Progress In Electromagnetics Research, *PIER 18, 245–260, 1998*

ELECTROMAGNETIC PULSES IN DISPERSIVE MEDIA

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1. INTRODUCTION

Sommerfeld [1] and Brillouin [2] were the first to analyse the propagation of harmonic pulses in a dispersive medium (see [3] for a recent discussion of the Sommerfeld-Brillouin analysis). There exist at least three reasons to revive this old problem. First the blossoming of digital technology leads to consider digital pulse propagation.Second the method of the steepest descents used to get the asymptotic behaviour of pulses is now superseded by a modern saddle point method. Finally the usual Fourier description of wave packets is not suited to this kind of problems and a Laplace description of pulses has to be used. Before explaining why, let us point out that the objective of this work is first to settle a procedure for tackling pulse propagation and second to analyse the general features of pulses along propagation.

A pulse of finite energy is described by a function null till some time t_0, with bounded variation for $t \geq t_0$. So, whatever the time variation of this function may be, either the function or a derivative must be discontinuous. That is, one has to look for a bounded solution of the wave equation with discontinuous boundary-initial data [1–5]. The Laplace transform was just developed to tackle these problems and it is only in some particular cases [6] that on can work with the Fourier description of waves. Note that for inverting Laplace transform-type integrals there exist numerical procedures [7] more performing that the fast Fourier transform.

Pulse velocity is an important question since in addition to the phase and group velocities one has also to consider wave front and signal velocities. We first remind how phase and group velocities are defined in the Fourier description of wave packets. A scalar pulse propagating along oz has the general form in which $f(k)$ is the excitation function

$$A(z,t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} f(k) \exp\{i[w(k)t - kz)]\} dk \tag{1}$$

The group velocity is valid [8, 9], for Fourier representations (1) that are confined to a narrow band of the spectrum. This suggests to look for a region of the (z,t)-space were the function $w(k)t - kz$ is stationary provided of course that this limitation supplies a reasonable approximation of $A(z,t)$. Such a domain exists at the saddle points of the integrand and we get for z, t, both large, more exactly for large t and z/t fixed an asymptotic approximation of $A(z,t)$ and it is only for this approximation that the concept of group velocity is valid.

Then, we assume that the excitation function has a pole at k_0: $f(k) = h(k)(k - k_0)^{-1}$ and that the exponent $w(k)t - kz \equiv tg(k)$ has a saddle point at k_s: $g'(k_s) = 0$. The different saddle point methods for the integration of (1) give the same first term for the asymptotic approximation and we get [6, 10]

$$A(z,t) \approx h(k_0) \exp\{itg(k_0)\} + [2\pi t g''(k_s)]^{-1/2} f(s) \exp\{ig(k_s)\} \tag{2}$$

The first term in (2) propagates with the phase velocity V_p and the second with the group velocity V_g and according to the definition of $g(k)$

$$V_p = w(k_0)k_0^{-1}, \qquad V_g = \partial_k w(k_s) \tag{3}$$

Note that V_p and V_g are defined respectively for the values of k at k_0 and k_s. In general the function $g(k)$ has many saddle points and

Brillouin [2] has introduced the concept of signal velocity V_s discussed later in this paper. In many cases $V_s = V_g$.

When $g(k)$ has several saddle points the classical method of the steepest descent for approximating (1) may present some difficulties since the paths of steepest descent may entangle some other singularities. And it is only recently that Oughstun and Al [3–5] using the modern saddle point method put forward by Olver [11, 12] were able to solve definitively the old Sommerfeld-Brillouin problem and to get a correct definition of the signal velocity.

We start this paper with the Laplace description of electromagnetic pulses in dispersive media.

