Progress In Electromagnetics Research, *PIER 17, 45–72, 1997*

BARGMAN TRANSFORMS AND PHASE SPACE FILTERS

H. N. Kritikos, J. H. Cho

Department of Electrical Engineering
University of Pennsylvania
200 S. 33 St.
Philadelphia, PA 19104, USA

J. G. Teti, Jr.

Lambda Science, Inc.
P.O. Box 238
Wayne, PA 19087-0238, USA

1. INTRODUCTION

The concepts of phase space find their origins in classical mechanics where dynamical systems are characterized in terms of the conjugate variables, the coordinate q and momentum p. H. Goldstein [1] has written one of the definitive treatises on classical mechanics and has extensively discussed the formulation of dynamical problems in phase space in terms of their Hamiltonians, Poisson Brackets etc. Phase Space concepts have become important in many fields of the physical sciences. They have found their way into Quantum Mechanics where they became central in the Schröndinger representation of quantum theory. In the area of quantum optics the field equations have been based on the phase space concepts. The formulation has been derived for the representation of fields in terms of Coherent States which have the form $g^{(p,q)}(x) = \frac{1}{\pi^{1/4}} e^{ipx} e^{-(x-q)^2/2}$. Extensive discussion of application of Coherent States in Quantum Optics have been given by J. R. Klauder and E. C. G. Sudarshan [2] and later J. R. Klauder and B. S. Skagerstam [3], B. Wells [4], R. Cloude [5], A. Perelomov [6], and J. Elliot and P. Dawber [7].

In a parallel development in the area of applied mathematics and signal theory many developments have taken place which have been derived from Phase Space concepts. It is interesting to note that the Coherent state $g^{(p,q)}(x)$ is known in the signal processing literature as a filter bank or a Gabor function. From the many contributions in this area notable are the ones by I. Daubechies[8], C. Meyer [9] and C. Heil and D. Walnut [10].

The coherent state analysis of a function supported in the real line R maps a function supported in the real line to a plane, $R \Rightarrow R \times R$. The Phase Space can be considered as a complex plane with the complex variable z defined as $z = q - ip$. This transformation is also known as the Bargman transform after V. Bargman [11] who showed that the phase space signature of a function is of the form $e^{-|z|^2/4} F(z)$, where $F(z)$ is an entire function. The transformation is therefore from the real line to the complex plane C, $R \Rightarrow C$. G. Folland [12] has given an excellent discussion on the Bargman transform. Some caution, however, has to be exercised with the notation. In order to be consistent with Folland the complex variable was defined as $z = q - ip$. This ensures that the functions $F(z)$ in the Bargman space are functions of z and not $\bar{z}$. I. Daubechies [8] also for similar reasons has chosen to represent the complex variable $z^{(D)} = p + iq$.

The representation of the signal in the phase plane offers the opportunity to carry out selective filtering. The filtering has a time-frequency character and offers another tool that can be used in the identification of targets. An important contribution has been made by J. Ramanathan and P. Topiwala [13] and I. Daubechies [14] has demonstrated that it is possible to carry the filtering in the time domain thus opening the possibility of real time implementation of these techniques. Many applications have appeared in the literature some of which have been reported by J. Teti and H. Kritikos [15, 16] regarding SAR signal processing.

The intent of this paper is to present a unified form of the Bargman transform so that workers in the field of Electromagnetic Scattering can use it effectively. The complex Bargman phase plane arising from windowed transforms is a natural evolution of the complex frequency plane and as such should find many applications in Scattering. The paper also presents a technique of time-frequency filtering which can be carried out in the time domain, thus offering the possibility of a real time implementation of feature extraction through phase space filtering.

2. THE HARMONIC OSCILLATOR

The harmonic oscillator occupies an important role in that its solutions are the Hermite functions which are the basis functions in coherent state analysis. The solutions of the harmonic oscillator can be obtained by first obtaining the Hamiltonian of the system in terms of the conjugate variables p (momentum) and q (coordinate). H. Goldstein [1] and D. Sattinger and O. Weaver [17] have discussed the Hamiltonian from a classical viewpoint. J. Elliot and P. Dawber [7], A. Perelomov [6], and J. Klauder and E. Sudarshan [2] have examined the quantum mechanical formulation.

The Hamiltonian in one dimension is

$$H = \frac{1}{2m}\left(p_o^2 + m^2\omega^2 q_o^2\right) \tag{2.1}$$

where m is the mass of the particle and ω is the angular frequency.

In the following development a normalized form of the conjugate coordinates p and q will be used resulting in the following equation

$$H = \frac{1}{2}\left(p^2 + q^2\right) \tag{2.2}$$

where

$$p = \frac{\partial}{i\partial x}, \quad q = x \tag{2.3}$$

with the commutation relations

$$[p,p] = [q,q] = 0, \quad [q,p] = i \tag{2.4}$$

Following A. Perelomov [6], J. Klauder and E. Sudarshan [2] we define the operators

$$a = \frac{(q+ip)}{\sqrt{2}} \tag{2.5}$$

and

$$a^* = \frac{(q-ip)}{\sqrt{2}} \tag{2.6}$$

with the commutation relations

$$[a, a^*] = 1 \tag{2.7}$$

Operators a and a^* are known as the annihilation and creation operators and they are useful in generating the Hermite functions which are the solutions of the Harmonic Oscillator.

D. Sattinger and O. Weaver [17] have shown that the Hermite functions φ_n can be generated as follows

$$\varphi_n(x) = \frac{(a^*)^n}{\sqrt{n!}} \varphi_0(x) \tag{2.8}$$

where

$$\varphi_0(x) = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}} \tag{2.9}$$

The Hermite functions also have the form

$$\varphi_n(x) = \frac{e^{\frac{x^2}{2}}}{\pi^{1/4}} \frac{H_n(x)}{\sqrt{2^n n!}} \tag{2.10}$$

where $H_n(x)$ is the Hermite Polynomial.

In terms of the differential operator they are also given by

$$\varphi_n(x) = \frac{(-1)^n e^{\frac{x^2}{2}}}{\pi^{1/4}\sqrt{2^n n!}} \left(\frac{d}{dx}\right)^n e^{-x^2} \tag{2.11}$$

or

$$\varphi_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \left(x - \frac{d}{dx}\right)^n e^{-x^2/2} \tag{2.12}$$

The creation and annihilation operators have the following well known properties

$$a\varphi_n = \sqrt{n}\varphi_{n-1} \tag{2.13}$$

and

$$a^*\varphi_n = \sqrt{n+1}\varphi_{n+1} \tag{2.14}$$

The Hamiltonian of eq. (2.2) is also called the Hermite operator and has the form

$$H = \frac{1}{2}\left(x^2 - \frac{d^2}{dx^2}\right) \tag{2.15}$$

or

$$H = \left(a^*a + \frac{1}{2}\right) = \left(aa^* - \frac{1}{2}\right) \tag{2.16}$$

One has therefore

$$H\varphi_n = \left(n + \frac{1}{2}\right)\varphi_n \tag{2.17}$$

which is the well known quantum mechanical equivalent of the Harmonic Oscillator. Notice also that

$$2a^*a = -\frac{d^2}{dx^2} + x^2 - 1 \tag{2.18}$$

This is also known as the Hermite operator.

