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ELECTRODYNAMICS, TOPSY-TURVY SPECIAL RELATIVITY, AND GENERALIZED MINKOWSKI CONSTITUTIVE RELATIONS FOR LINEAR AND NONLINEAR SYSTEMS

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1. INTRODUCTION AND RATIONALE

Many electrodynamicists with an engineering orientation quite often present their immediate work without pointing out its relation to the foundations of the theory. This might be justified in a succinct professional article, but unfortunately it also permeates to the classroom too. While most educators agree that electrodynamics and special relativity should be presented as one integrated entity, the efficient tools to carry out the task in a reasonable time are lacking. It is therefore proposed here to present Special Relativity in a topsy-turvy manner, taking the postulates and the consequences of the theory, as presented by Einstein [1], and usually followed in all textbooks, and turn them around. This produces a much more coherent structure, and is easier to teach and to learn, as proven by the present author.

Another conceptual aspect arises when the spatiotemporal and spectral representations are compared. It is shown below that the space-time transformations, and the Doppler effect (i.e., frequency-wavenumber transformations) in the spectral domain, are essentially equivalent. One could therefore start electrodynamics and special relativity in the spectral (Fourier transform) representation space, and obtain the spatiotemporal domain expressions as consequences. This duality allows for more flexibility in the presentation and subsequent understanding of the theory, but also involves the crucial argument whether electrodynamics is fundamentally "time-domain" or "frequency-domain." This question came to the front with the massive amount of research on time-domain methods, currently appearing in professional periodicals. The present assertion is that electrodynamics is fundamentally a spectral, i.e., "frequency-domain" theory. This is due to the spatial and temporal dispersion existing for most materials, and is associated with the fact that electrodynamics, by its very nature, is a nonlocal and noninstantaneous theory, allowing for signals to propagate from one event, or world point (spatiotemporal point), to another with a finite speed. Thus only in nondispersive systems which allow local instantaneous interaction, a time-domain approach is exactly applicable.

In a natural way, electrodynamics and special relativity tie up to the question of the constitutive relations in various inertial frames. Assuming the constitutive relations to be given in the comoving frame of reference where the medium is at rest, what will be the relevant relations in another inertial system where the medium is in motion (so called "laboratory system")? In the case of nonlinear systems, the problem is even more complicated, because the very question of a first principle statement of the constitutive relations in the comoving frame of reference is lacking. It is attempted here to define general nonlinear Minkowski constitutive relations. The problem of stating a general approach towards this problem is even more important than the question of the range of validity of the constitutive relations involved in individual problems.

Some of the ideas appearing in the present study have been mentioned in earlier works, and are brought here into a coherent context, focusing on the points stated in the title above.

2. SPATIOTEMPORAL DOMAIN ELECTRODYNAMICS AND SPECIAL RELATIVITY MODELS

The statement of Maxwell's equations for the electromagnetic field in the contemporary conventional form and involving the MKS system of units appears in many books, see Stratton [2], and Kong [3], to name a few. This is not the form used by the originator Maxwell, or by later researchers. The historical background is described by Berkson [4].

Here we present the basic equations in the form:

$$\begin{aligned}
\partial_{\mathbf{x}} \times \mathbf{E} &= -\partial_t \mathbf{B} - \mathbf{j}_m \\
\partial_{\mathbf{x}} \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{j}_e \\
\partial_{\mathbf{x}} \cdot \mathbf{D} &= \rho_e \\
\partial_{\mathbf{x}} \cdot \mathbf{B} &= \rho_m
\end{aligned} \tag{1}$$

where $\partial_{\mathbf{x}}$ and ∂_t, denote the space derivative (Del), and the time derivative, operators, respectively, and all the fields are space and time dependent, e.g., $\mathbf{E} = \mathbf{E}(\mathbf{X})$. Here $\mathbf{X} = \mathbf{X}(\mathbf{x}, ict)$ symbolizes the spacetime dependence, actually $\mathbf{X}$ denotes the event (world point) in the sense of a Minkowski-space location vector, where c is the universal constant of the speed of light, and i is the unit imaginary complex number $i^2 = -1$. For symmetry and completeness, in the present representation, the Maxwell equations include the usual (index e) as well as the fictitious magnetic (index m) current and charge density sources.

Special relativity theory has been announced by Einstein [1], however he considers there only free space ("vacuum") electrodynamics.

Einstein's Special Relativity theory postulates (relevant statements are denoted by S):

(S–1). Light speed c is a universal constant observed in all inertial frames.

(S–2). Maxwell's equations provide the model or "law of nature" for describing the electromagnetic field, i.e., the theory recognizes (1) above.

(S–3). Maxwell's equations existing for all observers in inertial frames of reference have the same functional structure (henceforth: covariance). This means that if (1) exists for an observer in one frame of reference, in another inertial frame (the "primed" frame), Maxwell's equations have the form:

$$\begin{aligned}
&\partial_{\mathbf{x}'} \times \mathbf{E}' = -\partial_{t'}\mathbf{B}' - \mathbf{j}'_m \\
&\partial_{\mathbf{x}'} \times \mathbf{H}' = \partial_{t'}\mathbf{D}' + \mathbf{j}'_e \\
&\partial_{\mathbf{x}'} \cdot \mathbf{D}' = \rho'_e \\
&\partial_{\mathbf{x}'} \cdot \mathbf{B}' = \rho'_m
\end{aligned} \tag{2}$$

where now $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$, and the primed space-time coordinate system is denoted by $\mathbf{X}' = \mathbf{X}'(\mathbf{x}', ict')$, which is also a Minkowski location four-vecor.

The consequences of the above three postulates follow:

(S–i). From (1), i.e., the constancy of the speed of light, the Lorentz space-time transformations $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$ are developed in the form:

$$\begin{aligned}
&\mathbf{x}' = \tilde{\mathbf{U}} \cdot (\mathbf{x} - \mathbf{v}t) \\
&t' = \gamma(t - \mathbf{x} \cdot \mathbf{v}/c^2)
\end{aligned} \tag{3}$$

Here $\mathbf{v}$ is the velocity by which the primed frame of reference is moving, as observed from the unprimed frame of reference, and we define

$$\begin{aligned}
&\gamma = (1 - \beta^2)^{-1/2}, \quad \beta = v/c, \quad v = |\mathbf{v}|, \\
&\tilde{\mathbf{U}} = \tilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}}, \quad \hat{\mathbf{v}} = \mathbf{v}/v
\end{aligned} \tag{4}$$

where the tilde denotes dyadics and $\tilde{\mathbf{I}}$ is the unit dyadic. From this follows the transformation $\partial_{\mathbf{X}'} = \partial_{\mathbf{X}'}[\partial_{\mathbf{X}}]$ of the space-time differential operators,

$$\begin{aligned}
&\partial_{\mathbf{x}'} = \tilde{\mathbf{U}} \cdot (\partial_{\mathbf{x}} + \mathbf{v}\partial_t/c^2) \\
&\partial'_t = \gamma(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})
\end{aligned} \tag{5}$$

from which we establish that the four-gradient operator,

$$\partial_{\mathbf{X}} = \partial_{\mathbf{X}}\left(\partial_{\mathbf{x}}, -\frac{i}{c}\partial_t\right) \tag{6}$$

is a four-vector.