2. LAPLACE DESCRIPTION OF ELECTROMAGNETIC PULSES

We consider an electromagnetic pulse originating at $z = 0$, at $t = 0$, propagating in the z-direction and solution of Maxwell's equations

$$
\begin{aligned}
c\partial_z E_y(z,t) - \partial_t B_x(z,t) &= 0 \\
c\partial_z H_x(z,t) - \partial_t D_y(z,t) &= 0
\end{aligned}
\tag{4}
$$

with similar equations for the components E_x, B_y, D_x, H_y of the electromagnetic field. This pulse propagates in a dispersive medium with the constitutive relations $B_x(z,t) = \mu H_x(z,t)$ and

$$
D_y(z,t) = \int_{-\infty}^{t} \beta(t-u) E_y(z,u)du \tag{5}
$$

The permeability μ is a constant while the permittivity kernel $\beta(t)$ is a monotonically decreasing function of t for $0 < t < \infty$.

Using the Laplace transform $f(s) = L[F(t)]$ we proved recently [13] that the solution of Eqs. (4) and (5) satisfying at $z = t = 0$ the initial-boundary conditions ($U(t)$ is the unit step function)

$$
H_x(z,0) = 0 \tag{6a}
$$

$$
E_y(0,t) = F(t)U(t) \tag{6b}
$$

is in the s-domain

$$
e_y(z,s) = e^{-sn(s)z/c} f(s)U(z) \qquad h_x(z,s) = -n(s)e_y(z,s) \tag{7}
$$

with $f(s) = L[F(t)]$, $\beta(s) = L[\beta(t)]$, $n^2(s) = \mu\beta(s)$.

But the refractive index must satisfy some conditions to make the solution (7) physically acceptable. First the velocity of the pulses (7) must be finite, in other words the second order partial differential equation satisfied by $e_y(z,s)$

$$\partial_z^2 e_y(z,s) - c^{-2} n^2(s) s^2 e_y(z,s) = 0 \tag{8}$$

must be hyperbolic. It results from Davis' work [14] that this condition is fulfilled if $n^2(s)$ is the quotient of two polynomials with real coefficients of the same degree

$$n^2(s) = p_0(s)/p_1(s), \quad \text{degree } p_0 = \text{degree } p_1 \tag{9}$$

In addition, the causality constraint requires that for $s = iw$, $n^2(s)$ satisfies the Kramers-Kronig relations [9, 10].

Then, omitting the subscript y the pulse $E(z,t)$ in the time domain is the inverse Laplace transform of the expression (7) obtained from the Bromwich integral $(k(s) = sn(s))$

$$E(z,t) = (2\pi i)^{-1} \int_{Br} e^{[st - k(s)z/c]} f(s) ds \tag{10}$$

the contour Br is a straight line from $L - i\infty$ to $L + i\infty$ where $L \geq 0$ is a real number and all the singularities of the integrand are on the left of L.

According to (9) as $s \Rightarrow \pm\infty$, $n(s) \Rightarrow n_0$. If now $t - n_0 z/c < 0$, the contour in the Bromwich integral (10) may be closed by a semi-circle in the right of infinite radius. This path excludes all the singularities of the integrand and consequently $E(z,t) = 0$. This proves that at a point z within the medium the field is zero as long as $t < n_0 z/c$ so that the wave front velocity is $V_w = c/n_0$. This means in particular that the unit step function $U(t - n_0 z/c)$ appears in the expression of $E(z,t)$. That V_w is constant is easy to understand since the medium cannot be dispersive before to be disturbed by the arrival of the pulse front.

In practice, the simple resonance Lorentz medium [8–10] represents quite well a standard dispersive medium, its refractive index is

$$n^2(s) = 1 + a^2(s^2 + ds + a_0^2)^{-1} \tag{11}$$

in which a, a_0, are respectively the electronic plasma frequency and the resonant frequency while $d > 0$ is a damping constant. This refractive index with $a_0 = 0$ is appropriate for dispersion in a metal or in the ionosphere. For propagation in a wave guide (and in a telegraph line in certain circumstances) the relation with $d = a_0 = 0$ is valid. one checks easily that the refractive index (11) satisfies the Davis and Kramers-Kronig relations.