3. THE PHASE SPACE TRANSFORM

Following G. B. Folland [12] and V. Bargman [11, 18] consider the operator V which maps a function from the real space x to the phase plane as follows:

$$Vf(x) \Rightarrow F(q,p),\ f(x) \subset L^2(R),\ x \subset R,\ (p,q) \subset R \times R \tag{3.1}$$

$$(Vf)(p,q) = \langle f(x), g^{(p,q)}(x)\rangle \tag{3.2}$$

where

$$g^{(p,q)}(x) = \frac{1}{\pi^{1/4}} e^{ipx} e^{-ipq/2} e^{-(x-q)^2/2} \tag{3.3}$$

G. Folland [12, p.22] and J. Klauder and B. Skagerstam [3] have introduced the function $g^{(p,q)}(x)$ as a coherent state vector and is generated by the operator

$$\rho(-q,p) = e^{it} e^{ipX - iqD} \tag{3.4}$$

where X and D are the coordinate and differential operators. The operator is the unitary representation of Heisenberg Group with the group law

$$(p,q,t)(p',q',t') = \left(p+p', q+q', t+t'+\frac{1}{2}(pq'-p'q)\right) \tag{3.5}$$

Folland [12, p.23] for convenience drops variable t and defines the symplectic form of the operator which is

$$\rho'(q,p) \leftrightarrow \rho(-q,p) \tag{3.6}$$

with the property

$$\rho'(p,q)f(x) = e^{ipx-ipq/2}f(x-q) \tag{3.7}$$

In the following discussion for convenience the prime will be omitted.

The coherent states can be produced using the Heisenberg operator acting on a generating function

$$g_0 = \frac{e^{-x^2/2}}{\pi^{1/4}} \tag{3.8}$$

as follows

$$g^{(p,q)}(x) = \rho(p,q)g_0 \tag{3.9}$$

Going over to a complex notation the combining of p and q in the form $z^{(F)} = q+ip$ leads to expressions where the complex conjugate forms appears in all of the expressions in Bargman space. In the following work to overcome this difficulty the expression $z = q-ip$ will be used. The analyzing vector is then

$$g^{(\bar{z})}(x) = \frac{1}{\pi^{1/4}}e^{\bar{z}x}e^{-|z|^2/4}e^{-(\bar{z})^2/4}e^{-x^2/2} \tag{3.10}$$

The transform, therefore, becomes

$$F(z) = \int_{-\infty}^{\infty} g^{(z)}f(x)dx \tag{3.11}$$

The operation can also be carried out in the frequency domain. For a Fourier transform of the form

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f(x)dx \tag{3.12}$$

the corresponding analyzing function is

$$\hat{g}^{(\bar{z})} = \frac{1}{\pi^{1/4}} e^{-|z|^2/4} e^{-(\bar{z})^2/4} e^{-(\bar{z}-k)^2/2} \tag{3.13}$$

The phase space signature is

$$F(z) = \int_{-\infty}^{\infty} \hat{g}^{(z)}(k) \hat{f}(k) dk \tag{3.14}$$

The original function can be recovered by

$$f(t) = \iint_S g^{(\bar{z})} F(z) d\mu, \quad d\mu = dpdq, \quad S = R \times R \tag{3.15}$$

The literature is rich with applications of this transform to many problems in the physical sciences. It does appear, however, in different forms depending on the definition of z. In some applications the complex variable is defined as $z = q + ip$. The basic results, however, are the same.

Transformations of this type were first introduced in the physics literature. J. Klauder and Sudarshan [2] has applied in quantum optics and I. Daubechie [8, 14] lately has applied them to signal processing.

4. THE BARGMAN TRANSFORM

V. Bargman [11, 18] and later G. Folland [12] have introduced a transform which is a derivative of the phase space transform and is defined as follows:

$$Bf(z) = \frac{1}{\pi^{1/4}} e^{-z^2/4} \int_{-\infty}^{\infty} e^{zx - x^2/2} f(x) dx \tag{4.1}$$

It is an isometry from the space of square integrable functions $L^2(R)$ to the space $L^2\left(C, e^{-|z|^2/4} dz\right)$ which is known as Fock space $\mathcal{F}$. It is defined as

$$\mathcal{F} = \Big\{ F : F \text{ is an entire function on } C, \quad \|F\|^2_{\mathcal{F}} = \int_{-\infty}^{\infty} |F(z)| e^{-|z|^2/4} dz < \infty \Big\} \tag{4.2}$$

The Fock Space is the space of entire analytic functions defined in the whole complex plane C. It is an important property that has been extensively utilized in applications reported in the literature.

The Bargman transform can also be defined in terms of the Bargman kernel as follows:

$$Bf(z) = \int_{-\infty}^{\infty} B(z,x)f(x)dx, \quad B^{-1}F(z) = \int_{-\infty}^{\infty} B(\bar{z},x)f(x)e^{-|z|^2/4}dz \tag{4.3}$$

where

$$B(z,x) = \frac{1}{\pi^{1/4}} e^{zx-x^2/2-z^2/4} \tag{4.4}$$

G. Folland [12] has an excellent discussion of this material there are, however, some differences in notation. In order to be consistent note that the variables used in this paper are related to the Folland ones by

$$p = \frac{p^{(F)}}{\sqrt{2\pi}}, \ q = \frac{q^{(F)}}{\sqrt{2\pi}}, \ x = \frac{x^{(F)}}{\sqrt{2\pi}} \tag{4.5}$$

Superscript (F) denotes the Folland notation.

In order to establish useful analytical tools it is instructive to determine the form of number of well-known operators in Fock space.

An operator O is transported in Bargman space by the operation

$$O_B = BOB^{-1} \tag{4.6}$$

A number of operators have been thus transported and have been reported in the literature, see Folland [12] I. Daubechies [8, 19], J. Ramanathan and P Topiwala [13], J. R. Klauder and B. S. Skagerstam [3], C. Heil and D. Walnut [10]. It can easily be shown by integration by parts (see Appendix I) that the following relations hold

$$x \Rightarrow \frac{d}{dz} + z/2 \tag{4.7}$$

$$\frac{d}{dx} \Rightarrow \frac{d}{dz} - z/2 \tag{4.8}$$

One can also easily show (see Appendix II) that

$$\int f(x)dx \Rightarrow e^{z^2/4}\left[\int e^{-z^2/4}F(z)dz + C\right], \quad C \text{ a Constant} \tag{4.9}$$

