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A UNIFIED REPRESENTATION OF THE DYADIC GREEN'S FUNCTIONS FOR PLANAR, CYLINDRICAL AND SPHERICAL MULTILAYERED BIISOTROPIC MEDIA

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1. INTRODUCTION

The dyadic Green's function technique is a very powerful and elegant method in the study of electromagnetic wave propagation and scattering [1–4]. Since the past two decades, considerable amount of research work have been devoted to the dyadic Green's functions in isotropic [1–8] and more complex media [9, 10] In recent years, there has been a lot of investigations and applications of the dyadic Green's functions for solving (reciprocal) chiral- and biisotropic-related problems [11-14].Various dyadic Green's functions have been constructed and utilized for multilayered chiral media of different geometries such as planarly stratified chiral media [15–17], cylindrically multilayered chiral media [18, 19] and spherically multilayered chiral media [20–23] Although these dyadic Green's functions have been investigated in great extent, they are mostly meant for a specific type of geometry only. Furthermore, most of the formulations require the laborious construction and inversion of large (4×4 or 8×8) matrices, thus inhibiting the lucid interpretation of their elements. For some of these dyadic Green's functions, their general expressions are extremely long and contain large number of dyads (32 or 64).

In this paper, we present a systematic and unified formulation of the dyadic Green's functions for multilayered biisotropic media. The media are stratified in a general direction which includes the canonical cases of planar, cylindrical and spherical geometries. The total number of layers can be arbitrary, so are the field and source locations. Based on the principle of scattering superposition, both electric and magnetic dyadic Green's functions are derived simultaneously in terms of unbounded and scattered parts. By applying the effective reflection and transmission concepts, the scattering coefficients for the scattered dyadic Green's functions are determined without cumbersome operations and this approach has also provided good physical insights to the scattering mechanism. Throughout the formulation, one needs to deal with matrices of size 2×2 only. The resulting general expressions for the dyadic Green's functions are written in very compact (16 dyads) and convenient forms. To demonstrate the application of these expressions, the three canonical multilayered biisotropic media are considered explicitly. In the following, the dyadic Green's functions are derived based on Post-Jaggard constitutive relations [24]. However, since there are other sets of constitutive relations available for characterizing a biisotropic medium, the dyadic Green's functions are also obtained for Drude-Born-Federov and Condon-Tellegen relations to illustrate their possible distinctions. Being a special case of biisotropic media, the multilayered isotropic media are considered in particular. Throughout the analysis, $e^{-i\omega t}$ time dependence is assumed and suppressed.

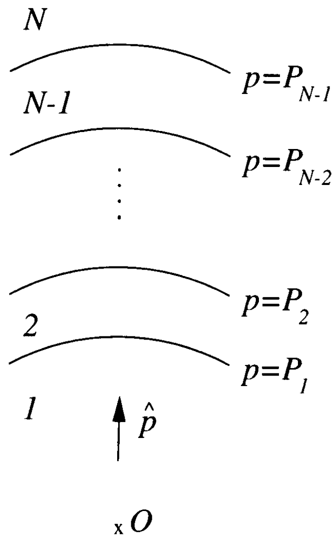

Figure 1. Geometry of a $\hat{p}$-stratified multilayered biisotropic media.

2. PROBLEM FORMULATION

Fig. 1 shows a portion of the N-layered biisotropic media. The media is assumed to be stratified in $\hat{p}$ direction, with each interface of stratification denoted by surface $p = P_f$ ($f = 1, 2, ..., N-1$). Within each layer f ($f = 1, 2, ..., N$), the medium is homogeneous and characterized by Post-Jaggard constitutive relations [24] as

$$\overline{D}_f = \epsilon_f \overline{E}_f + (\psi_f + i\xi_f)\overline{B}_f \tag{1}$$

$$\overline{H}_f = \frac{1}{\mu_f}\overline{B}_f - (\psi_f - i\xi_f)\overline{E}_f \tag{2}$$

where ϵ_f, μ_f, ξ_f and ψ_f represent respectively the medium permittivity, permeability, chirality admittance and nonreciprocity susceptance. Notice that we have included the layer number index in the subscripts. This convention will be followed throughout the paper.

