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MODAL EXPANSIONS AND ORTHOGONAL COMPLEMENTS IN THE THEORY OF COMPLEX MEDIA WAVEGUIDE EXCITATION BY EXTERNAL SOURCES FOR ISOTROPIC, ANISOTROPIC, AND BIANISOTROPIC MEDIA

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1. INTRODUCTION

Guided-wave electrodynamics in the modern understanding deals with the study of propagation, radiation, excitation, and interaction of waves in a variety of waveguiding structures. The term "complex media waveguide" applied in the title of the paper implies the medium complexity of two types:

(i) the physical complexity associated with medium properties diversified by the very nature (gas and solid state plasmas with drifting carriers; polarized and magnetized solids with different properties: piezoelectric, electrooptic, acoustooptic, magnetooptic, magnetoelastic; chiral, biisotropic, and bianisotropic media);

(ii) the *geometrical* complexity due to using composite and multilayered structures.

Electromagnetic theory has been developed up to its present state by extensive works and efforts of a great number of researchers and scientists. Besides pure scientific purposes, progress in classical electrodynamics at all stages of its advancement was encouraged by certain demands of technology.

At the first stage such a stimulating factor was related to practical needs of then incipient radar and antenna engineering. The consequent experience on electrodynamic properties of mostly passive nondispersive media specified by phenomenological constants, which had been gathered over a number of years, was accumulated in many scientific publications. Among them we should refer, for instance, to such famous and popular books as [1–5] which now constitute the theoretical foundation of classical electrodynamics. Much attention was given to the study of electromagnetic properties of gas plasma as a medium for wave propagation. Later on the plasma wave aspects were extended to the behavior of charge carriers in solids considered as a solid-state plasma. At present the literature devoted to the electromagnetic properties of plasmas is immense and the following books [6–11] with their bibliographies can give a good indication of the scope of plasma electrodynamics.

Another direction of electrodynamic aspects was inspired by developing the technology of microwave devices operating on wave principle. The first to be developed were vacuum devices using the space charge, cyclotron, and synchronous waves on an electron beam such as the traveling-wave tube, backward-wave tube, and others [12–14]. Later the similar idea to apply waves in solids for signal processing gave rise to new lines of solid-state electronics. They are due to applying the surface acoustic waves (SAW) in elastic piezo-dielectrics [15–17], the magnetostatic spin waves (MSW) in magnetized ferrites [18–23], and the space charge waves (SCW) in semiconductors with negative differential mobility of electrons [10, 24, 25]. These waves refer to the quasistatic part of the electromagnetic spectrum of waveguiding structures for which a relevant potential field (electric for SAW and SCW or magnetic for MSW) dominates over its curl counterpart. This fact caused some electrodynamic formulations to be revised in order to separate such potential fields and take into account the space-dispersive properties of these media described by the proper equations of medium motion [15, 22, 24, 25].

For the last decades the macroscopic electrodynamics of waveguiding structures has experienced two powerful stimulating actions. The first is associated with needs of fiber and integrated optics and began about twenty five years ago. A number of theoretical propositions in electrodynamics were reformulated, as applied to optical waveguides, and have been embodied in devices. The literature devoted to this topic is enormous including the well-known books [26–32].

Nowadays we observe the renewed interest in electrodynamic problems caused by efforts to apply chiral, biisotropic, and bianisotropic media for the control of electromagnetic radiation in waveguiding structures. Phenomenon of optical activity in certain natural substances generated by their handedness property (chirality) was already known last century. The present revival of scientific and technological attention to this problem is inspired by the modern progress of material science and technology in synthesizing artificial composite media. Such media possess unique properties to open new potential possibilities in their utilizing in optics and at microwaves. This has aroused a great wave of research followed by numerous publications, among them both the general books [33–35] comprising bianisotropic issues and the special books [36–40] devoted entirely to this subject.

Theoretical ground for many wave electrodynamics applications is the modal expansion method. In the case of the eigenmode excitation by external sources the question of completeness of the eigenfunction basis chosen inside the source region is of crucial importance in practice. Unfortunately, most authors solve this question rather superficially assuming intuitively that the set of eigenfunctions found as the general solution to the boundary-value problem without sources is complete also inside the source region. However, this is not the case in general.

From mathematical considerations given in Appendix A.1 it follows that the above statement is valid only for the desired functions $\psi(x)$ tangential to the Hilbert space spanned by the eigenfunction basis $\{\psi_k(x)\}$. Generally, for most functions $f(x)$ their series expansion in terms of the base functions (convergent in mean) is only a *projection* $\psi(x)$ of the function $f(x)$ on the Hilbert space. In addition, there may exist a nonzero function $c(x)$ orthogonal to this space, the so-called *orthogonal complement*, which in general must be added to the projection $\psi(x)$ in order for $f(x)$ to be considered as the complete required function (see Eq. (A.18) and relevant relations in Appendix A.2 involving the appropriate electrodynamic extensions). The above statement is fairly obvious for mathematicians but unfortunately was fully ignored in developing the modern topics of guided-wave electrodynamics by most authors, not counting Vainshtein [2] and Felsen and Marcuvitz [8]. Strange as it may seem, when developing the excitation theory of optical waveguides, many authors [26, 27, 29, 31, 32] have correctly applied the modal expansions to the transverse components of electromagnetic fields but entirely dropped the orthogonal complements due to the longitudinal exciting bulk currents. As will be shown, this causes the so-called effective surface currents to be lost. Similar situation also holds for electrodynamics of the waveguiding structures with chiral and bianisotropic media [34, 38–40] where the modal expansion method is practically undeveloped and the problem of the orthogonal complements and effective surface sources, worked out below, is more complicated.

The objective in writing this paper is to develop a unified electrodynamic theory of waveguide excitation by external sources (bulk and surface) applicable equally for any media and waveguiding structures. Particular attention will be given to the study of the unexpandable orthogonal complements to the eigenmode expansions which should be expressed in terms of the given exciting currents as well as the desired mode amplitudes of the modal expansions. To this end, we begin with Sec. 2 devoted to deriving the basic energy-power relations of electrodynamics applied to the lossy bianisotropic media including Poynting's theorem in the differential and integral forms involving the *self-power* and *cross-power* quantities (flows and losses) transmitted and dissipated by the eigenmodes of a waveguiding structure. Sec. 3 deals with a generalization of the known orthogonality relation for the waveguides without losses to the so-called *quasi-orthogonality* relation for lossy waveguides which describes, as a special case, the orthogonality of the reactive (nonpropagating) modes in lossless waveguides. In addition, an expression for the time-average energy stored by the active (propagating) modes is proved. Sec. 4 is concerned with the consideration of external sources (currents, fields, and medium perturbations) and electromagnetic fields inside the source region. The complete representation of the fields, besides their modal expansions, involves also the so-called *orthogonal complementary fields* which necessarily generate the *effective surface currents*. Sec. 5 contains two different approaches to the derivation of the equations of mode excitation by external sources. The first approach applied only to the lossless waveguides is based on an electrodynamic analogy with the known mathematical statements such as the method of variation of constants and the relations of functional analysis (see Appendix A). The second approach makes use of the reciprocity theorem in the complex-conjugate form to obtain the equations of mode excitation in the general form valid for both lossy and lossless waveguiding structures. Another alternative proof of the excitation equations for lossless waveguides starting directly from Maxwell's equations is adduced in Appendix B.

In this paper we restrict our consideration to the case of time-dispersive media whose electrodynamic properties (isotropic, anisotropic, bianisotropic) are characterized by the frequency-dependent constitutive parameters considered as phenomenologically given. More complicated case of space-dispersive media such as elastic piezodielectrics, magnetized ferrites, nondegenerate plasmas with drifting charge carriers whose electrodynamic description requires, besides Maxwell's equations, employing the proper equation of medium motion will be the subject of matter of the second part of the paper.

In conclusion there are a few words concerning the notation applied:

(i) tensors of rank 0 (scalars), 1 (vectors), 2 (dyadics), and more than 2 (tensors) are denoted as: A, $\mathbf{A}$, $\bar{\mathbf{A}}$, and $\bar{\bar{\mathbf{A}}}$, respectively;

(ii) their products are denoted as: AB (for two scalars); $\mathbf{A}\cdot\mathbf{B}$, $\mathbf{A}\times\mathbf{B}$, and $\mathbf{AB}$ (for scalar, vector, and dyad products of two vectors); $\mathbf{AB}\cdot\mathbf{CD} = \mathbf{AD}(\mathbf{B}\cdot\mathbf{C})$, $\bar{\mathbf{A}}\cdot\bar{\mathbf{B}} = A_{ij}B_{jk}$, and $\bar{\bar{\mathbf{A}}}\cdot\bar{\bar{\mathbf{B}}} = A_{ijk}B_{klm}$ (for scalar product of two vector dyads, dyadics, and tensors); $\mathbf{AB}:\mathbf{CD} = (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$, $\bar{\mathbf{A}}:\bar{\mathbf{B}} = A_{ij}B_{ji}$, and $\bar{\bar{\mathbf{A}}}:\bar{\bar{\mathbf{B}}} = A_{ijk}B_{kjl}$ (for double scalar product of two vector dyads, dyadics, and tensors).

2. GENERAL POWER-ENERGY RELATIONS OF ELECTRODYNAMICS FOR BIANISOTROPIC MEDIA

2.1 Poynting's Theorem

In macroscopic electrodynamics, the electromagnetic properties of a medium are described by two field-intensity vectors, $\mathbf{E}$ (the electric field) and $\mathbf{H}$ (the magnetic field), and two flux-density vectors, $\mathbf{D}$ (the electric induction) and $\mathbf{B}$ (the magnetic induction), which are related by means of Maxwell's equations (written in the rationalized mks system):

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{J}, \quad \nabla\cdot\mathbf{D} = \rho, \quad \nabla\cdot\mathbf{B} = 0. \tag{2.1}$$

Mobile charge effects in the medium are specified by the charge and current densities ρ and $\mathbf{J}$, whereas the bound charges arise as a result of polarization responses of the medium to electromagnetic actions characterized by the electric and magnetic polarization vectors $\mathbf{P}$ (the polarization vector) and $\mathbf{M}$ (the magnetization vector). These vectors yield the corresponding contributions to the electric and magnetic inductions:

$$\mathbf{D} = \epsilon_o\mathbf{E} + \mathbf{P} \quad \text{and} \quad \mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}). \tag{2.2}$$

The conventional procedure applied to Eqs. (2.1) reduces to Poynting's theorem in the form involving the instantaneous values of power-energy quantities:

$$\frac{\partial w}{\partial t} + \nabla\cdot\mathbf{S} = -I_{\mathbf{J}} - I_{\mathbf{P}} - I_{\mathbf{M}} \tag{2.3}$$

where $w = (\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})/2$ is the electromagnetic energy density and $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ is the electromagnetic energy flux density (Poynting's vector). The terms in the right-hand side of Eq. (2.3):

$$I_{\mathbf{J}} = \mathbf{J}\cdot\mathbf{E}, \tag{2.4}$$

$$I_{\mathbf{P}} = \frac{1}{2}\left(\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t} - \mathbf{P}\cdot\frac{\partial\mathbf{E}}{\partial t}\right), \tag{2.5}$$

$$I_{\mathbf{M}} = \frac{1}{2}\left(\mathbf{H}\cdot\frac{\partial\mu_o\mathbf{M}}{\partial t} - \mu_o\mathbf{M}\cdot\frac{\partial\mathbf{H}}{\partial t}\right), \tag{2.6}$$

reflect specific properties of the medium under study and take into account the instantaneous power of interaction between the electromagnetic fields $(\mathbf{E},\ \mathbf{H})$ and the charges — both mobile ones carrying the current $\mathbf{J}$ and bound ones generating the polarization $\mathbf{P}$ and the magnetization $\mathbf{M}$.

In the literature the energy term $\partial w/\partial t$ is conventionally identified with the sum $\mathbf{E}\cdot\partial\mathbf{D}/\partial t + \mathbf{H}\cdot\partial\mathbf{B}/\partial t$, which is true only if $\mathbf{D} = \bar{\boldsymbol{\epsilon}}\cdot\mathbf{E}$ and $\mathbf{B} = \bar{\boldsymbol{\mu}}\cdot\mathbf{H}$ where the tensors $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\mu}}$ are symmetric and time-independent. In this case only the first interaction term (2.4) is taken into account, whereas two others (2.5) and (2.6) are dropped without any justification. As will be evident from our subsequent examination including the second part of the paper, these terms play an important role in the power-energy theorem.

For time-harmonic fields (with time dependence in the form of $\exp(i\omega t)$) one is usually interested in time-average values of the power-energy quantities denoted as $\langle\cdots\rangle$. In this case $\langle\partial w/\partial t\rangle = 0$ so that Eq. (2.3) takes the following form involving the time-average values of quantities:

$$\boldsymbol{\nabla}\cdot\langle\mathbf{S}\rangle = -\langle I_{\mathbf{J}}\rangle - \langle I_{\mathbf{P}}\rangle - \langle I_{\mathbf{M}}\rangle. \tag{2.7}$$

Below we concentrate on bianisotropic media for which there is no equation of motion. Their properties are usually described by the constitutive equations establishing macroscopic local relations among field vectors. It should be emphasized that magneto-electric effects (for instance, optical activity), by their microscopic nature, are brought about by nonlocality of polarization response on electromagnetic actions [33, 35, 36, 41]. But their macroscopic manifestations are usually similar to those of actual time-dispersive media because for plane waves with the wave vector $\mathbf{k} = (\omega/c)\mathbf{n}$ all the constitutive tensor parameters of such media become solely frequency-dependent (see Ref. [41]).

There are a few forms of the constitutive relations for bianisotropic media [33–38]. Among them we choose the following form

$$\mathbf{D} = \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E} + \bar{\boldsymbol{\xi}} \cdot \mathbf{H}, \tag{2.8}$$

$$\mathbf{B} = \bar{\boldsymbol{\zeta}} \cdot \mathbf{E} + \bar{\boldsymbol{\mu}} \cdot \mathbf{H}, \tag{2.9}$$

as more convenient for our subsequent examination. Four constitutive medium parameters $\bar{\boldsymbol{\epsilon}}$, $\bar{\boldsymbol{\mu}}$, $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$ are considered as dyadic functions of frequency given phenomenologically. They comprise all special cases of the physical media without space dispersion:

(i) for the isotropic medium

$$\bar{\boldsymbol{\epsilon}} = \epsilon \bar{\mathbf{I}}, \quad \bar{\boldsymbol{\mu}} = \mu \bar{\mathbf{I}}, \quad \bar{\boldsymbol{\xi}} = \bar{\boldsymbol{\zeta}} = 0; \tag{2.10}$$

(ii) for the double anisotropic medium

$$\bar{\boldsymbol{\epsilon}} \neq \epsilon \bar{\mathbf{I}}, \quad \bar{\boldsymbol{\mu}} \neq \mu \bar{\mathbf{I}}, \quad \bar{\boldsymbol{\xi}} = \bar{\boldsymbol{\zeta}} = 0; \tag{2.11}$$

(iii) for the chiral (biisotropic) medium

$$\bar{\boldsymbol{\epsilon}} = \epsilon \bar{\mathbf{I}}, \quad \bar{\boldsymbol{\mu}} = \mu \bar{\mathbf{I}}, \quad \bar{\boldsymbol{\xi}} = (\chi - i\kappa)\sqrt{\epsilon_o \mu_o}\, \bar{\mathbf{I}}, \quad \bar{\boldsymbol{\zeta}} = (\chi + i\kappa)\sqrt{\epsilon_o \mu_o}\, \bar{\mathbf{I}}, \tag{2.12}$$

where χ and κ are Tellegen's parameter of nonreciprocity and Pasteur's parameter of chirality, respectively [35, 38]. It is known [33–35] that for a bianisotropic medium without losses the dyadics $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\mu}}$ are hermitian (self-adjoint) while the dyadics $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$ are hermitian conjugate (mutually adjoint), that is

$$\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}^{\dagger}, \quad \bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^{\dagger}, \quad \bar{\boldsymbol{\xi}} = \bar{\boldsymbol{\zeta}}^{\dagger}, \tag{2.13}$$

where superscript † denotes transpose and complex conjugate (hermitian conjugate). Relations (2.13) imply that in the general case of lossy media the antihermitian parts $\bar{\boldsymbol{\epsilon}}^{a} = (\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}^{\dagger})/2$, $\bar{\boldsymbol{\mu}}^{a} = (\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}^{\dagger})/2$ and the difference $(\bar{\boldsymbol{\xi}} - \bar{\boldsymbol{\zeta}}^{\dagger})$ are responsible for losses (dielectric, magnetic, and magneto-electric, respectively). If the medium has also the electric losses related to its conductive properties and specified by the conductivity dyadic $\bar{\boldsymbol{\sigma}}_c$, then in addition to Eqs. (2.8) and (2.9) there is another constitutive relation

$$\mathbf{J} = \bar{\boldsymbol{\sigma}}_c \cdot \mathbf{E}. \tag{2.14}$$

Let us calculate the terms in the right-hand side of Eq. (2.7) by using their definitions (2.4) through (2.6) and the constitutive relations (2.8), (2.9), and (2.14):

$$\begin{aligned}
\langle I_{\mathbf{J}}\rangle &\equiv \langle \mathbf{J}\cdot\mathbf{E}\rangle = \frac{1}{2}\mathrm{Re}\{\mathbf{J}\cdot\mathbf{E}^*\} = \frac{1}{2}\mathrm{Re}\{\mathbf{E}^*\cdot\bar{\boldsymbol{\sigma}}_c\cdot\mathbf{E}\}\\
&= \frac{1}{2}\bar{\boldsymbol{\sigma}}_c : \mathbf{E}\mathbf{E}^*, \qquad (2.15)\\
\langle I_{\mathbf{P}}\rangle &\equiv \frac{1}{2}\left\langle \mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t} - \mathbf{P}\cdot\frac{\partial\mathbf{E}}{\partial t}\right\rangle = \frac{1}{2}\left\langle \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial t}\right\rangle\\
&= \frac{1}{4}\mathrm{Re}\left\{\mathbf{E}^*\cdot(i\omega\mathbf{D}) - \mathbf{D}^*\cdot(i\omega\mathbf{E})\right\}\\
&= \frac{1}{4}\mathrm{Re}\left\{i\omega(\mathbf{E}^*\cdot\bar{\boldsymbol{\epsilon}}\cdot\mathbf{E} + \mathbf{E}^*\cdot\bar{\boldsymbol{\xi}}\cdot\mathbf{H} - \mathbf{E}\cdot\bar{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}^* - \mathbf{E}\cdot\bar{\boldsymbol{\xi}}^*\cdot\mathbf{H}^*)\right\}\\
&= \frac{1}{4}\mathrm{Re}\left\{i\omega[\mathbf{E}^*\cdot(\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}^\dagger)\cdot\mathbf{E} + 2\mathbf{E}^*\cdot\bar{\boldsymbol{\xi}}\cdot\mathbf{H}]\right\}\\
&= \frac{1}{2}\mathrm{Re}\left\{i\omega(\bar{\boldsymbol{\epsilon}}^a : \mathbf{E}\mathbf{E}^* + \bar{\boldsymbol{\xi}} : \mathbf{H}\mathbf{E}^*)\right\}, \qquad (2.16)\\
\langle I_{\mathbf{M}}\rangle &\equiv \frac{1}{2}\left\langle \mathbf{H}\cdot\frac{\partial\mu_o\mathbf{M}}{\partial t} - \mu_o\mathbf{M}\cdot\frac{\partial\mathbf{H}}{\partial t}\right\rangle\\
&= \frac{1}{2}\left\langle \mathbf{H}\cdot\frac{\partial\mu_o\mathbf{B}}{\partial t} - \mu_o\mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial t}\right\rangle = \frac{1}{4}\mathrm{Re}\left\{\mathbf{H}^*\cdot(i\omega\mathbf{B}) - \mathbf{B}^*\cdot(i\omega\mathbf{H})\right\}\\
&= \frac{1}{4}\mathrm{Re}\left\{i\omega(\mathbf{H}^*\cdot\bar{\boldsymbol{\mu}}\cdot\mathbf{H} + \mathbf{H}^*\cdot\bar{\boldsymbol{\zeta}}\cdot\mathbf{E} - \mathbf{H}\cdot\bar{\boldsymbol{\mu}}^*\cdot\mathbf{H}^* - \mathbf{H}\cdot\bar{\boldsymbol{\zeta}}^*\cdot\mathbf{E}^*)\right\}\\
&= \frac{1}{4}\mathrm{Re}\left\{i\omega[\mathbf{H}^*\cdot(\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}^\dagger)\cdot\mathbf{H} - 2\mathbf{E}^*\cdot\bar{\boldsymbol{\zeta}}^\dagger\cdot\mathbf{H}]\right\}\\
&= \frac{1}{2}\mathrm{Re}\left\{i\omega(\bar{\boldsymbol{\mu}}^a : \mathbf{H}\mathbf{H}^* - \bar{\boldsymbol{\zeta}}^\dagger : \mathbf{H}\mathbf{E}^*)\right\}. \qquad (2.17)
\end{aligned}$$

Substitution of Eqs. (2.15)–(2.17) into Eq. (2.7) gives the time-average Poynting theorem in the following form

$$\bar{\boldsymbol{\nabla}}\cdot\langle\mathbf{S}\rangle + \langle q\rangle = 0 \qquad (2.18)$$

involving the average Poynting vector

$$\langle\mathbf{S}\rangle = \frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\} \qquad (2.19)$$

and the average power loss density

$$\begin{aligned}
\langle q\rangle &= \langle I_{\mathbf{J}}\rangle + \langle I_{\mathbf{P}}\rangle + \langle I_{\mathbf{M}}\rangle\\
&= \frac{1}{2}\bar{\boldsymbol{\sigma}}_e : \mathbf{E}\mathbf{E}^* + \frac{1}{2}\bar{\boldsymbol{\sigma}}_m : \mathbf{H}\mathbf{H}^* + \frac{1}{2}\mathrm{Re}\{\bar{\boldsymbol{\sigma}}_{me} : \mathbf{H}\mathbf{E}^*\}
\end{aligned} \qquad (2.20)$$

where we have introduced the total tensor of *electric conductivity*

$$\bar{\boldsymbol{\sigma}}_e = \bar{\boldsymbol{\sigma}}_c + \bar{\boldsymbol{\sigma}}_d = \bar{\boldsymbol{\sigma}}_c + i\omega\,\bar{\boldsymbol{\epsilon}}^a \equiv \bar{\boldsymbol{\sigma}}_c + \frac{i\omega(\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}^\dagger)}{2} \tag{2.21}$$

associated with conductor $(\bar{\boldsymbol{\sigma}}_c)$ and dielectric $(\bar{\boldsymbol{\sigma}}_d = i\omega\,\bar{\boldsymbol{\epsilon}}^a)$ losses of a medium, the tensor of *magnetic conductivity*

$$\bar{\boldsymbol{\sigma}}_m = i\omega\,\bar{\boldsymbol{\mu}}^a \equiv \frac{i\omega(\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}^\dagger)}{2} \tag{2.22}$$

associated with magnetic losses of a medium, and the tensor of *magneto-electric conductivity*

$$\bar{\boldsymbol{\sigma}}_{me} = i\omega(\bar{\boldsymbol{\xi}} - \bar{\boldsymbol{\zeta}}^\dagger) \equiv i\omega\left[(\bar{\boldsymbol{\xi}}^a + \bar{\boldsymbol{\zeta}}^a) + (\bar{\boldsymbol{\xi}}^h - \bar{\boldsymbol{\zeta}}^h)\right] \tag{2.23}$$

consisting of both antihermitian (with superscript a) and hermitian (with superscript h) parts of the cross susceptibilities $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$. Unlike $\bar{\boldsymbol{\sigma}}_{me}$, the dyadics $\bar{\boldsymbol{\sigma}}_e = \bar{\boldsymbol{\sigma}}_c + i\omega\,\bar{\boldsymbol{\epsilon}}^a$ and $\bar{\boldsymbol{\sigma}}_m = i\omega\,\bar{\boldsymbol{\mu}}^a$ are hermitian so that they produce the real (positive) definite quadratic forms in Eq. (2.20).