It follows then that the creation operator is

$$a^* f(x) \Rightarrow z\frac{F(z)}{\sqrt{2}} \tag{4.10}$$

The annihilation operator is

$$af(x) \Rightarrow \sqrt{2}\frac{dF(z)}{dz} \tag{4.11}$$

The product of the annihilation and creation operators is

$$a^* af(x) \Rightarrow z\frac{dF(z)}{dz} \tag{4.12}$$

Starting with the Hamiltonian H one can easily show the following:

$$Hf(x) \Rightarrow \left(z\frac{d}{dz} - \frac{1}{2}\right) F(z) \tag{4.13}$$

The Heisenberg operator can also be transported in the Bargman space as follows

$$\rho(p_0, q_0)f(x) = \rho(w) \Rightarrow \beta(w)F(z), \quad w = q_0 - ip_0 \tag{4.14}$$

where

$$\beta(w)F(z) = e^{-|w|^2/4+(\bar{z}w+\bar{w}z)/4}F(z+w) \tag{4.15}$$

The Bargman transform of the Hermite functions from V. Bargman [11], G. Folland [12] are

$$\varphi_n(x) \Rightarrow \zeta_n(z), \quad \zeta_n(z) = \frac{z^n}{\sqrt{2^n n!}} \tag{4.16}$$

Operations on these functions by the harmonic oscillator operator produces the following action in Bargman Space

$$H\varphi_n(x) \Rightarrow (n+1/2)\zeta_n(z) \tag{4.17}$$

Similarly we also have

$$a^*\varphi_n(x) \Rightarrow \sqrt{n+1}\zeta_{n+1}(z) \tag{4.18}$$
$$a\varphi_n(x) \Rightarrow \sqrt{n}\zeta_{n-1}(z) \tag{4.19}$$

A number of useful transforms will be given below: They are: The delta function

$$\delta(x - x_0) \Rightarrow \frac{1}{\pi^{1/4}} e^{-z^2/4} e^{-x_0^2/2} e^{zx_0} \tag{4.20}$$

and the pulse train (see Appendix III).

$$\sum_{n=-\infty}^{\infty} \delta(x - nx_0) \Rightarrow \frac{e^{-z^2/4}}{\pi^{1/4}} \theta_3 \left(e^{-x_0^2/4}, -izx_0/2 \right) \tag{4.21}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n \delta(x - nx_0) \Rightarrow \frac{e^{-z^2/4}}{\pi^{1/4}} \theta_4 \left(e^{-x_0^2/4}, -izx_0/2 \right) \tag{4.22}$$

where θ_n are the theta functions defined by A. Rainville [20].

The step function

$$u(x - x_0) \Rightarrow \frac{\pi^{1/4}}{\sqrt{2}} e^{z^2/2} \left[1 - \phi \left(\frac{x_0 - z}{\sqrt{2}} \right) \right] \tag{4.23}$$

where ϕ is the error function.

$$\phi \left(\frac{x}{\sqrt{2}} \right) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-y^2/2} dy \tag{4.24}$$

The exponential has a transform

$$u(x) e^{-\lambda x} \Rightarrow \frac{\pi^{1/4}}{\sqrt{2}} e^{-|z|^2/4 - z^2/4} e^{(\lambda - z)^2/2} \left[1 - \phi \left(\frac{\lambda - z}{\sqrt{2}} \right) \right] \tag{4.25}$$

The monomials (see Appendix IV)

$$x^n \Rightarrow \frac{i^{-n} \pi^{1/4}}{\sqrt{2^{n-1}}} e^{z^2/4} H_n \left(\frac{iz}{\sqrt{2}} \right) \tag{4.26}$$

where H_n is the Hermite polynomial.

V. Bargman [11] and G. Folland [12] have shown that a reproducing kernel is also defined in Bargman space which is of the form

$$K(z, \bar{w}) = e^{z\bar{w}/2} \tag{4.27}$$

Any function $F(z)$ in Bargman space can be reproduced by

$$F(z) = \int K(z, \bar{w}) f(w) e^{-|w|^2/2} dw \tag{4.28}$$

The above transforms are tabulated in a table and shown separately. Many other transforms can also be recovered from the literature.

5. HERMITE FUNCTION EXPANSIONS OF COMMON SIGNALS

It is instructive to determine the Hermite function expansions of a number of commonly used functions in order to establish the character of Bargman space.

It has been shown (see G. Arfken [21]) that the generating function for the Hermite polynomial is

$$e^{-x^2+2tx} = \sum_{n=0}^{\infty} H_n(t) \frac{x^n}{n!} \tag{5.1}$$

The Hermite functions, therefore, can be generated by

$$e^{t^2/2} e^{-(x+t)^2} = \pi^{1/4} \sum_{n=0}^{\infty} \varphi_n(t) \sqrt{\frac{2^n}{n!}} x^n \tag{5.2}$$

A number of the elementary function expansions are given below: The delta function is

$$\delta(x - x_0) = \sum_{n=0}^{\infty} \varphi_n(x_0) \varphi_n(x) \tag{5.3}$$

The step function requires the evaluation of integrals of the form

$$\int_0^{\infty} e^{-x^2/2} H_n(x) dx$$

For n even G. Arfken [21] gives

$$\int_0^{\infty} e^{-x^2/2} H_n(x) dx = \sqrt{\frac{\pi}{2}} \frac{n!}{(n/2)!} \tag{5.4}$$

For n odd the result will be obtained by induction. Starting with

$$\int_0^\infty e^{-x^2/2} H_{2\mu+1}(x)dx = -\int_0^\infty e^{x^2/2} \frac{d^{2\mu+1}}{dx^{2\mu+1}} e^{-x^2} dx \tag{5.5}$$

integration by parts leads to

$$\int_0^\infty e^{-x^2/2} H_{2\mu+1}(x)dx = -H_{2\mu}(0) + \int_0^\infty x e^{-x^2/2} H_{2\mu}(x)dx \tag{5.6}$$

Since $\frac{dH_n(x)}{dx} = 2nH_{n-1}(x)$, an additional integration by parts leads to

$$\int_0^\infty e^{-x^2/2} H_{2\mu+1}(x)dx = 4\mu \int_0^\infty e^{-x^2/2} H_{2\mu-1}(x)dx \tag{5.7}$$

The recursion relation therefore is

$$\int_0^\infty e^{-x^2/2} H_{2\mu+1}(x)dx = 4\mu \int_0^\infty e^{-x^2/2} H_{2\mu-1}(x)dx \tag{5.8}$$

Since

$$\int_0^\infty e^{-x^2/2} H_1(x)dx = 1$$

and

$$\int_0^\infty e^{-x^2/2} H_3(x)dx = 4^1 1!2$$

and

$$\int_0^\infty e^{-x^2/2} H_5(x)dx = 4^2 2!2$$

by induction one has

$$\int_0^\infty e^{-x^2/2} H_{2\mu+1}(x)dt = 2^{2\mu+1} \mu! \tag{5.9}$$

Finally one obtains

$$\alpha_{2\mu} = \int_0^\infty \varphi_{2\mu}(x)dt = \frac{\pi^{1/4}}{\sqrt{2^{2\mu+1}}} \frac{\sqrt{2\mu!}}{\mu!} \tag{5.10}$$

and

$$\beta_{2\mu+1} = \int_0^\infty \varphi_{2\mu+1}(x)dx = \frac{\sqrt{2^{2\mu+1}}}{\pi^{1/4}} \frac{\mu!}{\sqrt{(2\mu+1)!}} \tag{5.11}$$

The step function is then finally given by

$$u(x) = \sum_{\mu=0}^{\infty} \alpha_{2\mu} \varphi_{2\mu}(x) + \sum_{\mu=0}^{\infty} \beta_{2\mu+1} \varphi_{2\mu+1}(x) \tag{5.12}$$

where the coefficients α_n and β_n are given by eqs. 5.10 and 5.11 respectively.

