbol{\kappa}, \mathbf{r})$ . Moreover, it also follows from that the eigenvalues of Equation (2) is real, and hence,  $\kappa_n^2$  is real. Therefore

$$\overline{\mathbf{A}}(\mathbf{r}, \mathbf{r}') = -2\Im m [\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')] = i [\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - \overline{\mathbf{G}}^*(\mathbf{r}, \mathbf{r}')] = -2 \sum_{\boldsymbol{\kappa}, n} \mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r}') \Im m \left( \frac{1}{\kappa_n^2 - k^2} \right) \quad (26)$$

One can rewrite

$$\frac{1}{\kappa_n^2 - k^2} = \frac{1}{2\kappa_n} \left( \frac{1}{\kappa_n - k} + \frac{1}{\kappa_n + k} \right) \quad (27)$$

If  $k \rightarrow k + i\eta$  to have a small imaginary part to mean loss, then

$$\Im m \left( \frac{1}{\kappa_n - k - i\eta} \right) = \Im m \left[ \frac{i\eta}{(\kappa_n - k)^2 + \eta^2} \right] = \frac{\eta}{(\kappa_n - k)^2 + \eta^2} \quad (28)$$

It can be shown that

$$\lim_{\eta \rightarrow 0} \frac{\eta}{x^2 + \eta^2} = \pi \delta(x) \quad (29)$$

Comparing, and using the above in Equation (26), it means that

$$\overline{\mathbf{A}}(\mathbf{r}, \mathbf{r}') = -2\Im m [\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')] = -2 \sum_{\boldsymbol{\kappa}, n} \mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r}') \frac{\pi}{2\kappa_n} [\delta(\kappa_n - k) + \delta(\kappa_n + k)] \quad (30)$$

Furthermore,  $\delta(ax) = a^{-1}\delta(x)$ ,  $\kappa_n = \mathcal{E}_{\boldsymbol{\kappa}, n}/\hbar c$ , and  $k = E/\hbar c$ , imply that

$$\Im m \left( \frac{1}{\kappa_n^2 - k^2} \right) = \frac{\pi E}{2k^2} [\delta(E - \mathcal{E}_{\boldsymbol{\kappa}, n}) - \delta(E + \mathcal{E}_{\boldsymbol{\kappa}, n})] \quad (31)$$

Consequently,

$$\overline{\mathbf{A}}(\mathbf{r}, \mathbf{r}') = -2\Im m [\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')] = -\frac{\pi E}{k^2} \sum_{\boldsymbol{\kappa}, n} \mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r}') \delta(E - \mathcal{E}_{\boldsymbol{\kappa}, n}) \quad (32)$$

Here,  $\delta(E + \mathcal{E}_{\boldsymbol{\kappa}, n})$  is ignored since  $E + \mathcal{E}_{\boldsymbol{\kappa}, n} \neq 0$ , and  $\mathcal{E}_{\boldsymbol{\kappa}, n} > 0$  for all  $\boldsymbol{\kappa}$ . Another point to note regards the longitudinal modes ( $n = 3$ ) with zero resonant frequency  $\mathcal{E}_{\boldsymbol{\kappa}, n} = 0$ . If one keeps track of modes only with  $\mathcal{E}_{\boldsymbol{\kappa}, n} > 0$ , they need not be included in the above expression.

If  $\mathbf{r} = \mathbf{r}'$ , and noticing that by taking the trace, one has  $\text{Tr} [\mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r})] = \sum_{i=1}^3 F_{ni}(\boldsymbol{\kappa}, \mathbf{r}) F_{ni}^*(\boldsymbol{\kappa}, \mathbf{r}) = |\mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r})|^2$ , it means that

$$D(\mathbf{r}, E) = -\frac{k^2}{\pi E} \text{Tr} [\overline{\mathbf{A}}(\mathbf{r}, \mathbf{r})] = \frac{2k^2}{\pi E} \text{Tr} \{ \Im m [\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r})] \} \quad (33)$$

where  $D(\mathbf{r}, E)$  is the local density of states as given by Equation (18). Hence, the LDOS can be found once the imaginary part of the dyadic Green's function is known. In the above, the spectral function and the Green's dyadic are implicit functions of  $\omega$  and hence,  $E$ .

#### 4.1. Photon Energy Density

A photon with frequency  $\omega$  has quantization energy given by  $E = \hbar\omega$ . It is the result of energy quantization of the quantum harmonic oscillator. If the quantum harmonic oscillator is in thermal equilibrium with another system, it can be shown that the average photon number in the photon number state of the oscillator is [44, 45]

$$\bar{n} = \frac{1}{e^{\hbar\omega/(k_B T)} - 1} \quad (34)$$

Then the average energy of the photon state is given by

$$\Theta(E = \hbar\omega) = \left(\bar{n} + \frac{1}{2}\right) \hbar\omega = \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/(k_B T)} - 1} \quad (35)$$

The above result can be derived from the quantum statistical mechanics of a quantum harmonic oscillator. In the limit when  $\hbar \rightarrow 0$ , the above becomes the classical result showing that  $\Theta(E) = k_B T$ : this is in agreement with the equipartition of energy theorem with  $0.5k_B T$  per degree of freedom. A harmonic oscillator has two degrees of freedom, one for storing kinetic energy while another one for storing potential energy [46].

If  $T \rightarrow 0$ , then  $\Theta(E) = (1/2)\hbar\omega$ , which is also a result from quantum mechanics. It implies that the quantum harmonic oscillator has nonzero energy even if it is in the ground state: this is called the zero point energy. This zero point energy is there even when  $\bar{n} = 0$ , or with no photon [44, 45] is present on average. This is the source of vacuum fluctuation of the electromagnetic field, and is the source of the Casimir force at zero temperature [13].

Hence, if the density of states of the system is known, the average photon energy of the system is given by

$$\langle \mathcal{E} \rangle = \int_0^\infty D(E)\Theta(E)dE \quad (36)$$

In the above,  $D(E)\Theta(E)dE$  is the average photon energy in the energy interval  $dE$ , and hence, when integrated yields the average energy of the system. If one lets  $E = \hbar\omega$ , and writes

$$D(E)\Theta(E)dE = D(E)\Theta(E)\hbar d\omega \quad (37)$$

then one can identify, after using Equation (16) for  $D(E)$  that

$$U(\omega) = \frac{k^3}{\pi^2 E} \Theta(E) \hbar = \frac{\omega^2}{\pi^2 c^3} \left( \frac{1}{2} \hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/(k_B T)} - 1} \right) \quad (38)$$

is just the black body energy per unit volume per unit radian frequency according to the generalized Planck's radiation law, except that the zero-point energy is absent in the original Planck's law.

If the local density of states is known, then the average photon energy per unit volume or the photon energy density at the location  $\mathbf{r}$  is given by

$$\langle \mathcal{E}(\mathbf{r}) \rangle = \int_0^\infty D(\mathbf{r}, E)\Theta(E)dE \quad (39)$$

Given that  $D(\mathbf{r}, E)\Theta(E)$  implies energy per unit energy per unit volume, it is prudent to define a field correlation function<sup>§</sup>

$$\bar{\mathcal{C}}(\mathbf{r}, \mathbf{r}', \omega) = \langle \mathcal{E}(\mathbf{r}, \omega) \mathcal{E}^\dagger(\mathbf{r}', \omega) \rangle = -\frac{\omega\mu_0}{\pi} \bar{\mathbf{A}}(\mathbf{r}, \mathbf{r}', \omega) \Theta(\hbar\omega) = \frac{2\omega\mu_0}{\pi} \Im m [\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega)] \Theta(\hbar\omega) \quad (40)$$

where  $E = \hbar\omega$ . In this manner, the LDOS can be directly related to this field correlation function as

$$D(\mathbf{r}, E)\Theta(E)dE = D(\mathbf{r}, E)\Theta(E)\hbar d\omega = \text{Tr}[\epsilon_0 \bar{\mathcal{C}}(\mathbf{r}, \mathbf{r}, \omega)] d\omega \quad (41)$$

In the above,  $\bar{\mathcal{C}}$  has the unit of field square per radian frequency. The factor of 1/2 is not needed here because the above represents the energy density in both the electric field and the magnetic field. For a

<sup>§</sup> This is due to hindsight that will be confirmed in Sect. 8.

homogeneous medium, they are equal to each other. With the above, we can define a density of states per unit radian frequency,  $\tilde{D}(\mathbf{r}, \omega)$ , so that

$$\tilde{D}(\mathbf{r}, \omega)\Theta(\hbar\omega)d\omega = D(\mathbf{r}, E)\Theta(E)dE = D(\mathbf{r}, E)\Theta(E)\hbar d\omega \quad (42)$$

or

$$\tilde{D}(\mathbf{r}, \omega) = D(\mathbf{r}, E)\hbar = \frac{2\omega\mu_0}{\pi} \text{Tr} \Im m [\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}, \omega)] \quad (43)$$

The relation expounded in Equation (40) can also be confirmed in Section 8, and are commensurate with FDT as shall be shown later.