2.2 Mode Power Transmission and Dissipation

In order to obtain expressions for the power carried by modes along a waveguiding structure involving complex (anisotropic and bian-isotropic) media and to find the dissipation of mode power it is necessary to go from the time-average Poynting theorem in differential form (2.18) to its integral form. For this purpose let us integrate Eq. (2.18) over the composite (multilayered) cross section $S = \sum S_i$ formed from a few medium parts S_i with interface contours L_i by using the two-dimensional divergence theorem (e.g., see Ref. [5], p.150)

$$\int_{S_i} \boldsymbol{\nabla} \cdot \mathbf{A}\, dS = \frac{\partial}{\partial z}\int_{S_i} \mathbf{z}_o \cdot \mathbf{A}\, dS = \oint_{L_i} \mathbf{n}_o \cdot \mathbf{A}\, dl \tag{2.24}$$

where $\mathbf{A}$ is the arbitrary field vector and $\mathbf{n}_o$ is the *outward* unit vector normal to the contour L_i and perpendicular to the longitudinal unit vector $\mathbf{z}_o$.

Application of the integral relation (2.24) to $\boldsymbol{\nabla} \cdot \langle \mathbf{S} \rangle$ gives

$$\int_S \boldsymbol{\nabla} \cdot \langle \mathbf{S} \rangle\, dS = \frac{\partial}{\partial z}\int_S \mathbf{z}_o \cdot \langle \mathbf{S} \rangle\, dS - \sum_i \oint_{L_i} \left[\mathbf{n}_i^+ \cdot \langle \mathbf{S}^+ \rangle + \mathbf{n}_i^- \cdot \langle \mathbf{S}^- \rangle\right] dl \tag{2.25}$$

where $\langle \mathbf{S}^{\pm} \rangle$ means values of the time-average Poynting vector taken at points of contour L_i lying on its different sides marked by the *inward* (for either adjacent medium) unit vectors $\mathbf{n}_i^{\pm}$. The parts of interfaces between two adjacent nonconducting media do not contribute to the line integrals in Eq. (2.25) owing to continuity in tangential components of the electric and magnetic fields. The only contribution may appear from the parts of L_i corresponding to conducting surfaces on which there is the known boundary condition [2, 5, 35]

$$\mathbf{E}_\tau = \bar{\mathbf{Z}}_s \cdot (\mathbf{H}_\tau \times \mathbf{n}_i) \tag{2.26}$$

where $\mathbf{E}_\tau$ and $\mathbf{H}_\tau$ are the electric and magnetic fields tangential to the surface and $\bar{\mathbf{Z}}_s$ is the surface impedance tensor. For the special case of the isotropic metallic surface with the conductivity σ and the skin depth $\delta = \sqrt{2/\omega\mu_o\sigma}$ we have [2, 5]

$$\bar{\mathbf{Z}}_s = (1+i)\mathcal{R}_s\bar{\mathbf{I}} \quad \text{where} \quad \mathcal{R}_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu_o}{2\sigma}}. \tag{2.27}$$

In this case the integrand of the line integral in Eq. (2.25) yields the surface loss power density $\langle q' \rangle$ in addition to the bulk loss power density $\langle q \rangle$ entering into Poynting's theorem (2.18).

The result of integrating Eq. (2.18) over the total cross section S of the waveguide and applying Eqs. (2.19), (2.20), and (2.25)–(2.27) give Poynting's theorem in the integral from

$$\frac{\partial P}{\partial z} + Q = 0 \tag{2.28}$$

where the total real power carried by electromagnetic fields in the direction of increasing coordinate z is equal to

$$P = \int_S \langle \mathbf{S} \rangle \cdot \mathbf{z}_o \, dS = \frac{1}{2} \text{Re} \int_S (\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{z}_o \, dS \tag{2.29}$$

and the total power loss per unit length caused by the bulk losses $Q^{(b)}$ (obtained by integrating $\langle q \rangle$ over $S = \sum S_i$) and the surface (skin) losses $Q^{(s)}$ (obtained by integrating $\langle q' \rangle$ along $L = \sum L_i$) is equal to

$$\begin{aligned} Q &= Q^{(b)} + Q^{(s)} = \int_S \langle q \rangle \, dS + \int_L \langle q' \rangle \, dl \\ &= \frac{1}{2} \int_S (\bar{\boldsymbol{\sigma}}_e : \mathbf{E}\mathbf{E}^*) \, dS + \frac{1}{2} \int_S (\bar{\boldsymbol{\sigma}}_m : \mathbf{H}\mathbf{H}^*) \, dS \\ &\quad + \frac{1}{2} \text{Re} \int_S (\bar{\boldsymbol{\sigma}}_{me} : \mathbf{H}\mathbf{E}^*) \, dS + \frac{1}{2} \int_L \mathcal{R}_s \, (\mathbf{H}_\tau \cdot \mathbf{H}_\tau^*) \, dl \end{aligned} \tag{2.30}$$

Power relation (2.28) is valid only for the source-free region of a wave-guiding structure where the electromagnetic fields can be expanded in terms of its eigenmodes (cf. Eqs. ($A.29$))

$$\mathbf{E}(\mathbf{r}_t, z) = \sum_k A_k \, \hat{\mathbf{E}}_k(\mathbf{r}_t) \, \mathrm{e}^{-\gamma_k z} = \sum_k a_k(z) \, \hat{\mathbf{E}}_k(\mathbf{r}_t), \tag{2.31}$$

$$\mathbf{H}(\mathbf{r}_t, z) = \sum_k A_k \, \hat{\mathbf{H}}_k(\mathbf{r}_t) \, \mathrm{e}^{-\gamma_k z} = \sum_k a_k(z) \, \hat{\mathbf{H}}_k(\mathbf{r}_t). \tag{2.32}$$

Every k th mode is specified by the propagation constant $\gamma_k = \alpha_k + i\beta_k$ and the eigenfunctions $\{\hat{\mathbf{E}}_k, \, \hat{\mathbf{H}}_k\}$ (where the hat sign over field vectors implies their dependence only on transverse coordinates $\mathbf{r}_t$, see Appendix A.2), which are regarded as known quantities found from solving the appropriate boundary-value problem. The amplitudes A_k are determined by the exciting sources and called the *excitation amplitudes*. Inside the source region they depend on z as a result of source actions but for the source-free region $A_k(z) = const.$, as in the case of Eqs. (2.31) and (2.32). It is often convenient instead of $A_k(z)$ to introduce the *mode amplitudes*

$$a_k(z) = A_k(z) \, \mathrm{e}^{-\gamma_k z} \tag{2.33}$$

which take into account the total z-dependence related both to the mode propagation $(\exp(-\gamma_k z))$ and to the exciting sources $(A_k(z))$, if any.

Let us employ the modal expansions (2.31) and (2.32) to calculate the power flow $P(z)$ and the power loss $Q(z)$ given by Eqs. (2.29) and (2.30) for the source-free region. The final result of calculations is the following:

$$\begin{aligned} P(z) &= \frac{1}{4} \int_S (\mathbf{E}^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{z}_0 \, dS \\ &= \frac{1}{4} \sum_k \sum_l N_{kl} \, a_k^*(z) a_l(z) \equiv \sum_k \sum_l P_{kl}(z) \, , \end{aligned} \tag{2.34}$$

$$\begin{aligned} Q(z) &= \frac{1}{2} \int_S (\mathbf{E}^* \cdot \bar{\boldsymbol{\sigma}}_e \cdot \mathbf{E}) \, dS + \frac{1}{2} \int_S (\mathbf{H}^* \cdot \bar{\boldsymbol{\sigma}}_m \cdot \mathbf{H}) \, dS + \\ &\quad + \frac{1}{4} \int_S (\mathbf{E}^* \cdot \bar{\boldsymbol{\sigma}}_{me} \cdot \mathbf{H} + \mathbf{H}^* \cdot \bar{\boldsymbol{\sigma}}_{me}^\dagger \cdot \mathbf{E}) \, dS + \frac{1}{2} \int_L \mathcal{R}_s \, (\mathbf{H}_\tau^* \cdot \mathbf{H}_\tau) \, dl \\ &= \frac{1}{4} \sum_k \sum_l M_{kl} \, a_k^*(z) a_l(z) \equiv \sum_k \sum_l Q_{kl}(z) \end{aligned} \tag{2.35}$$

where we have introduced the normalizing coefficients

$$N_{kl} = \int_S (\hat{\mathbf{E}}_k^* \times \hat{\mathbf{H}}_l + \hat{\mathbf{E}}_l \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0 \, dS \tag{2.36}$$

and the dissipative coefficients

$$\begin{aligned} M_{kl} = 2 \int_S (\hat{\mathbf{E}}_k^* \cdot \bar{\boldsymbol{\sigma}}_e \cdot \hat{\mathbf{E}}_l) \, dS + \, 2 \int_S (\hat{\mathbf{H}}_k^* \cdot \bar{\boldsymbol{\sigma}}_m \cdot \hat{\mathbf{H}}_l) \, dS \\ + \int_S (\hat{\mathbf{E}}_k^* \cdot \bar{\boldsymbol{\sigma}}_{me} \cdot \hat{\mathbf{H}}_l + \hat{\mathbf{H}}_k^* \cdot \bar{\boldsymbol{\sigma}}_{me}^{\dagger} \cdot \hat{\mathbf{E}}_l) \, dS + \, 2 \int_L \mathcal{R}_s \, (\hat{\mathbf{H}}_{\tau,k}^* \cdot \hat{\mathbf{H}}_{\tau,l}) \, dl \end{aligned} \tag{2.37}$$

constructed of the cross-section eigenfield vectors (marked with the hat sign above them, see Eq. (3.5)).

From Eqs. (2.36) and (2.37) it follows that the matrices $\{N_{kl}\}$ and $\{M_{kl}\}$ are hermitian, that is

$$N_{kl} = N_{lk}^* \qquad \text{and} \qquad M_{kl} = M_{lk}^* \, , \tag{2.38}$$

and have dimensions of watts and watts per meter, respectively, because the amplitudes A_k and a_k are dimensionless.

The quantities $P_k(z)$ and $Q_k(z)$ appearing in Eqs. (2.34) and (2.35) for $l = k$ in the following form

$$P_k(z) \equiv P_{kk}(z) = \frac{1}{4} \, N_{kk} \, a_k^*(z) a_k(z) = \frac{1}{4} \, N_k \, |A_k|^2 \, \mathrm{e}^{-2\alpha_k z} \tag{2.39}$$

$$Q_k(z) \equiv Q_{kk}(z) = \frac{1}{4} \, M_{kk} \, a_k^*(z) a_k(z) = \frac{1}{4} \, M_k \, |A_k|^2 \, \mathrm{e}^{-2\alpha_k z} \tag{2.40}$$

are the real *self powers* transmitted and dissipated at point z by the k th mode which was excited at point $z = 0$ with amplitude A_k .

Similarly, the quantities $P_{kl}(z)$ and $Q_{kl}(z)$ for $l \neq k$ equal to

$$P_{kl}(z) = \frac{1}{4} \, N_{kl} \, a_k^*(z) a_l(z) = \frac{1}{4} \, N_{kl} \, A_k^* A_l \, \mathrm{e}^{-\, (\gamma_k^* + \gamma_l) z} \tag{2.41}$$

$$Q_{kl}(z) = \frac{1}{4} \, M_{kl} \, a_k^*(z) a_l(z) = \frac{1}{4} \, M_{kl} \, A_k^* A_l \, \mathrm{e}^{-\, (\gamma_k^* + \gamma_l) z} \tag{2.42}$$

can be interpreted as the complex *cross powers* transmitted and dissipated at point z jointly by the k th and l th modes which were excited at point $z = 0$ with amplitudes A_k and A_l . Owing to (2.38), the quantities defined by Eqs. (2.41) and (2.42) are also hermitian:

$$P_{kl}(z) = P_{lk}^*(z) \qquad \text{and} \qquad Q_{kl}(z) = Q_{lk}^*(z) \, . \tag{2.43}$$

From Eqs. (2.41) through (2.43) it follows that in a lossy waveguiding structure every pair of modes always transmits and dissipates the real *combined cross powers*

$$P_{kl}^c(z) \equiv P_{kl}(z) + P_{lk}(z) = 2\,\mathrm{Re}\,P_{kl}(z) = \frac{1}{2}\,\mathrm{Re}\,\{N_{kl}\,a_k^*(z)a_l(z)\}\,, \tag{2.44}$$

$$Q_{kl}^c(z) \equiv Q_{kl}(z) + Q_{lk}(z) = 2\,\mathrm{Re}\,Q_{kl}(z) = \frac{1}{2}\,\mathrm{Re}\,\{M_{kl}\,a_k^*(z)a_l(z)\}. \tag{2.45}$$

Therefore, the double sums in Eqs. (2.34) and (2.35) yield the real (time-average) *total powers* transmitted and dissipated by all modes in a lossy waveguide:

$$\begin{aligned} P(z) &= \sum_k \sum_l P_{kl}(z) = \sum_k P_k(z) + \sum_k \sum_{l \neq k} P_{kl}^c(z) \\ &= \frac{1}{4} \sum_k N_k\,|a_k(z)|^2 + \frac{1}{2}\,\mathrm{Re} \sum_k \sum_{l>k} N_{kl}\,a_k^*(z)a_l(z), \end{aligned} \tag{2.46}$$

$$\begin{aligned} Q(z) &= \sum_k \sum_l Q_{kl}(z) = \sum_k Q_k(z) + \sum_k \sum_{l \neq k} Q_{kl}^c(z) \\ &= \frac{1}{4} \sum_k M_k\,|a_k(z)|^2 + \frac{1}{2}\,\mathrm{Re} \sum_k \sum_{l>k} M_{kl}\,a_k^*(z)a_l(z)\,, \end{aligned} \tag{2.47}$$

where $N_k \equiv N_{kk}$ and $M_k \equiv M_{kk}$.

In the next section we shall derive a relation named the *quasi-orthogonality* relation to link the cross powers P_{kl} and Q_{kl} for every pair of modes in a lossy waveguide or for every pair of the so-called *twin-conjugate* modes in a lossless waveguide.

3. ORTHOGONALITY AND QUASI-ORTHOGONALITY OF MODES IN LOSSLESS AND LOSSY WAVEGUIDES

3.1 Quasi-orthogonality Relation for Lossy Waveguides

Let us begin our examination with the general case of the composite (multilayered) waveguiding structure containing bianisotropic media with bulk (electric, magnetic, magneto-electric) losses and surface (skin) losses. Consider the k th and l th modes propagating in the source-free region of the waveguide which obey the curl Maxwell equations (2.1) rewritten by using the constitutive relations (2.8), (2.9), and (2.14) in the following form

$$\nabla \times \mathbf{E}_{k(l)} = -\, i\omega\, \bar{\boldsymbol{\mu}} \cdot \mathbf{H}_{k(l)} \;-\; i\omega\, \bar{\boldsymbol{\zeta}} \cdot \mathbf{E}_{k(l)}\ , \tag{3.1}$$

$$\nabla \times \mathbf{H}_{k(l)} = (\, \bar{\boldsymbol{\sigma}}_c + i\omega\, \bar{\boldsymbol{\epsilon}}\,) \cdot \mathbf{E}_{k(l)} \;+\; i\omega\, \bar{\boldsymbol{\xi}} \cdot \mathbf{H}_{k(l)}\, . \tag{3.2}$$

A conventional procedure applied to Eqs. (3.1) and (3.2) gives

$$\begin{aligned} \nabla \cdot (\mathbf{E}_k^* \times \mathbf{H}_l + \mathbf{E}_l \times \mathbf{H}_k^*) = &\; -2\, \mathbf{E}_k^* \cdot \bar{\boldsymbol{\sigma}}_e \cdot \mathbf{E}_l - 2\, \mathbf{H}_k^* \cdot \bar{\boldsymbol{\sigma}}_m \cdot \mathbf{H}_l \\ &\; -(\, \mathbf{E}_k^* \cdot \bar{\boldsymbol{\sigma}}_{me} \cdot \mathbf{H}_l + \mathbf{H}_k^* \cdot \bar{\boldsymbol{\sigma}}_{me}^{\dagger} \cdot \mathbf{E}_l) \end{aligned} \tag{3.3}$$

where we have used formulas (2.21) – (2.23).

Application of the two-dimensional divergence theorem (2.24) to the left-hand side of Eq. (3.3), by analogy with formula (2.25) and by using the boundary condition (2.26), results in the following expression

$$\begin{aligned} &\frac{\partial}{\partial z} \int_S (\mathbf{E}_k^* \times \mathbf{H}_l + \mathbf{E}_l \times \mathbf{H}_k^*) \cdot \mathbf{z}_0\, dS \;= \\ &= -\;2 \int_S (\mathbf{E}_k^* \cdot \bar{\boldsymbol{\sigma}}_e \cdot \mathbf{E}_l)\, dS - 2 \int_S (\mathbf{H}_k^* \cdot \bar{\boldsymbol{\sigma}}_m \cdot \mathbf{H}_l)\, dS \\ &\quad - \int_S (\mathbf{E}_k^* \cdot \bar{\boldsymbol{\sigma}}_{me} \cdot \mathbf{H}_l + \mathbf{H}_k^* \cdot \bar{\boldsymbol{\sigma}}_{me}^{\dagger} \cdot \mathbf{E}_l)\, dS - 2 \int_L \mathcal{R}_s\, (\mathbf{H}_{\tau,k}^* \cdot \mathbf{H}_{\tau,l})\, dl\, . \end{aligned} \tag{3.4}$$

Representation of the fields for the k th and l th modes in the form

$$\mathbf{E}_{k(l)}(\mathbf{r}_t, z) = \hat{\mathbf{E}}_{k(l)}(\mathbf{r}_t)\, \mathrm{e}^{-\gamma_{k(l)} z}, \qquad \mathbf{H}_{k(l)}(\mathbf{r}_t, z) = \hat{\mathbf{H}}_{k(l)}(\mathbf{r}_t)\, \mathrm{e}^{-\gamma_{k(l)} z} \tag{3.5}$$

and their substitution into Eq. (3.4) give, by comparing with formulas (2.36) and (2.37) for N_{kl} and M_{kl}, the desired relation

$$(\gamma_k^* + \gamma_l)\, N_{kl} = M_{kl} \tag{3.6}$$

referred to as the *quasi-orthogonality relation.* It will play the same role in deriving the excitation equations (see Eq. (5.46)) as the ordinary orthogonality relation.

Expressions (2.41) and (2.42) relate the normalizing and dissipative coefficients N_{kl} and M_{kl} to the cross-power flow P_{kl} and the cross-power loss Q_{kl}, respectively, transmitted and dissipated jointly by the

k th and l th modes. The use of these expressions allows us to rewrite the quasi-orthogonality relation (3.6) in the power form

$$(\gamma_k^* + \gamma_l)\, P_{kl} = \, Q_{kl} \, . \tag{3.7}$$

This formulation furnishes the following power interpretation of mode quasi-orthogonality: outside the source region every pair of modes, independently of other modes, transmits the complex cross-power flow P_{kl} rigidly coupled to the complex cross-power loss Q_{kl} by the factor $(\gamma_k^* + \gamma_l)$ consisting of the mode propagation constants $\gamma_{k(l)} \; = \; \alpha_{k(l)} + i\beta_{k(l)}$, and in doing so the combined cross powers $P_{kl}^c \; = \; P_{kl} + P_{lk}$ and $Q_{kl}^c \; = \; Q_{kl} + Q_{lk}$ always remain real. The quasi-orthogonality relation (3.7) means that outside the source region Poynting's theorem (2.28) takes place for any one of mode pairs (k, l) :

$$\frac{dP_{kl}}{dz} + \, Q_{kl} \, = \, 0 \, . \tag{3.8}$$

Besides, every single mode has the real self-power flow $P_k \equiv P_{kk}$ and self-power loss $Q_k \equiv Q_{kk}$ in the form of Eqs. (2.39) and Eq. (2.40). These self powers are coupled to each other by the same relations (3.7) and (3.8) which for $l = k$ yield the following expression for the attenuation constant:

$$\alpha_k = \frac{Q_k}{2\, P_k} = \frac{M_k}{2\, N_k} \, . \tag{3.9}$$

Hence, there occurs the following pattern of mode power transfer in the lossy waveguiding structures. Every k th mode propagates from the source region with the fixed value of amplitude A_k (the loss attenuation is taken into account by the amplitude constant α_k appearing in γ_k) which was excited by the sources. Outside them the mode, being a linearly independent solution to the boundary-value problem, does not interact with other modes owing to their linear independence. The k th mode transfers the self power P_k on its own and the cross powers P_{kl} in conjunction with the other l th modes which were also excited inside the source region and outside retain constant their excitation amplitudes A_l as well as the k th mode.