6. ELLIPTICAL FILTERS

The complete reconstruction of the signal requires that the phase space signature is known over the whole phase plane. Reconstructing an estimate of the signal from a limited portions of the space plane may, in some cases, be sufficient and may also point out certain features which serve to identify and distinguish the target from a larger class of similar objects.

Following Ramanathan and Topiwala [13] and I. Daubechies [8,14] the filter is defined as accessing only a finite portion of the phase plane having an area S and is given by

$$P_S f(x) = \iint_S g^{(z)} F(z) dp dq, \quad S_R \in R^2 \tag{6.1}$$

The shape of filtering area S is chosen so as to extract a portion of the signal energy which highlights only specific features of interest to the observer.

Ramanathan and Topiwala [13] and I. Daubechies [8] have shown that the operator P_S is trace class and consequently has eigenfunctions and their corresponding eigenvalues.

$$P_S \varphi_n(x) = \lambda_n \varphi_n(x) \tag{6.2}$$

They have also pointed out that the degrees of freedom (i.e. Approximately the number of terms) required to approximate a function within a finite area S is given by the expression.

$$\text{Degrees of freedom } \approx \frac{S}{2\pi} \tag{6.3}$$

This is a rather important result because it provides a simple measure of the complexity of the signal. In general the determination of the eigenfunctions is quite involved but doable. For the simple case, however, of S being a circular disk, I. Daubechie has shown [14] that the filtering operator commutes with the Hamiltonian of the harmonic oscillator. For this special case only they share the eigenfunctions. They are the Hermite functions given by equation 2.8 of the previous section. One has, therefore

$$P_{S_R}\varphi_n(x) = \lambda_n\varphi_n(x), \quad S_R \in R^2,\ p^2 + q^2 = zz^* \leq R \tag{6.4}$$

Functions φ_n are the Hermite functions and the eigenvalues are

$$\lambda_n = \frac{1}{n!}\gamma(n+1, R^2/2) \tag{6.5}$$

γ is the incomplete gamma function

R is the radius of the circular disk

The Filtered version of the signal in the time domain is

$$P_{S_R}f(x) \approx \sum_{n=0}^{R^2/2} \lambda_n\langle\varphi_n(x), f(x)\rangle\varphi_n(x) \tag{6.6}$$

In Phase Space one similarly obtains

$$P_{S_R}F(z) \approx \sum_{n=0}^{R^2/2} \lambda_n\langle\varphi_n(x), f(x)\rangle\zeta_n(z) \tag{6.7}$$

The center of the filter disk can be transported in phase plane in any location z_0 by using the modified Hermite functions

$$\varphi_n^{z_0}(x) = e^{ip_0x}\varphi_n(x - q_0) \Rightarrow \zeta_n(z - z_0) \tag{6.8}$$

where $z_0 = q_0 - ip_0$.

I. Daubechies [8] and Folland [12] has shown that the filter region can be transformed from a circular disc to an elliptical region defined by

$$\left(\frac{p}{\alpha}\right)^2 + (\alpha q)^2 = R^2 \tag{6.9}$$

where α is the stretching factor.

The phase space transform can be carried out as before by considering a dilated analyzing function

$$g^{(z_\alpha)} = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-|z_a|^2/4 - z_\alpha^2/4} e^{-x^2/2} e^{z_\alpha x} \tag{6.10}$$

where

$$z_\alpha = \alpha q - i\frac{p}{\alpha} \tag{6.11}$$

The Bargman transform is then

$$B_\alpha(z_\alpha) = \int_{-\infty}^{\infty} B_a(z_\alpha, x) f(x) dx \tag{6.12}$$

where the kernel is

$$B_a(z_\alpha, x) = e^{x z_\alpha - x^2/2 - z_\alpha^2/2} \tag{6.13}$$

The corresponding eigenfunctions become

$$\varphi_n^{(\alpha)}(x) = \frac{1}{\sqrt{\alpha}}\, \varphi_n\left(\frac{x}{\alpha}\right) \Rightarrow \zeta_n(z_\alpha) \tag{6.14}$$

G. Folland [12] calls this the FBI transform after Fourier Bros and Iagolnitzer. I. Daubechies [8] has shown that the new filter operator acting on the elliptical region shares the same eigenvalues as the one acting on the spherical disk. The filtered signal becomes

$$P_S^\alpha f(x) = \sum_{n=1}^{R^2/2} \lambda_n \langle f, \varphi_n^{(a)}(x)\rangle \varphi_n^{(a)}(x) \tag{6.15}$$

This is a very useful form because in many cases the gaussian window can be adjusted to capture the important features of the signal.

7. APPLICATIONS

From the many possible applications a few have been selected below which demonstrate the proposed analytical techniques.

a. Scattering by a dielectric slab.

D. Dudley [22] has shown that scattered signal from a dielectric slab has the form

$$\Gamma = -R - (1 - R^2)e^{i\omega t}\sum_{n=0}^{\infty}(-R)^n e^{-i\omega t} \tag{7.1}$$

where Γ is the round trip travel time in the slab, R is the reflection coefficient.

In the time domain one obtains

$$\Gamma(t) = -R\delta(t) - (1 - R^2)\sum_{n=0}^{\infty}(-R)^n\delta(t + (n-1)T) \tag{7.2}$$

The Bargman transform is

$$\Gamma(t) \Rightarrow Re^{-z^2/4} - (1 - R^2)e^{-z^2/4}\sum_{n=0}^{\infty}(-R)^n e^{z(n-1)T} e^{-(n-1)^2 T/2} \tag{7.3}$$

Figure 1 shows a contour plot of the magnitude of the Phase Space transform as obtained from equations 7.2 and 7.3. A disk with a relatively large footprint corresponding to 113 degrees of freedom was placed with its center at $t = 0$. This is the time when the second pulse arrives from the slab. The support of the filter is large enough to capture at least three of the infinite pulse train originating from the slab. Figure 2 shows a similar filter with a smaller footprint corresponding to 50 degrees of freedom. Only the second pulse is captured.

b. Scattering from a sphere.