#### 4.2. Physical Interpretation-Lossless Case

The spectral function is formed by the difference of two Green's function: it describes the response of a point source in equilibrium with its environment. If the Green's dyadic describes a point source field that is leaving the source point, the adjoint of the Green's dyadic describes a field that is returning back to the point source. If the Green's function satisfies causality in the time domain, it is also called the retarded Green's function, or the forward Green's function. The adjoint Green's function is non-causal, and it is also called the backward Green's function, the advanced Green's function, or a time reversed Green's function in the time domain. A forward Green's function fore propagates the field from a source point to a field point, while a backward Green's function back propagates the field from the field point to the source point. For instance, in the homogeneous medium case, one Green's function yields an outgoing field from the source point, while the other one absorbs an incoming field to the source point. The reactive fields [42] that represent stored energy in these two Green's functions cancel each other, and only the radiative fields remain in the difference.

In the lossless medium, when the frequency is off resonance,  $\kappa_n \neq k$ , the forward Green's function and the reversed Green's function cancel each other exactly, but when  $\kappa_n = k$ , they do not cancel each other giving rise to spikes in the spectral function. It yields the density of states of the system in Equation (14) and the local density of states in Equation (18). When the spectral function is multiplied by  $\Theta(E)$ , the average photon energy per state, it yields a quantity proportional to the energy density. Therefore, we have suggestively written the spectral function in Equation (40) so that it is related to the correlation function of the field at two points. This relationship also follows from Section 8 and also the fluctuation dissipation theorem. Hence, the spectral function motivates a result that can be drawn from the fluctuation dissipation theorem.

#### 5. LINE BROADENING AND LOSS

Notice that Equation (14) resembles a set of spectroscopic lines that are infinitely sharp because there is no loss in the system corresponding to infinite  $Q$ . If loss is introduced, such spectroscopic lines will be broadened due to finite  $Q$ . To see this, the delta function in (28) will be broadened if the loss is not infinitesimally small.

It is to be noted that Equation (11) is still valid even when  $k$  corresponds to a lossy medium or is a complex number. The generalized spectral function then becomes

$$\overline{\mathbf{A}}(\mathbf{r}, \mathbf{r}') = -2\Im m [\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')] = i [\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - \overline{\mathbf{G}}^*(\mathbf{r}, \mathbf{r}')] = -2 \sum_{\kappa, n} \mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_n^\dagger(\boldsymbol{\kappa}, \mathbf{r}') \gamma(\kappa_n, k) \quad (44)$$

where

$$\gamma(\kappa_n, k) = \Im m \left( \frac{1}{\kappa_n^2 - k^2} \right) = \Im m \left[ \frac{1}{2\kappa_n} \left( \frac{1}{\kappa_n - k} + \frac{1}{\kappa_n + k} \right) \right] \quad (45)$$

It can be shown that in the limit when  $V \rightarrow \infty$ , the summation becomes an integral, giving<sup>||</sup>

$$\sum_{\kappa, n} \gamma(\kappa_n, k) = 2 \times \frac{V}{8\pi^3} \int d\boldsymbol{\kappa} \gamma(\boldsymbol{\kappa}, k) = \frac{V}{2\pi} k' \quad (46)$$

<sup>||</sup> Derivation available upon request to the authors.

where  $k = k' + ik''$ ,  $k'$  and  $k''$  are real numbers, but the integral is independent of  $k''$ , the loss. The multiplication by 2 is necessary because there are two polarizations per mode. In the above,  $k = \omega\sqrt{\mu_0\epsilon_0}$  where for a lossy medium,  $\mu_0$  and  $\epsilon_0$  are complex numbers. But by the Kramers-Kronig relation [47], they have to be functions of frequency  $\omega$ : their real and imaginary parts are related by Hilbert transforms [42].

A generalized local density of states can be defined as

$$D(\mathbf{r}, E) = \sum_{\kappa, n} |\mathbf{F}_n(\boldsymbol{\kappa}, \mathbf{r})|^2 \Gamma(\mathcal{E}_{\kappa, n}, E) \quad (47)$$

where

$$\Gamma(\mathcal{E}_{\kappa, n}, E) = \frac{2k'^2}{\pi E} \gamma(\kappa_n, k)$$

with  $\kappa_n = \mathcal{E}_{\kappa, n}/(\hbar c)$ ,  $E = \hbar\omega$ , and  $c$  is the velocity of light in vacuum. In the above,  $\kappa_n$  as defined in Equation (11) remains real when  $k$  is complex due to material loss. In this manner,

$$\lim_{k'' \rightarrow 0} \Gamma(\mathcal{E}_{\kappa, n}, E) = \delta(E - \mathcal{E}_{\kappa, n})$$

Then a generalized density of states becomes

$$D(E) = \sum_{\kappa, n} \Gamma(\mathcal{E}_{\kappa, n}, E) \quad (48)$$

The above has a nice physical meaning since it can be shown that

$$D(E) = \sum_{\kappa, n} \Gamma(\mathcal{E}_{\kappa, n}, E) = \frac{2k'^2}{\pi E} \sum_{\kappa, n} \gamma(\kappa_n, k) = 2 \times \frac{V}{8\pi^3} \frac{2k'^2}{\pi E} \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \gamma(\boldsymbol{\kappa}, k) = \frac{k'^3 V}{\pi^2 E} \quad (49)$$

Therefore, the spectral function can still be related to the LDOS in the lossy case. The DOS in this case is related to  $k'$ , the real part of the  $k$  where the  $\gamma(\boldsymbol{\kappa}, k)$  function peaks on the real axis of  $\boldsymbol{\kappa}$ . If we choose  $V = 1$ , then the DOS is in terms of the number of states per unit volume per unit energy. We will show in Section 8 that this relationship exists even for a general lossy, anisotropic, inhomogeneous medium.

### 5.1. Physical Interpretation — Lossy Case

The above formula is similar to Equation (16) excepted that  $k$  is now replaced by  $k'$ , the location on the complex  $\boldsymbol{\kappa}$  plane where  $\gamma(\boldsymbol{\kappa}, k)$  peaks. The  $\Gamma$  function, which is the normalized  $\gamma$  function, behaves like a broadened delta function, and in the limit of no loss case, reverts back to a delta function. In the lossless case, each  $k$  is associated with modes with  $\boldsymbol{\kappa} = k$  as in Equations (14) and (18), but in the lossy case, there is a cluster of modes with  $\boldsymbol{\kappa}$  values associated with the peak of the  $\gamma$  function when  $\boldsymbol{\kappa} = k'$  as indicated in Equations (45) and (46).

In a lossy medium, the first term of the spectral function describes a decaying field, but the second term, a back propagating field, describes a growing field. This can be thought of as a lossy system in equilibrium with a thermal bath. The loss in the system is accompanied by Langevin sources [4] induced by the thermal excitation of the environment. These sources produce a field that grows, instead of decays with distance.

In either cases, the spectral function describes a system in thermal equilibrium. In the lossless case, the system is in thermal equilibrium with sources at infinity, while in the lossy case, the system is in equilibrium with the Langevin sources due to the loss in the medium. In both cases, the system is in time harmonic motion because of thermal equilibrium: as much energy is supplied to the system as it is lost, so that the harmonic oscillators in the vacuum, atoms, and molecules are in simple time-harmonic oscillation. (Notice that the spectral function is Hermitian because it describes an energy conserving system.)