3.2 Mode Orthogonality in Lossless Waveguides

The orthogonality relation for a lossless waveguiding structure is obtained from the general relation (3.6) as the special case of $M_{kl} = 0$

and has the following form

$$(\gamma_k^* + \gamma_l)\, N_{kl} = 0\,. \tag{3.10}$$

In spite of the absence of dissipation, in the eigenmode spectrum of the lossless waveguide, besides propagating modes with $\alpha_k \equiv 0$ and $\gamma_k = i\beta_k$, there are also modes having complex values of the propagation constant $\gamma_k = \alpha_k + i\beta_k$ with $\alpha_k \neq 0$. These modes exist in the cutoff regime of propagation and their attenuation is of reactive (nondissipative) character associated with the storage of reactive power. As a token of this, it seems reasonable to refer to such modes as the *reactive* modes to distinguish between them and the *active* (propagating) modes carrying an active (real) power.

In the literature the reactive (in our terminology) modes are variously termed the complex, cutoff, nonpropagating, and evanescent modes. The last term is usually attributed only to cutoff modes with pure decay ($\alpha_k \neq 0$) and without phase delay ($\beta_k = 0$). The latter feature of evanescent modes makes appropriate for them also the term "nonpropagating" because there is no phase propagation. But for the complex modes with $\alpha_k \neq 0$ and $\beta_k \neq 0$ their reactive decay as $\exp(-\,\alpha_k z)$ is accompanied by the phase variation in accordance with the wave factor $\exp[i(\omega t - \beta_k z)]$. For this reason it is more preferable to refer to the complex modes as reactive modes rather than nonpropagating ones. However, we shall apply both terms, the *reactive* and *nonpropagating* modes, as well as their antitheses, the *active* and *propagating* modes, to reflect the fact that the former do not transfer any self power, whereas the latter carry it.

Let us show that in any lossless waveguide, independently of its structure and media used, every reactive mode with number k has its own *twin* mode with number $\tilde{k}$ (marked by tilde) so that their propagation constants are related by the equality

$$\gamma_{\tilde{k}} = -\,\gamma_k^* \qquad \text{or} \qquad \alpha_{\tilde{k}} = -\,\alpha_k\,, \quad \beta_{\tilde{k}} = \beta_k\,. \tag{3.11}$$

Such mode twins with pair of numbers $(k, \tilde{k})$ that satisfy the relation (3.11) will be referred to as the *twin-conjugate* modes. As is seen from Eq. (3.11), these modes have the same phase velocity ($\beta_k = \beta_{\tilde{k}}$) but decay in opposite directions $(\alpha_k = -\,\alpha_{\tilde{k}})$.

The existence of twin-conjugate modes possessing the property expressed by Eq. (3.11) can be justified by means of the following reasoning. In our treatment of complex amplitude technique, we have chosen the wave factor in the form $\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})]$ where $k_z \equiv -i\gamma$. However, there is another alternative form $\exp[-i(\omega t - \tilde{\mathbf{k}}\cdot\mathbf{r})]$ with $\tilde{k}_z \equiv -i\tilde{\gamma}$ which differs from the first form in opposite sign of imaginary unity and having tilde above the wave vector. It is clearly evident that the alternative case can be obtained from our solution by applying complex conjugation, then

$$\tilde{k}_z \equiv -i\tilde{\gamma} = k_z^* \equiv i\gamma^* \qquad \text{whence} \qquad \tilde{\gamma} = -\gamma^*. \tag{3.12}$$

Equalities for gammas in Eqs. (3.11) and (3.12) are fully coincident not counting different positions of tilde (the former marking the mode number $\tilde{k}$ in subscripts will be used later on). This result substantiates the existence of twin-conjugate modes for which $\gamma_k = -\gamma_{\tilde{k}}^*$ or $k_{z,k} = k_{z,\tilde{k}}^*$. In other words, any dispersion equation obtained as a result of solving the boundary-value problem for lossless systems has the complex roots with complex-conjugate values of the longitudinal wavenumber k_z which appear in pairs. Such a pair of complex roots corresponds to the twin-conjugate modes.

Sign of the amplitude constant α_k can be used as the basis for classification of the reactive (nonpropagating) modes under two types, *forward* and *backward*, as is usually done for the active (propagating) modes but on the basis of a sign of the group velocity $v_{gr,k} = [d\beta_k(\omega)/d\omega]^{-1}$. In reference to the source region location between $z = 0$ and $z = L$, all the modes (active and reactive) can be classified into two categories:

(i) the *forward* modes marked by subscript $k = +n > 0$ (active with $v_{gr,+n} > 0$ or reactive with $\alpha_{+n} > 0$) which, being excited inside the source region, leave it (without or with reactive damping) across the right boundary and exist outside at $z > L$;

(ii) the *backward* modes marked by subscript $k = -n < 0$ (active with $v_{gr,-n} < 0$ or reactive with $\alpha_{-n} < 0$) which, being excited inside the source region, leave it (without or with reactive damping) across the left boundary and exist outside at $z < 0$.

3.2.1 Orthogonality and Normalization Relations for Active Modes

The active (propagating) modes exist in the pass band of lossless waveguides where they have zero amplitude attenuation $(\alpha_{k(l)} = 0)$, so that their propagation constants $\gamma_{k(l)} = i\beta_{k(l)}$ are pure imaginary. In this case the orthogonality relation (3.10) rewritten in the form

$$(\beta_k - \beta_l)\, N_{kl} = 0 \tag{3.13}$$

along with expression (2.36) for N_{kl} furnishes two alternatives:

$$N_{kl} \equiv \int_S (\hat{\mathbf{E}}_k^* \times \hat{\mathbf{H}}_l + \hat{\mathbf{E}}_l \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0 \, dS = 0 \qquad \text{for} \qquad l \neq k \tag{3.14}$$

or

$$N_k \equiv N_{kk} = 2\,\mathrm{Re} \int_S (\hat{\mathbf{E}}_k^* \times \hat{\mathbf{H}}_k) \cdot \mathbf{z}_0 \, dS \neq 0 \qquad \text{for} \qquad l = k\,. \tag{3.15}$$

Expression (3.14) is the orthogonality relation between the different propagating modes for which $\beta_k - \beta_l \neq 0$, whereas formula (3.15) defines the *norm* $N_k \equiv N_{kk}$ of the kth mode. It should be noted that Eq. (3.14) does not necessarily hold for different but degenerate modes with $\beta_k = \beta_l$. In this case one can employ the conventional technique commonly used for usual waveguides [2, 3] to ensure the orthogonality among degenerate modes by constructing from them such linear combinations that constitute a new orthogonal subset for which relation (3.14) is applicable. For this reason we shall no longer turn special attention to degenerate modes.

From Eqs. (2.41)) and (3.14) it follows that two different propagating modes (with numbers $k \neq l$) have zero cross-power flow $(P_{kl} = 0)$, i.e., they are orthogonal in power sense. Any mode carries along a waveguide only the self power P_k defined by formula (2.39), which gives the following power interpretation for the norm of active modes: N_k is equal to $4P_k^{\mathrm{o}}$ where P_k^{o} means the time-average power carried in the positive z-direction by the kth mode with unit amplitude ($|A_k| = 1$). In some instances it may be more convenient to normalize the mode amplitude to unit power ($|P_k^{\mathrm{o}}| = 1$ watt). Then $|N_k| = 4$ watts and according to Eq. (2.39)

$$P_k = \pm\,|a_k|^2 = \pm\,|A_k|^2 \tag{3.16}$$

where subscripts should be read as $k = \pm\,n$, with upper and lower signs corresponding to the forward and backward modes for which, respectively, $N_{+n} = 4$ watts and $N_{-n} = -\,4$ watts. In the special case of a reciprocal waveguide wherein for every forward mode there is a backward one with the same law of dispersion, their norms are related to each other by the equality

$$N_{+n} = -\,N_{-n}\,. \tag{3.17}$$

In conclusion, let us write the relation of orthonormalization for the active (propagating) modes in the following form

$$N_{kl} = N_{kk}\,\delta_{kl} \equiv \begin{cases} 0 & \text{for} \quad l \neq k\,, \\ N_{kk} \equiv N_k & \text{for} \quad l = k\,, \end{cases} \tag{3.18}$$

where the normalizing coefficient N_{kl} and the norm N_k are given by Eqs. (3.14) and (3.15), respectively.

3.2.2 Orthogonality and Normalization Relations for Reactive Modes

The reactive modes of a lossless waveguide are cutoff modes whose propagation constants $\gamma_{k(l)} = \alpha_{k(l)} + i\beta_{k(l)}$ are generally complex-valued or particularly real-valued for the evanescent modes with $\beta_{k(l)} = 0$. So the general relation of orthogonality (3.10) holds for them and by using Eq. (3.11) for the twin-conjugate modes gives two alternatives:

$$N_{kl} \equiv \int_S (\hat{\mathbf{E}}_k^* \times \hat{\mathbf{H}}_l + \hat{\mathbf{E}}_l \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0\, dS = 0 \qquad \text{for} \qquad l \neq \tilde{k} \tag{3.19}$$

or

$$N_k \equiv N_{k\tilde{k}} = \int_S (\hat{\mathbf{E}}_k^* \times \hat{\mathbf{H}}_{\tilde{k}} + \hat{\mathbf{E}}_{\tilde{k}} \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0\, dS \neq 0 \qquad \text{for} \qquad l = \tilde{k}\,. \tag{3.20}$$

Expression (3.19) fulfills a role of the orthogonality relation for the reactive modes. As is seen from here, every reactive k th mode is orthogonal to all the l th modes (reactive with $\alpha_l \neq 0$ and active with $\alpha_l = 0$) for which $\gamma_l + \gamma_k^* = \gamma_l - \gamma_{\tilde{k}} \neq 0$ and $N_{kl} = 0$, including itself since $\gamma_k + \gamma_k^* = 2\alpha_k \neq 0$ and $N_{kk} = 0$. The only mode nonorthogonal to the given k th mode is its own twin with number $\tilde{k}$ for which $\gamma_{\tilde{k}} + \gamma_k^* = 0$. Formula (3.20) defines the *norm* $N_k \equiv N_{k\tilde{k}}$ for the reactive k th mode which is constructed of the fields of twin-conjugate modes $(k, \tilde{k})$.

From Eqs. (2.39), (2.41), and (3.19) it follows that every reactive mode has no both the self power ($P_{kk} = 0$) and the cross powers with the other modes $(P_{kl} = 0)$ for which $\gamma_l \neq -\gamma_k^*$, i.e., these modes are orthogonal in power sense.

Each mode forming a twin-conjugate pair, being nonorthogonal to its twin, has its own norm defined by Eq. (3.20) as

$$N_k \equiv N_{k\tilde{k}} \qquad \text{or} \qquad N_{\tilde{k}} \equiv N_{\tilde{k}k}\,, \tag{3.21}$$

whence, according to the general property of hermitian symmetry for the normalizing coefficients expressed by equality (2.38), it follows that

$$N_k = N^*_{\tilde{k}} \,, \tag{3.22}$$

i.e., the reactive twin-conjugate modes have the complex-conjugate norms.

Although the reactive mode has no self power $(P_{kk} \equiv 0)$, the twin-conjugate modes in pair carry the real combined cross power (cf. Eq. (2.44))

$$P^c_{k\tilde{k}} \equiv P_{k\tilde{k}} + P_{\tilde{k}k} = 2\,\mathrm{Re}\,P_{k\tilde{k}} = \frac{1}{2}\,\mathrm{Re}\,\{N_k\, A^*_k A_{\tilde{k}}\} \equiv \frac{1}{2}\,\mathrm{Re}\,\{N_{\tilde{k}}\, A^*_{\tilde{k}} A_k\} \tag{3.23}$$

where subscripts should be read as $k = +n$ and $\tilde{k} = -n$. This is a consequence of relations (3.11) for the twin-conjugate modes and the definition of forward and backward reactive modes: if the kth mode is a forward one with $N_k = N_{+n} \equiv N_{+n,-n}$, then the $\tilde{k}$th mode is a backward one with $N_{\tilde{k}} = N_{-n} \equiv N_{-n,+n} = N^*_{+n}$.

As evident from Eq. (3.23), to transfer the real power by reactive modes it is necessary that both constituents of a twin-conjugate pair should have nonzero amplitudes (A_k and $A_{\tilde{k}}$) and to be in such a phase relationship that their combined cross power $P^c_{k\tilde{k}}$ would be other than zero. Similar situation usually takes place in the regular waveguide of finite length bounded by two irregularities and excited at frequencies below its cutoff frequency [1–3]. Reflections from these irregularities can form inside this length two evanescent (cutoff) modes with numbers $k = +n$ (forward mode) and $\tilde{k} = -n$ (backward mode) constituting the twin-conjugate pair for which $\beta_{\pm n} = 0$ and $\gamma_{+n} \equiv \alpha_{+n} = -\alpha_{-n} \equiv -\gamma^*_{-n}$. It is easy to see that the norms for the forward and backward evanescent modes are pure imaginary-valued and related to each other by the general relation (3.22). Superposition of fields for the two evanescent modes with opposite decay sense furnishes nonzero real cross-power flow along a short length of the cutoff waveguide.

In conclusion, let us write the relation of orthonormalization for the reactive (nonpropagating) modes in the following form

$$N_{kl} = N_{k\tilde{k}}\,\delta_{\tilde{k}l} \equiv \begin{cases} 0 & \text{for} \quad l \neq \tilde{k}\,, \\ N_{k\tilde{k}} \equiv N_k & \text{for} \quad l = \tilde{k}\,, \end{cases} \tag{3.24}$$

where the normalizing coefficient N_{kl} and the norm N_k are given by Eqs. (3.19) and (3.20), respectively. From comparison of Eqs. (3.18) and (3.24) it is seen that the latter relation is of general form because it comprises the former one for the active modes as a special case obtained by replacing subscript $\tilde{k}$ with k so that, in particular, the norm $N_k \equiv N_{kk}$ takes the form given by Eq. (3.15).

It is pertinent to note that all the above expressions for the norms and the relations of orthogonality and orthonormalization can contain the total field vectors in place of their cross section parts related to each other by Eqs. (3.5), i.e., the hat sign over the field vectors can be dropped. This is obvious for the active modes and follows from the equality $\gamma_k + \gamma^*_{\tilde{k}} = 0$ for the reactive twin-conjugate modes.

If in a waveguiding structure there are both the active (propagating) and reactive (nonpropagating) modes, the total power flow (2.46) carried by them, in accordance with the aforesaid, is given by the following expression

$$\begin{aligned} P = P_{act} + P_{react} &= \sum_{\substack{k \\ (active)}} P_k + {\sum_{\substack{k \\ (reactive)}}}' P^c_{k\tilde{k}} \\ &= \frac{1}{4} \sum_{\substack{k \\ (active)}} N_k \, |a_k(z)|^2 + \frac{1}{2} \operatorname{Re} {\sum_{\substack{k \\ (reactive)}}}' N_k \, a^*_k(z) a_{\tilde{k}}(z) \end{aligned} \tag{3.25}$$

where prime on the sum sign means summation of the twin-conjugate modes in pairs rather than that of the single reactive modes.

3.3 Time-average Stored Energy for Active Modes in Lossless Waveguides

The time-average Poynting theorem written in the form of Eq. (2.18) for time-harmonic fields does not contain a stored energy density. In order to find it one usually applies variational technique (e.g., see Ref. [29]). To this end, it is necessary to obtain a relation between variations of the electromagnetic fields $(\delta \mathbf{E}_k, \, \delta \mathbf{H}_k)$ for the k th mode and perturbations of the frequency and medium parameters $(\, \delta(\omega \, \bar{\boldsymbol{\epsilon}}\,),$ $\delta(\omega \, \bar{\boldsymbol{\mu}}\,), \; \delta(\omega \, \bar{\boldsymbol{\xi}}\,), \; \delta(\omega \, \bar{\boldsymbol{\zeta}}\,)\,)$ which bring about these variations.

The k th mode is governed by Maxwell's equations (3.1) and (3.2) with $\bar{\boldsymbol{\sigma}}_c = 0$ for a lossless medium whose other parameters satisfy the requirements (2.13). By taking variations in these equations we obtain

$$\nabla \times \delta \mathbf{E}_k = -\, i\omega\, \bar{\boldsymbol{\mu}} \cdot \delta \mathbf{H}_k -\, i\omega\, \bar{\boldsymbol{\zeta}} \cdot \delta \mathbf{E}_k -\, i\delta(\omega\, \bar{\boldsymbol{\mu}}) \cdot \mathbf{H}_k -\, i\delta(\omega\, \bar{\boldsymbol{\zeta}}) \cdot \mathbf{E}_k\,, \tag{3.26}$$

$$\nabla \times \delta \mathbf{H}_k = \ i\omega\, \bar{\boldsymbol{\epsilon}} \cdot \delta \mathbf{E}_k +\, i\omega\, \bar{\boldsymbol{\xi}} \cdot \delta \mathbf{H}_k +\, i\delta(\omega\, \bar{\boldsymbol{\epsilon}}\,) \cdot \mathbf{E}_k +\, i\delta(\omega\, \bar{\boldsymbol{\xi}}) \cdot \mathbf{H}_k\,. \tag{3.27}$$

A conventional procedure applied to Eqs. (3.26) and (3.27) reduces to the following relation

$$\begin{aligned}
&\nabla \cdot (\mathbf{E}_k^* \times \delta \mathbf{H}_k + \delta \mathbf{E}_k \times \mathbf{H}_k^*) \;= \\
&\qquad = -\, i\omega \left[\begin{array}{l} \mathbf{E}_k^* \cdot (\,\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}^{\,\dagger}) \cdot \delta \mathbf{E}_k \;+\; \mathbf{H}_k^* \cdot (\,\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}^{\,\dagger}) \cdot \delta \mathbf{H}_k \\ \qquad + \mathbf{E}_k^* \cdot (\,\bar{\boldsymbol{\xi}} - \bar{\boldsymbol{\zeta}}^{\,\dagger}) \cdot \delta \mathbf{H}_k + \mathbf{H}_k^* \cdot (\,\bar{\boldsymbol{\zeta}} - \bar{\boldsymbol{\xi}}^{\,\dagger}) \cdot \delta \mathbf{E}_k \end{array} \right] \\
&\qquad\quad -\; i \left[\begin{array}{l} \mathbf{E}_k^* \cdot \delta(\omega\, \bar{\boldsymbol{\epsilon}}\,) \cdot \mathbf{E}_k + \mathbf{H}_k^* \cdot \delta(\omega\, \bar{\boldsymbol{\mu}}) \cdot \mathbf{H}_k \\ \qquad + \mathbf{E}_k^* \cdot \delta(\omega\, \bar{\boldsymbol{\xi}}) \cdot \mathbf{H}_k + \mathbf{H}_k^* \cdot \delta(\omega\, \bar{\boldsymbol{\zeta}}) \cdot \mathbf{E}_k \end{array} \right]
\end{aligned}$$

where the terms inside the first square brackets vanish because of relations (2.13) for lossless media so that

$$\begin{aligned}
&\nabla\cdot(\mathbf{E}_k^* \times \delta \mathbf{H}_k + \delta \mathbf{E}_k \times \mathbf{H}_k^*) \;= \\
&\quad = -\, i \Big[\, \delta(\omega\, \bar{\boldsymbol{\epsilon}}\,) : \mathbf{E}_k \mathbf{E}_k^* + \delta(\omega\, \bar{\boldsymbol{\mu}}) : \mathbf{H}_k \mathbf{H}_k^* + 2\,\mathrm{Re}\,\{\delta(\omega\, \bar{\boldsymbol{\xi}}) : \mathbf{H}_k \mathbf{E}_k^* \Big]\,.
\end{aligned} \tag{3.28}$$

Electromagnetic fields of a propagating mode and their variations can be written on the basis of Eq. (3.5) as

$$\mathbf{E}_k \;=\; \hat{\mathbf{E}}_k\, \mathrm{e}^{-i\,\beta_k z}\,, \quad \delta \mathbf{E}_k = (\delta \hat{\mathbf{E}}_k - i\delta\beta_k z\, \hat{\mathbf{E}}_k)\, \mathrm{e}^{-i\,\beta_k z}\,, \tag{3.29}$$

$$\mathbf{H}_k = \hat{\mathbf{H}}_k\, \mathrm{e}^{-i\,\beta_k z}\,, \quad \delta \mathbf{H}_k = (\delta \hat{\mathbf{H}}_k - i\delta\beta_k z\, \hat{\mathbf{H}}_k)\, \mathrm{e}^{-i\,\beta_k z}. \tag{3.30}$$

When substituting Eqs. (3.29) and (3.30) into Eq. (3.28) and applying the integral relation (2.25) where the line integral vanishes owing to continuity in tangential components of the fields, the integration over the cross section S yields

$$\begin{aligned}
&\delta\beta_k\, \frac{1}{2} \mathrm{Re} \int_S (\mathbf{E}_k \times \mathbf{H}_k^*) \cdot \mathbf{z}_0\, dS \;= \\
&\quad = \frac{1}{4} \int_S \Big[\, \delta(\omega\, \bar{\boldsymbol{\epsilon}}\,) : \hat{\mathbf{E}}_k \hat{\mathbf{E}}_k^* + \delta(\omega\, \bar{\boldsymbol{\mu}}) : \hat{\mathbf{H}}_k \hat{\mathbf{H}}_k^* + 2\,\mathrm{Re}\{\delta(\omega\, \bar{\boldsymbol{\xi}}) : \hat{\mathbf{H}}_k \hat{\mathbf{E}}_k^*\} \Big]\, dS.
\end{aligned} \tag{3.31}$$

The left-hand side of Eq. (3.31) involves the time-average power flow P_k as a multiplier of $\delta\beta_k$. By using the known relation

$$P_k = v_{gr,k}\, W_k \qquad \text{where} \qquad v_{gr,k} = \left(\frac{\partial \beta_k(\omega)}{\partial \omega}\right)^{-1} \tag{3.32}$$

is the group velocity, we obtain from Eq. (3.31) the desired expression for the time-average energy stored per unit length of a waveguide (dropping the mode index k):

$$\begin{aligned} W = &\frac{1}{4}\int_S \left(\mathbf{E}^* \cdot \frac{\partial(\omega\,\bar{\boldsymbol{\epsilon}})}{\partial\,\omega} \cdot \mathbf{E} + \mathbf{H}^* \cdot \frac{\partial(\omega\,\bar{\boldsymbol{\mu}})}{\partial\,\omega} \cdot \mathbf{H}\right) dS \\ &+ \frac{1}{2}\,\mathrm{Re}\int_S \left(\mathbf{E}^* \cdot \frac{\partial(\omega\,\bar{\boldsymbol{\xi}})}{\partial\,\omega} \cdot \mathbf{H}\right) dS \\ \equiv &\int_S w\, dS\,. \end{aligned} \tag{3.33}$$

In accordance with expression (3.33), the time-average stored energy density for a lossless bianisotropic medium is given by the following formula

$$w = \frac{1}{4}\left(\mathbf{E}^* \;\; \mathbf{H}^*\right) \cdot \begin{pmatrix} \partial(\omega\,\bar{\boldsymbol{\epsilon}})/\partial\,\omega & \partial(\omega\,\bar{\boldsymbol{\xi}})/\partial\,\omega \\ \partial(\omega\,\bar{\boldsymbol{\zeta}})/\partial\,\omega & \partial(\omega\,\bar{\boldsymbol{\mu}})/\partial\,\omega \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{3.34}$$

which is a generalization of the conventional expression for lossless anisotropic media [7,10,22].