Assuming an electric current source $\overline{J}_s$ is impressed in layer s ($s = 1, 2, ..., N$). Due to linearity, the electric and magnetic fields in layer f can be related directly to the source in layer s through

$$\overline{E}_f(\overline{r}) = \iiint_{V'} dv' \, \overline{\overline{G}}_e^{(fs)}(\overline{r}, \overline{r}') \cdot \overline{J}_s(\overline{r}') \tag{3}$$

$$\overline{H}_f(\overline{r}) = \iiint\limits_{V'} dv' \,\overline{\overline{G}}_m^{(fs)}(\overline{r},\overline{r}') \cdot \overline{J}_s(\overline{r}') \tag{4}$$

where $\overline{\overline{G}}_e^{(fs)}$ and $\overline{\overline{G}}_m^{(fs)}$ are respectively the layered-media electric and magnetic dyadic Green's functions. Substituting (3) and (4) into the Maxwell equations, we obtain the Helmholtz-like dyadic equations as

$$\Box_f^2 \overline{\overline{G}}_e^{(fs)} = i\omega\mu_s \overline{\overline{I}}\delta(\overline{r}' - \overline{r})\delta_{fs} \tag{5}$$

$$\Box_f^2 \overline{\overline{G}}_m^{(fs)} = -i\omega\mu_s(\psi_s - i\xi_s)\overline{\overline{I}}\delta(\overline{r}' - \overline{r})\delta_{fs} + \nabla \times \overline{\overline{I}}\delta(\overline{r}' - \overline{r})\delta_{fs} \tag{6}$$

where the Helmholtz-like operator is defined as

$$\Box_f^2 \equiv \nabla \times \nabla \times -2\omega\mu_f\xi_f \nabla \times -k_{of}^2, \qquad k_{of}^2 = \omega^2\mu_f\epsilon_f \tag{7}$$

$\overline{\overline{I}}$ is the idemfactor, $\delta(\overline{r}' - \overline{r})$ is the Dirac delta function, and δ_{fs} is the Kronecker delta symbol. According to the principle of scattering superposition, each of the dyadic Green's functions can be considered as the superposition of unbounded and scattered parts, i.e.,

$$\overline{\overline{G}}_e^{(fs)} = \overline{\overline{G}}_{e0}\delta_{fs} + \overline{\overline{G}}_{eS}^{(fs)} \tag{8}$$

$$\overline{\overline{G}}_m^{(fs)} = \overline{\overline{G}}_{m0}\delta_{fs} + \overline{\overline{G}}_{mS}^{(fs)} \tag{9}$$

The unbounded dyadic Green's functions $\overline{\overline{G}}_{e0}$ and $\overline{\overline{G}}_{m0}$ account for the direct waves radiated by primary sources, while the scattered dyadic Green's functions $\overline{\overline{G}}_{eS}^{(fs)}$ and $\overline{\overline{G}}_{mS}^{(fs)}$ account for the waves scattered by layered-media interfaces.

For a biisotropic medium, the unbounded electric dyadic Green's function $\overline{\overline{G}}_{e0}$ can be expanded in terms of $\hat{p}$-propagating eigenfunctions [25] with the source point dyadic delta function term shown explicitly as

$$\begin{aligned}\overline{\overline{G}}_{e0}\delta_{fs} = &\frac{1}{i\omega\epsilon_s}\delta(\overline{r}' - \overline{r})\hat{p}\hat{p}' \\ &+ \int \sum C_s \Big\{ \Big[C_s^V \overline{V}_s^{>} \overline{V}_s'^{>} + C_s^W \overline{W}_s^{>} \overline{W}_s'^{>} \Big] U^{>} \\ &+ \Big[C_s^V \overline{V}_s^{<} \overline{V}_s'^{<} + C_s^W \overline{W}_s^{<} \overline{W}_s'^{<} \Big] U^{<} \Big\}\end{aligned} \tag{10}$$