But in the lossy case, when the system is excited to be in a simple time-harmonic motion, a cluster of modes are excited in unison to be in time-harmonic motion. When these harmonic oscillators are quantized to be associated with a quantum or packet of energy  $\hbar\omega$ , the energy is distributed over the cluster of modes with different  $\mathcal{E}_{\kappa, n}$ , and weighted with different coefficients according to  $\gamma$ . These modes can be given a probabilistic interpretation as in quantum mechanics. Because of the above physical interpretation, one can use Equations (36) to (39) to obtain the photon energy and photon energy density of a lossy system as well.

## 6. THE PERIODIC STRUCTURE CASE

The periodic structures can be assumed to be perfect electric conductors (PEC).<sup>¶</sup> In this case, the electric vector wave functions or eigenfunctions are defined to be the solutions to eigenvalue problem

$$\nabla \times \nabla \times \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) - \kappa_n^2 \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) = 0 \quad (50)$$

where  $\boldsymbol{\kappa}$  is the same as defined as above Equation (3). By choosing a periodic PEC (or PMC) structure, the above eigenvalue problem is similar to (2) except for the boundary condition to be satisfied on the surface of the PEC (or PMC). The above problem is easily proved to be Hermitian or self-adjoint, and hence, the eigenvalues are real. The above eigenvalue problem, in accordance with Bloch-Floquet theorem, will yield eigen-solutions where [53]

$$\mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) = \sum_{\mathbf{G}} \sum_{\lambda} F_{\mathbf{G}}^{\lambda} \mathbf{e}_{\mathbf{G}}^{\lambda} e^{i(\boldsymbol{\kappa} + \mathbf{G}) \cdot \mathbf{r}} \quad (51)$$

and  $\mathbf{G}$ 's are the reciprocal lattice vectors, and  $\lambda \in \{1, 2, 3\}$ , with  $\mathbf{e}_{\mathbf{G}}^1$ ,  $\mathbf{e}_{\mathbf{G}}^2$ ,  $\mathbf{e}_{\mathbf{G}}^3$  forming an orthogonal triad unit vectors, and  $\mathbf{e}_{\mathbf{G}}^3$  is parallel to  $\boldsymbol{\kappa} + \mathbf{G}$ . Hence,  $\lambda \in \{1, 2\}$  correspond to transverse modes, while  $\lambda = 3$  corresponds to the longitudinal mode.

The above is solved with the boundary condition that  $\hat{n} \times \mathbf{F}_{e,n} = 0$  on the PEC surface of the scatterer where  $\hat{n}$  is the unit surface normal. In this case, the eigenfunctions and eigenvalues have to be found numerically. It is to be noted that  $\kappa_n^2(\boldsymbol{\kappa})$  is the eigenvalue of Equation (50) with different choices of the Bloch-Floquet wave vector  $\boldsymbol{\kappa}$ . Hence, it is a function of  $\boldsymbol{\kappa}$ .

It can be shown that  $\mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r})$  can be orthonormalized such that

$$\int_V \mathbf{F}_{e,n}^{\dagger}(\boldsymbol{\kappa}, \mathbf{r}) \cdot \mathbf{F}_{e,n'}(\boldsymbol{\kappa}', \mathbf{r}) d\mathbf{r} = \delta_{\boldsymbol{\kappa}, \boldsymbol{\kappa}'} \delta_{nn'} \quad (52)$$

where  $n$  indicates the band of the solution for the same  $\boldsymbol{\kappa}$  as shall be explained.

In the homogeneous medium case, the transverse modes are degenerate, but in a periodic structure considered here, they need not be [53]. Furthermore, the higher frequency modes are generated in the Brillouin zone due to the aliasing or wrapping of high  $\boldsymbol{\kappa}$  modes into the primary zone. For unlike the homogeneous medium case, where  $n \in \{1, 2\}$ , for a fixed  $\boldsymbol{\kappa}$ , there could be many  $\kappa_n^2$  for each  $\boldsymbol{\kappa}$  value where  $n \in \{1, 2, 3, \dots\}$ .

One can also make  $\boldsymbol{\kappa}$  countably infinite by having the periodic structure nested within a larger periodic structure, so that  $\boldsymbol{\kappa}$  is discretized by a larger periodic boundary condition. One can also have only one unit cell within the larger periodic boundary condition. In this case, the problem becomes a single region scattering problem within a larger periodic boundary condition.

A similar expression to Equation (11) for the periodic structure case can be derived for the electric dyadic Green's function such that

$$\overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = \sum_{\boldsymbol{\kappa}, n} \frac{\mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_{e,n}^{\dagger}(\boldsymbol{\kappa}, \mathbf{r}')}{\kappa_n^2 - k^2} \quad (53)$$

Due to reciprocity of the dyadic Green's function, it implies that  $\mathbf{F}_{e,n}(-\boldsymbol{\kappa}, \mathbf{r}) = \mathbf{F}_{e,n}^*(\boldsymbol{\kappa}, \mathbf{r})$ .

The magnetic vector wave functions can be derived by taking the curl of the electric vector wave functions. To make them orthonormal [42], one defines

$$\mathbf{F}_{m,n}(\boldsymbol{\kappa}, \mathbf{r}) = \frac{1}{\kappa_n(\boldsymbol{\kappa})} \nabla \times \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) \quad (54)$$

where  $m$  above does not indicate an integer but 'magnetic'. It satisfies Equation (50) but with a different boundary condition that  $\hat{n} \times \nabla \times \mathbf{F}_{m,n} = 0$  since  $\nabla \times \mathbf{F}_{m,n} = \kappa_n \mathbf{F}_{e,n}$ .<sup>+</sup> The magnetic dyadic Green's function is derived to be

$$\overline{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') = \sum_{\boldsymbol{\kappa}, n} \frac{\mathbf{F}_{m,n}(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_{m,n}^{\dagger}(\boldsymbol{\kappa}, \mathbf{r}')}{\kappa_n^2 - k^2} \quad (55)$$

<sup>¶</sup> The perfect magnetic conductor (PMC) case can be similarly treated.

<sup>+</sup> An arbitrary phase factor  $e^{i\theta}$  can be added to the definition in Equation (54), and the orthonormality condition similar to Equation (52) is still preserved.

There is a need to include both types of Green's function, and the reason will be obvious when one derives the LDOS. We define the electric spectral function to be

$$\bar{\mathbf{A}}_e(\mathbf{r}, \mathbf{r}') = i \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') - \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}) \right] = -2\Im m \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') \right] \quad (56)$$

In a similar manner as before, one defines the spectral function, and show that

$$\Im m \left( \frac{1}{\kappa_n^2 - k^2} \right) = \frac{\pi E}{2k^2} [\delta(E - \mathcal{E}_{\kappa,n}) - \delta(E + \mathcal{E}_{\kappa,n})] \quad (57)$$

where  $\mathcal{E} = \hbar\kappa_n c$ , and  $\kappa_n(\boldsymbol{\kappa})$  which is a function of  $\boldsymbol{\kappa}$ . Therefore,

$$\bar{\mathbf{A}}_e(\mathbf{r}, \mathbf{r}', E) = -\frac{\pi E}{k^2} \sum_{\boldsymbol{\kappa}, n} \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_{e,n}^\dagger(\boldsymbol{\kappa}, \mathbf{r}') \delta(E - \mathcal{E}_{\kappa,n}) \quad (58)$$

A similar expression can be obtained for the magnetic spectral function. From the above, one derives the LDOS to be

$$D(\mathbf{r}, E) = -\frac{k^2}{2\pi E} \text{Tr} \left[ \bar{\mathbf{A}}_e(\mathbf{r}, \mathbf{r}) + \bar{\mathbf{A}}_m(\mathbf{r}, \mathbf{r}) \right] = \frac{1}{2} \sum_{\boldsymbol{\kappa}, n} \left[ |\mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r})|^2 + |\mathbf{F}_{m,n}(\boldsymbol{\kappa}, \mathbf{r})|^2 \right] \delta(E - \mathcal{E}_{\kappa,n}) \quad (59)$$

The above integrates over  $\mathbf{r}$  to become DOS. One can also express the LDOS directly in terms of the Green's dyadics, namely,

$$\begin{aligned} D(\mathbf{r}, E) &= \frac{k^2}{\pi E} \text{Tr} \left\{ \Im m \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}) + \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}) \right] \right\} \\ \tilde{D}(\mathbf{r}, \omega) &= \hbar D(\mathbf{r}, \omega) = \frac{\omega \mu_0 \epsilon_0}{\pi} \text{Tr} \left\{ \Im m \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}) + \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}) \right] \right\} \end{aligned} \quad (60)$$

The LDOS involves both the electric and magnetic vector wave functions. This is because LDOS gives the probability of detecting a photon in a complex structure and this probability is proportional to the strength of the electromagnetic field. In a periodic structure where standing wave can occur, the strength of the electric field may not have the same distribution as the strength of the magnetic field. The above can be integrated over  $\mathbf{r}$  to obtain the DOS.