In a series of papers [3–5], [15–17], Oughstun, Sherman, and Al use (11) to analyse the propagation of pulse-modulated sine waves of applied signal frequency w_c, that is with our notations they take as boundary condition (6b) $E(0, t) = u(t) \sin w_c t$ where $u(t)$ is the real-valued envelope of the input pulse at $z = 0$. In particular u(t) can be the unit step pulse $U(t)$, the rectangular pulse $U(t) - U(t-T)$ or the gaussian pulse $\exp[-(t-t_0)^2/T^2]$. With (11) the asymptotic approximations of (10) are so intricate that the excitation function $f(s)$ has to be simple enough to make calculations tractable.

As said in the introduction, our objective is to make people acquainted with electromagnetic pulse propagation in dispersive media specially for digital pulses. There is no interest to be bored with intricate mathematics and this leads us to work with the simplified refractive index

$$n^2(s) = 1 + a^2 s^{-2} \tag{12}$$

but per contra, we make no a-priori assumption on the boundary condition $E(0, t)$ that is on $f(s)$. We start with a discussion of two canonical pulses: Dirac and harmonic pulses.

3. DIRAC AND HARMONIC PULSES

3.1 Dirac Pulses

From now on, we use the following notations

$$k(s) = (s^2 + a^2)^{1/2}, \quad \beta = z/ct, \quad q = (1 - \beta^2)^{-1/2}, \quad g(s) = \beta k(s) - s \tag{13}$$

Then, for Dirac pulses $f(s) = 1$ and the Bromwich integral (10) becomes

$$E^\circ(z, t) = (2\pi i)^{-1} \int_{Br} e^{-tg(s)} ds \tag{14}$$

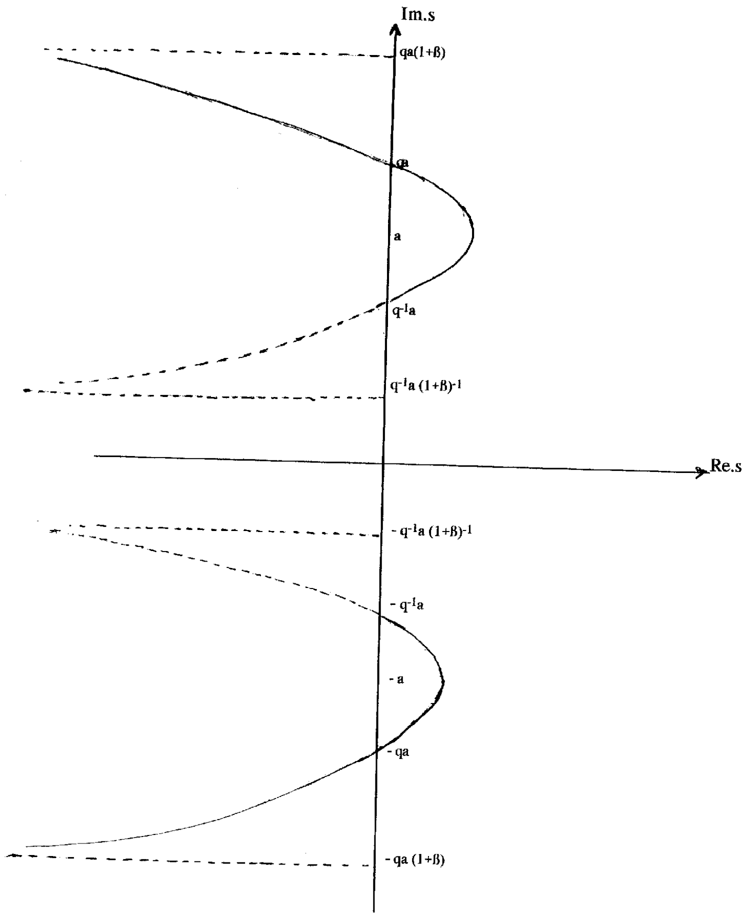

Figure 1. Paths of steepest descent for the function $g(s) = \beta(s^2 + a^2)^{1/2} - s$.