Here, the symbol $\int\sum$ represents the integration and/or summation operators in the expansion, with the coefficients of expansion given by $C_s C_s^V$ and $C_s C_s^W$. $U_{<}^{>} \equiv U(\pm p \mp p')$ denote the Heaviside unit step functions, which together with the superscripts '$>$' and '$<$' correspond to $p > p'$ and $p < p'$ respectively. $\overline{V}$ and $\overline{W}$ are the $\hat{p}$-propagating solenoidal eigenfunctions in biisotropic media, namely the left- and right-circularly polarized modes. These modes can be written in terms of commonly employed vector wave functions $\overline{M}$ and $\overline{N}$ [1] as

$$\overline{V}_s = \overline{M}(k_{vs}) + \overline{N}(k_{vs}), \qquad \overline{W}_s = \overline{M}(k_{ws}) - \overline{N}(k_{ws}) \tag{11}$$

with k_{vs} and k_{ws} being their corresponding wave numbers:

$$k_{\substack{v\\w}s} = \pm\omega\mu_s\xi_s + \sqrt{k_{os}^2 + (\omega\mu_s\xi_s)^2} \tag{12}$$

The explicit expressions for $\overline{V}$, $\overline{W}$ and the other notations mentioned above will be given later when considering the canonical cases of multilayered biisotropic media.

Using equation (10) and applying the relations

$$\nabla \times \overline{V}_s = k_{vs}\overline{V}_s, \qquad \nabla \times \overline{W}_s = -k_{ws}\overline{W}_s \tag{13}$$

the unbounded magnetic dyadic Green's function $\overline{\overline{G}}_{m0}$ can be expanded as

$$\begin{aligned} \overline{\overline{G}}_{m0}\delta_{fs} = &- \frac{\psi_s - i\xi_s}{i\omega\epsilon_s}\delta(\overline{r}' - \overline{r})\hat{p}\hat{p}' \\ &+ \int\sum C_s\Big\{\Big[C_s^V\eta_s^V\overline{V}_s^{>}\overline{V}'^{>}_s + C_s^W\eta_s^W\overline{W}_s^{>}\overline{W}'^{>}_s\Big]U^{>} \\ &+ \Big[C_s^V\eta_s^V\overline{V}_s^{<}\overline{V}'^{<}_s + C_s^W\eta_s^W\overline{W}_s^{<}\overline{W}'^{<}_s\Big]U^{<}\Big\} \end{aligned} \tag{14}$$

with

$$\eta_s^{\substack{V\\W}} = \frac{1}{i\omega\mu_s}\Big(\pm\frac{k_{vs} + k_{ws}}{2} - i\omega\mu_s\psi_s\Big) \tag{15}$$

acting as the biisotropic admittances. Note that there is an extra source point dyadic delta function term in addition to the solenoidal eigenfunction expansion. This is due to the fact that in contrast to the isotropic case, the magnetic dyadic Green's function is not solenoidal in biisotropic media characterized by Post-Jaggard constitutive relations, i.e., $\nabla \cdot \overline{\overline{G}}_{m0} \neq \overline{0}$, as one can deduce by taking the divergence of (2).

With the availability of the eigenfunction expansions for the unbounded dyadic Green's functions, we are ready to construct the expansions for the scattered dyadic Green's functions. In order to avoid laborious operations, some concepts of effective plane wave reflection and transmission are utilized to determine the scattering coefficients of these scattered dyadics.

3. SCATTERED DYADIC GREEN'S FUNCTION

In this section, we will first derive the expressions of the local (superscripted l) and global (superscripted g) reflection and transmission matrices for a $\hat{p}$-stratified multilayered biisotropic media, and then proceed to obtain the expressions of the scattered dyadic Green's functions in terms of these matrices. The local matrices correspond to the reflection and transmission at an interface separating two layers of biisotropic media, while the global matrices also incorporate the effects of multiple-reflections when the media consist of more layers. In the following, the derivation of these matrices would be based directly on the previously obtained eigenfunction expansions for the unbounded dyadic Green's functions, thus demonstrating the full advantage of these expansions. Since the $\overline{V}$ and $\overline{W}$ functions of (10) and (14) inherently contain the $\hat{p}$-propagating factors, both cases of outgoing and incoming waves are to be considered. These waves correspond to propagation in $+\hat{p}$ and $-\hat{p}$ direction respectively.