At this point, it is expedient to define field correlation functions as

$$\bar{\mathcal{C}}_e(\mathbf{r}, \mathbf{r}', \omega) = \left\langle \mathcal{E}(\mathbf{r}, \omega) \mathcal{E}^\dagger(\mathbf{r}', \omega) \right\rangle = -\frac{\omega \mu_0}{\pi} \bar{\mathbf{A}}_e(\mathbf{r}, \mathbf{r}', \omega) \Theta(\hbar\omega) = \frac{2\omega \mu_0}{\pi} \Im m \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}', \omega) \right] \Theta(\hbar\omega) \quad (61)$$

$$\bar{\mathcal{C}}_m(\mathbf{r}, \mathbf{r}', \omega) = \left\langle \mathcal{H}(\mathbf{r}, \omega) \mathcal{H}^\dagger(\mathbf{r}', \omega) \right\rangle = -\frac{\omega \epsilon_0}{\pi} \bar{\mathbf{A}}_m(\mathbf{r}, \mathbf{r}', \omega) \Theta(\hbar\omega) = \frac{2\omega \epsilon_0}{\pi} \Im m \left[ \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}', \omega) \right] \Theta(\hbar\omega) \quad (62)$$

Then the LDOS can be related to these correlation functions as

$$\tilde{D}(\mathbf{r}, \omega) \Theta(\hbar\omega) d\omega = \text{Tr} \left[ \frac{1}{2} \epsilon_0 \bar{\mathcal{C}}_e(\mathbf{r}, \mathbf{r}, \omega) + \frac{1}{2} \mu_0 \bar{\mathcal{C}}_m(\mathbf{r}, \mathbf{r}, \omega) \right] d\omega \quad (63)$$

In a homogeneous medium case,  $\epsilon_0 \bar{\mathcal{C}}_e(\mathbf{r}, \mathbf{r}) = \mu_0 \bar{\mathcal{C}}_m(\mathbf{r}, \mathbf{r})$ , and the above reduces to Equation (41).

In general,  $\kappa_n(\boldsymbol{\kappa}) = \kappa_n(k, \theta_k, \phi_k)$  is dependent on the direction of  $\boldsymbol{\kappa}$ , or anisotropic. This is unlike the homogeneous medium case,  $\kappa_n(\boldsymbol{\kappa}) = k$ , where it is much simpler or isotropic. Therefore, when loss is included, the prove of Equation (46), valid for homogeneous medium, is more difficult for normalization now. But a general spectral function with line-broadening, local density of states, and density of states equations can be assumed, and this can be confirmed as shall be confirmed later. Moreover, Equations (61) and (62) can be confirmed in Section 8 and are commensurate with FDT.

## 7. INHOMOGENEOUS MEDIUM CASE

### 7.1. Lossless, Reciprocal Case

In this case, the relative permittivity and permeability tensors  $\bar{\boldsymbol{\epsilon}}_r(\mathbf{r}) = \bar{\boldsymbol{\epsilon}}(\mathbf{r})/\epsilon_0$  and  $\bar{\boldsymbol{\mu}}_r(\mathbf{r}) = \bar{\boldsymbol{\mu}}(\mathbf{r})/\mu_0$ , respectively. They are Hermitian tensors [54]. The electric dyadic Green's function satisfies the equation

$$\nabla \times \bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') - k^2 \bar{\boldsymbol{\epsilon}}_r(\mathbf{r}) \cdot \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}') \quad (64)$$

An electric vector wave function is defined to be the solution to

$$\nabla \times \bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) - \kappa_n^2 \bar{\boldsymbol{\epsilon}}_r(\mathbf{r}) \cdot \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) = 0 \quad (65)$$

The medium can be arranged as a periodic structure, and the derivation of the eigenfunction above is guided by the Bloch-Floquet theorem. The eigenvalue  $\kappa_n(\boldsymbol{\kappa})$  is then a function of  $\boldsymbol{\kappa}$ . To this end, one defines the eigenfunctions  $\mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r})$  similar to the periodic structure case as in Equation (51). But the Bloch-Floquet modes are  $\bar{\boldsymbol{\epsilon}}_r(\mathbf{r})$  orthonormal since we can prove from Equation (65) that

$$\int_V d\mathbf{r} \mathbf{F}_{en'}^\dagger(\boldsymbol{\kappa}, \mathbf{r}) \cdot \bar{\boldsymbol{\epsilon}}_r(\mathbf{r}) \cdot \mathbf{F}_{e,n}(\boldsymbol{\kappa}', \mathbf{r}) = \delta_{nn'} \delta_{\boldsymbol{\kappa}, \boldsymbol{\kappa}'} \quad (66)$$

Using the above, it can be shown that

$$\begin{aligned} \bar{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') &= \sum_{\boldsymbol{\kappa}, n} \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_{e,n}^\dagger(\boldsymbol{\kappa}, \mathbf{r}') \cdot \bar{\boldsymbol{\epsilon}}_r(\mathbf{r}') \\ &= \sum_{\boldsymbol{\kappa}, n} \bar{\boldsymbol{\epsilon}}_r(\mathbf{r}) \cdot \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_{e,n}^\dagger(\boldsymbol{\kappa}, \mathbf{r}') = \sum_{\boldsymbol{\kappa}, n} \bar{\boldsymbol{\epsilon}}_r^{\frac{1}{2}}(\mathbf{r}) \cdot \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) \left[ \bar{\boldsymbol{\epsilon}}_r^{\frac{1}{2}}(\mathbf{r}') \cdot \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}') \right]^\dagger \end{aligned} \quad (67)$$

The electric dyadic Green's function, despite the change in the orthonormality condition expressed by Equation (66), can be defined as before in Equation (53), and the electric spectral function can be similarly defined.

Similarly, a magnetic dyadic Green's function can be defined as

$$\nabla \times \bar{\boldsymbol{\epsilon}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') - k^2 \bar{\boldsymbol{\mu}}_r(\mathbf{r}) \cdot \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') \quad (68)$$

with its magnetic vector wave function to be the solution to

$$\nabla \times \bar{\boldsymbol{\epsilon}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{F}_{m,n}(\boldsymbol{\kappa}, \mathbf{r}) - \kappa_n^2 \bar{\boldsymbol{\mu}}_r(\mathbf{r}) \cdot \mathbf{F}_{m,n}(\boldsymbol{\kappa}, \mathbf{r}) = 0 \quad (69)$$

and its corresponding magnetic spectral function.