(S–ii). From the axioms (2), (3) above, the transformation formulas for the fields are derived in the following form:

$$\begin{aligned}
\mathbf{E}' &= \tilde{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
\mathbf{B}' &= \tilde{\mathbf{V}} \cdot (\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2) \\
\mathbf{D}' &= \tilde{\mathbf{V}} \cdot (\mathbf{D} + \mathbf{v} \times \mathbf{H}/c^2) \\
\mathbf{H}' &= \tilde{\mathbf{V}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \\
\tilde{\mathbf{V}} &= \gamma\tilde{\mathbf{I}} + (1-\gamma)\hat{\mathbf{v}}\hat{\mathbf{v}}
\end{aligned} \tag{7}$$

where $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$ and $\mathbf{E} = \mathbf{E}(\mathbf{X})$, etc., and $\mathbf{X}$, and $\mathbf{X}'$ are related by the Lorentz transformation $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$ given above. Similarly, for the sources we derive the transformation formulas:

$$\begin{aligned}
\mathbf{j}'_{e,m} &= \tilde{\mathbf{U}} \cdot (\mathbf{j}_{e,m} - \mathbf{v}\rho_{e,m}) \\
\rho'_{e,m} &= \gamma(\rho_{e,m} - \mathbf{v} \cdot \mathbf{j}_{e,m}/c^2)
\end{aligned} \tag{8}$$

for the corresponding e or m sources, respectively.

Topsy-turvy (T) Special Relativity is stated in inverse order:

(T–1) Instead of assuming the constancy of the speed of light (S-1 above), we assume the validity of the Lorentz transformation (S-i), i.e., (3), (4).

(T–2) Here too we start with the same postulate (S-2) on the validity of equation (1).

(T–3) We postulate the validity of the formulas for the transformations of fields as given by (7), (8), i.e., what above constituted (S-ii).

The consequences are:

(T–i) From (T-1) we derive the constancy of the speed of light, i.e., (S-1) in the first model.

(T–ii) From (T-2), (T-3) we derive the covariance of Maxwell's equations, i.e., (S-3) of the previous model.

One might argue that the present model loses the motivation for universality and simplicity, displayed in the S-model. While this is true, it is compensated by the fact that the T-model is much easier to handle in the classroom [5], and the S-model can be mentioned in retrospect, showing how the two models are equivalent.

3. SPECTRAL DOMAIN ELECTRODYNAMICS

In order to effect an algebraization of Maxwell equations, a four-dimensional Fourier transform is used, which we denote by:

$$f(\mathbf{K}) = \int (d^4\mathbf{X}) f(\mathbf{X}) e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{9}$$

together with its inverse transform

$$f(\mathbf{X}) = (2\pi)^{-4} \int (d^4\mathbf{K}) f(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} \tag{10}$$

Here $d^4\mathbf{K}$ implies a four-dimensional integration with respect to the components of $\mathbf{k}$ and the (angular) frequency ω, which are grouped into one quadruple $\mathbf{K}$. We use the same symbol f on both sides of a formula, and whether the $\mathbf{X}$ or the $\mathbf{K}$ space is meant becomes clear from its argument. From the definition of the Fourier transform and in order to have the convenient form of the phase of a plane wave, we choose for the exponent,

$$\mathbf{K}\cdot\mathbf{X} = \mathbf{k}\cdot\mathbf{x} - \omega t \tag{11}$$

which implies that

$$\mathbf{K} = (\mathbf{k}, \mathbf{i}\omega/\mathbf{c}) \tag{12}$$

but this does not automatically mean that $\mathbf{K}$ is a four-vector. However, by applying the four-gradient operator (6) to (10), we get

$$\partial_{\mathbf{X}} f(\mathbf{X}) = (2\pi)^{-4} \int (d^4\mathbf{K}) f(\mathbf{K}) i\mathbf{K} e^{i\mathbf{K}\cdot\mathbf{X}} \tag{13}$$

By inspection of (13) it is concluded that according to (6) the left hand side (13) is a four-vector, therefore on the right hand side $\mathbf{K}$ as well must be a four-vector. Consequently the relevant transformations $\mathbf{K}' = \mathbf{K}'[\mathbf{K}]$ corresponding to the Lorentz transformations (3), are now given by

$$\begin{aligned} \mathbf{k}' &= \tilde{\mathbf{U}}\cdot(\mathbf{k} - \mathbf{v}\omega/c^2) \\ \omega' &= \gamma(\omega - \mathbf{k}\cdot\mathbf{v}) \end{aligned} \tag{14}$$

This is the celebrated relativistic Doppler effect. Obviously this is a two way street: we could have started from the Doppler effect (14), and through the Fourier transformation arrive at the Lorentz transformation (3). Therefore, without losing its general properties, the theory of Special Relativity could have been started in the spectral domain.

At this point it is worthwhile to realize that indeed this was the case: Abraham [6], see also Pauli [7], before the advent of Einstein's theory, already derived the relativistically correct results for reflection by a moving mirror.

Applying (10) to (1) yields

$$\begin{aligned} i\mathbf{k}\times\mathbf{E} &= i\omega\mathbf{B}-\mathbf{j}_m \\ i\mathbf{k}\times\mathbf{H} &= -i\omega\mathbf{D}+\mathbf{j}_e \\ i\mathbf{k}\cdot\mathbf{D} &= \rho_e \\ i\mathbf{k}\cdot\mathbf{B} &= \rho_m \end{aligned} \tag{15}$$

where $\mathbf{E}=\mathbf{E}(\mathbf{K})$ etc. We can of course apply (1) also to (2) and obtain

$$\begin{aligned} i\mathbf{k}'\times\mathbf{E}' &= i\omega'\mathbf{B}'-\mathbf{j}'_m \\ i\mathbf{k}'\times\mathbf{H}' &= -i\omega'\mathbf{D}'+\mathbf{j}'_e \\ i\mathbf{k}'\cdot\mathbf{D}' &= \rho'_e \\ i\mathbf{k}'\cdot\mathbf{B}' &= \rho'_m \end{aligned} \tag{16}$$