The integrand has two branch points at $s = \pm ia$, the saddle points are at $s_0 = \pm iqa$ and:

$$g(s_0) = \pm iq^{-1}a, \qquad g"(s_0) = (\pm i\beta^2 q^3 a)^{-1} \tag{15}$$

In the classical saddle point method [6, 10], the paths of steepest descent satisfy the condition $\Im g(s) = \Im g(s_0) = \pm q^{-1}a$ and they are shown on Fig. 1. They cross the branch line of k at $s = \pm iq^{-1}a$ and go over to the second sheet of the Riemann surface. For $\Re s \Rightarrow \infty$, the paths tend asymptotically to the lines $\pm iaq^{-1}(1+\beta)^{-1}$ on a sheet and $\pm iaq(1+\beta)$ on the other sheet.

In the Olver saddle point method [11, 12], the paths of integration satisfy the inequality $\Re\{g(s)-g(s_0)\}>0$ that is according to (13)

$$\Re\{(s^2+a^2)^{1/2}-s/\beta\}>0 \tag{16}$$

Writing $s=x+iy$, $G(x,y)=\Re\{(s^2+a^2)^{1/2}\}-x/\beta$ a simple calculation gives

$$s^2+a^2=re^{iu}, \quad r^2=b^2+4x^2y^2, \quad b=a^2+x^2-y^2, \quad \cos u=b/r \tag{17}$$

$$G(x,y)=\cos u/2\sqrt{r}-x/\beta=\pm(b+r)^{1/2}-x/\beta \tag{18}$$

For $x<0$ the condition (16) is satisfied if cos $u/2>0$. According to (17) this happens for $b>0$ but no path starting at saddle points is consistent with $b>0$. For $b<0$, cos $u/2>0$ if $\pi/2<u<\pi$ and one checks easily that the two branches of the hyperbola $y^2-x^2-q^2a^2=0$ in the half-plane $x<0$ are possible Olver paths (Fig. 2).

Then, deforming the Bromwich contour into the steepest descent or Olver paths, using (15) and the notations

$$B(z,t)=(\beta^2q^3a/2\pi t)^{1/2}, \quad v(z,t)=tq^{-1}a-\pi/4 \tag{19}$$

we get for large t [6, 10–12], the asymptotic approximation of $E^\circ(z,t)$ (see Eq. 2)

$$\begin{aligned} E^\circ(z,t) &\approx B(z,t)(e^{iv(z,t)}+e^{-iv(z,t)})U(t-z/c) \\ &\approx 2B(z,t)\cos v(z,t)U(t-z/c) \end{aligned} \tag{20}$$

which is reminiscent of the asymptotic approximation of the Bessel function J_0. In fact, E° has an exact analytical expression; let us write (14) $E^\circ(z,t)=L^{-1}\{e^{-zk(s)/c}\}$ where L^{-1} is the symbol of the inverse Laplace transform then, from Erdelyi's tables [18] we get

$$E^\circ(z,t)=-c\{\partial_z J_0[a(t^2-z^2/c^2)^{1/2}]\}U(t-z/c) \tag{21}$$

We postpone to the next section a comparison of the exact and approximate solutions.