The LDOS is related to the spectral functions as

$$\tilde{D}(\mathbf{r}, \omega) = \hbar D(\mathbf{r}, E) = -\frac{\omega \mu_0 \epsilon_0}{2\pi} \text{Tr} \left[ \bar{\boldsymbol{\epsilon}}_r^{\frac{1}{2}}(\mathbf{r}) \cdot \bar{\mathbf{A}}_e(\mathbf{r}, \mathbf{r}) \cdot \bar{\boldsymbol{\epsilon}}_r^{\frac{1}{2}}(\mathbf{r}) + \bar{\boldsymbol{\mu}}_r^{\frac{1}{2}}(\mathbf{r}) \cdot \bar{\mathbf{A}}_m(\mathbf{r}, \mathbf{r}) \cdot \bar{\boldsymbol{\mu}}_r^{\frac{1}{2}}(\mathbf{r}) \right] \quad (70)$$

The modification is necessary because of the new orthonormality condition expressed by Equation (66). Upon substituting Equations (53) and (55) into the above, and going through the manipulation as in Section 4, Equations (26) to (32), finally, the LDOS can be shown to be modified as

$$D(\mathbf{r}, E) = \frac{1}{2} \sum_{\boldsymbol{\kappa}, n} \left[ \mathbf{F}_{e,n}^\dagger(\boldsymbol{\kappa}, \mathbf{r}) \cdot \bar{\boldsymbol{\epsilon}}_r(\mathbf{r}) \cdot \mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) + \mathbf{F}_{m,n}^\dagger(\boldsymbol{\kappa}, \mathbf{r}) \cdot \bar{\boldsymbol{\mu}}_r(\mathbf{r}) \cdot \mathbf{F}_{m,n}(\boldsymbol{\kappa}, \mathbf{r}) \right] \delta(E - \mathcal{E}_{\boldsymbol{\kappa}, n}) \quad (71)$$

By using the orthonormality condition (66), the above integrates over  $\mathbf{r}$  to become the DOS. The LDOS can also be written directly in terms of the dyadic Green's functions as

$$\tilde{D}(\mathbf{r}, \omega) = \hbar D(\mathbf{r}, E) = \frac{\omega \mu_0 \epsilon_0}{\pi} \text{Tr} \left\{ \bar{\boldsymbol{\epsilon}}_r^{\frac{1}{2}}(\mathbf{r}) \cdot \Im m [\bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r})] \cdot \bar{\boldsymbol{\epsilon}}_r^{\frac{1}{2}}(\mathbf{r}) + \bar{\boldsymbol{\mu}}_r^{\frac{1}{2}}(\mathbf{r}) \cdot \Im m [\bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r})] \cdot \bar{\boldsymbol{\mu}}_r^{\frac{1}{2}}(\mathbf{r}) \right\} \quad (72)$$

With this modification, the photon energy density can be calculated accordingly. Notice that the LDOS becomes large when  $\bar{\boldsymbol{\epsilon}}_r(\mathbf{r})$  or  $\bar{\boldsymbol{\mu}}_r(\mathbf{r})$  is large. It is to be noted that the above analysis remains valid even if  $\bar{\boldsymbol{\epsilon}}_r(\mathbf{r})$  or  $\bar{\boldsymbol{\mu}}_r(\mathbf{r})$  are functions of  $\omega$  as long as we fix  $\omega$  during the analysis. However, the Kramers-Kronig relations [47] implies that a frequency dependent or dispersive medium is also necessary lossy.

In the above, we can similarly define correlation functions for the electric and magnetic fields as in Equations (61) and (62). Then we can relate them to the LDOS as

$$\tilde{D}(\mathbf{r}, \omega) \Theta(\hbar\omega) d\omega = \text{Tr} \left[ \frac{1}{2} \epsilon_0 \bar{\boldsymbol{\epsilon}}_r^{\frac{1}{2}}(\mathbf{r}) \cdot \bar{\mathcal{C}}_e(\mathbf{r}, \mathbf{r}, \omega) \cdot \bar{\boldsymbol{\epsilon}}_r^{\frac{1}{2}}(\mathbf{r}) + \frac{1}{2} \mu_0 \bar{\boldsymbol{\mu}}_r^{\frac{1}{2}}(\mathbf{r}) \cdot \bar{\mathcal{C}}_m(\mathbf{r}, \mathbf{r}, \omega) \cdot \bar{\boldsymbol{\mu}}_r^{\frac{1}{2}}(\mathbf{r}) \right] d\omega \quad (73)$$

## 7.2. Lossy, Reciprocal Case

The equation for the electric dyadic Green's function in this case is the same as before. The compendium equation for the vector wave function can be defined as before in Equations (65) and (69), but now,  $\bar{\boldsymbol{\mu}}_r(\mathbf{r})$  and  $\bar{\boldsymbol{\epsilon}}_r(\mathbf{r})$  are non-Hermitian, but symmetric tensors to represent a general lossy, reciprocal medium. In this case, the auxiliary or transpose equations have to be defined and  $\kappa_n(\boldsymbol{\kappa})$  is not a real number anymore [42, 48]. One can assume that the lossy medium forms periodic islands immersed in the lossless background. Then one can still use the Bloch-Floquet theorem as guidance in deriving the eigensolution. Again, one can also use periodic boundary condition to make  $\boldsymbol{\kappa}$  countably infinite.

The Bloch-Floquet modes are  $\bar{\boldsymbol{\epsilon}}_r(\mathbf{r})$  orthonormal since it can be shown that

$$\int_V d\mathbf{r} \mathbf{F}_{e,n}^t(\boldsymbol{\kappa}, \mathbf{r}) \cdot \bar{\boldsymbol{\epsilon}}_r(\mathbf{r}) \cdot \mathbf{F}_{e,n'}(\boldsymbol{\kappa}', \mathbf{r}) = \delta_{nn'} \delta_{\boldsymbol{\kappa}, \boldsymbol{\kappa}'} \quad (74)$$

Notice that the  $\dagger$  is now replaced by  $t$  for the orthonormality condition because this is the lossy reciprocal case. It can be shown that the electric dyadic Green's function can be expanded as

$$\bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = \sum_{\boldsymbol{\kappa}, n} \frac{\mathbf{F}_{e,n}(\boldsymbol{\kappa}, \mathbf{r}) \mathbf{F}_{e,n}^t(\boldsymbol{\kappa}, \mathbf{r}')}{\kappa_n^2 - k^2} \quad (75)$$

The above assumes the completeness of the eigenmodes even when the medium is lossy. The only quirk is that the eigenfunctions are not always normalizable compared to the lossless case. We can think of the lossy case as a perturbation of the lossless case. The above is also analogous to the complex symmetric generalized eigenvalue matrix systems in linear algebra where the right and left eigenvectors are orthogonal to each other via a matrix.\*

As before, the electric spectral function is defined to be

$$\bar{\mathbf{A}}_e(\mathbf{r}, \mathbf{r}') = i \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') - \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}) \right] = -2\Im m \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') \right] \quad (76)$$

The above form is more complicated compared to the lossless inhomogeneous medium case. Here,  $\kappa_n^2$  is complex in Equation (75), and we see that the above gives rise to line broadening as before. Moreover, at one given frequency  $\omega$ , a cluster of modes is excited. However, as shall be shown later, the spectral function can be related to the correlation function of the field, and hence, LDOS.

The permittivity and permeability tensors are of the form

$$\bar{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}' + i\boldsymbol{\epsilon}'', \quad \bar{\boldsymbol{\mu}} = \boldsymbol{\mu}' + i\boldsymbol{\mu}'' \quad (77)$$

where both the real and imaginary parts of the above tensors are real symmetric for reciprocal media and hence are Hermitian. For dispersive media, one can define effective permittivity and permeability tensors to be [4, 51, 52]

$$\bar{\boldsymbol{\epsilon}}_e = \frac{d\omega \boldsymbol{\epsilon}'}{d\omega}, \quad \bar{\boldsymbol{\mu}}_e = \frac{d\omega \boldsymbol{\mu}'}{d\omega} \quad (78)$$

where the above tensors are Hermitian. Since the above is derived using perturbation argument, it is only valid for the low loss case. Similar to the lossless inhomogeneous case, the generalized LDOS is defined as

$$\tilde{D}(\mathbf{r}, \omega) = \hbar D(\mathbf{r}, E) = \frac{\omega}{\pi} \text{Tr} \left\{ \mu_0 \bar{\boldsymbol{\epsilon}}_e^{\frac{1}{2}}(\mathbf{r}) \cdot \Im m \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}) \right] \cdot \bar{\boldsymbol{\epsilon}}_e^{\frac{1}{2}}(\mathbf{r}) + \epsilon_0 \bar{\boldsymbol{\mu}}_e^{\frac{1}{2}}(\mathbf{r}) \cdot \Im m \left[ \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}) \right] \cdot \bar{\boldsymbol{\mu}}_e^{\frac{1}{2}}(\mathbf{r}) \right\} \quad (79)$$

The above LDOS is not provable for a general inhomogeneous lossy medium, because the line broadening is rather complicated here. But we will justify it in the next section.