The back scattering amplitude of the sphere is given by the well known Mie series which are given by J. Bowman [23]

$$E^s(k)\frac{e^{ikr}}{kr}S(k) \tag{7.4}$$

where

$$S(k) = -i\sum_{n=1}^{\infty}(-1)^n\left(n + \frac{1}{2}\right)(b_n - a_n) \tag{7.5}$$

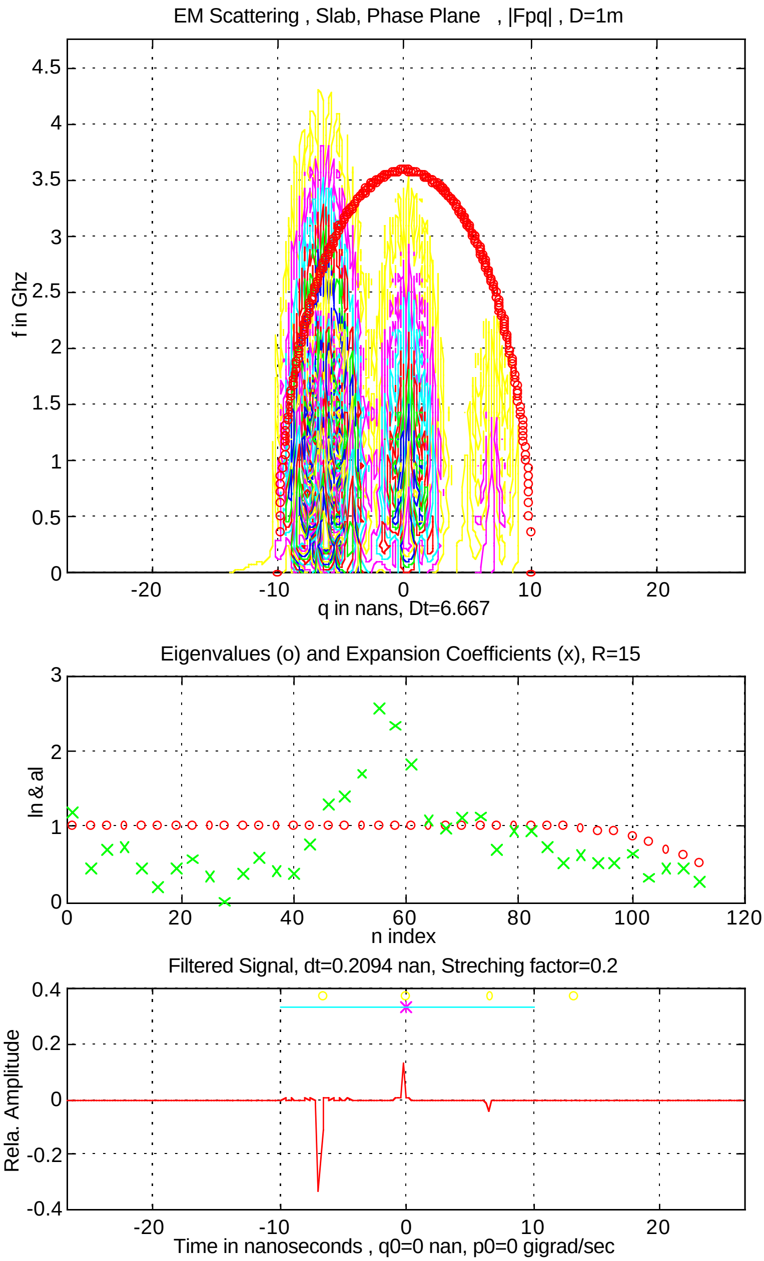

Figure 1. Electromagnetic scattering from slab. Filtered signal with $N = 112$ degrees of freedom. Reflection coefficient $R = 0.333$. Thickness $D = 1\,\mathrm{m}$. Circles '$\circ$' indicate eigenvalues λ_n. Crosses '$\times$' indicate expansion coefficients.

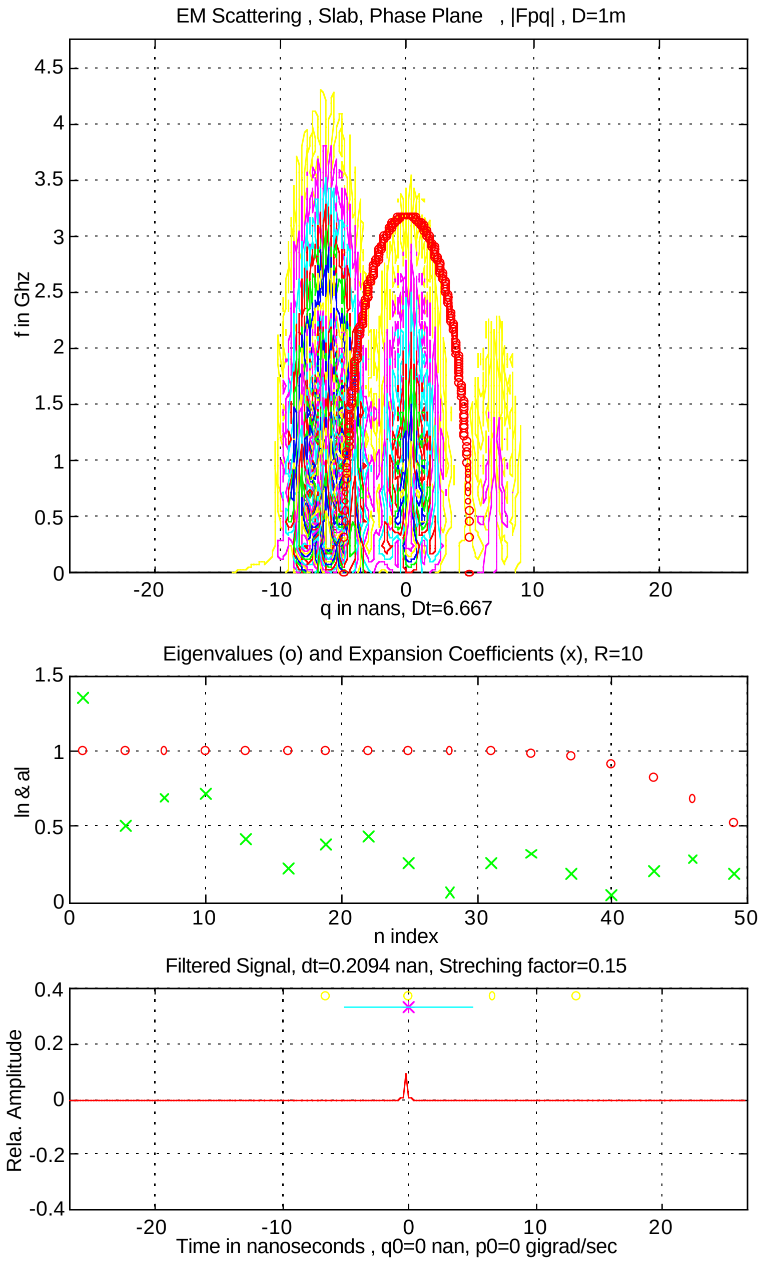

Figure 2. Electromagnetic scattering from slab. Filtered signal with $N = 50$ degrees of freedom. Reflection coefficient $R = 0.333$. Thickness $D = 1\,\mathrm{m}$.

and

$$a_n = \frac{j_n(x)}{h_n^{(1)}(x)} \tag{7.6}$$

$$b_n = \frac{d(xj_n(x))/dx}{d(xh_n^{(1)}(x))/dx} \tag{7.7}$$

$$x = ka$$

k is the wave number and a is the radius of the sphere.

Figure 3 shows a contour plot of the magnitude of the Phase Space transform as obtained from equation 7.5. A disk with a relatively large footprint corresponding to 113 degrees of freedom was placed with its center at $t = 0$. The support of the filter is large enough to capture at least the specular impulse function and the first creeping wave contribution. Figure 4 shows a similar filter with a smaller footprint corresponding to 50 degrees of freedom. The center of the filter is displaced at $t = \pi$ nan. The filter captures only the first creeping wave.