and here $\mathbf{E}'=\mathbf{E}'(\mathbf{K}')$, etc. A cardinal question arising at this point is whether the field transformation formulas (7), (8) hold in the representation space $\mathbf{K}$, and in what sense? Transforming the two sides of the first equation (7), we now get,

$$\begin{aligned} \mathbf{E}'(\mathbf{X}') = (\tilde{\mathbf{V}})\cdot(\mathbf{E}(\mathbf{X})+\mathbf{v}\times\mathbf{B}(\mathbf{X})) &= (2\pi)^{-4}\int(d^4\mathbf{K}')\mathbf{E}'(\mathbf{K}')e^{i\mathbf{K}'\cdot\mathbf{X}'} \\ &= \tilde{\mathbf{V}}\cdot(2\pi)^{-4}\int(d^4\mathbf{K})(\mathbf{E}(\mathbf{K})+\mathbf{v}\times\mathbf{B}(\mathbf{K}))e^{i\mathbf{K}\cdot\mathbf{X}} \end{aligned} \tag{17}$$

and the question before us is whether the integrands are identical, which is not obvious, [8]. By identifying the dummy integration variables as the proper spectral domain variables, obeying $\mathbf{K}'=\mathbf{K}'[\mathbf{K}]$ as above in (14), the exponentials become identical. Furthermore, it is easily shown that the change of variable involves a Jacobian whose value is unity, hence we get

$$\int(d^4\mathbf{K})[\mathbf{E}'(\mathbf{K}')-\tilde{\mathbf{V}}\cdot(\mathbf{E}(\mathbf{K})+\mathbf{v}\times\mathbf{B}(\mathbf{K}))]e^{i\mathbf{K}\cdot\mathbf{X}}=0 \tag{18}$$

implying that in general the expression in brackets in the integrand vanishes, hence (7), (8) apply in the transform space too, provided (14) is understood.

4. CONSTITUTIVE RELATIONS IN THE COMOVING FRAME

So far we have not raised the question of deriving solutions for Maxwell's equations, or mentioned constitutive relations. It is also noted that up to this point the symmetry between the spatiotemporal and spectral domains is preserved, in the sense that the transformations are reciprocal and one can switch from one representation to the other without any loss of information.

Clearly Maxwell's equations (1) or (15) are indeterminate, in the sense that we have more variables than equations. Therefore we need to heuristically add an appropriate set of relations for the problem at hand, which will render the system determinate. Such equations, referred to as constitutive relations, depend on the physical nature of the materials involved and are not considered an integral part of the Maxwell equations. There are two methods by which constitutive relations enter our problem: (a) Constitutive relations can be postulated, and, (b) Constitutive relations can be derived. For example, a heuristic theoretical argument, or curve fitting applied to empirical results can suggest a model for constitutive relations. As an example of a derived constitutive relation, consider the case of a one particle model for a lossless, unmagnetized cold plasma (i.e., the simplest model for the ionosphere). In addition to (1) we assume the validity of Newton's law for a particle of charge q and mass m, moving with a velocity $\mathbf{v}$ in an electric field $\mathbf{E}$. After linearization (replacing of total time derivatives with partial ones) we obtain

$$q\mathbf{E} = m\partial_t \mathbf{v} \tag{19}$$

In order to continue we need either to transform into the spectral domain, using $\partial_t \mathbf{v} \Leftrightarrow -i\omega\mathbf{v}$, or define the inverse operator of the time derivative and solve for $\mathbf{v}$ in (19) in the form

$$\mathbf{v} = (q/m)\partial_t^{-1}\mathbf{E} \tag{20}$$

In general the calculus of such operators is quite complicated, see for example Felsen and Marcuvitz [9], who also present instructive examples of deriving constitutive operators for a one-component fluid model of a plasma field, in which more complicated operator calculus is involved. From (20) the current density is defined as $\mathbf{j} = Nq\mathbf{v} = (Nq^2/m)\partial_t^{-1}\mathbf{E}$ and substituted into (1), and after some manipulation, the first equation (1) attains the form

$$\begin{aligned} &\partial_{\mathbf{x}} \times \mathbf{H} = \partial_t \varepsilon_0 \varepsilon_r(\partial_t^{-1})\mathbf{E} \\ &\varepsilon_r(\partial_t^{-1}) = 1 + \omega_p^2 \partial_t^{-1}\partial_t^{-1}, \quad \omega_p^2 = Nq^2/m\varepsilon_0 \end{aligned} \tag{21}$$

involving the defined plasma frequency ω_p and a differential operator. The practical meaning of a differential operator as in the special case (21), is that the operator should be substituted in the equation (first eq. (21) in the present case), and both sides judiciously manipulated and multiplied by the inverse operator, in order to finally obtain a conventional differential equation. Thus in the first equation (21) we obtain $\partial_t\partial_t\partial_t^{-1}\partial_t^{-1} = 1$ and the inverse time derivative vanishes, while additional time derivatives appear. It follows that the success of such a procedure depends on deriving inverse operators which we know how to handle [9]. Henceforth it will be assumed that this is the case. We have therefore derived something similar to a constitutive relation,

$$\mathbf{D}(\mathbf{X}) = \varepsilon(\partial_{\mathbf{X}})\mathbf{E}(\mathbf{X}) \tag{23}$$

but in a differential operator form. In the spectral domain the operator becomes algebraic, with the substitution $\partial_t^{-1} = -1/i\omega$, as a part of the general analogy $\partial_{\mathbf{X}} \Leftrightarrow i\mathbf{K}$, and the corresponding constitutive relation is found,

$$\mathbf{D}(\mathbf{K}) = \varepsilon(i\mathbf{K})\mathbf{E}(\mathbf{K}) \tag{24}$$

where for (21) we obtain,

$$\varepsilon(i\mathbf{K}) = \varepsilon_0\varepsilon_r = \varepsilon_0(1 - \omega_p^2/\omega^2) \tag{25}$$

It is therefore clear that spatiotemporal dispersivity of the constitutive parameters, i.e., their dependence on $\mathbf{K}$, corresponds to the appearance of differential operators involving components of $\partial_{\mathbf{X}}$ in the spatiotemporal domain.

The sequence leading from the spectral representation (24) back to the spatiotemporal domain representation (23) is straightforward. The

Fourier transform (10) is applied to (24), yielding:

$$\begin{aligned}\mathbf{D}(\mathbf{X}) &= (2\pi)^{-4}\int(d^4\mathbf{K})\mathbf{D}(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}}\\ &= (2\pi)^{-4}\int(d^4\mathbf{K})\mathbf{E}(\mathbf{K})\varepsilon(\partial_{\mathbf{X}})e^{i\mathbf{K}\cdot\mathbf{X}}\\ &= \varepsilon(\partial_{\mathbf{X}})(2\pi)^{-4}\int(d^4\mathbf{K})\mathbf{E}(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}}\\ &= \varepsilon(\partial_{\mathbf{X}})\mathbf{E}(\mathbf{X})\end{aligned} \tag{26}$$

and provided we know how to deal with the resultant operator $\varepsilon(\partial_X)$, the problem is considered as solved.