3.1 Outgoing Reflection and Transmission Matrices

Consider a primary outgoing wave in the layer q of a two-layered biisotropic media, Fig. 2a. Due to the presence of the interface at $p = P_q$, the total electric fields in layers q and $q+1$ can be written with reference to the unbounded electric dyadic Green's function (10) as

$$\overline{E}_q = \int \sum \left\{ \left[\overline{V}_q^{>}, \overline{W}_q^{>}\right] \cdot \overline{g}^{>} + \left[\overline{V}_q^{<}, \overline{W}_q^{<}\right] \cdot \overline{\overline{R}}_{q,q+1}^{l} \cdot \overline{g}^{>} \right\} \tag{16}$$

$$\overline{E}_{q+1} = \int \sum \left[\overline{V}_{q+1}^{>}, \overline{W}_{q+1}^{>}\right] \cdot \overline{\overline{T}}_{q,q+1}^{l} \cdot \overline{g}^{>} \tag{17}$$

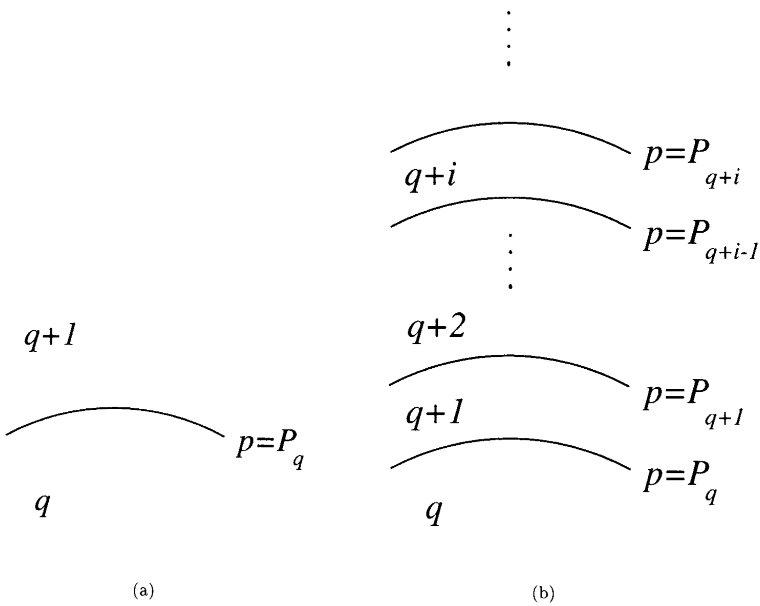

Figure 2. Reflection and transmission of an outgoing wave in biisotropic media with (a) two and (b) more layers.

Here, $\overline{g}^{>}$ is a 2×1 column vector specifying the amplitudes of the $\overline{V}^{>}$ and $\overline{W}^{>}$ outgoing waves in the form

$$\overline{g}^{>} = \begin{bmatrix} C_q C_q^V \overline{V}'^{>}_q \cdot \hat{a} \\ C_q C_q^W \overline{W}'^{>}_q \cdot \hat{a} \end{bmatrix} \tag{18}$$

where for generality, we assume that these waves are excited by a point source oriented in $\hat{a}$ direction. Furthermore, $\overline{\overline{R}}^l_{q,q+1}$ and $\overline{\overline{T}}^l_{q,q+1}$ are the 2×2 local reflection and transmission matrices to be determined for the outgoing waves traveling from layer q towards layer $q+1$:

$$\overline{\overline{R}}^l_{q,q+1} = \begin{bmatrix} R^{VV}_{q,q+1} & R^{VW}_{q,q+1} \\ R^{WV}_{q,q+1} & R^{WW}_{q,q+1} \end{bmatrix}, \qquad \overline{\overline{T}}^l_{q,q+1} = \begin{bmatrix} T^{VV}_{q,q+1} & T^{VW}_{q,q+1} \\ T^{WV}_{q,q+1} & T^{WW}_{q,q+1} \end{bmatrix} \tag{19}$$