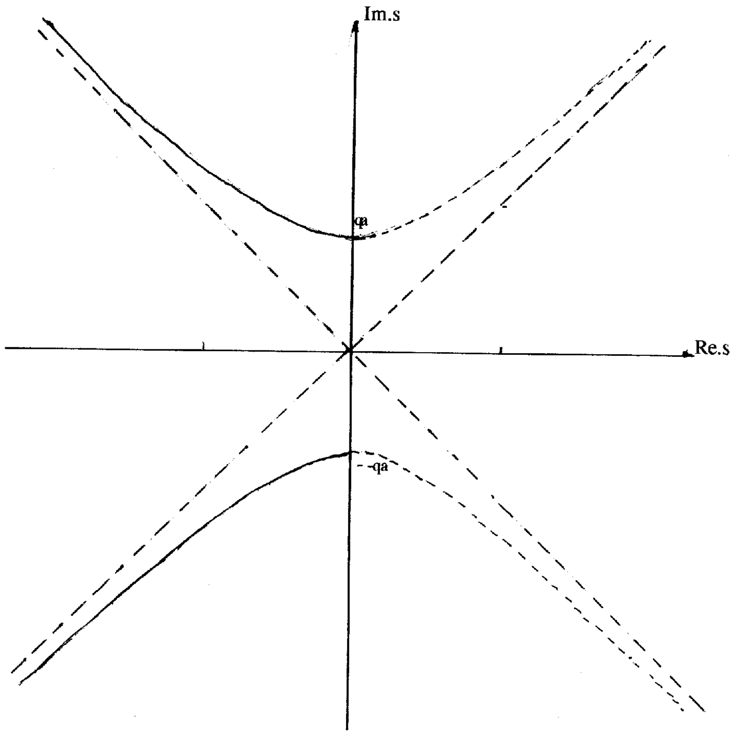

Figure 2. Olver path for the function $g(s) = \beta(s^2 + a^2)^{1/2} - s$.

3.2 Harmonic Pulses

For an harmonic pulse $F(t) = \exp(iw_c t)$, $f(s) = (s - iwc)^{-1} ds$, the Bromwich integral (10) becomes

$$E(z,t) = (2\pi i)^{-1} \int_{Br} e^{-tg(s)} (s - iw_c)^{-1} ds \qquad (22)$$

In addition to the branch points of $g(s)$ the integrand has a pole at $s = iw_c$ with real w_c . Then, using Olver or steepest descent paths (but in this case one has to assume $w_c > qa$ so that no deformation of the steepest descent path around the pole is necessary) we get as asymptotic approximation of (22) [10–12] with $h(z,t) = w_c t - zc^{-1}(w_c^2 - a^2)^{1/2}$

$$E(z,t) \approx e^{ih(z,t)}U[h(z,t)] - iB(z,t)$$
$$[(w_c - qa)^{-1}e^{iv(z,t)} + (w_c + qa)^{-1}e^{-iv(z,t)}]U(t - z/c) \tag{23}$$

The harmonic pulse (22) has also an exact expression. Using the well known property of the Laplace transform to change a convolution product into an ordinary product [19] we get from (14) and (22)

$$\begin{aligned} E(z,t) &= E^\circ(z,t) * \exp(iw_c t) \\ &= \exp(iw_c t) \int_0^t E^\circ(z,u) \exp(-iw_c u) du \end{aligned} \tag{24}$$

Substituting (21) into (24) gives with $u = (u^2 - z^2c^{-2})^{1/2}$

$$E(z,t) = -ce^{iw^\circ t}\partial_z \left\{ \int_{z/c}^t J_0[a(u^2 - z^2/c^2)^{1/2}] \exp(-iw_c u) du \right\} U(t - z/c) \tag{25}$$

This result is interesting from a mathematical point of view and the expression (25) could be used for numerical calculations. But from a physical point of view the approximation (23) is more useful because physics happens at the singularities of the integrand in the Bromwich integral and (25) has the disadvantage of not displaying the physical significance of saddle points and poles. At the opposite in (23) the first term corresponding to the pole of the excitation function represents a steady state that gradually builts up as the transient supplied by the second term and corresponding to the singularities of the exponent function dies out.