As mentioned, the constitutive equations can also be heuristically postulated. Here the problem is that the physicality of the definition must be justified for each case individually, a point which is sometimes overlooked. As long as the statement on the constitutive relation is initiated in $\mathbf{K}$, the spectral space, no problem arises. For example, consider a dielectric scalar constitutive relation defined in the form,

$$\mathbf{D}(\mathbf{K}) = \varepsilon(\mathbf{K})\mathbf{E}(\mathbf{K}) \tag{27}$$

The problem can be dealt with either in the spectral domain, using (15), or, if we know how to use the ensuing operator, we can recast (27) in the form (24), i.e., in terms involving components of $i\mathbf{K}$ and then invoke the process shown in (26), and using (1) work in the spatiotemporal domain. Another way of dealing with forms like (27) is to represent the inverse transformation as a four-fold convolution, with $\mathbf{X}_1$ as the four-dimensional integration (dummy) variable

$$\mathbf{D}(\mathbf{X}) = \int(d^4\mathbf{X}_1)\varepsilon(\mathbf{X}_1)\mathbf{E}(\mathbf{X}-\mathbf{X}_1) \tag{28}$$

and substitute (28) in (1), thus working again in the spatiotemporal domain. Two problems arise in this context: Firstly, solving (1) with (28) involves an integro-differential set of equations, which might be more complicated than computing a solution in the spectral domain, and secondly, one must be careful with the limits of integration in (28) so that causality in the relativistic sense is preserved. There are two spatiotemporal events involved in (1), namely $\mathbf{X}$ and $\mathbf{X}_1$. The field $\mathbf{D}(\mathbf{X})$ is effected by $\mathbf{E}(\mathbf{X}-\mathbf{X}_1)$ at a different point, i.e., this shows that the electromagnetic field is non-local and non-instantaneous, which is not surprising at all, it just is a manifestation of the finite speed of propagation of signals encountered in electrodynamics. This prescribes that $|\mathbf{X} - \mathbf{X}_1| \leq 0$. i.e., the events are within the light cone, in the special-relativistic jargon. This proviso can be met by prescribing the limits of integration, or by judiciously defining $\varepsilon(\mathbf{X}_1)$ so that it is only non-vanishing in the range satisfying $|\mathbf{X} - \mathbf{X}_1| \leq 0$.

5. THE QUESTION OF SPATIOTEMPORAL DOMAIN CONSTITUTIVE RELATIONS

In many physical situations the attributes of media vary in space and time, and one would like to be able to take these properties into account. There are two avenues to approach this problem: (a) If it is possible, one divides the problem into homogeneous sub-domains, derives infinite domain solutions for the individual sub-domains, and connects all these sub-domains by satisfying the appropriate boundary and initial conditions. (b) If the inhomogeneity of the medium varies slowly in space and time, one can resort to ray methods, e.g., Hamiltonian ray theory [10], further considered subsequently. The question as to the range of validity of spatiotemporal domain methods is central to the present study.

It is noticed that the indeterminate Maxwell equations are completely equivalent in the spatiotemporal and the spectral domains. Thus (1) and (2), and (15) and (16), respectively, are equivalent and inversive, each in its pertinent inertial frame of reference. The transformation formulas (7), (8) display the same property. However, the constitutive relations do *not* possess this symmetry.

Using a simple example (thus avoiding fancy tensor relations that might be more general, but also distract us from the fundamental issues in question), a dispersive constitutive relation (27) has been postulated, and its corresponding forms in the spatiotemporal domain, (23) and (28), were considered. Although in (27) $\varepsilon(\mathbf{K})$ amounts to an inhomogeneous medium in the spectral domain, we are still dealing with a homogeneous material system in the spatiotemporal domain.

Let us consider imposing on (1) a constitutive relation describing an inhomogeneous medium, in the form,

$$\mathbf{D}(\mathbf{X}) = \varepsilon(-i\mathbf{X})\mathbf{E}(\mathbf{X}) \tag{29}$$

Substitution in (19) yields the analog of (23) in the form,

$$\mathbf{D}(\mathbf{K}) = \varepsilon(\partial_{\mathbf{K}})\mathbf{E}(\mathbf{K}) \tag{30}$$

where the operator

$$\partial_{\mathbf{K}} = (\partial_{\mathbf{k}}, -ic\partial_\omega) \tag{31}$$

can be shown to be a four-vector [5], with relativistic appropriate transformation formulas (cf. (3), (5), (14)),

$$\begin{aligned} \partial_{\mathbf{k}'} &= \tilde{\mathbf{U}} \cdot (\partial_{\mathbf{k}} + \mathbf{v}\partial_\omega) \\ \partial'_\omega &= \gamma(\partial_\omega + \mathbf{v} \cdot \partial_{\mathbf{k}}/c^2) \end{aligned} \tag{32}$$

The other alternative is to represent (29) in the spectral domain, in analogy with (28),

$$\mathbf{D}(\mathbf{K}) = \int (d^4\mathbf{K}_1)\varepsilon(\mathbf{K}_1)\mathbf{E}(\mathbf{K} - \mathbf{K}_1) \tag{33}$$

In this form it appears that the spectral event $\mathbf{K}$ interacts with other values $\mathbf{K}_1$ .Whether one considers (30), involving differentiations in the spectral domain, or (33) with its $\mathbf{K}$-domain non-local interactions, in any case these forms seem to have no bearing on physical models as we know them. It is noted that interactions in spectral space are possible in the case of nonlinear systems, discussed below, but not in the present case of linear systems.

The special case of nondispersive materials deserves special attention. Such circumstances exists for free space (vacuum), but might also be considered in other special cases as an adequate approximation. Here $\varepsilon(\mathbf{K})$ in (27) becomes a constant, which is equivalent to $\varepsilon(\mathbf{X}_1) = \varepsilon\delta(\mathbf{X}_1 - \mathbf{X})$ in (28), and the four-dimensional impulse function causes the integral to collapse, resulting in $\mathbf{D}(\mathbf{X}) = \varepsilon\mathbf{E}(\mathbf{X})$. But this special case refers once more to a homogeneous medium!