8. DISCUSSION OF RESULTS

The Bargman transforms (see Table 1) were originally introduced in the physics community and were closely connected with the Weyl Heisenberg group that served as one of the foundations of modern quantum mechanics. It has found many applications notable of which are the ones in Quantum Optics. Our interest in these transforms lies in that they are widowed Fourier transforms and as such they represent a forward step in the traditional Fourier analysis of transients. They give rise to a moving windowed Fourier transform or as they are presently called to time frequency analysis.

The significant property of the transforms is that the Bargman-Fock space is a space of entire analytic functions. Entire functions have no poles and are analytic in the whole z plane. They can be represented as series of monomials of the form $\sum \beta_n z^n$, β_n is a constant. This opens a multitude of possibilities which although have not been fully exploited in this paper they remain open for future work. They are analytic interpolation, extrapolation and the reconstruction of the signal from limited data etc. The complex Bargman plane can attain the same significance as the complex ω plane. This is a rich area for future work.

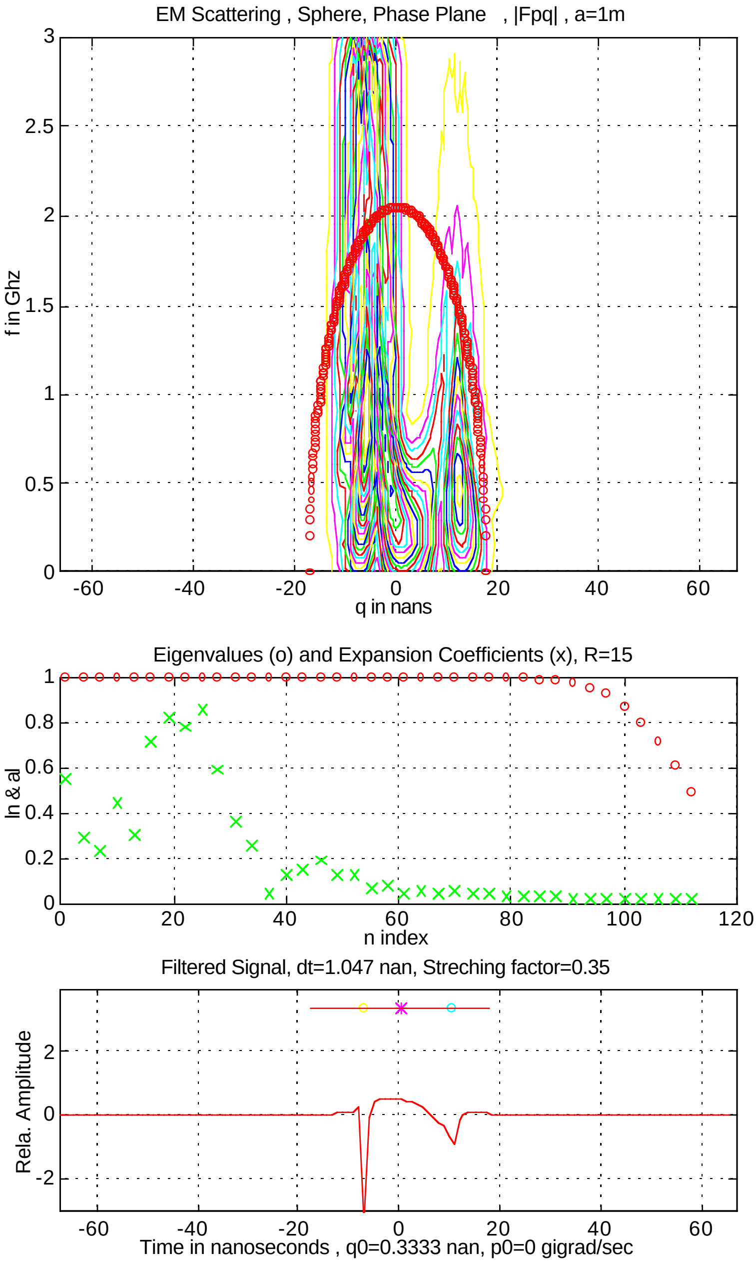

Figure 3. Electromagnetic scattering from sphere. Filtered signal with $N = 112$ degrees of freedom. Radius $a = 1$ m.

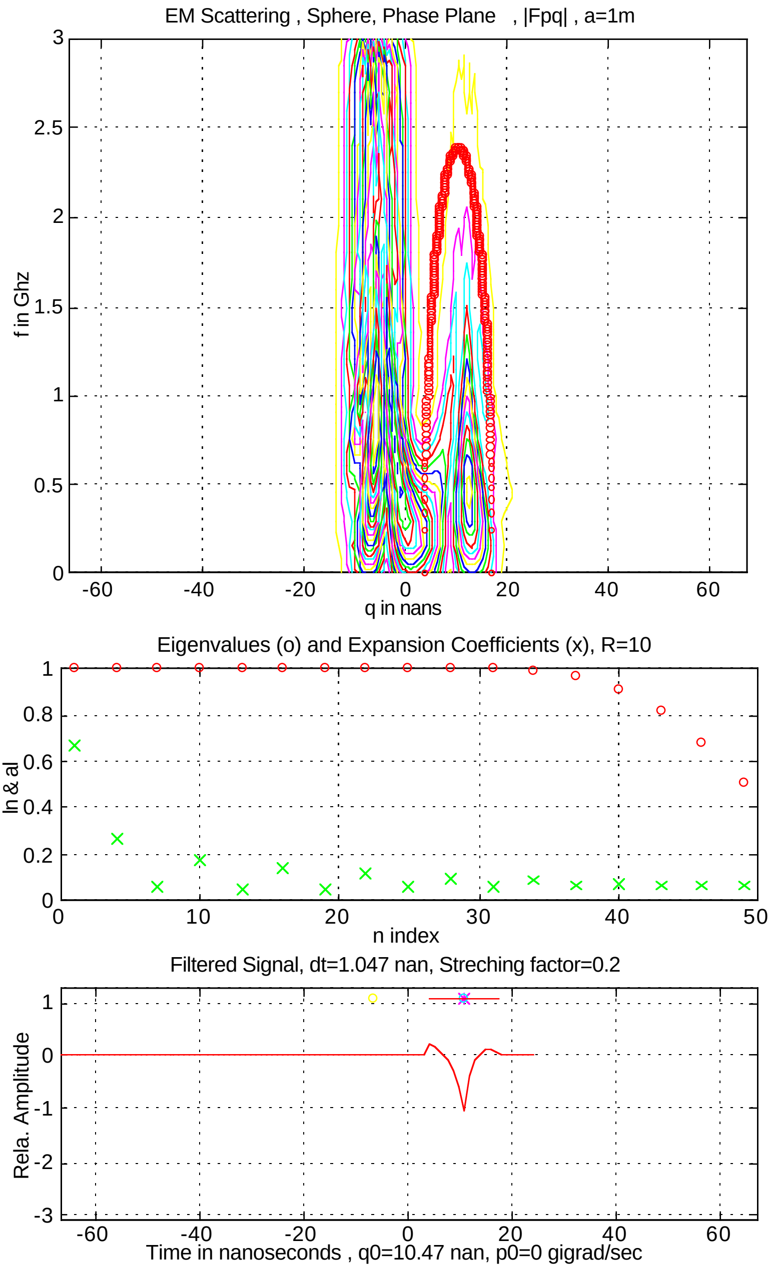

Figure 4. Electromagnetic scattering from sphere. Filtered signal with $N = 50$ degrees of freedom. Filtered is displaced by π nans. Radius $a = 1\,\mathrm{m}$.