4. ORTHOGONAL COMPLEMENTS AND EFFECTIVE SURFACE CURRENTS INSIDE SOURCE REGION

4.1 Bulk and Surface Exciting Sources

Up to the present, the external sources exciting the composite (multilayered) waveguiding structures involving isotropic, anisotropic, and bianisotropic media have been dropped. From this point onward, the special attention will be given to investigating the behavior of modes inside the source region. In doing so, we assume that all the eigenfields $\{\mathbf{E}_k, \mathbf{H}_k\}$ in the form of Eq. (3.5), including their eigenfunctions of cross-section coordinates $\{\hat{\mathbf{E}}_k, \hat{\mathbf{H}}_k\}$ (marked by hat over them) and their eigenvalues of propagation constants $\gamma_k = \alpha_k + i\beta_k$, are known from solving the corresponding boundary-value problem. As shown in Appendix A, these eigenfields constitute an infinite countable set of the vector functions quadratically integrable on the cross section S of a waveguiding structure. This set can be taken as a basis of the proper Hilbert space to expand the required fields $\mathbf{E}$ and $\mathbf{H}$ not only outside sources, as was done by Eqs. (2.31) and (2.32), but also inside the region of external sources. In general, this eigenvector basis is not complete inside the source region since it cannot take into account entirely the potential fields of the sources. This requires to supplement the *modal expansions* $\mathbf{E}_a$ and $\mathbf{H}_a$ with unknown modal amplitudes $A_k(z)$ by the *orthogonal complements* $\mathbf{E}_b$ and $\mathbf{H}_b$ (see Eqs. (A.27) and (A.28)). Hence, the desired issues to be obtained inside the source region are both the longitudinal dependence of modal amplitudes and the orthogonal complements to the modal expansions.

In the most general case there exist three physical reasons to excite the waveguiding structure under examination:

(a) the external currents – electric $\mathbf{J}^e_{ext}$ and magnetic $\mathbf{J}^m_{ext}$,
(b) the external fields – electric $\mathbf{E}_{ext}$ and magnetic $\mathbf{H}_{ext}$,
(c) the external perturbations of bianisotropic medium parameters $\Delta\bar{\boldsymbol{\epsilon}}$, $\Delta\bar{\boldsymbol{\mu}}$, $\Delta\bar{\boldsymbol{\xi}}$, and $\Delta\bar{\boldsymbol{\zeta}}$.

Owing to these medium perturbations, the total electric $(\mathbf{E}+\mathbf{E}_{ext})$ and magnetic $(\mathbf{H}+\mathbf{H}_{ext})$ fields create the excess electric $\Delta\mathbf{D}$ and magnetic $\Delta\mathbf{B}$ inductions linked by the constitutive relations (2.8) and (2.9), that is

$$\Delta\mathbf{D} = \Delta\bar{\boldsymbol{\epsilon}}\cdot(\mathbf{E}+\mathbf{E}_{ext}) + \Delta\bar{\boldsymbol{\xi}}\cdot(\mathbf{H}+\mathbf{H}_{ext})\,, \tag{4.1}$$

$$\Delta\mathbf{B} = \Delta\bar{\boldsymbol{\zeta}}\cdot(\mathbf{E}+\mathbf{E}_{ext}) + \Delta\bar{\boldsymbol{\mu}}\cdot(\mathbf{H}+\mathbf{H}_{ext})\,. \tag{4.2}$$

These excess inductions bring about the induced displacement currents – electric $\mathbf{J}^e_{ind} = i\omega\Delta\mathbf{D}$ and magnetic $\mathbf{J}^m_{ind} = i\omega\Delta\mathbf{B}$ which, being added to the external currents $\mathbf{J}^e_{ext}$ and $\mathbf{J}^m_{ext}$, yield the *bulk exciting currents*

$$\mathbf{J}^e_b = \mathbf{J}^e_{ext} + \mathbf{J}^e_{ind} = \mathbf{J}^e_{ext} + i\omega\Delta\mathbf{D}\,, \quad \mathbf{J}^m_b = \mathbf{J}^m_{ext} + \mathbf{J}^m_{ind} = \mathbf{J}^m_{ext} + i\omega\Delta\mathbf{B} \tag{4.3}$$

entering into the curl Maxwell equations (2.1) in the following form

$$\boldsymbol{\nabla}\times\mathbf{E} = -i\omega\mathbf{B} - \mathbf{J}^m_b\,, \tag{4.4}$$

$$\boldsymbol{\nabla}\times\mathbf{H} = i\omega\mathbf{D} + \mathbf{J}^e_b\,. \tag{4.5}$$

The conduction current $\mathbf{J}$ of a conductive medium defined by Eq. (2.14) is now assumed to be incorporated with the electric displacement current $i\omega\mathbf{D}$, whereas the induction vectors $\mathbf{D}$ and $\mathbf{B}$ are taken, as before, to be related to the intrinsic electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ inside the medium in question by the same constitutive relations (2.8) and (2.9). So the permittivity tensor $\bar{\boldsymbol{\epsilon}}$ is now regarded as a sum $(\bar{\boldsymbol{\epsilon}} + \bar{\boldsymbol{\sigma}}_c/i\omega)$ whose antihermitian part defines the total tensor of electric conductivity $\bar{\boldsymbol{\sigma}}_e = \bar{\boldsymbol{\sigma}}_c + \bar{\boldsymbol{\sigma}}_d$ given by Eq. (2.21).

Besides the bulk currents $\mathbf{J}_b^e$ and $\mathbf{J}_b^m$, there may exist the *surface exciting currents* $\mathbf{J}_s^e$ and $\mathbf{J}_s^m$ which give discontinuities of the tangential components of fields at the surface whereon these sources are located, written in the form of the following boundary conditions:

$$\mathbf{n}_s^+ \times \mathbf{E}^+ + \mathbf{n}_s^- \times \mathbf{E}^- = -\mathbf{J}_s^m, \tag{4.6}$$

$$\mathbf{n}_s^+ \times \mathbf{H}^+ + \mathbf{n}_s^- \times \mathbf{H}^- = \mathbf{J}_s^e. \tag{4.7}$$

Here the field vectors with superscripts $\pm$ mean their values taken at points of the source location contour L_s lying on its different sides marked by the *inward* (for either adjacent medium) unit vectors $\mathbf{n}_s^\pm$.

4.2 Orthogonal Complementary Fields and Effective Surface Currents

The general electrodynamic eigenmode treatment (see Appendix A.2) based on the well-known mathematical formulations (see Appendix A.1) yields the complete representation of the desired field vector $\mathbf{F}$ inside sources as a sum of the the modal expansion $\boldsymbol{\Psi}$ giving a *projection* of $\mathbf{F}$ onto the Hilbert space and the complement $\mathbf{C}$ *orthogonal* to the Hilbert space (see Eq. (A.25)). Thus, the electromagnetic fields inside the source region have the complete representation given by Eqs. (A.27) and (A.28), namely (cf. Eqs. (2.31) and (2.32)):

$$\mathbf{E}(\mathbf{r}_t, z) = \mathbf{E}_a(\mathbf{r}_t, z) + \mathbf{E}_b(\mathbf{r}_t, z) = \sum_k a_k(z)\,\hat{\mathbf{E}}_k(\mathbf{r}_t) + \mathbf{E}_b(\mathbf{r}_t, z), \tag{4.8}$$

$$\mathbf{H}(\mathbf{r}_t, z) = \mathbf{H}_a(\mathbf{r}_t, z) + \mathbf{H}_b(\mathbf{r}_t, z) = \sum_k a_k(z)\,\hat{\mathbf{H}}_k(\mathbf{r}_t) + \mathbf{H}_b(\mathbf{r}_t, z), \tag{4.9}$$

where $\mathbf{E}_b(\mathbf{r}_t, z)$ and $\mathbf{H}_b(\mathbf{r}_t, z)$ are the required orthogonal complements. The mode amplitude $a_k(z) = A_k(z)\exp(-\gamma_k z)$ allows for the total dependence on z due to both the unperturbed propagation of the k th mode with a constant γ_k and the perturbed amplitude $A_k(z)$ as a result of source actions. Using Eqs. (2.8) and (2.9) gives the similar expressions for the induction vectors

$$\mathbf{D}(\mathbf{r}_t, z) = \mathbf{D}_a(\mathbf{r}_t, z) + \mathbf{D}_b(\mathbf{r}_t, z) = \sum_k a_k(z)\, \hat{\mathbf{D}}_k(\mathbf{r}_t) + \mathbf{D}_b(\mathbf{r}_t, z), \tag{4.10}$$

$$\mathbf{B}(\mathbf{r}_t, z) = \mathbf{B}_a(\mathbf{r}_t, z) + \mathbf{B}_b(\mathbf{r}_t, z) = \sum_k a_k(z)\, \hat{\mathbf{B}}_k(\mathbf{r}_t) + \mathbf{B}_b(\mathbf{r}_t, z). \tag{4.11}$$

In order to find the orthogonal complements let us substitute Eqs. (4.8) through (4.11) into Maxwell's equations (4.4) and (4.5) with taking into account the fact that the eigenfields (3.5) satisfy the homogeneous (with no sources) Maxwell equations (3.1) and (3.2). Some transformations yield

$$\sum_k \frac{dA_k}{dz}\,(\mathbf{z}_0 \times \mathbf{E}_k) = -\boldsymbol{\nabla} \times \mathbf{E}_b \,-\, i\omega \mathbf{B}_b \,-\, \mathbf{J}_b^m\,, \tag{4.12}$$

$$\sum_k \frac{dA_k}{dz}\,(\mathbf{z}_0 \times \mathbf{H}_k) = -\boldsymbol{\nabla} \times \mathbf{H}_b \,+\, i\omega \mathbf{D}_b \,+\, \mathbf{J}_b^e\,. \tag{4.13}$$

The left-hand side of Eqs. (4.12) and (4.13) has only transverse components. From here it necessarily follows that there must exist nonzero orthogonal complementary fields. Otherwise (when $\mathbf{E}_b = \mathbf{H}_b = \mathbf{D}_b = \mathbf{B}_b = 0$) these equations become physically contradictory because then they require the longitudinal components of the arbitrary bulk currents $\mathbf{J}_b^e$ and $\mathbf{J}_b^m$ to be always equal to zero, which is of course not the case.

Hence, the required orthogonal complementary fields should be chosen so as to make the longitudinal component of the right-hand part of Eqs. (4.12) and (4.13) vanish, that is

$$\boldsymbol{\nabla} \cdot (\mathbf{z}_0 \times \mathbf{E}_b) - \mathbf{z}_0 \cdot (i\omega \mathbf{B}_b + \mathbf{J}_b^m) = 0\,, \tag{4.14}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{z}_0 \times \mathbf{H}_b) + \mathbf{z}_0 \cdot (i\omega \mathbf{D}_b + \mathbf{J}_b^e) = 0\,, \tag{4.15}$$

where the identity $\mathbf{z}_0 \cdot (\boldsymbol{\nabla} \times \mathbf{a}) = -\boldsymbol{\nabla} \cdot (\mathbf{z}_0 \times \mathbf{a})$ has been used.

Since the field parts $\mathbf{E}_b$ and $\mathbf{H}_b$ form the orthogonal complement to the Hilbert space spanned by the base eigenvectors $\{\mathbf{E}_k, \mathbf{H}_k\}$, they must be orthogonal to the fields of any eigenmode in power sense given by the relation similar to Eqs. (3.14) and (3.19) (cf. Eq. (A.26))):

$$\int_S (\mathbf{E}_k^* \times \mathbf{H}_b + \mathbf{E}_b \times \mathbf{H}_k^*) \cdot \mathbf{z}_0 \, dS \equiv \equiv \int_S \Big[(\mathbf{z}_0 \times \mathbf{E}_b) \cdot \mathbf{H}_k^* - (\mathbf{z}_0 \times \mathbf{H}_b) \cdot \mathbf{E}_k^* \Big] dS = 0 \,. \tag{4.16}$$

In virtue of arbitrary choice of the kth eigenmode taken from the base set, zero equality in Eq. (4.16) can occur if and only if

$$\mathbf{z}_0 \times \mathbf{E}_b = 0 \qquad \text{or} \qquad \mathbf{E}_b = \mathbf{z}_0 \, E_b \,, \tag{4.17}$$

$$\mathbf{z}_0 \times \mathbf{H}_b = 0 \qquad \text{or} \qquad \mathbf{H}_b = \mathbf{z}_0 \, H_b \,, \tag{4.18}$$

i.e., both complementary fields are longitudinal. In order for their magnitude to be found, it is necessary to insert Eqs. (2.8), (2.9), (4.17), and (4.18) into Eqs. (4.14) and (4.15), then

$$\mathbf{z}_0 \cdot \mathbf{D}_b \equiv \epsilon_{zz} \, E_b + \xi_{zz} \, H_b = - \frac{1}{i\omega} \, J_{bz}^e \,, \tag{4.19}$$

$$\mathbf{z}_0 \cdot \mathbf{B}_b \equiv \zeta_{zz} \, E_b + \mu_{zz} \, H_b = - \frac{1}{i\omega} \, J_{bz}^m \,. \tag{4.20}$$

From here it finally follows that

$$E_b = - \frac{J_{bz}^e - \nu^m J_{bz}^m}{i\omega\epsilon_{zz} \, (1 - \nu^e \nu^m)} \,, \tag{4.21}$$

$$H_b = - \frac{J_{bz}^m - \nu^e J_{bz}^e}{i\omega\mu_{zz} \, (1 - \nu^e \nu^m)} \,, \tag{4.22}$$

where we have denoted

$$\nu^e = \frac{\zeta_{zz}}{\epsilon_{zz}} = \frac{\bar{\boldsymbol{\zeta}} : \mathbf{z}_0\mathbf{z}_0}{\bar{\boldsymbol{\epsilon}} : \mathbf{z}_0\mathbf{z}_0} \,, \qquad \nu^m = \frac{\xi_{zz}}{\mu_{zz}} = \frac{\bar{\boldsymbol{\xi}} : \mathbf{z}_0\mathbf{z}_0}{\bar{\boldsymbol{\mu}} : \mathbf{z}_0\mathbf{z}_0} \,. \tag{4.23}$$

Hence, both the orthogonal complementary fields are longitudinal and produced by the longitudinal components of the bulk exciting currents.

Existence of the complementary fields $\mathbf{E}_b$ and $\mathbf{H}_b$ immediately reduces to appearance of the so-called *effective surface currents* $\mathbf{J}_{s,ef}^e$ and $\mathbf{J}_{s,ef}^m$. Consider the bulk source region having the cross section

S_b with a boundary contour L_b and write the complete electric field inside and outside this area:

$$\mathbf{E}(\mathbf{r}_t, z) = \begin{cases} \sum_k A_k(z)\, \mathbf{E}_k(\mathbf{r}_t, z) + \mathbf{E}_b(\mathbf{r}_t, z) \equiv \mathbf{E}^- & -\text{inside} \quad S_b\,, \\ \sum_k A_k(z)\, \mathbf{E}_k(\mathbf{r}_t, z) \equiv \mathbf{E}^+ & -\text{outside}\ S_b\,. \end{cases} \tag{4.24}$$

Analogous expressions can be written for the magnetic field $\mathbf{H}(\mathbf{r}_t, z)$.

The eigenfields $\mathbf{E}_k$ and $\mathbf{H}_k$, being obtained for the situation without sources, are generally continuous at points of the line L_b. Then the tangential components of the complete fields $\mathbf{E}$ and $\mathbf{H}$ prove discontinuous:

$$\mathbf{n}_b^+ \times \mathbf{E}^+ + \mathbf{n}_b^- \times \mathbf{E}^- = -\,\mathbf{n}_b \times \mathbf{E}_b\,, \tag{4.25}$$

$$\mathbf{n}_b^+ \times \mathbf{H}^+ + \mathbf{n}_b^- \times \mathbf{H}^- = -\,\mathbf{n}_b \times \mathbf{H}_b\,, \tag{4.26}$$

where $\mathbf{n}_b = \mathbf{n}_b^+ = -\,\mathbf{n}_b^-$ is the *outward* unit vector normal to both the line L_b and the longitudinal unit vector $\mathbf{z}_0$. The comparison of these relations with the boundary conditions (4.6) and (4.7) yields the desired effective surface currents

$$\mathbf{J}^e_{s,ef} = -\,\mathbf{n}_b \times \mathbf{H}_b \Big|_{L_b} = -\,\boldsymbol{\tau}\, \frac{J^m_{bz}(L_b) -\, \nu^e J^e_{bz}(L_b)}{i\omega\mu_{zz}\,(1 - \nu^e\nu^m)}\,, \tag{4.27}$$

$$\mathbf{J}^m_{s,ef} = \mathbf{n}_b \times \mathbf{E}_b \Big|_{L_b} = \boldsymbol{\tau}\, \frac{J^e_{bz}(L_b) -\, \nu^m J^m_{bz}(L_b)}{i\omega\epsilon_{zz}\,(1 - \nu^e\nu^m)}\,, \tag{4.28}$$

where $J^e_{bz}(L_b)$ and $J^m_{bz}(L_b)$ mean the longitudinal components of the bulk currents taken at points lying on the boundary L_b of their existence area S_b and $\boldsymbol{\tau} = \mathbf{z}_0 \times \mathbf{n}_b$ is the unit vector tangential to the contour L_b.

The general expressions (4.21) and (4.22) for the complementary fields $\mathbf{E}_b$ and $\mathbf{H}_b$ and the general expressions (4.27) and (4.28) for the effective surface currents $\mathbf{J}^e_{s,ef}$ and $\mathbf{J}^m_{s,ef}$ take the following simplified form in special cases of:

(i) the isotropic medium with parameters (2.10) ($\nu^e = \nu^m = 0$)

$$\mathbf{E}_b = -\,\frac{\mathbf{z}_0}{i\omega\epsilon}\, J^e_{bz} \qquad \text{and} \qquad \mathbf{H}_b = -\,\frac{\mathbf{z}_0}{i\omega\mu}\, J^m_{bz}\,, \tag{4.29}$$

$$\mathbf{J}^m_{s,ef} = \frac{\boldsymbol{\tau}}{i\omega\epsilon}\, J^e_{bz}(L_b) \qquad \text{and} \qquad \mathbf{J}^e_{s,ef} = -\,\frac{\boldsymbol{\tau}}{i\omega\mu}\, J^m_{bz}(L_b)\,; \tag{4.30}$$

(ii) the anisotropic medium with parameters (2.11) ($\nu^e = \nu^m = 0$)

$$\mathbf{E}_b = -\frac{\mathbf{z}_0}{i\omega\epsilon_{zz}} J^e_{bz} \quad \text{and} \quad \mathbf{H}_b = -\frac{\mathbf{z}_0}{i\omega\mu_{zz}} J^m_{bz}, \tag{4.31}$$

$$\mathbf{J}^m_{s,ef} = \frac{\boldsymbol{\tau}}{i\omega\epsilon_{zz}} J^e_{bz}(L_b) \quad \text{and} \quad \mathbf{J}^e_{s,ef} = -\frac{\boldsymbol{\tau}}{i\omega\mu_{zz}} J^m_{bz}(L_b). \tag{4.32}$$

Formulas (4.29) and (4.31) are in agreement with those obtained first by Vainshtein [2] and Felsen and Marcuvitz [8], respectively. As for the effective surface currents (4.30) and (4.32), Vainshtein did not consider them at all but the excitation integrals in the theory of Felsen and Marcuvitz allow for them implicitly, which will be shown later (see Sec. 5.1).

Therefore, in the absence of medium bianisotropy the bulk currents, *electric* $\mathbf{J}^e_b$ and *magnetic* $\mathbf{J}^m_b$, generate the effective surface currents, respectively, *magnetic* $\mathbf{J}^m_{s,ef}$ and *electric* $\mathbf{J}^e_{s,ef}$. As is seen from Eqs. (4.27) and (4.28), the bianisotropic properties of a medium intermix the contributions from the bulk currents into the effective surface currents owing to the longitudinal components ξ_{zz} and ζ_{zz}.

The newly obtained effective surface currents $\mathbf{J}^e_{s,ef}$ and $\mathbf{J}^m_{s,ef}$, as well as the actual surface currents $\mathbf{J}^e_s$ and $\mathbf{J}^m_s$ entering into the boundary conditions (4.6) and (4.7), make contributions to the excitation amplitudes A_k together with the bulk currents $\mathbf{J}^e_b$ and $\mathbf{J}^m_b$.

The next step should be done toward deriving the differential equations to find the functions $A_k(z)$ inside the region of bulk and surface sources. For this purpose we shall apply three independent approaches set forth in the next section and Appendix B.

5. EQUATIONS OF MODE EXCITATION

5.1 Approach Based on the Electrodynamic Method of Variation of Constants

Mathematical method of variation of constants is applied to solve an inhomogeneous differential equation (with driving terms) by representing its general solution in the form of a superposition of the known linearly independent solutions of the proper homogeneous equation with coefficients which are no longer considered to be constant and assumed to be the desired functions of an independent variable [42]. Electrodynamic analog of the mathematical method of variation of constants is built by representing the fields $\mathbf{E}(\mathbf{r}_t, z)$ and $\mathbf{H}(\mathbf{r}_t, z)$ inside the source region in the form of the expansions (4.8) and (4.9) in terms of eigenfunctions of the proper homogeneous boundary-value problem (without sources) whose amplitude coefficients $A_k(z)$ are the desired functions of z rather than constants, as they are outside sources.