If we reject forms like (30), (33) as inadequate for describing physically meaningful constitutive relations, the unavoidable conclusion is that spatiotemporal constitutive relations like (29) are invalid. This sweeping conclusion still allows for approximation methods like the Hamiltonian ray theory mentioned above. The implementation of such constitutive relations will be discussed below. It follows that Maxwell's equations are to be viewed fundamentally as a model for homogeneous systems, and any departure from this statement can only be valid in the context of an appropriate approximation.

6. GENERALIZED MINKOWSKI CONSTITUTIVE RELATIONS FOR LINEAR SYSTEMS

Sommerfeld [11] indicates that historically the extension of Maxwell's theory to moving systems played a central role, and the fact that Minkowski [12] (See Pauli [7] for more references) finally answered this problem. For relevant studies and literature references see also Post [13], Hebenstreit [14, 15], and Hebenstreit and Suchy [16]. Generally speaking, many researchers asked the suggestive question: If constitutive parameters (e.g., dielectric and permeability parameters) are given in a medium's comoving (rest) frame of reference, *what are the corresponding constitutive parameters for an observer in the laboratory system where the medium is moving?* On the other hand, the philosophy of Minkowski was different: Given Maxwell's equations and constitutive relations in the comoving system of reference, *what are the relations between the fields in the laboratory system?* I.e., constitutive relations, according to Minkowski, should render Maxwell's equations as a determinate system, but there is no physical import to the constitutive parameters *per se*. Minkowski's constitutive relations are mentioned by Pauli [7], Sommerfeld [11], and Censor [5], but all these derivations actually aim at showing that moving media manifest bi-anisotropic behavior of the form

$$\begin{aligned}\mathbf{D} &= \tilde{\alpha}_1 \cdot \mathbf{E} + \tilde{\alpha}_2 \cdot \mathbf{H} \\ \mathbf{B} &= \tilde{\alpha}_3 \cdot \mathbf{E} + \tilde{\alpha}_4 \cdot \mathbf{H}\end{aligned} \tag{34}$$

where $\tilde{\alpha}_1 .... \tilde{\alpha}_4$ denote appropriate dyadics. It is not always possible or convenient to represent the additional constitutive relations in the form (34). This has been encountered before [17], and this approach is adopted here too. Namely, we strive to transform the relations for the fields in the comoving frame to the laboratory frame, but do not bother to find explicit expressions of the form (34). Given in the primed, comoving frame, sufficient constitutive relations whose general implicit form is

$$F_i(\mathbf{D}', \mathbf{B}', \mathbf{E}', \mathbf{H}', \mathbf{j}'_e, \rho'_e, \mathbf{j}'_m, \rho'_m; \partial_{\mathbf{X}'}) = 0 \tag{35}$$

where i, the number of scalar constitutive relations, depends on the number of equations and independent scalar variables involved, $\partial_{\mathbf{X}'}$, signifies that constitutive differential operators in the sense of (23) are involved. In order to contract the notation, (35) is written in the shorthand form,

$$\mathbf{F}(\mathbf{Z}'(\mathbf{X}'); \partial_{\mathbf{X}'}) = 0 \tag{36}$$

with $\mathbf{Z}'(\mathbf{X}')$ symbolizing all the fields involved in (35). The relativistic covariance of Maxwell's equations, i.e., the equivalence of (1) and (2) as explained above, allows us to start either from (1) or (2). Following the above given argument, in the spatiotemporal domain we start from (2) in the primed comoving system of the medium at rest. To obtain a determinate system of equations constitutive expressions as symbolized by (35), (36) are introduced. In the spatiotemporal domain they stand for expressions like (21), or more generally like (23), given now in terms of the comoving frame, i.e., the primed frame of reference. Hence we are dealing here with differential operators. As in (2), we have now in (35), (36) functions $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$, etc., where $\mathbf{X}' = \mathbf{X}'(\mathbf{x}', ict')$. Carrying out this scheme, a determinate system is now defined in the primed frame. We now use the transformation equations for the fields, (7), (8), and the transformations $\partial_{\mathbf{X}'} = \partial_{\mathbf{X}'}[\partial_{\mathbf{X}'}]$ for the differential operators, (5). This finally yields a corresponding set of equations for the laboratory unprimed frame of reference in terms of variables $\mathbf{X} = \mathbf{X}(\mathbf{x}, ict)$. This constitutes the general implementation of Minkowski's approach towards electrodynamics in moving media.

The corresponding spectral domain $\mathbf{K} = (\mathbf{k}, i\omega/c)$ analysis is now straightforward. The constitutive expressions symbolized by (36) now take the form

$$\mathbf{F}(\mathbf{Z}'(\mathbf{K}'); \mathbf{K}') = 0 \tag{37}$$

Here we start with (16), involving variables $\mathbf{K}' = (\mathbf{k}', i\omega'/c)$, which transforms to (15) in terms of unprimed fields and independent variables $\mathbf{K} = (\mathbf{k}, i\omega/c)$. For the present case (37) is assumed to be given in terms of algebraic forms, and fields $\mathbf{Z}' = \mathbf{Z}'(\mathbf{K}')$ as functions of $\mathbf{K}' = (\mathbf{k}', i\omega'/c)$, to which we apply the field transformation formulas (7), (8), in the spectral domain. The dielectric parameters (i.e. coefficients appearing in (37)) are functions of $\mathbf{K}' = (\mathbf{k}', i\omega'/c)$, and are substituted by the Doppler effect formulas $\mathbf{K}' = \mathbf{K}'[\mathbf{K}]$, namely (14), so that finally a determinate system involving unprimed fields in terms of the laboratory system components of $\mathbf{K}$ is obtained.

In the Minkowski approach to electrodynamics in moving media, a crucial idea is involved, which needs close scrutiny: Generally speaking, simply substituting the transformation formulas for variables and operators only means that we have recast the expressions in terms of other variables, i.e., a mere change of variables has been effected. This does not automatically mean that the new set of equations represents the physical measurable quantities in the unprimed frame. Here is an example: By starting with a given expression for the electric field $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$, and substituting the Lorentz transformation for the coordinates $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$, we end up with a new expression $\mathbf{E}' = \mathbf{E}'(\mathbf{X}'[\mathbf{X}])$ which might be denoted as $\overline{\mathbf{E}}(\mathbf{X})$. Obviously all we have achieved is merely a change of variables, and the new representation $\overline{\mathbf{E}}(\mathbf{X})$ does not define the electric field in the unprimed frame. If we are interested in $\mathbf{E}(\mathbf{X})$, as measurable by an observer in the unprimed frame of reference, we have to use (7) and then substitute the Lorentz transformation,, i.e.,

$$\mathbf{E}(\mathbf{X}) = \tilde{\mathbf{V}} \cdot (\mathbf{E}'(\mathbf{X}'[\mathbf{X}]) - \mathbf{v} \times \mathbf{B}'(\mathbf{X}'[\mathbf{X}])) \tag{38}$$

The question one has to ask is whether the Minkowski approach is merely a change of variables as in the above example leading to $\overline{\mathbf{E}}(\mathbf{X})$, or is it also a change of physical observation from one (the primed, comoving) frame of reference to another (the unprimed, laboratory) frame of reference. In view of the equivalence of (1) and (2), referred to as the covariance of Maxwell's equations, the latter applies, i.e., the Minkowski approach automatically takes the transformations of fields as exemplified by (38), into account. The same argument applies to the spectral domain as well.