Now according to (13): $g(iw) = i[(ct)^{-1}z(w^2 - a^2)^{1/2} - w]$ and the phase and group velocities are

$$\begin{aligned} V_p &= z/ct = cw_c(w_c^2 - a^2)^{1/2} \\ V_g^{-1} &= ct/z = \partial w(w^2 - a^2)^{-1/2} = w(w^2 - a^2)^{-1/2} \end{aligned} \tag{26}$$

since as discussed in the introduction V_g is defined at the point where the phase $g(iw)$ is stationary. One sees at once that the steady state propagates with the phase velocity, let us now look for the V_s velocity of the transient. Generalizing Brillouin's definition. Oughstun and Al [3–5], [7], [15–17] define the signal velocity by the relation $V_s = \beta_0 c$ where β_0 is such as the real part of $ig(s)$ has the same value for the pole of $f(s)$ and for the saddle points of $g(s)$ that is

$$g(\pm iqa, \beta_0) = g(iw_c, \beta_0) \tag{27}$$

or according to (13): $q^{-1}a = \beta_0(w_c^2 - a^2)^{-1/2} - w_c$ supplying the equation

$$\beta_0 w_c^2 - 2\beta_0 w_c (w_c^2 - a^2)^{1/2} + w_c^2 - a^2 = 0 \tag{28}$$

with the solution $\beta_0 = w_c^{-1}(w_c^2 - a^2)^{1/2}$ so that making $w = w_c$ in (26) we get $V_s = V_g$. The transient propagates with the group velocity of a wave packet centered on the pulsation w° of the steady state. This result is often assumed as long as $V_g \leq V_p$,without justification [8–9].

Remark 1. The previous results can be generalized to complex w_c for a damped harmonic pulse.

Remark 2. With the refractive index (11) used by Sommerfeld and Brillouin the situation is more intricate since there are now four branch points. It is only recently that Oughstun and coworkers using the Olver saddle point method were able to describe correctly the famous Sommerfeld-Brillouin precursors whose velocity is no more the group velocity (for a complete discussion, see [3–5], [7], [15–17] where further references can be found).

4. ARBITRARY PULSES IN DISPERSIVE MEDIA

4.1 General Behaviour

The results obtained for the canonical pulses yield some general information on the behaviour of arbitrary pulses in dispersive media.

1. Still assuming propagation in the z-direction any pulse can be represented by a Bromwich integral

$$E(z,t) = (2\pi i)^{-1} \int_{Br} e^{-tg(s)} f(s) ds \tag{29}$$

$f(s)$ is the excitation function and $g(s) = \beta s n(s) - s, \beta = z/ct$ and $n(s)$ is the refractive index of the medium in which propagation takes place.

2. E is the sum of a steady-state E_s generated by the singularities of $f(s)$ and of a transient E_t due to the singularities of $g(s)$. We assume that $f(s)$ has only poles and using the Olver saddle point method, one may in many cases find integration paths such as the singularities of $f(s)$ and $g(s)$ do not interfere in the evaluation of E_s and E_t.

3. The steady state propagates with the phase velocity $V_p = cn^{-1}(s_0)$ where s_0 is a pure imaginary pole of $f(s)$. The group and signal velocities are defined by the relations

$$V_g(s) = c[n(s) + sn'(s)]^{-1} \tag{30a}$$

$$V_s = c\beta_0, \qquad \Re g(s_f, \beta_0) = \Re g(s_0, \beta_0) \tag{30b}$$

in which s_f is the point where the phase $g(iw)$ is stationary.When the solution of (30b) is $\beta_0 = c^{-1}V_g(s_0)$ the transient propagates with the group velocity.

4. The asymptotic pulse which is in fact what generally people observe is supplied by the saddle point approximations of (29). Other approximations are possible but they yield a less clear picture of pulses. Here is an example borrowed from [20].

The sound pressure $p(z,t)$ in a loud speaker horn is given by (29) with the expression (13) of $g(s)$ and the excitation function in which K is an amplitude

$$f(s) = Kw^2s^{-1}(s^2 + w^2)^{-1}[a - (a^2 + s^2)^{1/2}] \tag{31}$$

The singularities of the integrand are two simple poles at $s = \pm iw$ and two branch points at $s = \pm ia$. We assume $w > a$ so that the Bromwich contour reduces to the contour of Fig. 3. Then, with $m = (w^2 - a^2)^{1/2}$ the steady pressure is [20]

$$p_s(z,t) = -2c^{-1}K[a\cos(wt - mz/c) + m\sin(wt - mz/c)]U(t - mz/cw) \tag{32}$$