In accordance with the conventional mathematical technique, the method of variation of constants is to give differential equations for the mode amplitudes in the following form

$$\frac{dA_k(z)}{dz} = f_k(z) , \qquad k = 1, 2, \ldots \tag{5.1}$$

where the functions $f_k(z)$ take into account the longitudinal distribution of exciting sources (bulk and surface). Integration of Eq. (5.1) yields the required dependence

$$A_k(z) \,=\, A_k^{\circ} \,+ \int f_k(z)\, dz \,\equiv\, A_k^{\circ} \,+\, \Delta A_k(z) \,. \tag{5.2}$$

The integration constant A_k° should be determined from a boundary condition given at one of two boundaries ($z = 0$ or $z = L$) of the source region depending on the type of modes for a lossless waveguiding structure:

(i) for the *forward* modes (active and reactive, $k = +n$) supplied at the left input

$$A_{+n}(0) \,\neq\, 0 \qquad \text{or} \qquad A_{+n}(0) \,=\, 0 \,, \tag{5.3}$$

(ii) for the *backward* modes (active and reactive, $k = -n$) supplied at the right input

$$A_{-n}(L) \,\neq\, 0 \qquad \text{or} \qquad A_{-n}(L) \,=\, 0 \,. \tag{5.4}$$

Substitution of Eq. (5.2) into Eqs. (4.8) and (4.9) allows us to represent the complete solution for the electromagnetic fields inside the source region as the sum of the general solution $(\mathbf{E}_{gen}, \mathbf{H}_{gen})$ to the homogeneous boundary-value problem (without exciting sources) involving the constant amplitude coefficients A_k° and the particular solution $(\mathbf{E}_{par}, \mathbf{H}_{par})$ to the proper inhomogeneous problem (with exciting sources), namely:

$$\begin{aligned}\mathbf{E}(\mathbf{r}_t,z) &= \mathbf{E}_{gen}+\mathbf{E}_{par}\\ &\equiv \sum_k A_k^{\rm o}\,\mathbf{E}_k(\mathbf{r}_t,z)\;+\;\left[\sum_k \Delta A_k(z)\,\mathbf{E}_k(\mathbf{r}_t,z)+\mathbf{E}_b(\mathbf{r}_t,z)\right],\\ \mathbf{H}(\mathbf{r}_t,z) &= \mathbf{H}_{gen}+\mathbf{H}_{par}\\ &\equiv \sum_k A_k^{\rm o}\,\mathbf{H}_k(\mathbf{r}_t,z)\;+\;\left[\sum_k \Delta A_k(z)\,\mathbf{H}_k(\mathbf{r}_t,z)+\mathbf{H}_b(\mathbf{r}_t,z)\right].\end{aligned}$$

Thus, the general technique of solving the electrodynamic problem of waveguide excitation by external sources based on the method of variation of constants gives rise to the representation of the desired electromagnetic fields as the sum of the general and particular solutions adopted in the theory of linear differential equations. The next task is to obtain a specific form for the excitation equation like Eq. (5.1).

To derive the equation of mode excitation let us vector-multiply both sides of Eqs. (4.8) and (4.9) by $\hat{\mathbf{H}}_l^*$ and $-\hat{\mathbf{E}}_l^*$, respectively, and add them. Then after scalar-multiplying the result of summation by $\mathbf{z}_0$ and integrating over the cross section S of a waveguide we obtain

$$\begin{aligned}&\int_S(\hat{\mathbf{E}}_l^*\times\mathbf{H}+\mathbf{E}\times\hat{\mathbf{H}}_l^*)\cdot\mathbf{z}_0\,dS=\\ &\qquad=\sum_k a_k\int_S(\hat{\mathbf{E}}_l^*\times\hat{\mathbf{H}}_k+\hat{\mathbf{E}}_k\times\hat{\mathbf{H}}_l^*)\cdot\mathbf{z}_0\,dS+\\ &\qquad\quad+\int_S(\hat{\mathbf{E}}_l^*\times\mathbf{H}_b+\mathbf{E}_b\times\hat{\mathbf{H}}_l^*)\cdot\mathbf{z}_0\,dS\;.\end{aligned} \tag{5.5}$$

The last integral in the right-hand side of Eq. (5.5) vanishes because of the orthogonality relation (4.16) or (A.26). According to the orthonormalization relations (3.18) and (3.24), the integral under the sign of summation is equal to $N_l\delta_{lk}$ for the active (propagating) modes and to $N_l\delta_{\tilde{l}k}$ for the reactive (nonpropagating) modes. Hence, from Eq. (5.5) we obtain (cf. Eq. (A.31)):

(i) for the active modes (with replacing subscripts $l\to k$)

$$\begin{aligned}a_k &= \frac{1}{N_k}\int_S(\hat{\mathbf{E}}_k^*\times\mathbf{H}+\mathbf{E}\times\hat{\mathbf{H}}_k^*)\cdot\mathbf{z}_0\,dS\;\equiv\;A_k\,\mathrm{e}^{-i\,\beta_k z}\\ \text{or}\qquad A_k &= \frac{1}{N_k}\int_S(\mathbf{E}_k^*\times\mathbf{H}+\mathbf{E}\times\mathbf{H}_k^*)\cdot\mathbf{z}_0\,dS\end{aligned} \tag{5.6}$$

where the norm N_k is defined by formula (3.15),

(ii) for the reactive modes (with replacing subscripts $l \to \tilde{k}$ and $\tilde{l} \to k$)

$$\begin{aligned} a_k &= \frac{1}{N_{\tilde{k}}} \int_S (\hat{\mathbf{E}}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \hat{\mathbf{H}}^*_{\tilde{k}}) \cdot \mathbf{z}_0 \, dS \equiv A_k \, \mathrm{e}^{-\gamma_k z} \\ \text{or} \qquad A_k &= \frac{1}{N_{\tilde{k}}} \int_S (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}}) \cdot \mathbf{z}_0 \, dS \end{aligned} \tag{5.7}$$

where subscript $\tilde{k}$ corresponds to the mode which together with the k th mode constitute the twin-conjugate pair and have the propagation constant $\gamma_{\tilde{k}} = -\gamma^*_k$ and the norm $N_{\tilde{k}} = N^*_k$ defined by formula (3.20). From comparison of Eqs. (5.6) and (5.7) it is seen that the latter expression can be considered as the general form valid not only for the reactive modes but also for the active modes with replacing $\tilde{k}$ by k.

It is pertinent to note that expression (5.6) for a_k is in agreement with Eq. (A.31) obtained by minimizing the mean-square difference D_n (defined by Eq. (A.33)) between the mode series expansion $\mathbf{\Psi}$ and the partial sum $\mathbf{S}_n$ of the n th order (given by Eq. (A.32)) to provide convergence in mean for the modal expansion.

Formulas (5.6) and (5.7) give a rule to find the mode excitation amplitude if the electromagnetic fields are known. However, this is usually not the case because the exciting currents (bulk and surface) are assumed to be given rather than the fields. In order to go from the fields to the currents, let us differentiate the general relation (5.7) with respect to z:

$$\begin{aligned} N_{\tilde{k}} \frac{dA_k}{dz} &= \frac{\partial}{\partial z} \int_S (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}}) \cdot \mathbf{z}_0 \, dS \\ &= \int_S \boldsymbol{\nabla} \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}}) \, dS \, + \\ &\quad + \sum_i \oint_{L_i} \begin{bmatrix} \mathbf{n}^+_i \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}})^+ \\ + \, \mathbf{n}^-_i \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}})^- \end{bmatrix} dl \end{aligned} \tag{5.8}$$

where the last equality is written by using the relation similar to Eq. (2.25).

The complete fields $\mathbf{E}$ and $\mathbf{H}$ inside the source region satisfy the inhomogeneous Maxwell equations (4.4) and (4.5), whereas the fields

$\mathbf{E}^*_{\tilde{k}}$ and $\mathbf{H}^*_{\tilde{k}}$ of the $\tilde{k}$th mode obey the following homogeneous equations

$$\nabla \times \mathbf{E}^*_{\tilde{k}} = i\omega \mathbf{B}^*_{\tilde{k}}\,, \tag{5.9}$$

$$\nabla \times \mathbf{H}^*_{\tilde{k}} = -i\omega \mathbf{D}^*_{\tilde{k}}\,. \tag{5.10}$$

By using the constitutive relations (2.8), (2.9) and Eqs. (4.4), (4.5), (5.9), and (5.10) it is easy to prove that

$$\begin{aligned}\nabla \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}}) = &- (\mathbf{J}^e_b \cdot \mathbf{E}^*_{\tilde{k}} + \mathbf{J}^m_b \cdot \mathbf{H}^*_{\tilde{k}}) - \\ &- i\omega \left[\begin{array}{l} (\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}^{\dagger}) : \mathbf{E}\mathbf{E}^*_{\tilde{k}} + (\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}^{\dagger}) : \mathbf{H}\mathbf{H}^*_{\tilde{k}} \\ \qquad + (\bar{\boldsymbol{\xi}} - \bar{\boldsymbol{\zeta}}^{\dagger}) : \mathbf{H}\mathbf{E}^*_{\tilde{k}} + (\bar{\boldsymbol{\zeta}} - \bar{\boldsymbol{\xi}}^{\dagger}) : \mathbf{E}\mathbf{H}^*_{\tilde{k}} \end{array} \right]\end{aligned} \tag{5.11}$$

where the square bracket equals zero for a lossless medium owing to Eq. (2.13).

The contour integrals in the right-hand side of Eq. (5.8) include two contributions:

(i) from the *actual surface currents* $\mathbf{J}^e_s$ and $\mathbf{J}^m_s$ which are located on a contour L_s and meet the boundary conditions (4.6) and (4.7),

(ii) from the *effective surface currents* $\mathbf{J}^e_{s,ef}$ and $\mathbf{J}^m_{s,ef}$ given by Eqs. (4.27) and (4.28) which are located on a contour L_b bounding the bulk current area S_b and meet the boundary conditions (4.25) and (4.26).

On the strength of the aforesaid we can write

$$\begin{aligned}&\sum_i \oint_{L_i} \Big[\mathbf{n}^+_i \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}})^+ \,+\, \mathbf{n}^-_i \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}})^- \Big] dl \\ &= \int_{L_s} \Big[\mathbf{n}^+_s \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}})^+ \,+\, \mathbf{n}^-_s \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}})^- \Big] dl \\ &+ \int_{L_b} \Big[\mathbf{n}^+_b \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}})^+ \,+\, \mathbf{n}^-_b \cdot (\mathbf{E}^*_{\tilde{k}} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*_{\tilde{k}})^- \Big] dl \\ &= -\int_{L_s} (\mathbf{J}^e_s \cdot \mathbf{E}^*_{\tilde{k}} + \mathbf{J}^m_s \cdot \mathbf{H}^*_{\tilde{k}})\, dl \,-\, \int_{L_b} (\mathbf{J}^e_{s,ef} \cdot \mathbf{E}^*_{\tilde{k}} + \mathbf{J}^m_{s,ef} \cdot \mathbf{H}^*_{\tilde{k}})\, dl\,.\end{aligned} \tag{5.12}$$

Eqs. (5.8), (5.11), and (5.12) finally give the desired equations of mode excitation written as

(i) for the excitation amplitudes $A_k(z), \quad k = \pm n$:

$$\begin{aligned}\frac{dA_k}{dz} &= -\frac{1}{N_{\tilde{k}}}\int_{S_b}(\mathbf{J}_b^e\cdot\mathbf{E}_{\tilde{k}}^* + \mathbf{J}_b^m\cdot\mathbf{H}_{\tilde{k}}^*)\,dS \\ &\quad -\frac{1}{N_{\tilde{k}}}\int_{L_s}(\mathbf{J}_s^e\cdot\mathbf{E}_{\tilde{k}}^* + \mathbf{J}_s^m\cdot\mathbf{H}_{\tilde{k}}^*)\,dl \\ &\quad -\frac{1}{N_{\tilde{k}}}\int_{L_b}(\mathbf{J}_{s,ef}^e\cdot\mathbf{E}_{\tilde{k}}^* + \mathbf{J}_{s,ef}^m\cdot\mathbf{H}_{\tilde{k}}^*)\,dl\,;\end{aligned} \tag{5.13}$$

(ii) for the mode amplitudes $a_k(z) = A_k(z)\exp(-\gamma_k z), \quad k = \pm n$:

$$\begin{aligned}\frac{da_k}{dz} + \gamma_k\, a_k &= -\frac{1}{N_{\tilde{k}}}\int_{S_b}(\mathbf{J}_b^e\cdot\hat{\mathbf{E}}_{\tilde{k}}^* + \mathbf{J}_b^m\cdot\hat{\mathbf{H}}_{\tilde{k}}^*)\,dS \\ &\quad -\frac{1}{N_{\tilde{k}}}\int_{L_s}(\mathbf{J}_s^e\cdot\hat{\mathbf{E}}_{\tilde{k}}^* + \mathbf{J}_s^m\cdot\hat{\mathbf{H}}_{\tilde{k}}^*)\,dl \\ &\quad -\frac{1}{N_{\tilde{k}}}\int_{L_b}(\mathbf{J}_{s,ef}^e\cdot\hat{\mathbf{E}}_{\tilde{k}}^* + \mathbf{J}_{s,ef}^m\cdot\hat{\mathbf{H}}_{\tilde{k}}^*)\,dl\,.\end{aligned} \tag{5.14}$$

The excitation equations (5.13) and (5.14) written for the amplitudes of reactive modes hold true also for an active mode if one assumes $\tilde{k} = k$ and $\gamma_k = i\beta_k$. The excitation integrals in the right-hand side of these equations represent the complex power of interaction between the external currents (bulk and surface) and the eigenfields of the k th mode (for active ones) or those of its twin-conjugate $\tilde{k}$ th mode (for reactive ones).

As distinct from the theory developed, Vainshtein [2] fully excluded from consideration the reactive (nonpropagating) modes and the effective surface currents and restricted his analysis only to the reciprocal waveguides with isotropic media. In this case every forward-propagating mode $(k = +n)$ has a backward counterpart $(k = -n)$ of the same type so that their common norm is defined by Vainshtein as

$$N_n = \int_S (\hat{\mathbf{E}}_{+n}\times\hat{\mathbf{H}}_{-n} - \hat{\mathbf{E}}_{-n}\times\hat{\mathbf{H}}_{+n})\cdot\mathbf{z}_0\,dS\,.$$

Unlike the definition (3.15), Vainshtein's norm has no power sense and does not allow a generalization to nonreciprocal waveguides to be made.

Theory of Felsen and Marcuvitz [8], unlike Vainshtein's theory, takes into consideration anisotropic (not bianisotropic) media but also does not allow for the reactive modes. The excitation integral in their equation similar to our Eq. (5.14) has a visually different form which does not involve the effective surface currents explicitly. In order for their implicit existence to be displayed, let us convert our excitation integral containing the bulk currents.

To this end, it is necessary to transform the products of longitudinal components such as $J^e_{bz}\, E^*_{kz}$ and $J^m_{bz}\, H^*_{kz}$ (where $k = \tilde{k}$ for reactive modes). The use of the constitutive relations $\mathbf{D} = \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}$ and $\mathbf{B} = \bar{\boldsymbol{\mu}} \cdot \mathbf{H}$ for a double-anisotropic medium in Eqs. (5.9) and (5.10) written for the k th mode gives the longitudinal projections of these equations:

$$(\boldsymbol{\nabla} \times \mathbf{E}^*_k)_z \equiv (\boldsymbol{\nabla}_t \times \mathbf{E}^*_{kt}) \cdot \mathbf{z}_0 = i\omega\, (\mu^*_{zx} H^*_{kx} + \mu^*_{zy} H^*_{ky} + \mu^*_{zz} H^*_{kz})\,, \tag{5.15}$$

$$(\boldsymbol{\nabla} \times \mathbf{H}^*_k)_z \equiv (\boldsymbol{\nabla}_t \times \mathbf{H}^*_{kt}) \cdot \mathbf{z}_0 = -\, i\omega\, (\epsilon^*_{zx} E^*_{kx} + \epsilon^*_{zy} E^*_{ky} + \epsilon^*_{zz} E^*_{kz})\,. \tag{5.16}$$

Taking into account that for a lossless medium $\epsilon^*_{ij} = \epsilon_{ji}$ and $\mu^*_{ij} = \mu_{ji}$, on the basis of Eqs. (5.15) and (5.16) we can obtain the following expressions

$$\begin{aligned} J^e_{bz}\, E^*_{kz} &= -\,\frac{1}{i\omega\epsilon_{zz}}\,(\boldsymbol{\nabla}_t \times \mathbf{H}^*_{kt}) \cdot \mathbf{J}^e_{bz} - \frac{\epsilon_{xz} E^*_{kx} + \epsilon_{yz} E^*_{ky}}{\epsilon_{zz}}\, J^e_{bz} \\ &= -\,\frac{1}{i\omega}\,\boldsymbol{\nabla}_t \cdot \left(\mathbf{H}^*_{kt} \times \frac{\mathbf{J}^e_{bz}}{\epsilon_{zz}}\right) - \frac{1}{i\omega}\left(\boldsymbol{\nabla}_t \times \frac{\mathbf{J}^e_{bz}}{\epsilon_{zz}}\right) \cdot \mathbf{H}^*_{kt} - \frac{\boldsymbol{\epsilon}_{tz}}{\epsilon_{zz}}\, J^e_{bz}\,, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} J^m_{bz}\, H^*_{kz} &= \frac{1}{i\omega\mu_{zz}}\,(\boldsymbol{\nabla}_t \times \mathbf{E}^*_{kt}) \cdot \mathbf{J}^m_{bz} - \frac{\mu_{xz} H^*_{kx} + \mu_{yz} H^*_{ky}}{\mu_{zz}}\, J^m_{bz} \\ &= \frac{1}{i\omega}\,\boldsymbol{\nabla}_t \cdot \left(\mathbf{E}^*_{kt} \times \frac{\mathbf{J}^m_{bz}}{\mu_{zz}}\right) + \frac{1}{i\omega}\left(\boldsymbol{\nabla}_t \times \frac{\mathbf{J}^m_{bz}}{\mu_{zz}}\right) \cdot \mathbf{E}^*_{kt} - \frac{\boldsymbol{\mu}_{tz}}{\mu_{zz}}\, J^m_{bz}\,, \end{aligned} \tag{5.18}$$

where following [8] we have introduced the auxiliary vectors

$$\boldsymbol{\epsilon}_{tz} = \mathbf{x}_0\, \epsilon_{xz} + \mathbf{y}_0\, \epsilon_{yz} \qquad \text{and} \qquad \boldsymbol{\mu}_{tz} = \mathbf{x}_0\, \mu_{xz} + \mathbf{y}_0\, \mu_{yz}\,. \tag{5.19}$$

It is easy to see that the terms in Eqs. (5.17) and (5.18) containing the transverse divergence operator $\boldsymbol{\nabla}_t \cdot$, after integrating over the bulk current area S_b, yield the following results

$$\frac{1}{i\omega}\int_{S_b}\boldsymbol{\nabla}_t\cdot\left(\mathbf{E}^*_{kt}\times\frac{\mathbf{J}^m_{bz}}{\mu_{zz}}\right)dS=\frac{1}{i\omega}\oint_{L_b}\mathbf{n}_b\cdot\left(\mathbf{E}^*_{kt}\times\frac{\mathbf{J}^m_{bz}}{\mu_{zz}}\right)dl$$
$$\equiv-\int_{L_b}\mathbf{J}^e_{s,ef}\cdot\mathbf{E}^*_k\,dl\ ,\qquad(5.20)$$
$$-\frac{1}{i\omega}\int_{S_b}\boldsymbol{\nabla}_t\cdot\left(\mathbf{H}^*_{kt}\times\frac{\mathbf{J}^e_{bz}}{\epsilon_{zz}}\right)dS=-\frac{1}{i\omega}\oint_{L_b}\mathbf{n}_b\cdot\left(\mathbf{H}^*_{kt}\times\frac{\mathbf{J}^e_{bz}}{\epsilon_{zz}}\right)dl$$
$$\equiv-\int_{L_b}\mathbf{J}^m_{s,ef}\cdot\mathbf{H}^*_k\,dl\ ,\qquad(5.21)$$

where expressions (4.32) for the effective surface currents have been used.

As is quite evident, the terms (5.20) and (5.21), being inserted in the excitation integral with the bulk currents by means of equalities (5.17) and (5.18), fully compensate for the contribution from the excitation integral with the effective surface currents entering into Eq. (5.13) and (5.14). Then the excitation equation (5.14) written for propagating modes takes the form entirely coincident with that of Felsen and Marcuvitz [8] (in different notation):

$$\begin{aligned}\frac{da_k}{dz}+\gamma_k\,a_k=&-\frac{1}{N_k}\int_{S_b}(\mathbf{J}^e_{b,ef}\cdot\hat{\mathbf{E}}^*_{kt}+\mathbf{J}^m_{b,ef}\cdot\hat{\mathbf{H}}^*_{kt})\,dS\\&-\frac{1}{N_k}\int_{L_s}(\mathbf{J}^e_s\cdot\hat{\mathbf{E}}^*_k+\mathbf{J}^m_s\cdot\hat{\mathbf{H}}^*_k)\,dl\end{aligned}\qquad(5.22)$$

where following [8] we have introduced the effective bulk currents

$$\mathbf{J}^e_{b,ef}=\mathbf{J}^e_{bt}+\frac{1}{i\omega}\left(\boldsymbol{\nabla}_t\times\frac{\mathbf{J}^m_{bz}}{\mu_{zz}}\right)-\frac{\boldsymbol{\epsilon}_{tz}}{\epsilon_{zz}}\,J^e_{bz}\ ,\qquad(5.23)$$
$$\mathbf{J}^m_{b,ef}=\mathbf{J}^m_{bt}-\frac{1}{i\omega}\left(\boldsymbol{\nabla}_t\times\frac{\mathbf{J}^e_{bz}}{\epsilon_{zz}}\right)-\frac{\boldsymbol{\mu}_{tz}}{\mu_{zz}}\,J^m_{bz}\ ,\qquad(5.24)$$

with the transverse vectors $\boldsymbol{\epsilon}_{tz}$ and $\boldsymbol{\mu}_{tz}$ being defined by formulas (5.19). The currents (5.23) and (5.24) were introduced by Felsen and Marcuvitz in different designations but of the same structure.