7. NONLINEAR CONSTITUTIVE RELATIONS

We start this discussion in the comoving frame where the medium is at rest, and for the time being, this will be the unprimed frame of reference. Loosely speaking, nonlinear media are characterized by constitutive parameters depending on fields. This gives rise to a plethora of new phenomena, both academically interesting *per se*, and of interest for applications. Paramount are the phenomena of harmonic generation, which one finds also in nonlinear lumped elements (e.g., magnetic materials which become saturated when flux increases, or electronic devices possessing nonlinear voltage-current characteristic curves), and new wave-specific phenomena of self-focusing. In the latter, due to the field dependent constitutive parameters, the wave, depending on the intensity profile, "creates for itself" a "lens", thus a self-focusing phenomenon appears. Constitutive parameters for nonlinear media are mostly (heuristically) defined, thus belonging to category (a) above. Attempts of defining nonlinear constitutive parameters from first principles usually lead to Volterra's series of functionals [17, 18]. For additional literature references and related work see [19-28]. A postulated model [29], applied to numerical simulation of rays in nonlinear media, was used in conjunction with experimental data [30] given in the literature, and close agreement of the experimental and simulation results was found.

Our prototype linear model for constitutive relations is given by (23), or its spectral equivalent in one of the forms (24) or (27), whichever is more convenient at a given time, or the spatiotemporal convolution integral equivalent (28). The corresponding nonlinear model (which is usually postulated, i.e., belongs to category (a) above) should satisfy a "correspondence principle" and reduce to the linear case in the limit of vanishing nonlinearity. Moreover, the model should indicate the various modes of nonlinear interaction. This will facilitate the investigation of effects caused by specific modes, and in many cases indicate a hierarchy of such effects. If this is achieved, certain leading effects can be investigated.

Accordingly, instead of the convolution integral (28) we seek now a more general relation, which for simplicity of the discussion we fashion after (23) or (28), i.e., dielectric media,

$$\mathbf{D}(\mathbf{X}) = \sum_{n=1}^{\infty} \mathbf{D}^n(\mathbf{X}) = \sum_{n=1}^{\infty} \mathbf{P}^n\{\mathbf{X}, \mathbf{E}\} \tag{39}$$

where the series suggests a hierarchy of increasingly complex nonlinear interactions, such that the most significant ones are the leading terms, and $\mathbf{P}^n\{\mathbf{X}, \mathbf{E}\}$ are adequate functionals depending on the coordinates $\mathbf{X}$ and the fields $\mathbf{E}$. A quite natural candidate for such a model is the Volterra series of functionals, which is the functional counterpart and generalization of the Taylor series for functions, thus providing an adequate model for a hierarchical system that in practice can be truncated after a certain number of terms:

$$\begin{aligned} \mathbf{D}^{(n)}(\mathbf{X}) = \int (d^4\mathbf{X}_1) \cdots \int (d^4\mathbf{X}_n) \tilde{\varepsilon}^{(n)}(\mathbf{X}_1, \cdots, \mathbf{X}_n) \\ \because \mathbf{E}(\mathbf{X} - \mathbf{X}_1) \cdots \mathbf{E}(\mathbf{X} - \mathbf{X}_n) \end{aligned} \tag{40}$$

In (40) we have n four-fold integrations which for $n = 1$ reduce to the linear case (28), the n-th order constitutive parameter $\tilde{\varepsilon}^{(n)}$ is now a dyadic (in the generic sense, some would refer to it as a tensor) acting on the indicated fields. The symbol $\because$ denotes all the inner multiplications of the constitutive dyadic and the fields. Relativistic causality as discussed above must be incorporated into the model, i.e., $|\mathbf{X}-\mathbf{X}_n| \leq 0$. Inasmuch as the Volterra functionals have an inverse Fourier transformation, the relation in $\mathbf{K}$ space follows:

$$\begin{aligned}\mathbf{D}^{(n)}(\mathbf{K}) =&(2\pi)^{4(1-n)}\int(d^4\mathbf{K}_1)\cdots\int(d^4\mathbf{K}_{n-1})\tilde{\varepsilon}^{(n)}(\mathbf{K}_1\cdots\mathbf{K}_n)\\&\because\mathbf{E}(\mathbf{K}_1)\cdots\mathbf{E}(\mathbf{K}_n)\end{aligned}\tag{41}$$

This now is an $n-1$ four-fold integration expression which for $n=1$ reduces to the algebraic linear case (27). A scrutiny of (41) reveals that $\mathbf{K}_n$ is undefined, indeed, (41) must be supplemented by the constraint

$$\mathbf{K}=\mathbf{K}_1+\cdots\mathbf{K}_n\tag{42}$$

i.e., $\mathbf{k}=\mathbf{k}_1+\cdots+\mathbf{k}_n$, $\omega=\omega_1+\cdots+\omega_n$. This is a remarkable relation. In the quantum-mechanical context it is an expression of conservation of energy (for frequencies) and momenta (for wave vectors). Moreover, as far as nonlinear processes are concerned, (42) is a statement of the production of harmonics and mixing of frequencies, and the production of new propagation vectors for these waves. The Volterra model is therefore very plausible for the purposes of modeling nonlinear constitutive relations.

An outline of the proof for the Fourier transform leading from (40) to (41) in conjunction with (42) is given in Appendix A.