For $w^2 \gg a^2$, $f(s)$ can be approximate by $K \quad s^{-1}[a - (a^2 + s^2)^{1/2}]$ and with $u = (t^2 - z^2/c^2)^{1/2}$ the transient pressure is [20] in terms of Bessel functions

$$\begin{aligned} p_t(z,t) \approx 2c^{-1}aK\Big[(1 + az/c)\int_t^\infty u^{-1}J_1(u)dt \\ + a^2c^{-2}z^2\int_t^\infty u^{-2}J_2(u)dt\Big]U(t - z/c) \end{aligned} \tag{33}$$

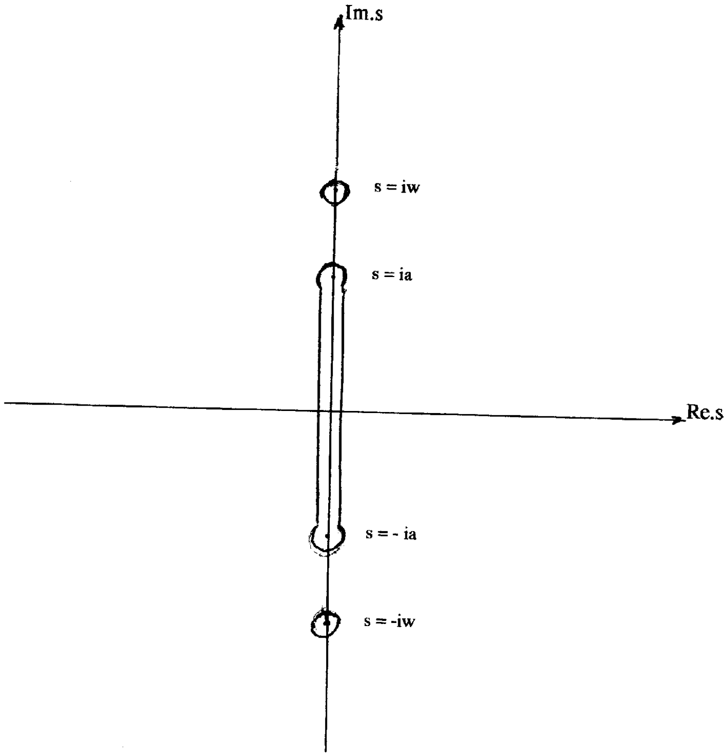

Figure 3. Bromwich contour for acoustic pressure in a horn.

No clear physical picture emerges from this approximation.

5. It was assumed that $f(s)$ has only poles, the situation is more intricate when $f(s)$ has branch points as for instance when the boundary condition (6b) is $E(0,t) = J_0(wt)U(t)$ where J_0 is the Bessel function so that $f(s) = (s^2 + w^2)^{1/2}$. The integrand in (29) has now four branch points at $s = \pm iw$ and $s = \pm ia$ and it is shown in [20] how to deform the Bromwich contour to get in this case the asymptotic approximation of (29). Because of its interest for the recently discovered Bessel beams [20] we shall discuss this approximation elsewhere.

4.2 Digital Pulses

As stated in the introduction, the blossoming of digital technology leads to consider the propagation of digital pulses in dispersive media.

So we take as excitation function $F(t)$ in the boundary condition (6b) a finite or infinite sequence of discontinuous pulses. We use in this section the notations (19). Here are some examples [22].