The Kramers-Kronig relations [47] requires that a lossy medium is also dispersive, and the lossless medium be non-dispersive. Hence, the above reduces to Equation (72) in the lossless medium case. The above analysis remains valid even if  $\bar{\boldsymbol{\epsilon}}_r(\mathbf{r})$  or  $\bar{\boldsymbol{\mu}}_r(\mathbf{r})$  are functions of  $\omega$ . As a consequence, the eigenvalue  $\kappa_n(\boldsymbol{\kappa}, \omega)$  is a function of both  $\boldsymbol{\kappa}$  and  $\omega$ , so are the eigenfunctions  $\mathbf{F}_n(\boldsymbol{\kappa}, \omega, \mathbf{r})$  of the electric and magnetic types. Equations (74) and (75) are valid for dispersive media as long as  $\omega$  is fixed.

\* If the auxiliary equation is chosen such that its medium is described by the  $\bar{\boldsymbol{\epsilon}}^\dagger$  and  $\bar{\boldsymbol{\mu}}^\dagger$ , namely, the conjugate transpose of the original problem corresponding to an active medium, then the  $\mathbf{F}_{e,n}^t$  above can be replaced by  $\mathbf{F}_{a,e,n}^\dagger$  where  $\mathbf{F}_{a,e,n}$  is the solution of the auxiliary equation.

## 8. SPECTRAL FUNCTION AND FIELD CORRELATION

For the lossless media, we have shown that the spectral functions is related to the local density of states, and hence, they can be related to the field correlation functions. This is because the self correlation of the field is related to local energy density, and hence, the local density of states.

For the lossy media case, it is not clear that the spectral function is related to the local density of states, but we can relate the spectral function to field correlation functions directly, as we shall show in this section. In this manner, the spectral function can be related to local energy density and hence, the local density of states.

### 8.1. Statistical Average of Random Fields

Before proceeding with this section, one needs to define some properties of the correlations of random fields and sources. Imagine that random Langevin sources are generating random fields due to thermal excitation of these sources by the environment. These random fields are stationary time processes, namely the time correlations of these fields depend only on time delay between the fields, or  $t - t'$  [8, 24]. Then the frequency correlations between these fields depend only on  $\delta(\omega - \omega')$ , namely, they are uncorrelated in frequency. When the Langevin sources are in thermal equilibrium with its environment, and as shall be shown, the correlation of the fields at different locations  $\mathbf{r}$  and  $\mathbf{r}'$  are related via the spectral function.

If the Fourier transform of the random field  $\mathbf{E}(\mathbf{r}, t)$  is defined as

$$\mathbf{E}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} dt \mathbf{E}(\mathbf{r}, t) e^{i\omega t} \quad (80)$$

then the  $\mathbf{E}(\mathbf{r}, \omega)$  has the dimension  $1/\omega$  times field. The general form of the correlation of the electric fields at two locations  $\mathbf{r}$  and  $\mathbf{r}'$  is

$$\langle \mathbf{E}(\mathbf{r}, \omega) \mathbf{E}^\dagger(\mathbf{r}', \omega') \rangle = \delta(\omega - \omega') \langle \mathcal{E}(\mathbf{r}, \omega) \mathcal{E}^\dagger(\mathbf{r}', \omega') \rangle = \delta(\omega - \omega') \bar{\mathcal{C}}_e(\mathbf{r}, \mathbf{r}', \omega) \quad (81)$$

Similarly, the correlation function for the magnetic field is of the form

$$\langle \mathbf{H}(\mathbf{r}, \omega) \mathbf{H}^\dagger(\mathbf{r}', \omega') \rangle = \delta(\omega - \omega') \langle \mathcal{H}(\mathbf{r}, \omega) \mathcal{H}^\dagger(\mathbf{r}', \omega') \rangle = \delta(\omega - \omega') \bar{\mathcal{C}}_m(\mathbf{r}, \mathbf{r}', \omega) \quad (82)$$

In the above  $\bar{\mathcal{C}}_e$  and  $\bar{\mathcal{C}}_m$  are proportional to energy density per unit radian frequency.

### 8.2. Connection of the Spectral Function to Medium Loss

The random fields in a medium at thermal equilibrium are generated by Langevin current sources [38]. To this end, one rewrites electric dyadic Green's function equation in operator notation as:

$$(\bar{\mathcal{D}} - k^2 \bar{\mathcal{E}}_r) \bar{\mathcal{G}}_e = \bar{\mathcal{I}} \quad (83)$$

where

$$\bar{\mathcal{D}} \Rightarrow [\nabla \times \bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}) \nabla \times], \quad \bar{\mathcal{E}}_r \Rightarrow [\bar{\boldsymbol{\epsilon}}_r(\mathbf{r})] \quad (84)$$

These are the operators (or general Hilbert space representation of the operators) originally defined in coordinate space. For lossy media,  $\bar{\mathcal{D}}$  and  $\bar{\mathcal{E}}_r$  are non Hermitian, and the expressions right of the arrows are their coordinate space representations.

Equivalently, the Green's dyadic operator is then

$$\bar{\mathcal{G}}_e = (\bar{\mathcal{D}} - k^2 \bar{\mathcal{E}}_r)^{-1} \quad (85)$$

Using these notations, an interesting expression for the spectral function operator can be derived. It can be shown that

$$\bar{\Gamma} = i \left[ (\bar{\mathcal{G}}_e^a)^{-1} - \bar{\mathcal{G}}_e^{-1} \right] = i \left[ \bar{\mathcal{D}}^a - \bar{\mathcal{D}} - k^2 (\bar{\mathcal{E}}_r^a - \bar{\mathcal{E}}_r) \right] = 2\Im m(\bar{\mathcal{D}}) - 2k^2 \Im m(\bar{\mathcal{E}}_r) \quad (86)$$

The above  $\bar{\Gamma}$  would be zero for a lossless medium for which  $\bar{\boldsymbol{\mu}}_r(\mathbf{r})$  and  $\bar{\boldsymbol{\epsilon}}_r(\mathbf{r})$  are Hermitian. Hence  $\bar{\Gamma}$  is related to the loss of the system. In the above, the adjoint operator denoted by superscript  $a$  is as defined in Equation (22) where the 3-vector and Hilbert space nature of the space is accounted for.

Multiplying the above by  $\bar{\mathcal{G}}_e$  and  $\bar{\mathcal{G}}_e^a$  from the left and right, respectively, the electric spectral function operator is given by:

$$\bar{\mathcal{A}}_e = i(\bar{\mathcal{G}}_e - \bar{\mathcal{G}}_e^a) = \bar{\mathcal{G}}_e \bar{\Gamma} \bar{\mathcal{G}}_e^a \quad (87)$$

The above can also be written more suggestively so that

$$(\bar{\mathcal{D}} - k^2 \bar{\mathcal{E}}_r) \bar{\mathcal{A}}_e = \bar{\Gamma} \bar{\mathcal{G}}_e^a \quad (88)$$

It implies that the spectral function operator  $\bar{\mathcal{A}}_e$  is generated by distributed sources on the right-hand side that are related to the Langevin sources. The right-hand side of the above can be interpreted as induced sources due to the back propagation of field via  $\bar{\mathcal{G}}_e^a$  into the lossy part of the medium represented by  $\bar{\Gamma}$ .

However, the spectral function operator can also be defined more simply as  $\bar{\mathcal{A}}_e = i(\bar{\mathcal{G}}_e - \bar{\mathcal{G}}_e^a)$ . As shall be shown, the spectral function is proportional to the field correlation function, and it appears that the fields are correlated by the difference of a forward Green's function and a backward Green's function, making it appear like a radiating source occurring concurrently with an absorbing source.