8. DIFFERENTIAL OPERATOR REPRESENTATION FOR NONLINEAR CONSTITUTIVE RELATIONS

A quite natural question at this stage concerns the generalization of the linear forms (35)-(37) (now without motion, i.e., in the unprimed frame which is also the frame in which the medium is at rest) to the case of nonlinear media. Here we observe that at least symbolically, the differential form (23) is equivalent to its integral counterpart (28). It follows that the generalization of (28), as expressed in the Volterra functionals series (39), (40), should now be written as

$$\begin{aligned}\mathbf{D}^{(n)}(\mathbf{X}) =&\tilde{\varepsilon}^{(n)}(\partial_{\mathbf{X}_1}|_{\mathbf{X}_1=\mathbf{X}},\cdots,\partial_{\mathbf{X}_n}|_{\mathbf{X}_n=\mathbf{X}})\\&\because\mathbf{E}(\mathbf{X}_1)\cdots\mathbf{E}(\mathbf{X}_n)\end{aligned}\tag{43}$$

which entails differential operations on the fields indicated, followed by substitutions $\mathbf{X}_1 = \mathbf{X}, \ldots, \mathbf{X}_n = \mathbf{X}$ for all independent variables $\mathbf{X}_n$ involved. As far as the present author is aware, this seems to constitute a novel representation for nonlinear constitutive differential operators. Its implementation to specific cases of interest will be considered in a later study. According to (43), we now recast the generalized form (36) symbolically as

$$\mathbf{F}^{(n)}(\mathbf{Z}(\mathbf{X}); \partial_{\mathbf{X}_1}|_{\mathbf{X}_1=\mathbf{X}}, \cdots, \partial_{\mathbf{X}_n}|_{\mathbf{X}_n=\mathbf{X}}) = 0 \tag{44}$$

Here we understand (44) to constitute an array $\mathbf{F}^{(n)}$ where the superscript denotes the order of nonlinear interaction, as in (39)-(41), involving $\mathbf{Z}(\mathbf{X})$ as a product of n fields to which the appropriate differential operators are applied, in terms of multivariate dyadic differential operators, and finally $\mathbf{X}_1 = \mathbf{X}$ etc. are inserted. Once (44) is available, the Volterra series representation as in (40) is straightforward, and can be reconstructed from (44).

In the spectral domain we trace back the same argument as used for the linear case, with the appropriate modifications.

9.MINKOWSKI'S CONSTITUTIVE RELATIONS AND NONLINEAR SYSTEMS

The strategy discusses above for linear systems can be straightforwardly extended to nonlinear Volterra systems as well. Similarly to the linear system, we start with Maxwell's equations (2) in the primed comoving system of reference, but instead of the linear constitutive relations (35), or in its contracted notation (36), we now invoke the nonlinear analog (44), in the comoving frame, i.e., in the form:

$$\mathbf{F}^{(n)}(\mathbf{Z}'(\mathbf{X}'); \partial_{\mathbf{X}'_1}|_{\mathbf{X}'_1=\mathbf{X}'}, \cdots, \partial_{\mathbf{X}'_n}|_{\mathbf{X}'_n=\mathbf{X}'}) = 0 \tag{46}$$

Like the linear case before, the various transformation formulas for fields and operators are applied, to finally derive the corresponding determinate system of equations in the laboratory (unprimed) frame of reference.

Obviously for the spectral domain treatment, the analogy leads to a more complicated form. This is due to the fact that unlike the linear case, where (37) becomes algebraic, in the nonlinear case the transform (41) and similar forms involving fields will still be functionals with respect to the fields involved. Consequently the implementation for specific cases might lead to more complicated forms. However, the least we can say is that the general feasibility of the Minkowski approach carries over to the present case too.

10. RAY-THEORETIC APPROXIMATIONS FOR CONSTITUTIVE RELATIONS

We have argued above that for fundamental reasons spatiotemporal dependence of the constitutive parameters in dispersive media must be excluded, in general. See (29) and the related discussion. Obviously mixed forms like

$$\mathbf{D}(\mathbf{K}, \mathbf{X}) = \varepsilon(\mathbf{K}, \mathbf{X})\mathbf{E}(\mathbf{K}, \mathbf{X}) \tag{47}$$

are even more problematic, because they involve the fundamental uncertainty principle, which plays an important role in quantum theory for the very same reasons. Accordingly, we cannot simultaneously specify the values of both $\mathbf{K}, \mathbf{X}$. However, in the context of the eiconal approximation and Hamiltonian ray theory, such forms are legitimate for media slowly varying in space and time [10, 31, 32].

In the context of the eiconal approximation, we start with an *ansatz* that the solutions for the fields be represented by a product of a slowly varying amplitude and an oscillatory exponential, thus describing a *quasi* harmonic locally and instantaneously plane wave form for all the fields appearing in (1), e.g.,

$$\mathbf{E}(\mathbf{K}, \mathbf{X}) = \mathbf{E}_0(\mathbf{K}, \mathbf{X})e^{i\theta(\mathbf{K}, \mathbf{X})} \tag{48}$$

etc., constituting a narrow band wave packet characterized by a carrier $e^{i\theta(\mathbf{K}, \mathbf{X})}$ possessing a central value $\mathbf{K}$, and a slowly tapering envelope $\mathbf{E}_0(\mathbf{K}, \mathbf{X})$. The phase function $\theta(\mathbf{K}, \mathbf{X})$ is defined in terms of a four-dimensional line integral,

$$\theta(\mathbf{K}, \mathbf{X}) = \int_{\mathbf{X}_0}^{\mathbf{X}} \overline{\mathbf{K}}(\mathbf{K}, \mathbf{X}_1) \cdot d\mathbf{X}_1 \tag{49}$$

where now $\overline{\mathbf{K}}(\mathbf{K}, \mathbf{X}_1)$ changes slowly in the spatiotemporal domain. It is assumed that

$$\partial_{\mathbf{X}}\theta(\mathbf{K}, \mathbf{X}) = \overline{\mathbf{K}}(\mathbf{K}, \mathbf{X}) \tag{50}$$

i.e., that the integral (49) is independent of the path of integration chosen between the limits.. Subject to these restrictions, one can introduce spatiotemporally varying constitutive relations for linear media, similarly to (36),

$$\mathbf{F}(\mathbf{Z}(\mathbf{K},\mathbf{X});\partial_{\mathbf{X}};\mathbf{K},\mathbf{X}) = 0 \tag{51}$$

where $\mathbf{Z}(\mathbf{K},\mathbf{X})$ possess the form given in (48) and the last $\mathbf{K},\mathbf{X}$ in (51) indicates that the constitutive relations can depend on these parameters, in addition to the differential operators involved. By substitution of (48), with (49) understood, for the fields appearing in the Maxwell equations (1), and performing the indicated differentiations, one obtains (15), into which constitutive relations similar to (37) can be substituted, now having the form,

$$\mathbf{F}(\mathbf{Z}(\mathbf{K},\mathbf{X});\mathbf{K},\mathbf{X}) = 0 \tag{52}$$

where the operations $\partial_{\mathbf{X}}$ produce factors $i\overline{\mathbf{K}}(\mathbf{K},\mathbf{X})$ which are already taken into account in (52).

Similarly for nonlinear constitutive relations one must modify (44) appropriately. This has been verified previously for periodic *quasi* harmonic solutions [17–25, 28, 29].