Function	**Transform**
$f(x)$	Bf
$\delta(x-x_0)$	$\frac{1}{\pi^{1/4}}e^{-z^2/4}e^{-x_0^2/2}e^{zx_0}$
$\sum_{n=-\infty}^{\infty}\delta(x-nx_0)$	$\frac{e^{-z^2/4}}{\pi^{1/4}}\theta_3(e^{-x_0^2/4},-izx_0/2)$
$\sum_{n=-\infty}^{\infty}(-1)^n\delta(x-nx_0)$	$\frac{e^{-z^2/4}}{\pi^{1/4}}\theta_4(e^{-x_0^2/4},-izx_0/2)$
$u(x-x_0)$	$\frac{\pi^{1/4}}{\sqrt{2}}e^{z^2/2}\left[1-\phi\left(\frac{x_0-z}{\sqrt{2}}\right)\right]$
$u(x)e^{-\lambda x}$	$\frac{\pi^{1/4}}{\sqrt{2}}e^{-z^2/4}e^{(\lambda-z)^2/2}\left[1-\phi\left(\frac{\lambda-z}{\sqrt{2}}\right)\right]$
x^n	$\frac{i^{-n}\pi^{1/4}}{\sqrt{2^{n-1}}}e^{z^2/4}H_n\left(\frac{-iz}{\sqrt{2}}\right)$
$\varphi_n(x)$	$\zeta_n(z)=\frac{z^n}{\sqrt{2^n n!}}$
$\varphi_n^{(\alpha)}(x)$	$\zeta_n^{(\alpha)}(z_\alpha),\ z_\alpha=\alpha q-ip/\alpha$

Operator	**Transported Operator**
O	BOB^{-1}
x	$\frac{d}{dz}+\frac{z}{2}$
$\frac{d}{dx}$	$\frac{d}{dz}-\frac{z}{2}$
$g(x)=\int f(x)dx$	$G(z)=e^{z^2/4}\left[\int e^{-z^2/4}F(z)dz+C\right]$, C is a constant
a^*	$\frac{z}{\sqrt{2}}$
a	$\sqrt{2}\frac{d}{dz}$
a^*a	$z\frac{d}{dz}$
$H=a^*a+\frac{1}{2}$	$z\frac{d}{dz}+\frac{1}{2}$
$\rho(p_0,q_0)=\rho(w)$	$\beta(w)\to\beta(w)F(z)=e^{\|w\|^2/4+(z\bar{w}+\bar{z}w)/4}F(z+w)$
$H\varphi_n(x)$	$(n+1/2)\zeta_n(z)$
$a^*\varphi_n(x)$	$\sqrt{n+1}\zeta_{n+1}(z)$
$a\varphi_n(x)$	$\sqrt{n}\zeta_{n-1}(z)$

Table 1. Bargman Transforms.

As was pointed out in the text the analyzing function g_0 was the ground state of the Harmonic oscillator. This was the original choice dictated by applications in Physics and in particular in quantum optics. The analyzing function can be any function in L^2. The choices for new physically significant analyzing functions are enormous and have hardly been investigated.

The filtering techniques are very effective and as shown by the examples easy to implement. The proposed filtering techniques can be implemented in real time by taking inner products of the return signal (impulse response) with the hermite functions φ_n. The filtering takes place by weighting the inner products with the appropriate eigenvalues as demonstrated in eqs 6.5 and 6.6. The filtering techniques were very effective in isolating features such as the creeping waves from a sphere and single reflected pulses from a slab.

The Bargman analysis has to be used with a certain amount of caution, however, for transients which are zero before a certain observation time. The time windows have to be carefully chosen to avoid convergence problems. These questions have been addressed by B. Frielander and B. Porat [24] who have developed a new formulation that overcomes these problems.

APPENDIX I

The transform of the operator x is given by the expression

$$\begin{aligned} B(xf(x))(z) =& \frac{1}{\pi^{1/4}} e^{-z^2/4} \int_{-\infty}^{\infty} e^{zx-x^2/2} [xf(x)] dx && \text{(AI.1)} \\ =& \frac{1}{\pi^{1/4}} e^{-z^2/4} \int_{-\infty}^{\infty} \left[\frac{d}{dz} e^{zx}\right] e^{-x^2/2} f(x) dx && \text{(AI.2)} \\ =& \frac{1}{\pi^{1/4}} e^{-z^2/4} \frac{d}{dz} e^{z^2/4} \left[e^{-z^2/4} \int_{-\infty}^{\infty} e^{zx} e^{-x^2/2} f(x) dx\right] && \text{(AI.3)} \\ =& e^{-z^2/4} \frac{d}{dz} e^{z^2/4} [F(z)] && \text{(AI.4)} \end{aligned}$$

with

$$F(z) = \frac{1}{\pi^{1/4}} e^{-z^2/4} \int_{-\infty}^{\infty} e^{zx} e^{-x^2/2} f(x) dx \qquad \text{(AI.5)}$$

The integral therefore becomes

$$B(xf(x))(z) = \left(\frac{d}{dz} + \frac{z}{2}\right) F(z) \tag{AI.6}$$

A similar derivation shows that

$$B\left(\frac{d}{dx}f(x)\right)(z) = \frac{1}{\pi^{1/4}} e^{-z^2/4} \int_{-\infty}^{\infty} e^{zx-x^2/2} \left[\frac{d}{dx}f(x)\right] dx \tag{AI.7}$$

Integrating by parts one obtains

$$B\left(\frac{d}{dx}f(x)\right)(z) = -\frac{1}{\pi^{1/4}} e^{-z^2/4} \int_{-\infty}^{\infty} \frac{d}{dx}\left[e^{zx-x^2/2}\right] f(x)dx \tag{AI.8}$$

$$= -\frac{1}{\pi^{1/4}} e^{-z^2/4} \int_{-\infty}^{\infty} (z-x)\left[e^{zx-x^2/2}\right] f(x)dx \tag{AI.9}$$

using the result derived above one finally obtains

$$B\left(\frac{d}{dx}f(x)\right)(z) = \left(\frac{d}{dz} - \frac{z}{2}\right) F(z) \tag{AI.10}$$

APPENDIX II

Consider the integral of a function

$$g(x) = \int f(x)dx \tag{AII.1}$$

From Appendix I the derivative of $g(x)$ is

$$\frac{d}{dx}g(x) \to \left(\frac{d}{dz} - \frac{z}{2}\right) G(z) \tag{AII.2}$$

One has therefore

$$\frac{d}{dz}G(z) - \frac{z}{2}G(z) = F(z) \tag{AII.3}$$

The above differential equation is similar to the one

$$\frac{dy(x)}{dx} + f(x)y(x) = r(x) \tag{AII.4}$$

which has a solution

$$y(x) = e^{-h(x)} \left[\int e^{h(x)} r(x) dx + C \right] \tag{AII.5}$$

Using this result the finally we obtain

$$G(z) = e^{z^2/4} \left[\int e^{-z^2/4} F(z) dz + C \right] \tag{AII.6}$$

where C is an integration constant.