It is to be noted that in Equation (87), the operators in the 3-vector functional Hilbert space, and the  $\dagger$  sign implies the adjoint operator defined in this infinite dimensional space. With this caution, the coordinate representation of the spectral function operator  $\bar{\mathcal{A}}_e$  in Equation (87) becomes

$$\begin{aligned} \bar{\mathbf{A}}_e(\mathbf{r}, \mathbf{r}') &= \bar{\epsilon}_r^{-1}(\mathbf{r}) \int d\mathbf{r}'' \nabla \times \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}'') \cdot \bar{\boldsymbol{\mu}}_r(\mathbf{r}'') \cdot [2\Im m \bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}'')] \cdot \nabla'' \times \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}'') \\ &\quad - \int d\mathbf{r}'' \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'') \cdot [2k^2 \Im m \bar{\epsilon}_r(\mathbf{r}'')] \cdot \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}'') \end{aligned} \quad (89)$$

In the above, one has made use of that

$$[\bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')]^t = \bar{\epsilon}_r^{-1}(\mathbf{r}') \cdot \nabla' \times \bar{\mathbf{G}}_m(\mathbf{r}', \mathbf{r}) \quad (90)$$

The above can be derived using reciprocity theorem.<sup>‡</sup> It is the generalization of (1.4.14b) in [42] to inhomogeneous, anisotropic media. Furthermore, it can be shown that

$$-\bar{\boldsymbol{\mu}}_r \Im m (\bar{\boldsymbol{\mu}}_r^{-1}) \bar{\boldsymbol{\mu}}_r^\dagger = \Im m (\bar{\boldsymbol{\mu}}_r) \quad (91)$$

Therefore, Equation (89) can be rewritten as

$$\begin{aligned} \bar{\mathbf{A}}_e(\mathbf{r}, \mathbf{r}') &= -2\Im m \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = -\bar{\epsilon}_r^{-1}(\mathbf{r}) \int d\mathbf{r}'' \nabla \times \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}'') \cdot [2\Im m \bar{\boldsymbol{\mu}}_r(\mathbf{r}'')] \cdot [\bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}'')]^\dagger \cdot \nabla'' \times \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}'') \\ &\quad - \int d\mathbf{r}'' \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'') \cdot [2k^2 \Im m \bar{\epsilon}_r(\mathbf{r}'')] \cdot \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}'') \end{aligned} \quad (92)$$

### 8.3. Connection of the Field Correlations to Langevin Sources

Alternatively, Equation (92) above can be related to the correlations of electric fields. To this end, one defines electric field due to the electric Langevin sources as

$$\mathbf{E}_e(\mathbf{r}, \omega) = i\omega\mu_0 \int d\mathbf{r}' \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{J}(\mathbf{r}', \omega) \quad (93)$$

The average of the outer product of electric fields due to the electric type sources at two different frequencies is then given by the field correlation

$$\langle \mathbf{E}_e(\mathbf{r}, \omega) \mathbf{E}_e^\dagger(\mathbf{r}', \omega') \rangle = \omega^2 \mu_0^2 \int d\mathbf{r}'' \int d\mathbf{r}''' \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'', \omega) \cdot \langle \mathbf{J}(\mathbf{r}'', \omega) \mathbf{J}^\dagger(\mathbf{r}', \omega') \rangle \cdot \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}', \omega') \quad (94)$$

Similarly, one defines the electric field due to magnetic Langevin sources as

$$\mathbf{E}_m(\mathbf{r}, \omega) = \bar{\epsilon}_r^{-1} \int d\mathbf{r}'' \nabla \times \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}'', \omega) \cdot \mathbf{M}(\mathbf{r}'', \omega) \quad (95)$$

<sup>‡</sup> Derivation available upon request.

The average of outer product of electric fields due to the magnetic type sources is then

$$\begin{aligned} \langle \mathbf{E}_m(\mathbf{r}, \omega) \mathbf{E}_m^\dagger(\mathbf{r}', \omega') \rangle &= \bar{\epsilon}_r^{-1}(\mathbf{r}) \int d\mathbf{r}'' \int d\mathbf{r}''' \nabla \times \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}'', \omega) \cdot \langle \mathbf{M}(\mathbf{r}'', \omega) \mathbf{M}^\dagger(\mathbf{r}', \omega') \rangle \\ &\quad \cdot [\bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}''')]^\dagger \cdot \nabla''' \times \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}', \omega) \end{aligned} \quad (96)$$

In the above, one has made use of Equation (90), the fact that  $\bar{\mathbf{G}}^t(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{G}}(\mathbf{r}', \mathbf{r})$  from reciprocity, and that  $\bar{\boldsymbol{\mu}}_r$  is a symmetric tensor. Consequently,

$$\begin{aligned} \langle \mathbf{E}(\mathbf{r}, \omega) \mathbf{E}^\dagger(\mathbf{r}', \omega') \rangle &= \langle \mathbf{E}_e(\mathbf{r}, \omega) \mathbf{E}_e^\dagger(\mathbf{r}', \omega') \rangle + \langle \mathbf{E}_m(\mathbf{r}, \omega) \mathbf{E}_m^\dagger(\mathbf{r}', \omega') \rangle \\ &= \omega^2 \mu_0^2 \int d\mathbf{r}'' \int d\mathbf{r}''' \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'', \omega) \cdot \langle \mathbf{J}(\mathbf{r}'', \omega) \mathbf{J}^\dagger(\mathbf{r}', \omega') \rangle \cdot \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}', \omega') \\ &\quad + \bar{\epsilon}_r^{-1}(\mathbf{r}) \cdot \int d\mathbf{r}'' \int d\mathbf{r}''' \nabla \times \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}'', \omega) \cdot \langle \mathbf{M}(\mathbf{r}'', \omega) \mathbf{M}^\dagger(\mathbf{r}', \omega') \rangle \\ &\quad \cdot [\bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}''')]^\dagger \cdot \nabla''' \times \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}', \omega) \end{aligned} \quad (97)$$

Notice that the above is of the same structural form as Equation (92).

#### 8.4. Connection of Spectral Function to the Field Correlations

By comparing Equations (92) and (97), and in order for them to be proportional to each other, it implies that the Langevin currents have correlations of the form

$$\langle \mathbf{J}(\mathbf{r}, \omega) \mathbf{J}^\dagger(\mathbf{r}', \omega') \rangle = \delta(\omega - \omega') \delta(\mathbf{r} - \mathbf{r}') \frac{\omega \epsilon_0}{\pi} \Im m [\bar{\epsilon}_r(\mathbf{r})] \Theta(\hbar\omega) \quad (98)$$

and

$$\langle \mathbf{M}(\mathbf{r}, \omega) \mathbf{M}^\dagger(\mathbf{r}', \omega') \rangle = \delta(\omega - \omega') \delta(\mathbf{r} - \mathbf{r}') \frac{\omega \mu_0}{\pi} \Im m [\bar{\boldsymbol{\mu}}_r(\mathbf{r})] \Theta(\hbar\omega) \quad (99)$$

The above implies that Langevin sources are uncorrelated in frequency and space. It also implies that the field correlation function in Equation (97) becomes

$$\begin{aligned} \langle \mathbf{E}(\mathbf{r}, \omega) \mathbf{E}^\dagger(\mathbf{r}', \omega') \rangle &= \delta(\omega - \omega') \left\{ \bar{\epsilon}_r^{-1}(\mathbf{r}) \int d\mathbf{r}'' \nabla \times \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}'', \omega) \cdot \frac{\omega \mu_0}{\pi} \Im m [\bar{\boldsymbol{\mu}}_r(\mathbf{r})] \Theta(\hbar\omega) \cdot [\bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}'')]^\dagger \cdot \nabla'' \right. \\ &\quad \left. \times \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}', \omega) + \omega^2 \mu_0^2 \int d\mathbf{r}'' \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'', \omega) \cdot \frac{\omega \epsilon_0}{\pi} \Im m [\bar{\epsilon}_r(\mathbf{r})] \Theta(\hbar\omega) \cdot \bar{\mathbf{G}}_e^\dagger(\mathbf{r}', \mathbf{r}', \omega') \right\} \end{aligned} \quad (100)$$