The above discussions regarding the general Minkowski constitutive relations carry over in a straightforward manner, and need not be discussed in detail.

12. CONCLUDING REMARKS

It would be presumptuous to profess that the whole of electrodynamics and special relativity can be summarized in this short paper. What was attempted here was to bring together under one roof all the fundamental issues mentioned in the title of this paper, and indicate their interdependence.

The attention of the reader is drawn to an alternative method of dealing with special relativity, particular as regards relativistic electrodynamics in a manner more amenable to the needs of students.

The origin of constitutive relations and their validity is often underplayed both in textbooks and research. This is especially important for research where it is attempted to analyze dispersive systems in the time, or more generally, in the spatiotemporal domain. It is argued above that constitutive relations are basically *differential operators* in the spatiotemporal domain. This includes nonlinear Volterra systems as defined above. In the spectral domain, algebraic expressions are obtained for linear systems, and *integral* constitutive relations are obtained for Volterra systems.

Akin to the questions regarding constitutive relations is the problem of electrodynamics in moving systems. This problem was for decades considered central to the formulation of electromagnetism. Minkowski [12] in 1908 realized that the problem is not to define constitutive relations for the laboratory frame of reference, which will describe the medium in terms of properties resulting from the motion, but to provide a set of sufficient number of equations to solve for the number of field components (dependent variables) involved. The feasibility of generalizing Minkowski's approach to linear and nonlinear constitutive relations of arbitrary form is demonstrated.

Finally, we return to spatiotemporally varying media, and the feasibility of solving wave propagation problems involving such media. The solution of a problem as a patchwork of finite domains for which we have an infinite domain solution, and the "stitching together" of such solutions into an overall solution by using boundary and initial conditions is obvious. This prescribes that subdomains be treated as uniform, so for a general spatiotemporally varying system, this is an approximation. Another approach is the treatment of spatiotemporally slowly varying media in the ray regime, as summarized above. This will allow, within the range of validity of the approximation, to deal with constitutive relations varying in both $\mathbf{K}, \mathbf{X}$ domains.

APPENDIX A: FOURIER TRANSFORMATION OF THE VOLTERRA SERIES

In order to avoid cumbersome notation, consider (40) for $n = 2$, i.e.,

$$\begin{aligned}\mathbf{D}^{(2)}(\mathbf{X}) = \int (d^4\mathbf{X}_1) \int (d^4\mathbf{X}_2) \tilde{\varepsilon}^{(2)}(\mathbf{X}_1, \mathbf{X}_2) \\ \because \mathbf{E}(\mathbf{X} - \mathbf{X}_1) \cdots \mathbf{E}(\mathbf{X} - \mathbf{X}_2)\end{aligned} \tag{A1}$$

and show that accordingly we obtain as in (41) in the form,

$$\mathbf{D}^{(2)}(\mathbf{K}) = (2\pi)^{-4} \int (d^4\mathbf{K}_1) \tilde{\varepsilon}^{(2)}(\mathbf{K}_1, \mathbf{K}_2) \because \mathbf{E}(\mathbf{K}_1)\mathbf{E}(\mathbf{K}_2) \tag{A2}$$

The proof for $n > 2$ follows the same pattern. Incorporating (42) for the present case into (A2), it is now recast in the form,

$$\begin{aligned}\mathbf{D}^{(2)}(\mathbf{K}) =&(2\pi)^{-4}\int(d^4\mathbf{K}_1)\int(d^4\mathbf{K}_2)\delta(\mathbf{K}_1+\mathbf{K}_2-\mathbf{K})\\ &\tilde{\varepsilon}^{(2)}(\mathbf{K}_1,\mathbf{K}_2)\because\mathbf{E}(\mathbf{K}_1)\mathbf{E}(\mathbf{K}_2)\end{aligned}\tag{A3}$$

The spectral representation of the four-dimensional unit impulse function is given by,

$$\delta(\mathbf{K})=(2\pi)^{-4}\int(d^4\mathbf{X})e^{-i\mathbf{K}\cdot\mathbf{X}}\tag{A4}$$

Substituting for the unit impulse function in (A3) we further have,

$$\begin{aligned}\mathbf{D}^{(2)}(\mathbf{K}) =&(2\pi)^{-8}\int(d^4\mathbf{X})e^{-i\mathbf{K}\cdot\mathbf{X}}\int(d^4\mathbf{K}_1)\int(d^4\mathbf{K}_2)e^{i(\mathbf{K}_1+\mathbf{K}_2)\cdot\mathbf{X}}\\ &\tilde{\varepsilon}^{(2)}(\mathbf{K}_1,\mathbf{K}_2)\because\mathbf{E}(\mathbf{K}_1)\mathbf{E}(\mathbf{K}_2)\end{aligned}\tag{A5}$$

hence according to (9), we obtain,

$$\begin{aligned}\mathbf{D}^{(2)}(\mathbf{X}) =&(2\pi)^{-8}\int(d^4\mathbf{K}_1)\int(d^4\mathbf{K}_2)e^{i(\mathbf{K}_1+\mathbf{K}_2)\cdot\mathbf{X}}\\ &\tilde{\varepsilon}^{(2)}(\mathbf{K}_1,\mathbf{K}_2)\because\mathbf{E}(\mathbf{K}_1)\mathbf{E}(\mathbf{K}_2)\end{aligned}\tag{A6}$$

Each of the integrals in (A6) defines a transform of a product, according to (10), which in the $\mathbf{X}$ domain corresponds to a convolution integral of the kind shown in (28), hence finally (A1) is obtained.

From (A6), if we slightly change the notation from $\tilde{\varepsilon}^{(2)}(\mathbf{K}_1,\mathbf{K}_2)$ to $\tilde{\varepsilon}^{(2)}(i\mathbf{K}_1,i\mathbf{K}_2)$, then in the integral one can replace $\tilde{\varepsilon}^{(2)}(i\mathbf{K}_1,i\mathbf{K}_2)$ by $\tilde{\varepsilon}^{(2)}(\partial_{\mathbf{X}_1}|_{\mathbf{X}_1=\mathbf{X}}\partial_{\mathbf{X}_2}|_{\mathbf{X}_2=\mathbf{X}})$ according to (43), and take it outside the integrations. The remaining integrals, according to (10), finally yield (43) in the explicit form,

$$\mathbf{D}^{(2)}(\mathbf{X})=\tilde{\varepsilon}^{(2)}(\partial_{\mathbf{X}_1}|_{\mathbf{X}_1=\mathbf{X}}\partial_{\mathbf{X}_2}|_{\mathbf{X}_2=\mathbf{X}})\because\mathbf{E}(\mathbf{X}_1)\mathbf{E}(\mathbf{X}_2)\tag{A7}$$

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