From Eq. (5.22) it follows that the *effective bulk currents* $\mathbf{J}^e_{b,ef}$ and $\mathbf{J}^m_{b,ef}$, being formed as mixtures of the longitudinal and transverse components of the *actual* electric and magnetic currents, interact only with the transverse eigenfield components $\hat{\mathbf{E}}_{kt}$ and $\hat{\mathbf{H}}_{kt}$ of the kth mode, but in doing so take into account the contribution from the *effective surface currents* $\mathbf{J}^e_{s,ef}$ and $\mathbf{J}^m_{s,ef}$ defined by Eqs. (4.32). It should be mentioned that the contour integral in Eq. (5.22) allowing for the contribution from the *actual surface currents* $\mathbf{J}^e_s$ and $\mathbf{J}^m_s$ is absent in the appropriate equation of Felsen and Marcuvitz [8].

In view of fundamental importance of the mode excitation equations, Appendix B displays another derivation for the lossless waveguiding structures based on the direct use of Eqs. (4.12) and (4.13) which are an exact consequence of Maxwell's equations (4.4) and (4.5). The general case of lossy waveguides is studied below on the basis of the reciprocity theorem in the complex-conjugate form.

5.2 Approach Based on the Reciprocity Theorem

5.2.1 Derivation of the Conjugate Reciprocity Theorem

The basis of deriving the reciprocity theorem in complex-conjugate form is constituted by two systems of Maxwell's equations like Eqs. (4.4) and (4.5):

$$\nabla \times \mathbf{E}_1 = -i\omega\mathbf{B}_1 - \mathbf{J}^m_{b1}\,, \qquad \nabla \times \mathbf{E}^*_2 = i\omega\mathbf{B}^*_2 - \mathbf{J}^{m*}_{b2}, \tag{5.25}$$

$$\nabla \times \mathbf{H}_1 = i\omega\mathbf{D}_1 + \mathbf{J}^e_{b1}\,, \qquad \nabla \times \mathbf{H}^*_2 = -i\omega\mathbf{D}^*_2 + \mathbf{J}^{e*}_{b2}, \tag{5.26}$$

written for two different electromagnetic processes (marked with subscripts 1 and 2) excited by different external currents (bulk and surface), with the frequency and the constitutive parameters of a waveguiding medium entering into relations (2.8) and (2.9) assumed to be the same.

Application of the conventional technique to Eqs. (5.25) and (5.26) yields

$$\begin{aligned} \nabla\cdot(\mathbf{E}_1 \times \mathbf{H}^*_2 + \mathbf{E}^*_2 \times \mathbf{H}_1) = \\ = -\,(\mathbf{J}^e_{b1}\cdot\mathbf{E}^*_2 + \mathbf{J}^{e*}_{b2}\cdot\mathbf{E}_1) - (\mathbf{J}^m_{b1}\cdot\mathbf{H}^*_2 + \mathbf{J}^{m*}_{b2}\cdot\mathbf{H}_1) \\ -\,i\omega\Big[(\mathbf{D}_1\cdot\mathbf{E}^*_2 - \mathbf{D}^*_2\cdot\mathbf{E}_1) + (\mathbf{B}_1\cdot\mathbf{H}^*_2 - \mathbf{B}^*_2\cdot\mathbf{H}_1)\Big]\,. \end{aligned} \tag{5.27}$$

After transformation with using the constitutive relations (2.8) and (2.9) the last term in the right-hand side of Eq. (5.27) accepts the following form

$$i\omega\Big[(\mathbf{D}_1\cdot\mathbf{E}_2^* - \mathbf{D}_2^*\cdot\mathbf{E}_1) + (\mathbf{B}_1\cdot\mathbf{H}_2^* - \mathbf{B}_2^*\cdot\mathbf{H}_1)\Big] =$$
$$= i\omega\left[\begin{array}{l}(\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}^{\dagger}) : \mathbf{E}_1\mathbf{E}_2^* + (\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}^{\dagger}) : \mathbf{H}_1\mathbf{H}_2^* \\ \qquad\qquad + (\bar{\boldsymbol{\xi}} - \bar{\boldsymbol{\zeta}}^{\dagger}) : \mathbf{H}_1\mathbf{E}_2^* + (\bar{\boldsymbol{\zeta}} - \bar{\boldsymbol{\xi}}^{\dagger}) : \mathbf{E}_1\mathbf{H}_2^*\end{array}\right].$$

In accordance with the aforesaid in Sec. 4.1, the permittivity tensor $\bar{\boldsymbol{\epsilon}}$ is regarded here as a sum $(\bar{\boldsymbol{\epsilon}} + \bar{\boldsymbol{\sigma}}_c/i\omega)$ so that its antihermitian part determines the total tensor of electric conductivity, $\bar{\boldsymbol{\sigma}}_e = \bar{\boldsymbol{\sigma}}_c + \bar{\boldsymbol{\sigma}}_d$ taking into account both dielectric $(\bar{\boldsymbol{\sigma}}_d)$ and conductor $(\bar{\boldsymbol{\sigma}}_c)$ losses of a medium. Magnetic losses $(\bar{\boldsymbol{\sigma}}_m)$ are taken into account by the antihermitian part of the permeability tensor $\bar{\boldsymbol{\mu}}$, whereas the tensor $\bar{\boldsymbol{\sigma}}_{me} = i\omega(\bar{\boldsymbol{\xi}} - \bar{\boldsymbol{\zeta}}^{\dagger})$ reflects the magneto-electric losses due to bianisotropic properties of a medium. The use of Eqs. (2.21)–(2.23) converts relation (5.27) into the differential form of the conjugate reciprocity theorem

$$\boldsymbol{\nabla}\cdot\mathbf{S}_{12} + q_{12} = r_{12}^{(b)} \tag{5.28}$$

where we have denoted

$$\mathbf{S}_{12} = \mathbf{E}_1\times\mathbf{H}_2^* + \mathbf{E}_2^*\times\mathbf{H}_1\,, \tag{5.29}$$

$$\begin{aligned} q_{12} = {} & 2\,(\bar{\boldsymbol{\sigma}}_e : \mathbf{E}_1\mathbf{E}_2^* + \bar{\boldsymbol{\sigma}}_m : \mathbf{H}_1\mathbf{H}_2^*) + \\ & (\bar{\boldsymbol{\sigma}}_{me} : \mathbf{H}_1\mathbf{E}_2^* + \bar{\boldsymbol{\sigma}}_{me}^{\dagger} : \mathbf{E}_1\mathbf{H}_2^*)\,, \end{aligned} \tag{5.30}$$

$$r_{12}^{(b)} = -\,(\mathbf{J}_{b1}^{e}\cdot\mathbf{E}_2^* + \mathbf{J}_{b2}^{e*}\cdot\mathbf{E}_1) - (\mathbf{J}_{b1}^{m}\cdot\mathbf{H}_2^* + \mathbf{J}_{b2}^{m*}\cdot\mathbf{H}_1)\,. \tag{5.31}$$

Superscript (b) reflects belonging an appropriate quantity to bulk properties of a system, while the surface properties will be marked by superscript (s).

To obtain the integral form of the reciprocity theorem it is necessary to integrate Eq. (5.28) over the cross section S of a waveguiding structure with using the integral relation similar to Eq. (2.25) which involves the contour integrals taking into account two physical phenomena:

(i) the skin losses expressed by the boundary condition (2.26) with the surface impedance tensor (2.27) given along a contour L,

(ii) the discontinuity in tangential components of the fields caused both by the *actual surface currents* $\mathbf{J}_s^e$ and $\mathbf{J}_s^m$ located on a contour L_s with the boundary conditions (4.6) and (4.7) and by the *effective surface currents* $\mathbf{J}_{s,ef}^e$ and $\mathbf{J}_{s,ef}^m$ located on a contour L_b with the boundary conditions (4.25) and (4.26).

For the sake of brevity it is convenient to write both surface currents as the overall surface sources

$$\mathbf{J}^e_\Sigma = \mathbf{J}^e_s + \mathbf{J}^e_{s,ef} \qquad \text{and} \qquad \mathbf{J}^m_\Sigma = \mathbf{J}^m_s + \mathbf{J}^m_{s,ef} \tag{5.32}$$

located along the combined contour $L_\Sigma = L_s + L_b$.

Substitution of Eq. (5.29) into the integral relation (2.25) yields

$$\begin{aligned}
\int_S \nabla \cdot \mathbf{S}_{12}\, dS &= \frac{\partial}{\partial z}\int_S \Big(\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1\Big) \cdot \mathbf{z}_0\, dS \\
&\quad - \int_{L+L_s} \left[\begin{matrix} \mathbf{n}_s^+ \cdot (\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1)^+ \\ + \mathbf{n}_s^- \cdot (\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1)^- \end{matrix}\right] dl \\
&\quad + \int_{L_b} \mathbf{n}_b \cdot \left[\begin{matrix} (\mathbf{E}_{a1} \times \mathbf{H}_{b2}^* + \mathbf{E}_{b2}^* \times \mathbf{H}_{a1}) \\ + (\mathbf{E}_{b1} \times \mathbf{H}_{a2}^* + \mathbf{E}_{a2}^* \times \mathbf{H}_{b1}) \end{matrix}\right] dl \\
&= \frac{\partial}{\partial z}\int_S \Big(\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1\Big) \cdot \mathbf{z}_0\, dS \\
&\quad + 2\int_L \mathcal{R}_s\, (\mathbf{H}_{\tau 1} \cdot \mathbf{H}_{\tau 2}^*)\, dl \\
&\quad + \int_{L_s} \left[\begin{matrix} (\mathbf{J}^e_{s1} \cdot \mathbf{E}_2^* + \mathbf{J}^m_{s1} \cdot \mathbf{H}_2^*) \\ + (\mathbf{J}^{e*}_{s2} \cdot \mathbf{E}_1 + \mathbf{J}^{m*}_{s2} \cdot \mathbf{H}_1) \end{matrix}\right] dl \\
&\quad + \int_{L_b} \left[\begin{matrix} (\mathbf{J}^e_{s,ef1} \cdot \mathbf{E}_{a2}^* + \mathbf{J}^m_{s,ef1} \cdot \mathbf{H}_{a2}^*) \\ + (\mathbf{J}^{e*}_{s,ef2} \cdot \mathbf{E}_{a1} + \mathbf{J}^{m*}_{s,ef2} \cdot \mathbf{H}_{a1}) \end{matrix}\right] dl\,.
\end{aligned}$$

Here we have used: (a) the boundary condition (2.26) on the contour L with surface impedance (2.27), (b) the boundary conditions (4.6) and (4.7) with the actual surface currents $\mathbf{J}^e_{s1(2)}$ and $\mathbf{J}^m_{s1(2)}$ given on the contour L_s, (c) the effective surface currents $\mathbf{J}^e_{s,ef1(2)} = -\,\mathbf{n}_b \times \mathbf{H}_{b1(2)}$ and $\mathbf{J}^m_{s,ef1(2)} = \mathbf{n}_b \times \mathbf{E}_{b1(2)}$ defined on the contour L_b. Therefore, the line integrals in the previous formula yield two resulting contributions:

(i) from the skin losses on the contour L of a conducting surface

$$q'_{12} = 2\,\mathcal{R}_s\, (\mathbf{H}_{\tau 1} \cdot \mathbf{H}_{\tau 2}^*)\,, \tag{5.33}$$

(ii) from the overall surface currents on the contour $L_\Sigma = L_s + L_b$

$$r_{12}^{(s)} = -\,(\mathbf{J}^e_{\Sigma 1} \cdot \mathbf{E}_2^* + \mathbf{J}^{e*}_{\Sigma 2} \cdot \mathbf{E}_1) - (\mathbf{J}^m_{\Sigma 1} \cdot \mathbf{H}_2^* + \mathbf{J}^{m*}_{\Sigma 2} \cdot \mathbf{H}_1)\,. \tag{5.34}$$

Therefore, the reciprocity theorem in the integral form is given by the relation

$$\frac{dP_{12}(z)}{dz} + Q_{12}(z) = R_{12}(z) \tag{5.35}$$

where we have introduced the following integral quantities (complex-valued) (cf. Eqs. (2.29) and (2.30))

$$\begin{aligned} P_{12}(z) &\equiv \int_S \mathbf{S}_{12}(\mathbf{r}_t, z)\cdot \mathbf{z}_0\, dS \\ &= \int_S (\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1)\cdot \mathbf{z}_0\, dS\,, \end{aligned} \tag{5.36}$$

$$\begin{aligned} Q_{12}(z) &= Q_{12}^{(b)}(z) + Q_{12}^{(s)}(z) \equiv \int_S q_{12}(\mathbf{r}_t, z)\, dS + \int_L q'_{12}(\mathbf{r}_t, z)\, dl \\ &= 2\int_S (\bar{\boldsymbol{\sigma}}_e : \mathbf{E}_1\mathbf{E}_2^*)\, dS + 2\int_S (\bar{\boldsymbol{\sigma}}_m : \mathbf{H}_1\mathbf{H}_2^*)\, dS \\ &\quad + \int_S (\bar{\boldsymbol{\sigma}}_{me} : \mathbf{H}_1\mathbf{E}_2^* + \bar{\boldsymbol{\sigma}}_{me}^{\dagger} : \mathbf{E}_1\mathbf{H}_2^*)\, dS \\ &\quad + 2\int_L \mathcal{R}_s\, (\mathbf{H}_{\tau 1}\cdot \mathbf{H}_{\tau 2}^*)\, dl\,, \end{aligned} \tag{5.37}$$

$$\begin{aligned} R_{12}(z) &= R_{12}^{(b)}(z) + R_{12}^{(s)}(z) \equiv \int_{S_b} r_{12}^{(b)}(\mathbf{r}_t, z)\, dS + \int_{L_\Sigma} r_{12}^{(s)}(\mathbf{r}_t, z)\, dl \\ &= -\int_{S_b} \Big[(\mathbf{J}_{b1}^{e}\cdot \mathbf{E}_2^* + \mathbf{J}_{b1}^{m}\cdot \mathbf{H}_2^*) + (\mathbf{J}_{b2}^{e*}\cdot \mathbf{E}_1 + \mathbf{J}_{b2}^{m*}\cdot \mathbf{H}_1) \Big]\, dS \\ &\quad - \int_{L_\Sigma} \Big[(\mathbf{J}_{\Sigma 1}^{e}\cdot \mathbf{E}_2^* + \mathbf{J}_{\Sigma 1}^{m}\cdot \mathbf{H}_2^*) + (\mathbf{J}_{\Sigma 2}^{e*}\cdot \mathbf{E}_1 + \mathbf{J}_{\Sigma 2}^{m*}\cdot \mathbf{H}_1) \Big]\, dl\,. \end{aligned} \tag{5.38}$$

It is easy to see that with no sources (when $R_{12} = 0$) the second system (with subscript 2) of Maxwell's equations (5.25) and (5.26) describes the same fields as the first (marked by subscript 1) only with taking complex conjugation. This makes it possible to replace subscript 2 with 1 and what is more to drop them. In this case the integral reciprocity theorem (5.35) turns into the integral Poynting theorem (2.28) in which the real power flow P and the real power loss (bulk and surface) Q per unit length of a waveguide are equal to

$$P = \frac{1}{4}\int_S \mathbf{S}_{11}\cdot \mathbf{z}_0\, dS \equiv \int_S \langle\, \mathbf{S}\,\rangle \cdot \mathbf{z}_0\, dS\,, \tag{5.39}$$

$$Q = \frac{1}{4}\int_S q_{11}\, dS + \frac{1}{4}\int_L q'_{11}\, dl \equiv \int_S \langle\, q\,\rangle\, dS + \int_L \langle\, q'\,\rangle\, dl\,, \tag{5.40}$$

where their expressions in terms of fields are given by Eq. (2.29) and (2.30).

5.2.2 Derivation of the Equations of Mode Excitation

Inside the source region the reciprocity theorem in the integral form (5.35) is the basis for obtaining the excitation equations. To this end, the fields marked by subscript 1 (which will be dropped for the exciting currents) are assumed to be the desired fields excited by the bulk and surface sources ($\mathbf{J}_{b1}^{e,m} \equiv \mathbf{J}_{b}^{e,m} \neq 0$ and $\mathbf{J}_{\Sigma 1}^{e,m} \equiv \mathbf{J}_{\Sigma}^{e,m} \neq 0$) and represented in the form of expressions (4.8) and (4.9) (with replacing summation index k by l), whereas those marked by subscript 2 are the known fields of the k th mode outside the source region $(\mathbf{J}_{b2}^{e,m} = \mathbf{J}_{\Sigma 2}^{e,m} = 0)$ given in the form of Eq. (3.5).

Substitution of Eqs. (4.8) and (4.9) into Eqs. (5.36), (5.37), and (5.38) yields the following expressions:

$$\begin{aligned} P_{1k}(z) &\equiv \int_S \mathbf{S}_{1k}(\mathbf{r}_t, z) \cdot \mathbf{z}_0 \, dS = \int_S (\mathbf{E}_1 \times \mathbf{H}_k^* + \mathbf{E}_k^* \times \mathbf{H}_1) \cdot \mathbf{z}_0 \, dS \\ &= \sum_l N_{kl} \, A_l(z) \, \mathrm{e}^{-(\gamma_k^* + \gamma_l) z}, \end{aligned} \tag{5.41}$$

$$\begin{aligned} Q_{1k}(z) &\equiv \int_S q_{1k}(\mathbf{r}_t, z) \, dS + \int_L q'_{1k}(\mathbf{r}_t, z) \, dl \\ &= \sum_l M_{kl} \, A_l(z) \, \mathrm{e}^{-(\gamma_k^* + \gamma_l) z}, \end{aligned} \tag{5.42}$$

$$\begin{aligned} R_{1k}(z) &= R_{1k}^{(b)}(z) + R_{1k}^{(s)}(z) \equiv \int_{S_b} r_{1k}^{(b)}(\mathbf{r}_t, z) \, dS + \int_{L_\Sigma} r_{1k}^{(s)}(\mathbf{r}_t, z) \, dl \\ &= R_k^{(b)}(z) \, \mathrm{e}^{-\gamma_k^* z} + R_k^{(s)}(z) \, \mathrm{e}^{-\gamma_k^* z} \equiv R_k(z) \, \mathrm{e}^{-\gamma_k^* z}, \end{aligned} \tag{5.43}$$

where the normalizing and dissipative coefficients N_{kl} and M_{kl} have the form of Eqs. (2.36) and (2.37) and the quantity $R_k(z) = R_k^{(b)}(z) + R_k^{(s)}(z)$ consists of two exciting integrals (bulk and surface):

$$R_k^{(b)}(z) = \int_{S_b} \left(\mathbf{J}_b^e \cdot \hat{\mathbf{E}}_k^* + \mathbf{J}_b^m \cdot \hat{\mathbf{H}}_k^* \right) dS \,, \tag{5.44}$$

$$\begin{aligned} R_k^{(s)}(z) &= -\int_{L_\Sigma} \left(\mathbf{J}_\Sigma^e \cdot \hat{\mathbf{E}}_k^* + \mathbf{J}_\Sigma^m \cdot \hat{\mathbf{H}}_k^* \right) dl \\ &= -\int_{L_s} \left(\mathbf{J}_s^e \cdot \hat{\mathbf{E}}_k^* + \mathbf{J}_s^m \cdot \hat{\mathbf{H}}_k^* \right) dl \end{aligned}$$

$$- \int_{L_b} \left(\mathbf{J}^e_{s,ef} \cdot \hat{\mathbf{E}}^*_k + \mathbf{J}^m_{s,ef} \cdot \hat{\mathbf{H}}^*_k \right) dS \,. \tag{5.45}$$

These integrals involve the cross-section eigenfield vectors (marked with hat above them) and their dependence on z is due to that of the external currents $\mathbf{J}^{e,m}_b(z)$ and $\mathbf{J}^{e,m}_\Sigma(z)$.

It should be pointed out that expressions (5.41) and (5.42) come only from the field contributions of the mode expansions $\mathbf{E}_a$ and $\mathbf{H}_a$ since the orthogonal complementary fields $\mathbf{E}_b$ and $\mathbf{H}_b$, being proportional to the longitudinal component of external currents, do not contribute into P_{1k} and cannot influence the intrinsic losses in a medium related to Q_{1k}.

Substitution of Eqs. (5.41), (5.42), and (5.43) into the integral reciprocity theorem (5.35) (with replacing 2 by k) gives a relation

$$\sum_l \left\{ N_{kl} \frac{dA_l}{dz} - \Big[(\gamma^*_k + \gamma_l) N_{kl} - M_{kl} \Big] A_l \right\} \mathrm{e}^{-\gamma_l z} = R_k \equiv R^{(b)}_k + R^{(s)}_k . \tag{5.46}$$

The quasi-orthogonality relation of the general form (3.6) make the square bracket in Eq. (5.46) vanish so that it reduces to the desired set of the excitation equations written in the following form:

(i) for the excitation amplitudes $A_l(z)$

$$\sum_l N_{kl} \frac{dA_l(z)}{dz} \mathrm{e}^{-\gamma_l z} = R^{(b)}_k(z) + R^{(s)}_k(z) \,, \qquad k = 1, 2, \ldots \tag{5.47}$$

(ii) for the mode amplitudes $a_l(z) = A_l(z)\, \mathrm{e}^{-\gamma_l z}$

$$\sum_l N_{kl} \left[\frac{da_l(z)}{dz} + \gamma_l a_l(z) \right] = R^{(b)}_k(z) + R^{(s)}_k(z) \,, \qquad k = 1, 2, \ldots \tag{5.48}$$

Discussion of the excitation equations obtained will be put off until the similar equations for the waveguiding structures with space-dispersive media are developed in the second part of the paper.

Up to this point the waveguiding structures under examination have been assumed to be closed with a screening metallic boundary, whose spectrum of eigenmodes is always discrete.