In these matrices, the diagonal elements denote the self-coupling of waves, that is, $\overline{V}$ to $\overline{V}$ or $\overline{W}$ to $\overline{W}$ coupling, while the off-diagonal terms denote the cross-coupling from $\overline{W}$ to $\overline{V}$ and from $\overline{V}$ to $\overline{W}$. Following the similar arguments, the total magnetic fields in layers q

and $q+1$ can be written by referring to the unbounded magnetic dyadic Green's function (14) as

$$\overline{H}_q = \int\sum\left\{\left[\eta_q^V \overline{V}_q^{>}, \eta_q^W \overline{W}_q^{>}\right]\cdot\overline{g}^{>} + \left[\eta_q^V \overline{V}_q^{<}, \eta_q^W \overline{W}_q^{<}\right]\cdot\overline{\overline{R}}^{l}_{q,q+1}\cdot\overline{g}^{>}\right\} \tag{20}$$

$$\overline{H}_{q+1} = \int\sum\left[\eta_{q+1}^V \overline{V}_{q+1}^{>}, \eta_{q+1}^W \overline{W}_{q+1}^{>}\right]\cdot\overline{\overline{T}}^{l}_{q,q+1}\cdot\overline{g}^{>} \tag{21}$$

To find $\overline{\overline{R}}^{l}_{q,q+1}$ and $\overline{\overline{T}}^{l}_{q,q+1}$, we impose boundary conditions involving both electric and magnetic fields at $p = P_q$ and obtain

$$\overline{\overline{E}}^{>}_{q+1}\cdot\overline{\overline{T}}^{l}_{q,q+1} = \overline{\overline{E}}^{>}_{q} + \overline{\overline{E}}^{<}_{q}\cdot\overline{\overline{R}}^{l}_{q,q+1} \tag{22}$$

$$\overline{\overline{H}}^{>}_{q+1}\cdot\overline{\overline{T}}^{l}_{q,q+1} = \overline{\overline{H}}^{>}_{q} + \overline{\overline{H}}^{<}_{q}\cdot\overline{\overline{R}}^{l}_{q,q+1} \tag{23}$$

where

$$\overline{\overline{E}}^{\gtrless}_{f} = \int\sum\int dS\begin{bmatrix}\overline{t}_1\\ \overline{t}_2\end{bmatrix}\cdot\left[\,\overline{V}_f^{\gtrless}\quad \overline{W}_f^{\gtrless}\,\right] \tag{24}$$

$$\overline{\overline{H}}^{\gtrless}_{f} = \overline{\overline{E}}^{\gtrless}_{f}\cdot\begin{bmatrix}\eta_f^V & 0\\ 0 & \eta_f^W\end{bmatrix} \tag{25}$$

In the above, $\overline{t}_1$ and $\overline{t}_2$ are two linearly independent vectors transverse to $\hat{p}$ which when dot-integrated with $\overline{V}$ and $\overline{W}$ over $\int dS$ lead to extraction of a particular mode corresponding to a particular index in $\int\sum$. Note that this step is crucial in imposing restriction to our general $\hat{p}$-stratified multilayered media. In particular, the layered-media interfaces should be of the geometry on which one can apply some orthogonalities by selecting some convenient vector functions. Typical examples conforming to this requirement are the canonical Dupin surfaces [26] of planar, cylindrical and spherical geometries. Some convenient $\overline{t}_1$ and $\overline{t}_2$ will be given later for these three surfaces. (In reality, similar restrictions have been imposed on the eigenfunction expansions of the unbounded dyadic Green's functions [1], [25].) Solving equations (22) and (23), we obtain

$$\overline{\overline{R}}^{l}_{q,q+1} = \left[\overline{\overline{E}}^{<}_{q} - \overline{\overline{E}}^{>}_{q+1}\cdot\left(\overline{\overline{H}}^{>}_{q+1}\right)^{-1}\cdot\overline{\overline{H}}^{<}_{q}\right]^{-1}\cdot$$

$$\left[\overline{\overline{E}}^{>}_{q+1}\cdot\left(\overline{\overline{H}}^{>}_{q+1}\right)^{-1}\cdot\overline{\overline{H}}^{>}_{q}-\overline{\overline{E}}^{>}_{q}\right]\Bigg|_{P_q} \tag{26}$$