The above implies that the right-hand side is proportional to the spectral function times  $\Theta(\hbar\omega)$ . Consequently, the field correlation and the spectral function are related as indicate before in Equations (40), (61), and (62). Namely,

$$\begin{aligned} \langle \mathbf{E}(\mathbf{r}, \omega) \mathbf{E}^\dagger(\mathbf{r}', \omega') \rangle &= \delta(\omega - \omega') \langle \mathcal{E}(\mathbf{r}, \omega) \mathcal{E}^\dagger(\mathbf{r}', \omega') \rangle = \delta(\omega - \omega') \bar{\mathcal{C}}_e(\mathbf{r}, \mathbf{r}', \omega) \\ &= \delta(\omega - \omega') \frac{\omega \mu_0}{\pi} \Im m [\bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')] \Theta(\hbar\omega) \end{aligned} \quad (101)$$

Or that  $\bar{\mathcal{C}}_e(\mathbf{r}, \mathbf{r}', \omega)$  is connected to the electric spectral functions as

$$\bar{\mathcal{C}}_e(\mathbf{r}, \mathbf{r}', \omega) = \frac{\omega \mu_0}{\pi} \Im m [\bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')] \Theta(\hbar\omega) \quad (102)$$

The assertions in Equations (98) and (99) are justified for the following reasons:

- The spectral function consists of two bilinear terms, one of which is linearly proportional to the permittivity loss and the other one linearly proportional to the permeability loss. We can also derive Equations (98) and (99) by turning off the loss of each kind respectively and equating Equations (92) and (97) with the appropriate proportionality constant.

- The Green's dyadic is undefined for a lossless medium unless infinitesimal small loss is introduced. Hence, one thinks of Equation (97), and hence, Equation (100) being valid for the case of infinitesimally small loss media as well. Therefore, Equation (102) is valid for the lossless case as well.
- For the lossless case, Equations (39), (61), and (62) are confirmations that the above derivation leading to Equation (102) is correct. Furthermore, the spectral function can be related to the local density of states, and hence energy density after multiplying by  $\Theta(\hbar\omega)$ . This is further affirmation that Equation (102) is at least valid when  $\mathbf{r} = \mathbf{r}'$ .

A correlation function for the magnetic field can similarly be defined as

$$\left\langle \mathbf{H}(\mathbf{r}, \omega) \mathbf{H}^\dagger(\mathbf{r}', \omega') \right\rangle = \delta(\omega - \omega') \bar{\mathcal{C}}_m(\mathbf{r}, \mathbf{r}', \omega) \quad (103)$$

where  $\bar{\mathcal{C}}_m(\mathbf{r}, \mathbf{r}', \omega)$  can be connected to the magnetic spectral functions as

$$\bar{\mathcal{C}}_m(\mathbf{r}, \mathbf{r}', \omega) = \frac{\omega \epsilon_0}{\pi} \Im m \left[ \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') \right] \Theta(\hbar\omega) \quad (104)$$

The above results are also corroborated by the fluctuation dissipation theorem [8, 24].

### 8.5. Physical Interpretation

Now, however, Equations (92) and (100) have a nice physical interpretation. It implies that there are two types of contributions to the electric field correlation function or the electric spectral function: one that comes from magnetic type Langevin current sources and the second that comes from electric type Langevin current sources. Since the complex conjugation of a field corresponds to time reversal or back propagation, a field at point  $\mathbf{r}'$  back propagates into the lossy medium to the source point at  $\mathbf{r}''$  via the backward Green's function  $\bar{\mathbf{G}}_e^\dagger$ . The Langevin source at point  $\mathbf{r}''$  can only correlate or be coherent with the same Langevin source. The same Langevin source radiates a field that propagates from the source point  $\mathbf{r}''$  to the field point at  $\mathbf{r}$  via the forward Green's function  $\bar{\mathbf{G}}_e$ . Different Green's functions are used depending on the source type. The first term corresponds to magnetic type sources while the second term corresponds to electric type sources.

As can be seen, the above shows that the fields at two locations  $\mathbf{r}$  and  $\mathbf{r}'$  are correlated if they come from the same Langevin source. Most important, the above also shows that the two fields at two locations  $\mathbf{r}$  and  $\mathbf{r}'$  are correlated directly by the spectral function as in the left-hand side of Equation (92), multiplied by  $\Theta(\hbar\omega)$  and a multiplicative constant, which is a much simpler representation of the correlation function. The spectral function is non-singular at  $\mathbf{r} = \mathbf{r}'$  and is seen to be set up by distributed Langevin sources.

### 8.6. Energy Correlation Functions

In the above, we have derived the field correlation functions, and show their relationships to the spectral functions. To be more precise, when the energy density is needed, the energy correlation function needs to be derived as follows. An energy correlation function for the electric field, in connection to Equation (78), is then defined as

$$\left\langle \bar{\boldsymbol{\epsilon}}_e^{\frac{1}{2}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}, \omega) \mathbf{E}^\dagger(\mathbf{r}', \omega') \cdot \bar{\boldsymbol{\epsilon}}_e^{\frac{1}{2}}(\mathbf{r}') \right\rangle = \delta(\omega - \omega') \bar{\mathcal{W}}_e(\mathbf{r}, \mathbf{r}', \omega) \quad (105)$$

In the above,  $\bar{\mathcal{W}}_e(\mathbf{r}, \mathbf{r}', \omega)$  should have the dimension of energy density per radian frequency, and it can be connected to the electric spectral functions as

$$\bar{\mathcal{W}}_e(\mathbf{r}, \mathbf{r}', \omega) = \frac{\omega \mu_0}{\pi} \bar{\boldsymbol{\epsilon}}_e^{\frac{1}{2}}(\mathbf{r}) \cdot \Im m \left[ \bar{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') \right] \cdot \bar{\boldsymbol{\epsilon}}_e^{\frac{1}{2}}(\mathbf{r}') \Theta(\hbar\omega) \quad (106)$$

The above can be related to the LDOS when  $\mathbf{r} = \mathbf{r}'$ .

Similarly, a correlation function for the magnetic field is defined as

$$\left\langle \bar{\boldsymbol{\mu}}_e^{\frac{1}{2}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}, \omega) \mathbf{H}^\dagger(\mathbf{r}', \omega') \cdot \bar{\boldsymbol{\mu}}_e^{\frac{1}{2}}(\mathbf{r}') \right\rangle = \delta(\omega - \omega') \bar{\mathcal{W}}_m(\mathbf{r}, \mathbf{r}', \omega) \quad (107)$$

With the understanding that  $\overline{\mathcal{W}}_m(\mathbf{r}, \mathbf{r}', \omega)$  should have the dimension of energy density per radian frequency, it can be connected to the magnetic spectral functions as

$$\overline{\mathcal{W}}_m(\mathbf{r}, \mathbf{r}', \omega) = \frac{\omega \epsilon_0}{\pi} \overline{\boldsymbol{\mu}}_e^{\frac{1}{2}}(\mathbf{r}) \cdot \Im m [\overline{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}')] \cdot \overline{\boldsymbol{\mu}}_e^{\frac{1}{2}}(\mathbf{r}') \Theta(\hbar\omega) \quad (108)$$

The above can be combined and are related to the LDOS defined in Equation (79).

## 9. CONCLUSION

The Green's dyadics in terms of eigenmode expansions have been derived for general media including lossy, inhomogeneous, anisotropic media. Then the electromagnetic spectral functions are defined. The spectral function as defined has the physical meaning of describing a source in equilibrium with its environment. It consists of a causal or radiative part, subtracting a non-causal, or absorptive part, representing the radiation and absorption of fields by Langevin sources.

For lossless media, it can be shown that the spectral function is related to the local density of states. For lossy media, due to line broadening, the spectral function describes a source at  $\omega$  that excites a cluster of modes in time harmonic motion, even though these modes may not be degenerate. Furthermore, by relating the spectral function to the field correlation function, the spectral function, including that for lossy anisotropic inhomogeneous media, can be related to the local density of states.

Many relations derived using FDT can also be derived using spectral function. One of them is that the correlation of the fields at two points is related via the spectral function which is the imaginary part of the Green's function. The second one is that the correlation of two Langevin currents are related via the imaginary part of their permittivity or permeability depending on the current type. Hence, our results are commensurate with FDT: namely, the same conclusions from spectral functions can be arrived at from FDT. It also justifies the use of FDT for the calculation of Casimir force in a lossy medium.

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