In conclusion, it is pertinent to show features of the excitation theory peculiar to open waveguiding structures (without losses) in which an outside homogeneous medium extends to infinity in one or both transverse directions. As is known [27, 29, 31], for the open waveguides in addition to the discrete part of the spectrum of bound modes (with the outside medium fields localized near outer boundaries of the waveguiding layer), there is a continuous part of the spectrum related to radiation modes (with the fields extending to infinity in the outside medium). Unlike the eigenfields $\mathbf{E}_k(\mathbf{r}_t, z)$ and $\mathbf{H}_k(\mathbf{r}_t, z)$ of discrete modes marked by the integer-valued subscript $k = 1, 2, \ldots$ and expressed by Eqs. (3.5), the fields of a radiation mode

$$\mathbf{E}(\mathbf{r}_t, z; \mathbf{k}_t^{\mathrm{o}}) = \hat{\mathbf{E}}(\mathbf{r}_t; \mathbf{k}_t^{\mathrm{o}})\,\mathrm{e}^{-i\beta(\mathbf{k}_t^{\mathrm{o}})z}, \quad \mathbf{H}(\mathbf{r}_t, z; \mathbf{k}_t^{\mathrm{o}}) = \hat{\mathbf{H}}(\mathbf{r}_t; \mathbf{k}_t^{\mathrm{o}})\,\mathrm{e}^{-i\beta(\mathbf{k}_t^{\mathrm{o}})z} \tag{5.49}$$

are specified by the transverse wave vector $\mathbf{k}_t^{\mathrm{o}} = \mathbf{x}_0\, k_x^{\mathrm{o}} + \mathbf{y}_0\, k_y^{\mathrm{o}}$ of the outside passive medium.

In this case the modal expansions of the total fields $\mathbf{E}_a(\mathbf{r}_t, z)$ and $\mathbf{H}_a(\mathbf{r}_t, z)$ inside the source region, besides the series expansions in terms of discrete modes, involve also the integral expansions in terms of radiation modes:

$$\begin{aligned}\mathbf{E}_a(\mathbf{r}_t, z) = &\sum_k A_k(z)\,\hat{\mathbf{E}}_k(\mathbf{r}_t)\,\mathrm{e}^{-i\beta_k z}\\ &+ \int A(z; \mathbf{k}_t^{\mathrm{o}})\,\hat{\mathbf{E}}(\mathbf{r}_t; \mathbf{k}_t^{\mathrm{o}})\,\mathrm{e}^{-i\beta(\mathbf{k}_t^{\mathrm{o}})z}\, d\mathbf{k}_t^{\mathrm{o}}\,, \end{aligned} \tag{5.50}$$

$$\begin{aligned}\mathbf{H}_a(\mathbf{r}_t, z) = &\sum_k A_k(z)\,\hat{\mathbf{H}}_k(\mathbf{r}_t)\,\mathrm{e}^{-i\beta_k z}\\ &+ \int A(z; \mathbf{k}_t^{\mathrm{o}})\,\hat{\mathbf{H}}(\mathbf{r}_t; \mathbf{k}_t^{\mathrm{o}})\,\mathrm{e}^{-i\beta(\mathbf{k}_t^{\mathrm{o}})z}\, d\mathbf{k}_t^{\mathrm{o}}\,, \end{aligned} \tag{5.51}$$

where integrating over k_x^{o} and k_y^{o} is taken along the real axes from $-\infty$ to ∞.

The orthonormalization relation for radiation modes can be written by analogy with relation (3.18) for discrete modes in the following form

$$\begin{aligned} N(\mathbf{k}_t^{\mathrm{o}}, \mathbf{k}_t^{\mathrm{o}\prime}) \equiv &\\ \equiv &\int_S \Big[(\hat{\mathbf{E}}^*(\mathbf{r}_t; \mathbf{k}_t^{\mathrm{o}}) \times \hat{\mathbf{H}}(\mathbf{r}_t; \mathbf{k}_t^{\mathrm{o}\prime}) + \hat{\mathbf{E}}(\mathbf{r}_t; \mathbf{k}_t^{\mathrm{o}\prime}) \times \hat{\mathbf{H}}^*(\mathbf{r}_t; \mathbf{k}_t^{\mathrm{o}}) \Big] \cdot \mathbf{z}_0\, dS\\ = &\ \delta(\mathbf{k}_t^{\mathrm{o}} - \mathbf{k}_t^{\mathrm{o}\prime})\, N(\mathbf{k}_t^{\mathrm{o}}) \end{aligned} \tag{5.52}$$

where the Dirac delta function $\delta(\mathbf{k}_t^{\mathrm{o}} - \mathbf{k}_t^{\mathrm{o}\prime}) = \delta(k_x^{\mathrm{o}} - k_x^{\mathrm{o}\prime})\,\delta(k_y^{\mathrm{o}} - k_y^{\mathrm{o}\prime})$ replaces the Kronecker delta function δ_{kl}. It should be mentioned that since $\delta(\mathbf{k}_t^{\mathrm{o}}-\mathbf{k}_t^{\mathrm{o}\prime})$ has dimensions of (length)^2, the dimensionality of the norm $N(\mathbf{k}_t^{\mathrm{o}})$ and the excitation amplitude $A(z;\mathbf{k}_t^{\mathrm{o}})$ for the radiation modes is equal to watts/m^2 and m^2, respectively, as distinct from the bounded modes for which the similar quantities are taken in watts and as dimensionless.

The equation for the excitation amplitude $A(z;\mathbf{k}_t^{\mathrm{o}})$ of the radiation mode has the form similar to Eq. (5.13):

$$\begin{aligned}\frac{dA(\mathbf{k}_t^{\mathrm{o}})}{dz} = &- \frac{1}{N(\mathbf{k}_t^{\mathrm{o}})}\int_{S_b}\Big(\mathbf{J}_b^e\cdot\mathbf{E}^*(\mathbf{k}_t^{\mathrm{o}}) + \mathbf{J}_b^m\cdot\mathbf{H}^*(\mathbf{k}_t^{\mathrm{o}})\Big)\,dS \\ &- \frac{1}{N(\mathbf{k}_t^{\mathrm{o}})}\int_{L_\Sigma}\Big(\mathbf{J}_\Sigma^e\cdot\mathbf{E}^*(\mathbf{k}_t^{\mathrm{o}}) + \mathbf{J}_\Sigma^m\cdot\mathbf{H}^*(\mathbf{k}_t^{\mathrm{o}})\Big)\,dl\end{aligned} \tag{5.53}$$

where the coordinate variables are dropped for simplicity.

The orthogonal complementary fields $\mathbf{E}_b = \mathbf{z}_0 E_b$ and $\mathbf{H}_b = \mathbf{z}_0 H_b$ obtained in the form of Eqs. (4.21) and (4.22) remain valid for open waveguides.

6. CONCLUSION

We have shown a unified treatment of the electrodynamic theory of the guided wave excitation by external sources applied to any waveguiding structure involving the complex media with bianisotropic properties. Allowing for losses in such media has reduced to the power loss density in Poynting's theorem due to the *magneto-electric conductivity* $\bar{\bar{\boldsymbol{\sigma}}}_{me}$ defined by Eq. (2.23), in addition to the usual electric and magnetic conductivities.

Application of the desired field expansions in terms of eigenmode fields gives the *self-power* and *cross-power* quantities (flows and losses) transmitted and dissipated by the eigenmodes of a lossy waveguide, as well as the time-average energy density stored by the propagating modes in a lossless waveguides which involve the additional contributions from bianisotropic properties of a medium.

The basis of developing the excitation theory for lossy waveguides is the novel relation (3.6) called the *quasi-orthogonality relation* whose general form is always true including the propagating (active) and nonpropagating (reactive) modes in lossless waveguides considered as a special case. Among the external sources exciting the waveguiding structure we have included the bulk sources (currents, fields, and medium perturbations) and the actual surface currents. Inside the source region the modal expansions (2.31) and (2.32) have proved to be incomplete and must be supplemented with the *orthogonal complementary fields* (4.21) and (4.22), as it is done by Eqs. (4.8) and (4.9). In general, these complementary fields generate the *effective surface currents* (4.27) and (4.28). So in the most general case the external source region contains the bulk currents $\mathbf{J}_b^{e,m}$, the actual surface currents $\mathbf{J}_s^{e,m}$, and the effective surface currents $\mathbf{J}_{s,ef}^{e,m}$ brought about by the longitudinal components of the bulk currents.

The equations of mode excitation in the form of (5.47) or (5.48) have been derived by using three approaches based on: (i) the direct derivation from Maxwell's equations (see Appendix B), (ii) the electrodynamic analogy with the mathematical method of variation of constants (see Sec. 5.1), (iii) the reciprocity theorem in the complex-conjugate form (see Sec. 5.2).

All the results obtained are valid only for the time-dispersive media specified by macroscopically-local and frequency-dependent parameters. An extension of the theory to space-dispersive media which require for their description the special equations of motion with regard for nonlocal effects will be examined in the second part of the paper where the orthogonal complementary fields are explained as a part of the contribution from the potential fields of external sources.

APPENDIX A. BASIC RELATIONS OF FUNCTIONAL ANALYSIS AND THEIR ELECTRODYNAMIC ANALOGS

A.1 Mathematical Formulation (in notation of [42])

Unlike [42], we shall examine the general case of *nonorthogonal* base functions which gives the orthogonal basis as a special case.

Consider a countable set of complex functions $\psi_1(x), \psi_2(x), \ldots$ quadratically integrable in the sense of Lebesgue on a given set S of points (x). The class $L_2(S)$ of such functions (regarded as vectors) constitutes an infinite-dimensional *unitary* functional (vector) space if, in addition to two binary operations of the vector sum $\psi_k(x) + \psi_l(x)$ and the product $a_k\psi_k(x)$ by a complex scalar a_k, one defines the inner product of $\psi_k(x)$ and $\psi_l(x)$ as

$$(\psi_k, \psi_l) = \int_S \psi_k^*(x)\gamma(x)\psi_l(x)\, dx \qquad (A.1)$$

where the weighting function $\gamma(x)$ is a given real nonnegative function quadratically integrable on S, in particular, may be $\gamma(x) \equiv 1$.

If Gram's determinant $\det[(\psi_k, \psi_l)]$ built up on the inner products of the form $(A.1)$ differs from zero, the functions $\psi_k(x), k=1,2,...$ are linearly independent in L_2 and can be chosen as a basis of the unitary functional space, with their mutual orthogonality not being necessarily required in general. The given set of functions $\psi_k(x)$ spans a linear manifold comprising all linear combinations of $\psi_1(x), \psi_2(x), \ldots$.

Let us compose a partial sum of the nth order

$$s_n(x) = \sum_{k=1}^{n} a_k^{(n)} \psi_k(x) \tag{A.2}$$

with scalar coefficients $a_k^{(n)}$ not yet defined. Given a function $\psi(x)$ fully belonging to the linear manifold spanned by $\psi_1(x), \psi_2(x), \ldots$, these coefficients can be found from the requirement that the weighted mean-square difference

$$D_n = \int_S \gamma(x) |s_n(x) - \psi(x)|^2 \, dx \tag{A.3}$$

between $s_n(x)$ and $\psi(x)$ would be minimum. With the help of $(A.2)$ the quantity D_n can be rewritten in the following form

$$D_n = \int_S \left[\sum_{k=1}^{n} a_k^{(n)*} \psi_k^*(x) - \psi^*(x) \right] \gamma(x) \left[\sum_{l=1}^{n} a_l^{(n)} \psi_l(x) - \psi(x) \right] dx \, . \tag{A.4}$$

Then the conditions of its minimality with respect to the set of coefficients $a_k^{(n)}$ are written as

$$\frac{\partial D_n}{\partial a_k^{(n)*}} = \int_S \psi_k^*(x) \gamma(x) \left[\sum_{l=1}^{n} a_l^{(n)} \psi_l(x) - \psi(x) \right] dx = 0 \, , \tag{A.5}$$

$$\frac{\partial^2 D_n}{\partial a_k^{(n)*} \partial a_k^{(n)}} = \int_S \psi_k^*(x) \gamma(x) \psi_k(x) \, dx > 0 \, . \tag{A.6}$$

$(A.6)$ complies with the requirement of quadratic integrability initially imposed on the base functions $\psi_k(x)$, while the condition $(A.5)$ yields the following system of equations to find $a_k^{(n)}$:

$$\sum_{l=1}^{n} N_{kl} a_l^{(n)} = R_k \, , \qquad k = 1, \, 2, \ldots \tag{A.7}$$

where we have denoted

$$N_{kl} = \int_S \psi_k^*(x)\gamma(x)\psi_l(x)\,dx \equiv (\psi_k, \psi_l)\,, \tag{A.8}$$

$$R_k = \int_S \psi_k^*(x)\gamma(x)\psi(x)\,dx \equiv (\psi_k, \psi)\,. \tag{A.9}$$

Metric convergence in L_2 is defined as *convergence in mean* (with index 2) of the sequence of partial sums $s_n(x)$ (with coefficients $a_k^{(n)}$ from Eqs. $(A.7)$) to the function $\psi(x)$, i.e., $s_n(x) \stackrel{\text{mean}}{\longrightarrow} \psi(x)$ as $n \to \infty$, which occurs if and only if

$$D_n \equiv \int_S \gamma(x)|s_n(x) - \psi(x)|^2\,dx \to 0 \quad \text{as} \quad n \to \infty\,. \tag{A.10}$$

Using Eqs. $(A.7)$ through $(A.9)$ and the equality $N_{kl} = N_{lk}^*$ allows Eq. $(A.4)$ to take the following form

$$D_n = \sum_{k=1}^{n}\sum_{l=1}^{n} N_{kl} a_k^{(n)*} a_l^{(n)} - \sum_{k=1}^{n} R_k a_k^{(n)*} - \sum_{l=1}^{n} R_l^* a_l^{(n)} + \int_S \gamma(x)|\psi(x)|^2\,dx = -\sum_{k=1}^{n}\sum_{l=1}^{n} N_{kl} a_k^{(n)*} a_l^{(n)} + \int_S \gamma(x)|\psi(x)|^2\,dx.$$

From here for limiting case $(A.10)$, when $a_k^{(n)}(z) \to a_k(z)$ as $n \to \infty$, it follows that

$$(\psi, \psi) \equiv \int_S \psi^*(x)\gamma(x)\psi(x)\,dx = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} N_{kl} a_k^* a_l\,. \tag{A.11}$$

Relation $(A.11)$ is realizable only for functions $\psi(x)$ quadratically integrable on S (with the weighting function $\gamma(x)$), i.e., for which there exists an integral on the left. This relation expresses completeness of the set of the base functions $\psi_k(x)$ (also quadratically integrable on S) inside the class of functions $\psi(x)$. The completeness property establishes the space L_2 as the Hilbert space for which the series expansion

$$\psi(x) \stackrel{\text{mean}}{=} \sum_{k=1}^{\infty} a_k \psi_k(x) \tag{A.12}$$

interpreted in the sense of convergence in mean given by formula $(A.10)$ is valid. Uniqueness of this expansion arises from the following reasoning.

By contradiction, let two different series expansions $\sum_k a'_k \psi_k(x)$ and $\sum_k a''_k \psi_k(x)$ correspond to the same function $\psi(x)$ in the sense of convergence in mean. To determine the expansion coefficients a'_k and a''_k there are two systems of form $(A.7)$ with the same right-hand sides R_k. When resulted from them, the difference system of equations $\sum_l N_{kl}(a'_l - a''_l) = 0$, $k = 1, 2, \ldots$ gives $a'_l \equiv a''_l$ by virtue of $N_{kl} \neq 0$, i.e., the initial series expansions coincide. If on the contrary one assumes that the same series expansion $\sum_k a_k \psi_k(x)$ corresponds to two different functions $\psi'(x)$ and $\psi''(x)$, then the difference function $\psi^-(x) = \psi'(x) - \psi''(x)$ has the expansion coefficients identically equal to zero. So the right-hand side of the completeness relation $(A.11)$ vanishes, which necessarily provides $\psi^-(x) \equiv 0$, i.e., the initial functions coincide.

The completeness relation $(A.11)$ is a generalization of the conventional Parseval identity (see Eq. $(A.16)$) to the case of nonorthogonal bases. All the aforestated convince us that linear independence and completeness of the set of base functions $\psi_k(x)$ are fundamental properties of the basis, whereas their mutual orthogonality is not obligatory requirement and merely facilitates the problem of finding the expansion coefficients a_k. Indeed, for the orthogonal basis

$$(\psi_k, \psi_l) \equiv \int_S \psi_k^*(x)\gamma(x)\psi_l(x)\,dx = 0 \quad \text{for} \quad k \neq l \qquad (A.13)$$

so that

$$N_{kl} = N_k \delta_{kl} \quad \text{with} \quad N_k = \int_S \psi_k^*(x)\gamma(x)\psi_k(x)\,dx \equiv \|\psi_k\|^2 \qquad (A.14)$$

where $\|\psi_k\| = \sqrt{(\psi_k, \psi_k)} \equiv \sqrt{N_k}$ is conventionally called the norm of a function $\psi_k(x)$ [42]. In addition, we extend this term to quantities N_{kl} recognizing the *self norm* $N_k \equiv N_{kk}$ for $l = k$ and the *cross norm* N_{kl} for $l \neq k$.

Hence, in the special case of the orthogonal basis satisfying Eq. $(A.14)$: (i) the system of coupled equations $(A.7)$ falls apart into separate equations yielding

$$a_k = \frac{R_k}{N_k} \equiv \frac{(\psi_k, \psi)}{(\psi_k, \psi_k)} = \frac{1}{N_k} \int_S \psi_k^*(x)\gamma(x)\psi(x)\,dx\,, \qquad (A.15)$$

(ii) the general completeness relation $(A.11)$ gives the conventional Parseval identity

$$(\psi, \psi) \equiv \int_S \psi^*(x)\gamma(x)\psi(x)\,dx = \sum_{k=1}^{\infty} N_k |a_k|^2 \,. \qquad (A.16)$$

It should be remembered that the use of the known Gram-Schmidt orthogonalization process [42], in principle, allows one to construct the orthonormal basis.

The above completeness property of a basis expressed by relation $(A.11)$ or $(A.16)$ concerns only such functions $\psi(x)$ that fully belong to the linear manifold spanned by the functions $\psi_1(x), \psi_2(x), \ldots$. However, for the most general functions $f(x)$ this is not the case.

Any given function $f(x)$ quadratically integrable on S (with the weighting function $\gamma(x)$, in general) can formally be associated with the function $\psi(x)$ represented by series $(A.12)$ if one assumes that its coefficients a_k satisfying Eqs. $(A.7)$ through $(A.9)$ are due to $f(x)$ and not to $\psi(x)$, i.e., the quantities $R_k(x)$ contain $f(x)$ in place of $\psi(x)$ under the integral sign of Eq. $(A.9)$. Let us prove that the difference $c\,(x) = f(x) - \psi(x)$ is orthogonal to every base function $\psi_k(x)$ in the sense of relation $(A.13)$:

$$\begin{aligned}(\psi_k, c) &\equiv (\psi_k, f - \psi) = (\psi_k, f) - (\psi_k, \psi) \\ &= (\psi_k, f) - \sum_l (\psi_k, \psi_l) a_l = R_k - \sum_l N_{kl} a_l = 0\end{aligned} \qquad (A.17)$$

where the relations $N_{kl} = (\psi_k, \psi_l)$ and $R_k = (\psi_k, f)$ have been used.

Thus, any arbitrary function $f(x)$ not belonging fully to the Hilbert space (spanned, for instance, by eigenfunctions of a boundary-value problem) can be represented in the following form

$$f(x) = \psi(x) + c\,(x) = \sum_k a_k \psi_k(x) + c\,(x)\,. \qquad (A.18)$$

Here the function $\psi(x)$ written as a series expansion in terms of base functions (convergent in mean) and considered as tangential to the given Hilbert space is called the *projection* of $f(x)$ on this space, while $c\,(x)$ is a function orthogonal to the Hilbert space and referred to as the *orthogonal complement* because $(\psi_k, c) = 0$. For such a function $f(x)$ instead of the generalized Parseval identity $(A.11)$ there exists the generalized Bessel inequality

$$\begin{aligned}(f,f) \equiv \|\psi + c\|^2 \equiv \int_S f^*(x)\gamma(x)f(x)\,dx \\ \geq \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} N_{kl} a_k^* a_l \quad \text{or} \quad \geq \sum_{k=1}^{\infty} N_k |a_k|^2\end{aligned} \qquad (A.19)$$

where the last single sum corresponds to the orthogonal basis.

A.2 Electrodynamic Treatment (in notation of [8])

Let us consider a relevant aspect of the electrodynamic modal theory on the basis of analogy with the foregoing mathematical relations.

Given an infinite set of eigenfunctions of a boundary-value problem defined on the cross section S of a waveguiding structure with the transverse radius vector $\mathbf{r}_t$ and the longitudinal axis z, any eigenfunction $\boldsymbol{\Psi}_k(\mathbf{r}_t)$ and its adjoint (hermitian conjugate) $\boldsymbol{\Psi}_k^\dagger(\mathbf{r}_t)$ are denoted in the two-vector notation as

$$\boldsymbol{\Psi}_k(\mathbf{r}_t) = \begin{pmatrix}\hat{\mathbf{E}}_k(\mathbf{r}_t) \\ \hat{\mathbf{H}}_k(\mathbf{r}_t)\end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi}_k^\dagger(\mathbf{r}_t) = \begin{pmatrix}\hat{\mathbf{E}}_k^*(\mathbf{r}_t) & \hat{\mathbf{H}}_k^*(\mathbf{r}_t)\end{pmatrix} \qquad (A.20)$$

where the hat over field vectors means the absence of their dependence on z.

By analogy with Eq. $(A.1)$, the inner product of two eigenfunctions $\boldsymbol{\Psi}_k(\mathbf{r}_t)$ and $\boldsymbol{\Psi}_l(\mathbf{r}_t)$ can be defined in the following form

$$(\boldsymbol{\Psi}_k, \boldsymbol{\Psi}_l) = \int_S \boldsymbol{\Psi}_k^\dagger(\mathbf{r}_t) \cdot \bar{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Psi}_l(\mathbf{r}_t)\, dS \qquad (A.21)$$

with the weighting function given in the form of a special dyadic

$$\bar{\boldsymbol{\Gamma}} = \begin{pmatrix} 0 & -\mathbf{z}_0 \times \bar{\mathbf{I}} \\ \mathbf{z}_0 \times \bar{\mathbf{I}} & 0 \end{pmatrix} \qquad (A.22)$$

where $\mathbf{z}_0$ is the unit vector of the axis z and $\bar{\mathbf{I}}$ is the unit dyadic such that $(\mathbf{z}_0 \times \bar{\mathbf{I}}) \cdot \mathbf{a} = -\,\mathbf{a} \cdot (\bar{\mathbf{I}} \times \mathbf{z}_0) = \mathbf{z}_0 \times \mathbf{a}$ for any vector $\mathbf{a}$. The weighting dyadic $\bar{\boldsymbol{\Gamma}}$ is constructed so as to make the double scalar product $\boldsymbol{\Psi}_k^\dagger \cdot \bar{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Psi}_l = \boldsymbol{\Psi}_k^\dagger \cdot \left(\bar{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Psi}_l\right) = \left(\boldsymbol{\Psi}_k^\dagger \cdot \bar{\boldsymbol{\Gamma}}\right) \cdot \boldsymbol{\Psi}_l \equiv \bar{\boldsymbol{\Gamma}} : \boldsymbol{\Psi}_l \boldsymbol{\Psi}_k^\dagger$ under the integral sign of Eq. $(A.\ 21)$ be equal to $(\hat{\mathbf{E}}_k^* \times \hat{\mathbf{H}}_l + \hat{\mathbf{E}}_l \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0$.