$$\overline{\overline{T}}^{l}_{q,q+1}=\left[\overline{\overline{E}}^{>}_{q+1}-\overline{\overline{E}}^{<}_{q}\cdot\left(\overline{\overline{H}}^{<}_{q}\right)^{-1}\cdot\overline{\overline{H}}^{>}_{q+1}\right]^{-1}\cdot \left[\overline{\overline{E}}^{>}_{q}-\overline{\overline{E}}^{<}_{q}\cdot\left(\overline{\overline{H}}^{<}_{q}\right)^{-1}\cdot\overline{\overline{H}}^{>}_{q}\right]\Bigg|_{P_q} \tag{27}$$

where $|_{P_q}$ indicates that all p's are to be replaced by P_q, i.e., the interface of layers q and $q+1$. In general, one should assign $p=P_{\min(s,f)}$ for matrices subscripted with (s,f) in order to select the interface according to our convention. Having determined the local reflection and transmission matrices, we are ready to derive the expressions of the global reflection and transmission matrices for media consisting of more than two layers. In the following, the amplitudes of the outgoing and incoming waves in each layer are denoted respectively by $\overline{a}^{>}$ and $\overline{a}^{<}$ subscripted with the corresponding layer number index.

With reference to Fig. 2b, at the interface $p=P_q$, the incoming wave in layer q is related to the outgoing wave via the outgoing global reflection matrix which takes into account the presence of all layers beyond layer q:

$$\overline{a}^{<}_{q}=\overline{\overline{R}}^{g}_{q,q+1}\cdot\overline{a}^{>}_{q} \tag{28}$$

Moreover, the constraint condition requires that this incoming wave is a consequence of the local transmission of the incoming wave in layer $q+1$ plus the local reflection of the outgoing wave in layer q, i.e.,

$$\overline{a}^{<}_{q}=\overline{\overline{T}}^{l}_{q+1,q}\cdot\overline{a}^{<}_{q+1}+\overline{\overline{R}}^{l}_{q,q+1}\cdot\overline{a}^{>}_{q} \tag{29}$$

Similarly, at the interface $p=P_{q+1}$, the incoming wave in layer $q+1$ can be related to the outgoing wave as

$$\overline{a}^{<}_{q+1}=\overline{\overline{R}}^{g}_{q+1,q+2}\cdot\overline{a}^{>}_{q+1} \tag{30}$$

while at the interface $p=P_q$, the outgoing wave in layer $q+1$ satisfies the constraint condition of

$$\overline{a}^{>}_{q+1}=\overline{\overline{T}}^{l}_{q,q+1}\cdot\overline{a}^{>}_{q}+\overline{\overline{R}}^{l}_{q+1,q}\cdot\overline{a}^{<}_{q+1} \tag{31}$$

Manipulating equations (28)–(31), we obtain the recursive relations

$$\overline{\overline{R}}^{g}_{q,q+1}=\overline{\overline{R}}^{l}_{q,q+1}+\overline{\overline{T}}^{l}_{q+1,q}\cdot\overline{\overline{R}}^{g}_{q+1,q+2}\cdot\overline{\overline{S}}_{q,q+1} \tag{32}$$

$$\overline{a}^{>}_{q+1}=\overline{\overline{S}}_{q,q+1}\cdot\overline{a}^{>}_{q} \tag{33}$$

where

$$\overline{\overline{S}}_{q,q+1} = \left[\overline{\overline{I}} - \overline{\overline{R}}^{l}_{q+1,q} \cdot \overline{\overline{R}}^{g}_{q+1,q+2}\right]^{-1} \cdot \overline{\overline{T}}^{l}_{q,q+1} \tag{34}$$

and $\overline{\overline{R}}^{g}_{N,N+1} = \overline{\overline{0}}$ since no reflection exists beyond layer N. Equation (33) can be generalized to relate the outgoing wave in layer $q+i$ to that in layer q via the outgoing global transmission matrix which includes the effects of multiple-reflections in-between:

$$\overline{\overline{T}}^{g}_{q,q+i} = \overline{\overline{S}}_{q+i-1,q+i} \cdots \overline{\overline{S}}_{q+1,q+2} \cdot \overline{\overline{S}}_{q,q+1}, \qquad i = 1, 2, ... \tag{35}$$