1. sequence of m Dirac pulses (ns-laser pulses)

$$F(t) = \sum_{j=0}^{m} \delta(t - jb), \qquad f(s) = (1 - e^{-2mbs})(1 - e - bs)^{-1} \tag{34}$$

From Sec. 3.1 we get at once, with cc denoting the complex conjugate quantity,

$$E(z,t) \approx \{B(z,t)(1 - e^{-2imqab})(1 - e^{-iqab})^{-1} e^{iv(z,t)} U(t - z/c)\} + \{cc\} \tag{35}$$

2. sequence of m positive square pulses

$$\begin{aligned} F(t) &= \sum_{0}^{m} \{U[t - (2j+1)b] - U[t - (2j+2)b]\} \\ f(s) &= s^{-1}(1 - e^{-2mbs})(1 + e{-}bs)^{-1} \end{aligned} \tag{36}$$

then,

$$E(z,t) \approx \{(iqa)^{-1} B(z,t)(1 - e^{-2imqab})(1 + e^{-iqab})^{-1} e^{iv(z,t)} U(t - z/c)\} - \{cc\} \tag{37}$$

3. sequence of alternative square pulses

$$\begin{aligned} F(t) &= \sum_{0}^{m} \{U[t - (2j+1)b] - U[t - (2j+2)b]\} \\ f(s) &= s^{-1}(1 - e^{-bs})(1 - e^{-2mbs})(e^{bs} + e{-}bs)^{-1} \end{aligned} \tag{38}$$

$$\begin{aligned} E(z,t) \approx \{2[iqa\cos(qab)]^{-1} B(z,t)(1 - e^{-iqab}) \\ (1 - e^{-2imqab}) e^{iv(z,t)} U(t - z/c)\} - \{cc\} \end{aligned} \tag{39}$$

4. infinite sequence of rectangular pulses whose amplitude decreases with time [19]

$$\begin{aligned} F(t) &= \sum_{0}^{\infty} [1/2 + (-1)^j / 2^{j+1}] \{U(tjb) - U[t - (j+1)b]\} \\ f(s) &= (2s)^{-1}(4 - e^{-bs})(2 + e{-}bs)^{-1} \end{aligned} \tag{40}$$

$f(s)$ has a pole at $s = 0$ and we get

$$E(z,t) \approx 1/2e^{-az/ct}U(t-z/c) + \{(2iqa)^{-1}B(z,t)(4-e^{-iqab}) \\ (2+e^{-iqab})^{-1}e^{iv(z,t)}U(t-z/c)\} - \{cc\} \tag{41}$$

These results are easily translated to other types of digital pulses. Of course for finite sequences there exists no steady state and the transients propagate with the velocity of light that is u ith the wave front velocity since the solution of (28) for infinite w is $\beta^\circ = 1$. This means that a medium with the refractive index (12) recovers its original structure between two successive pulses.This would not be the case for instance with the refractive index (11).

As previously, using the expression (21) of $E^\circ(z,t)$, the convolution product $E^\circ(z,t) * F(t)$ would supply the exact expression of the sequences $E(z,t)$.

5. DISCUSSION

The procedure for analysing the propagation of electromagnetic pulse is now obvious: write the Bromwich integral, locate the singularities of the integrand, use a saddle point method for approximating the Bromwich integral. The first two steps are easy but in the third one, the task to locate the saddle points and to find the solutions of the equation $g'(s) = 0$ may be difficult and require some approximations so that only approximate positions of the saddle points might be known. In addition, if one uses the method of steepest descents one has to check carefully whether the paths of steepest descent entangle other singularities. This could require some deformation of these paths with as consequence the introduction of further terms in the asymptotic approximation [10]. This difficulty does not generally exist in the Olver saddle point method but the construction of Olver paths which "can often be solved by an intelligent guess" [11] demands sometimes a lot of ingenuity.

The main difference between packets of time-harmonic waves and pulses is the existence in this last case of transients that propagate just behind the wave front and which are the main cause of the pulse distortion (see for example [3] for a discussion of the harmonic pulse distortion due to the Sommerfeld-Brillouin precursors). So for practical applications, the behaviour of transients as function of z and t is of the utmost importance and may require a huge quantity of numerical calculations [3–5, 7, 15–17]. An interesting example concerning the propagation of a radar pulse in sea water is given in [23]. It is shown that targets at considerably greater depths become detectable with a receiver that detects the arrival of the transients as well as that of the low frequency carrier.

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