Thus, the *cross norm* N_{kl} for the kth and lth modes and the *self norm* $N_k \equiv N_{kk}$ for the kth mode, according to Eqs. $(A.8)$ and $(A.21)$, can be represented as

$$\begin{aligned} N_{kl} \equiv (\boldsymbol{\Psi}_k, \boldsymbol{\Psi}_l) &= \int_S \boldsymbol{\Psi}_k^\dagger \cdot \bar{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Psi}_l \, dS \\ &= \int_S (\hat{\mathbf{E}}_k^* \times \hat{\mathbf{H}}_l + \hat{\mathbf{E}}_l \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0 \, dS \end{aligned} \qquad (A.23)$$

and

$$\begin{aligned} N_k \equiv (\boldsymbol{\Psi}_k, \boldsymbol{\Psi}_k) &= \int_S \boldsymbol{\Psi}_k^\dagger \cdot \bar{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Psi}_k \, dS \\ &= 2\,\mathrm{Re} \int_S (\hat{\mathbf{E}}_k^* \times \hat{\mathbf{H}}_k) \cdot \mathbf{z}_0 \, dS \,. \end{aligned} \qquad (A.24)$$

By analogy with the series expansion $(A.18)$, an arbitrary function $\mathbf{F}(\mathbf{r}_t, z)$ quadratically integrable on S can be represented as a sum of the modal expansion $\boldsymbol{\Psi}(\mathbf{r}_t, z)$ in terms of eigenfunctions $\boldsymbol{\Psi}_k(\mathbf{r}_t)$ (the *projection* of $\mathbf{F}(\mathbf{r}_t, z)$ tangent to Hilbert space and convergent in mean) and the *orthogonal complement* $\mathbf{C}(\mathbf{r}_t, z)$:

$$\mathbf{F}(\mathbf{r}_t, z) = \boldsymbol{\Psi}(\mathbf{r}_t, z) + \mathbf{C}(\mathbf{r}_t, z) = \sum_k a_k(z)\, \boldsymbol{\Psi}_k(\mathbf{r}_t) + \mathbf{C}(\mathbf{r}_t, z) \qquad (A.25)$$

where we have denoted

$$\mathbf{F}(\mathbf{r}_t, z) = \begin{pmatrix} \mathbf{E}(\mathbf{r}_t, z) \\ \mathbf{H}(\mathbf{r}_t, z) \end{pmatrix}, \quad \boldsymbol{\Psi}(\mathbf{r}_t, z) = \begin{pmatrix} \mathbf{E}_a(\mathbf{r}_t, z) \\ \mathbf{H}_a(\mathbf{r}_t, z) \end{pmatrix},$$

$$\mathbf{C}(\mathbf{r}_t, z) = \begin{pmatrix} \mathbf{E}_b(\mathbf{r}_t, z) \\ \mathbf{H}_b(\mathbf{r}_t, z) \end{pmatrix}$$

and by analogy with Eq. $(A.17)$ the orthogonal complement $\mathbf{C}$ satisfy the relation

$$\begin{aligned} (\boldsymbol{\Psi}_k, \mathbf{C}) &\equiv \int_S \boldsymbol{\Psi}_k^\dagger \cdot \bar{\boldsymbol{\Gamma}} \cdot \mathbf{C} \, dS \\ &= \int_S (\hat{\mathbf{E}}_k^* \times \mathbf{H}_b + \mathbf{E}_b \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0 \, dS = 0 \end{aligned} \qquad (A.26)$$

Eqs. $(A.25)$ and $(A.26)$ allow the electromagnetic fields to be represented in the following form

$$\mathbf{E}(\mathbf{r}_t, z) = \mathbf{E}_a(\mathbf{r}_t, z) + \mathbf{E}_b(\mathbf{r}_t, z) = \sum_k a_k(z)\,\hat{\mathbf{E}}_k(\mathbf{r}_t) + \mathbf{E}_b(\mathbf{r}_t, z) \tag{A.27}$$

$$\mathbf{H}(\mathbf{r}_t, z) = \mathbf{H}_a(\mathbf{r}_t, z) + \mathbf{H}_b(\mathbf{r}_t, z) = \sum_k a_k(z)\hat{\mathbf{H}}_k(\mathbf{r}_t) + \mathbf{H}_b(\mathbf{r}_t, z) \tag{A.28}$$

where the orthogonal complementary fields $\mathbf{E}_b(\mathbf{r}_t, z)$ and $\mathbf{H}_b(\mathbf{r}_t, z)$ as well as the mode amplitudes $a_k(z)$ of the modal expansions

$$\mathbf{E}_a(\mathbf{r}_t, z) = \sum_k a_k(z)\,\hat{\mathbf{E}}_k(\mathbf{r}_t) \quad \text{and} \quad \mathbf{H}_a(\mathbf{r}_t, z) = \sum_k a_k(z)\,\hat{\mathbf{H}}_k(\mathbf{r}_t) \tag{A.29}$$

should be determined. The amplitudes $a_k(z)$, in principle, can be found from the equations similar to Eqs. $(A.7)$ for the nonorthogonal basis or to Eq. $(A.15)$ for the orthogonal basis, with R_k being given as follows

$$\begin{aligned} R_k \equiv (\mathbf{\Psi}_k, \mathbf{\Psi}) = (\mathbf{\Psi}_k, \mathbf{F}) &= \int_S \mathbf{\Psi}_k^\dagger \cdot \bar{\mathbf{\Gamma}} \cdot \mathbf{F}\, dS \\ &= \int_S (\hat{\mathbf{E}}_k^* \times \mathbf{H} + \mathbf{E} \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0\, dS\,, \end{aligned} \tag{A.30}$$

in particular, by analogy with Eq. $(A.15)$

$$a_k = \frac{R_k}{N_k} \equiv \frac{(\mathbf{\Psi}_k, \mathbf{F})}{(\mathbf{\Psi}_k, \mathbf{\Psi}_k)} = \frac{1}{N_k} \int_S (\hat{\mathbf{E}}_k^* \times \mathbf{H} + \mathbf{E} \times \hat{\mathbf{H}}_k^*) \cdot \mathbf{z}_0\, dS\,. \tag{A.31}$$

It is of great importance in electrodynamic applications that such a procedure of determining the mode amplitude $a_k(z)$ based on Eq. $(A.7)$ or $(A.15)$ allows us instead of the series expansion $\mathbf{\Psi}(\mathbf{r}_t, z)$ in terms of eigenmodes to apply its finite sum of the n th order

$$\mathbf{S}_n(\mathbf{r}_t, z) = \sum_{k=1}^{n} a_k^{(n)}(z)\mathbf{\Psi}_k(\mathbf{r}_t) \tag{A.32}$$

like Eq. $(A.2)$, which yields the least mean-square error

$$D_n = \int_S \Big[\mathbf{S}_n^\dagger(\mathbf{r}_t, z) - \mathbf{\Psi}^\dagger(\mathbf{r}_t, z)\Big] \cdot \bar{\mathbf{\Gamma}} \cdot \Big[\mathbf{S}_n(\mathbf{r}_t, z) - \mathbf{\Psi}(\mathbf{r}_t, z)\Big]\, dS \tag{A.33}$$

analogously to Eq. $(A.4)$.

The above general reasoning concerning the convergence in mean, completeness, and orthogonality properties of base functions can be extended to the electrodynamic basis of eigenfunctions so that, in particular, the generalized Parseval identity $(A.11)$ and Bessel inequality $(A.19)$ take the following form

$$\begin{aligned}(\mathbf{\Psi}, \mathbf{\Psi}) \equiv \|\mathbf{\Psi}\|^2 &= \int_S \mathbf{\Psi}^\dagger \cdot \bar{\mathbf{\Gamma}} \cdot \mathbf{\Psi}\, dS \\ &= \int_S (\mathbf{E}_a^* \times \mathbf{H}_a + \mathbf{E}_a \times \mathbf{H}_a^*) \cdot \mathbf{z}_0\, dS \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} N_{kl} a_k^* a_l \quad \text{or} \quad = \sum_{k=1}^{\infty} N_k |a_k|^2 \end{aligned} \qquad (A.34)$$

and

$$\begin{aligned}(\mathbf{F}, \mathbf{F}) \equiv \|\mathbf{\Psi} + \mathbf{C}\|^2 &= \int_S \mathbf{F}^\dagger \cdot \bar{\mathbf{\Gamma}} \cdot \mathbf{F}\, dS \\ &= \int_S (\mathbf{E}^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{z}_0\, dS \\ &\geq \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} N_{kl} a_k^* a_l \quad \text{or} \quad \geq \sum_{k=1}^{\infty} N_k |a_k|^2 \end{aligned} \qquad (A.35)$$

where the last single sums correspond to the orthogonal basis.

As noted above, the orthogonality property of a basis is not mandatory but its existence facilitates the determination of the expansion coefficients $a_k(z)$. Such a property is inherent in lossless physical systems, whereas losses destroy the "pure" orthogonality and convert it into the so-called *quasi-orthogonality* (see Sec. 3.1).

APPENDIX B. DIRECT DERVATION OF THE EQUATIONS OF MODE EXCITATION FROM MAXWELL'S EQUATIONS

Starting point to derive the equation of mode excitation is formulas (4.12) and (4.13) which are a result of transforming Maxwell's equations (4.4) and (4.5) inside the source region. Let us rewrite Eqs. (4.12) and (4.13) for transverse components:

$$\sum_l \frac{dA_l}{dz}\,(\mathbf{z}_0 \times \mathbf{E}_l) = -\boldsymbol{\nabla} \times \mathbf{E}_b - i\omega\mu_0\mathbf{M}_{bt} - \mathbf{J}^m_{bt}\,, \qquad (B.1)$$

$$\sum_l \frac{dA_l}{dz}\,(\mathbf{z}_0 \times \mathbf{H}_l) = -\boldsymbol{\nabla} \times \mathbf{H}_b \;+\; i\omega\mathbf{P}_{bt} \;+\; \mathbf{J}^e_{bt}\,. \qquad (B.2)$$

Here, in accordance with Eq. (2.2), the orthogonal complements for the polarization $\mathbf{P}_b$ and magnetization $\mathbf{M}_b$ are defined as

$$\mathbf{P}_b = \mathbf{D}_b - \epsilon_0\mathbf{E}_b \qquad \text{and} \qquad \mu_0\mathbf{M}_b = \mathbf{B}_b - \mu_0\mathbf{H}_b \qquad (B.3)$$

so that, as follows from Eqs. (4.14), (4.15), (4.17), and (4.18), their longitudinal components contribute to the complementary fields:

$$\mathbf{E}_b \;\equiv\; \mathbf{z}_0\,E_b \;=\; -\,\frac{1}{i\omega\epsilon_0}\,(\,\mathbf{J}^e_{bz} \;+\; i\omega\mathbf{P}_{bz})\,, \qquad (B.4)$$

$$\mathbf{H}_b \;\equiv\; \mathbf{z}_0\,H_b = -\,\frac{1}{i\omega\mu_0}(\mathbf{J}^m_{bz} + i\omega\mu_0\mathbf{M}_{bz})\,. \qquad (B.5)$$

If we scalar-multiply Eqs. $(B.1)$ and $(B.2)$ by $\mathbf{H}^*_k$ and $-\mathbf{E}^*_k$, respectively, and add the results, then after integrating over S we obtain

$$\begin{aligned}
&\sum_l \frac{dA_l}{dz}\int_S (\mathbf{E}^*_k \times \mathbf{H}_l + \mathbf{E}_l \times \mathbf{H}^*_k)\cdot\mathbf{z}_0\,dS \\
&= \int_S (\mathbf{E}^*_k \cdot \boldsymbol{\nabla} \times \mathbf{H}_b - \mathbf{H}^*_k \cdot \boldsymbol{\nabla} \times \mathbf{E}_b)\,dS \\
&- \int_S (\mathbf{J}^e_{bt}\cdot\mathbf{E}^*_{kt} + \mathbf{J}^m_{bt}\cdot\mathbf{H}^*_{kt})\,dS \\
&- i\omega\int_S (\mathbf{P}_{bt}\cdot\mathbf{E}^*_{kt} + \mu_0\mathbf{M}_{bt}\cdot\mathbf{H}^*_{kt})\,dS\,.
\end{aligned} \qquad (B.6)$$

Now it is necessary to transform the first integral in the right-hand side of Eq. $(B.6)$. The terms of its integrand can be rearranged as follows

$$\begin{aligned}
\mathbf{E}^*_k \cdot \boldsymbol{\nabla} \times \mathbf{H}_b &= \mathbf{z}_0 \cdot (\mathbf{E}^*_k \times \nabla H_b) \\
&= \mathbf{H}_b \cdot \nabla \times \mathbf{E}^*_k \;-\; \mathbf{z}_0 \cdot \boldsymbol{\nabla} \times (\mathbf{E}^*_k\,H_b) \\
&= \; i\omega\mu_0\,(\mathbf{H}^*_k + \mathbf{M}^*_k)\cdot\mathbf{H}_b \;-\; \mathbf{z}_0 \cdot \boldsymbol{\nabla} \times (\mathbf{E}^*_k\,H_b)\,, \\
\mathbf{H}^*_k \cdot \boldsymbol{\nabla} \times \mathbf{E}_b &= \; \mathbf{z}_0 \cdot (\mathbf{H}^*_k \times \nabla E_b) \\
&= \; \mathbf{E}_b \cdot \nabla \times \mathbf{H}^*_k \;-\; \mathbf{z}_0 \cdot \nabla \times (\mathbf{H}^*_k\,E_b) \\
&= \; -\,i\omega\,(\epsilon_0\mathbf{E}^*_k + \mathbf{P}^*_k)\cdot\mathbf{E}_b \;-\; \mathbf{z}_0 \cdot \nabla \times (\mathbf{H}^*_k\,E_b)\,,
\end{aligned}$$

where in the last equalities for the k th mode we have used Eqs. (5.9) and (5.10). Then the first integral in Eq. (B.6) turns into the sum of three integrals:

$$\begin{aligned}&\int_{S_b} (\mathbf{E}_k^* \cdot \boldsymbol{\nabla} \times \mathbf{H}_b - \mathbf{H}_k^* \cdot \boldsymbol{\nabla} \times \mathbf{E}_b)\, dS \\ &= \int_{S_b} \boldsymbol{\nabla} \times (\mathbf{H}_k^*\, E_b - \mathbf{E}_k^*\, H_b) \cdot \mathbf{z}_0\, dS \\ &+ i\omega \int_{S_b} (\mathbf{E}_k^* \cdot \epsilon_0 \mathbf{E}_b + \mathbf{H}_k^* \cdot \mu_0 \mathbf{H}_b)\, dS \\ &+ i\omega \int_{S_b} (\mathbf{P}_k^* \cdot \mathbf{E}_b + \mu_0 \mathbf{M}_k^* \cdot \mathbf{H}_b)\, dS\,,\end{aligned} \qquad (B.7)$$

The first integral in the right-hand side of Eq. (B.7) is transformed by using the Stokes theorem [42] into the following form

$$\begin{aligned}&\int_{S_b} \boldsymbol{\nabla} \times (\mathbf{H}_k^*\, E_b - \mathbf{E}_k^*\, H_b) \cdot \mathbf{z}_0\, dS = \oint_{L_b} (\mathbf{H}_k^*\, E_b - \mathbf{E}_k^*\, H_b) \cdot \boldsymbol{\tau}\, dl \\ &= \oint_{L_b} \Big[\, (\mathbf{n}_b \times \mathbf{H}_b) \cdot \mathbf{E}_k^* - (\mathbf{n}_b \times \mathbf{E}_b) \cdot \mathbf{H}_k^* \Big]\, dl\end{aligned} \qquad (B.8)$$

where $\mathbf{n}_b$ and $\boldsymbol{\tau} = \mathbf{z}_0 \times \mathbf{n}_b$ are the unit vectors, respectively, normal (outward) and tangential to the contour L_b bounding the bulk current area S_b .

The second integral in the right-hand side of Eq. (B.7) is rearranged by using Eqs. (B.4) and (B.5) to the following form

$$\begin{aligned}&i\omega \int_{S_b} (\mathbf{E}_k^* \cdot \epsilon_0 \mathbf{E}_b + \mathbf{H}_k^* \cdot \mu_0 \mathbf{H}_b)\, dS = -\int_{S_b} (J_{bz}^e\, E_{kz}^* + J_{bz}^m\, H_{kz}^*)\, dS \\ &- i\omega \int_{S_b} (P_{bz}\, E_{kz}^* + \mu_0 M_{bz}\, H_{kz}^*)\, dS\,.\end{aligned} \qquad (B.9)$$

After inserting Eqs. (B.8) and (B.9) into Eq. (B.7) we obtain

$$\begin{aligned}
&\int_{S_b}\Big(\mathbf{E}_k^*\cdot\nabla\times\mathbf{H}_b-\mathbf{H}_k^*\cdot\nabla\times\mathbf{E}_b\Big)\,dS\\
&=\oint_{L_b}\Big[\,(\mathbf{n}_b\times\mathbf{H}_b)\cdot\mathbf{E}_k^*-(\mathbf{n}_b\times\mathbf{E}_b)\cdot\mathbf{H}_k^*\Big]dl\\
&-\int_{S_b}(J^e_{bz}\,E^*_{kz}+J^m_{bz}\,H^*_{kz})\,dS\\
&-\,i\omega\int_{S_b}(P_{bz}\,E^*_{kz}+\mu_0M_{bz}\,H^*_{kz})\,dS\\
&+\,i\omega\int_{S_b}(\mathbf{P}_k^*\cdot\mathbf{E}_b+\mu_0\mathbf{M}_k^*\cdot\mathbf{H}_b)dS
\end{aligned}\qquad(B.10)$$

The first integral in the right-hand side of Eq. (B.10) involves the effective surface currents $\mathbf{J}^e_{s,ef}=-\mathbf{n}_b\times\mathbf{H}_b$ and $\mathbf{J}^m_{s,ef}=\mathbf{n}_b\times\mathbf{E}_b$ defined by Eqs. (4.27) and (4.28). With allowing for this and employing the expression for the normalizing coefficient

$$N_{kl}\;=\;\int_S(\mathbf{E}_k^*\times\mathbf{H}_l+\mathbf{E}_l\times\mathbf{H}_k^*)\cdot\mathbf{z}_0\,dS\,,$$

the substitution of Eq. (B.10) into Eq. (B.6) yields

$$\begin{aligned}
&\sum_l N_{kl}\,\frac{dA_l}{dz}=-\int_{S_b}(\mathbf{J}_b^e\cdot\mathbf{E}_k^*+\mathbf{J}_b^m\cdot\mathbf{H}_k^*)\,dS\\
&-\int_{L_b}(\mathbf{J}^e_{s,ef}\cdot\mathbf{E}_k^*+\mathbf{J}^m_{s,ef}\cdot\mathbf{H}_k^*)\,dl\\
&-\,i\omega\int_{S_b}\Big[\,(\mathbf{P}_b\cdot\mathbf{E}_k^*-\mathbf{P}_k^*\cdot\mathbf{E}_b)+(\mu_0\mathbf{M}_b\cdot\mathbf{H}_k^*-\mu_0\mathbf{M}_k^*\cdot\mathbf{H}_b)\Big]\,dS\,.
\end{aligned}\qquad(B.11)$$

The last integral in the right-hand side of Eq. (B.11) vanishes because of

$$\begin{aligned}
&\Big[\,(\mathbf{P}_b\cdot\mathbf{E}_k^*-\mathbf{P}_k^*\cdot\mathbf{E}_b)+(\mu_0\mathbf{M}_b\cdot\mathbf{H}_k^*-\mu_0\mathbf{M}_k^*\cdot\mathbf{H}_b)\Big]\\
&=\Big[\,(\mathbf{D}_b\cdot\mathbf{E}_k^*-\mathbf{D}_k^*\cdot\mathbf{E}_b)+(\mathbf{B}_b\cdot\mathbf{H}_k^*-\mathbf{B}_k^*\cdot\mathbf{H}_b)\Big]\\
&=\Big[(\bar\epsilon-\bar\epsilon^{\,\dagger}):\mathbf{E}_b\mathbf{E}_k^*\,+\,(\bar\mu-\bar\mu^{\,\dagger}):\mathbf{H}_b\mathbf{H}_k^*\\
&+\,(\bar\xi-\bar\zeta^{\,\dagger}):\mathbf{H}_b\mathbf{E}_k^*\,+\,(\bar\zeta-\bar\xi^{\,\dagger}):\mathbf{E}_b\mathbf{H}_k^*\Big]=0
\end{aligned}$$

where the constitutive relations (2.8), (2.9), and (2.13) have been used for lossless bianisotropic media.

For the most general case of the reactive kth mode from the orthonormalization relation (3.24) we have $N_{kl} = N_k \delta_{\tilde{k}l}$. Then formula $(B.11)$ finally gives the excitation equation for the $\tilde{k}$th mode:

$$\begin{aligned} \frac{dA_{\tilde{k}}}{dz} = &- \frac{1}{N_k} \int_{S_b} (\mathbf{J}_b^e \cdot \mathbf{E}_k^* + \mathbf{J}_b^m \cdot \mathbf{H}_k^*)\, dS \\ &- \frac{1}{N_k} \int_{L_b} (\mathbf{J}_{s,ef}^e \cdot \mathbf{E}_k^* + \mathbf{J}_{s,ef}^m \cdot \mathbf{H}_k^*)\, dl\,. \end{aligned} \qquad (B.12)$$

This formula is in agreement with the similar equation (5.13) obtained by another method, not counting the absence of the actual surface currents which can be considered as enclosed implicitly into the bulk currents.

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