Hence, using equations (32) and (35) repeatedly, one can find $\overline{\overline{R}}^{g}_{q,q+1}$ and $\overline{\overline{T}}^{g}_{q,q+i}$ for all q and i. These global matrices will be used later to express the scattered dyadic Green's functions. Note from above that the incorporation of $\hat{p}$-propagating factors into the local reflection and transmission matrices has made the unified treatment of general $\hat{p}$-stratified multilayered media possible, thus providing the same formulas for global matrices of different $\hat{p}$. In particular, for the planarly layered media, one does not require to introduce explicitly a propagator matrix since the exponential phase factors have been taken into account in the local matrices. Furthermore, it is seen that all the above matrix formulas are dependent on $\overline{\overline{E}}^{\gtrless}$ of (24) only. Therefore, when considering the multilayered media of a specific geometry, one needs to determine the corresponding $\overline{\overline{E}}^{\gtrless}$ only whereas the other formulas are still applicable without any modification.

3.2 Incoming Reflection and Transmission Matrices

Consider now a primary incoming wave in layer q traveling towards inner layer(s). As can be seen, this case is actually similar to the outgoing wave case, except that all propagating directions should be reversed and the outer layer indices should be replaced with the inner ones. Hence, to avoid repetition, the expressions for the incoming local and global reflection and transmission matrices are given directly as

$$\begin{aligned}\overline{\overline{R}}^{l}_{q,q-1} = {}& \left[\overline{\overline{E}}^{>}_{q} - \overline{\overline{E}}^{<}_{q-1} \cdot \left(\overline{\overline{H}}^{<}_{q-1}\right)^{-1} \cdot \overline{\overline{H}}^{>}_{q}\right]^{-1} \cdot \\ & \left[\overline{\overline{E}}^{<}_{q-1} \cdot \left(\overline{\overline{H}}^{<}_{q-1}\right)^{-1} \cdot \overline{\overline{H}}^{<}_{q} - \overline{\overline{E}}^{<}_{q}\right]\Bigg|_{P_{q-1}}\end{aligned} \tag{36}$$

$$\overline{\overline{T}}^{l}_{q,q-1} = \left[\overline{\overline{E}}^{<}_{q-1} - \overline{\overline{E}}^{>}_{q} \cdot \left(\overline{\overline{H}}^{>}_{q}\right)^{-1} \cdot \overline{\overline{H}}^{<}_{q-1}\right]^{-1} \cdot \left[\overline{\overline{E}}^{<}_{q} - \overline{\overline{E}}^{>}_{q} \cdot \left(\overline{\overline{H}}^{>}_{q}\right)^{-1} \cdot \overline{\overline{H}}^{<}_{q}\right]\Bigg|_{P_{q-1}} \tag{37}$$

$$\overline{\overline{R}}^{g}_{q,q-1} = \overline{\overline{R}}^{l}_{q,q-1} + \overline{\overline{T}}^{l}_{q-1,q} \cdot \overline{\overline{R}}^{g}_{q-1,q-2} \cdot \overline{\overline{S}}_{q,q-1} \tag{38}$$

$$\overline{\overline{T}}^{g}_{q,q-i} = \overline{\overline{S}}_{q-i+1,q-i} \cdots \overline{\overline{S}}_{q-1,q-2} \cdot \overline{\overline{S}}_{q,q-1}, \qquad i = 1, 2, \ldots \tag{39}$$

$$\overline{\overline{S}}_{q,q-1} = \left[\overline{\overline{I}} - \overline{\overline{R}}^{l}_{q-1,q} \cdot \overline{\overline{R}}^{g}_{q-1,q-2}\right]^{-1} \cdot \overline{\overline{T}}^{l}_{q,q-1} \tag{40}$$

Notice that these equations resemble very closely those for the outgoing wave. Moreover, since $\overline{\overline{R}}^{g}_{1,0} = \overline{\overline{0}}$, one can use equations (38) and (39) recursively to find $\overline{\overline{R}}^{g}_{q,q-1}$ and $\overline{\overline{T}}^{g}_{q,q-i}$ for all q and i.