APPENDIX III

The delta function series gives by simple substitution in eq. 4.19 the following results

$$\sum_{n=-\infty}^{\infty} \delta(x - nx_0) \Rightarrow \frac{e^{-z^2/4}}{\pi^{1/4}} \sum_{n=-\infty}^{\infty} e^{-x_0^2 n^2/2} e^{nzx_0} \tag{AIII.1}$$

also

$$\sum_{n=-\infty}^{\infty} (-1)^n \delta(x - nx_0) \Rightarrow \frac{e^{-z^2/4}}{\pi^{1/4}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-x_0^2 n^2/2} e^{nzx_0} \tag{AIII.2}$$

Comparing the above results with the definitions for the theta functions (see E. Rainville [20]) which are

$$\theta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz} \tag{AIII.3}$$

$$\theta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} \tag{AIII.4}$$

one easily derives equations 4.20 and 4.21.

APPENDIX IV

The Bargman transform of the monomial x^n is given by

$$B(x^n) = \frac{e^{-z^2/4}}{\pi^{1/4}} \int_{-\infty}^{\infty} x^n e^{zx} e^{-x^2/2} dx \tag{AIV.1}$$

Rearranging the exponent we have the integral become

$$B(x^n) = \frac{e^{z^2/4}}{\pi^{1/4}} \int_{-\infty}^{\infty} x^n e^{-(x-z)^2/2} dx \tag{AIV.2}$$

or

$$B(x^n) = \frac{e^{z^2/4}}{\pi^{1/4}} \int_{-\infty}^{\infty} (\sqrt{2}x)^n e^{-\left(x-\frac{z}{\sqrt{2}}\right)^2} \sqrt{2} dx \tag{AIV.3}$$

and finally becomes

$$B(x^n) = \frac{e^{z^2/4}}{\pi^{1/4}} 2^{(n+1)/2} \int_{-\infty}^{\infty} x^n e^{-\left(x-\frac{z}{\sqrt{2}}\right)^2} dx \tag{AIV.4}$$

Using the identity

$$B(x^n) = \int_{-\infty}^{\infty} x^n e^{-(x-\beta)^2} dx = \frac{\sqrt{\pi}}{(2i)^n} H_n(i\beta) \tag{AIV.5}$$

from I. Gradshteyn and I. Ryzhik [25] we finally obtain

$$B(x^n) = \frac{\sqrt{2}}{(i\sqrt{2})^n} \frac{e^{z^2/4}}{\pi^{-1/4}} H_n\left(\frac{iz}{\sqrt{2}}\right) \tag{AIV.6}$$

REFERENCES

1. Goldstein, H., *Classical Mechanics,* Addison Wesley, Reading Mass, 1950.
2. Klauder, J. R., and E. C. G. Sudarshan, *Fundamentals of Quantum Optics,* W.B. Benjamin, Inc., 1968.
3. Klauder, J. R., and B. S. Skagerstam, *Coherent States,* World Scientific, Singapore, 1985.
4. Wells, B. "The role of symmetry in modern physics problems," in *Electromagnetic Symmetry,* edited by C. Baum and H. Kritikos, Taylor Francis New York, 1995.
5. Cloude, S. R., "Lie groups in electromagnetic wave propagation and scattering," in *Electromagnetic Symmetry,* edited by C. Baum and H. Kritikos. Taylor Francis New York, 1995.
6. Perelomov, A., *Generalized Coherent States and their Applications,* Springer-Verlag, Berlin, 1986.
7. Elliot, J. P., and P. G. Dawber, *Symmetry in Physics,* Vol. I,II, Oxford University Press, New York, 1990.

8. Daubechies, I., *Ten Lectures on Wavelets,* SIAM, Philadelphia, 1992.
9. Meyer, Y., *Algorithms and Applications,* SIAM, Philadelphia, 1993.
10. Heil, C., and D. Walnut, "Continuous and discrete wavelet transforms," SIAM Rev. 31, 626-666,1989.
11. Bargman, V., "On a Hilbert space of analytic functions and associated integral transform," *Communications of Pure and Applied Mathematics,* Vol XIV, 187-214,1961.
12. Folland, G. B., *Harmonic Analysis in Phase Space,* Princeton University Press, Princeton 1989.
13. Ramanathan, J. and P. Topiwala, "Time-frequency localization via the Weyl correspondence," *SIAM J. Mat. Anal.,* Vol. 24. No. 5., 1378-1393., Sept. 1993.
14. Daubechies, I., "Time-frequency localization operators: A geometric phase space approach," *IEEE Transactions on Information Theory,* Vol. 34, No. 4, July 1988.
15. Teti, J. G., Jr. and H. N. Kritikos, "SAR ocean image decomposition using the Gabor expansion," *IEEE Transactions on Geoscience and Remote Sensing,* vol. 30, no. 1, 192-96, January 1992.
16. Teti, J. G., Jr. and H. N. Kritikos, "SAR ocean image representation using wavelets," *IEEE Transactions on Geoscience and Remote Sensing,* vol. 30, no. 5, 1089-94, September 1992.
17. Sattinger, D. H. and O. L. Weaver, *Lie Groups and Algebras with Applications to Physics Geometry, and Mechanics,* Springer Verlag, New York, 1991.
18. Bargman, V., "On a Hilbert space of analytic functions and an associated integral transform Part II. A family or related function spaces application to distribution theory," *Communications of Pure and Applied Mathematics,* Vol XX, 1-101, 1967.
19. Daubechies, I., "Frames in the Bargman space of entire functions," *Communications of Pure and Applied Mathematics,* Vol. XLI, 51-164 (1988).
20. Rainville, R. *Special Functions,* MaxMillan, New York, 1960.
21. Arfken, G., *Mathematical Methods for Physicists,* Academic Press, New York 1970.
22. Dudley, D. G., "Progress in the identification of electromagnetic system, *IEE Antennas and Propagation Society Newsletter,* August 1988.
23. Bowman, J. J. et al., *Electromagnetic and Acoustic Scattering by Simple Shapes,* North Holland Publishing Co., Amsterdam, 1969.

24. Frielander, B. and B. Porat, "Detection of transient signals by the Gabor representation," *IEEE Trans. ASSP,* Vol. 37, No. 2, 169-170, Feb., 1989.
25. Gradsheteyn, I. S., and I. M. Ryzhik, *Tables of Integrals Series and Products.*
26. Teti, J. G., Jr., W. A. Berger and R. P. Gorman, "A multifeature decision space approach to radar target identification," *IEEE Transactions on Aerospace and Electronic Systems,* vol. 32, No. 1, 480, January 1996.