3.3 Compact General Expressions of Scattered Dyadic Green's Functions

With the availability of the outgoing and incoming global reflection and transmission matrices, one can readily use them to determine the scattering coefficients which relate the scattered fields in layer f to the primary fields excited by a point source in layer s. Furthermore, since both field and source locations are arbitrary, i.e., f and s could satisfy any case of $f = s$, $f > s$ or $f < s$, one can unify the results corresponding to each of these cases using the Kronecker delta symbol and the Heaviside unit step functions. Then, the complete final solutions for both scattered electric and magnetic dyadic Green's functions are

$$\begin{aligned}\overline{\overline{G}}^{(fs)}_{eS} = \int \sum C_s \Big\{ & \left[\overline{V}^{>}_{f}, \overline{W}^{>}_{f}\right] \cdot \overline{\overline{A}} \cdot \left[C^{V}_{s} \overline{V}'^{>}_{s}, C^{W}_{s} \overline{W}'^{>}_{s}\right]^{T} \\ & + \left[\overline{V}^{>}_{f}, \overline{W}^{>}_{f}\right] \cdot \overline{\overline{B}} \cdot \left[C^{V}_{s} \overline{V}'^{<}_{s}, C^{W}_{s} \overline{W}'^{<}_{s}\right]^{T} \\ & + \left[\overline{V}^{<}_{f}, \overline{W}^{<}_{f}\right] \cdot \overline{\overline{C}} \cdot \left[C^{V}_{s} \overline{V}'^{>}_{s}, C^{W}_{s} \overline{W}'^{>}_{s}\right]^{T} \\ & + \left[\overline{V}^{<}_{f}, \overline{W}^{<}_{f}\right] \cdot \overline{\overline{D}} \cdot \left[C^{V}_{s} \overline{V}'^{<}_{s}, C^{W}_{s} \overline{W}'^{<}_{s}\right]^{T} \Big\}\end{aligned} \tag{41}$$

$$\begin{aligned}\overline{\overline{G}}^{(fs)}_{mS} = \int \sum C_s \Big\{ & \left[\eta^{V}_{f} \overline{V}^{>}_{f}, \eta^{W}_{f} \overline{W}^{>}_{f}\right] \cdot \overline{\overline{A}} \cdot \left[C^{V}_{s} \overline{V}'^{>}_{s}, C^{W}_{s} \overline{W}'^{>}_{s}\right]^{T} \\ & + \left[\eta^{V}_{f} \overline{V}^{>}_{f}, \eta^{W}_{f} \overline{W}^{>}_{f}\right] \cdot \overline{\overline{B}} \cdot \left[C^{V}_{s} \overline{V}'^{<}_{s}, C^{W}_{s} \overline{W}'^{<}_{s}\right]^{T}\end{aligned}$$

$$+\left[\eta_f^V \overline{V}_f^<, \eta_f^W \overline{W}_f^<\right] \cdot \overline{\overline{C}} \cdot \left[C_s^V \overline{V}'^>_s, C_s^W \overline{W}'^>_s\right]^T$$
$$\left. + \left[\eta_f^V \overline{V}_f^<, \eta_f^W \overline{W}_f^<\right] \cdot \overline{\overline{D}} \cdot \left[C_s^V \overline{V}'^<_s, C_s^W \overline{W}'^<_s\right]^T \right\} \quad (42)$$

where the scattering coefficient matrices are given explicitly by

$$\begin{aligned}
\overline{\overline{A}} = & \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s-1} \cdot \overline{\overline{R}}^g_{s,s+1}\right]^{-1} \cdot \overline{\overline{R}}^g_{s,s-1} \cdot \overline{\overline{R}}^g_{s,s+1} \delta_{fs} \\
& + \overline{\overline{T}}^g_{s,f} \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s-1} \cdot \overline{\overline{R}}^g_{s,s+1}\right]^{-1} U(f-s) \\
& + \overline{\overline{R}}^g_{f,f-1} \cdot \overline{\overline{T}}^g_{s,f} \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s+1} \cdot \overline{\overline{R}}^g_{s,s-1}\right]^{-1} \cdot \overline{\overline{R}}^g_{s,s+1} U(s-f) \quad (43) \\
\overline{\overline{B}} = & \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s-1} \cdot \overline{\overline{R}}^g_{s,s+1}\right]^{-1} \cdot \overline{\overline{R}}^g_{s,s-1} \delta_{fs} \\
& + \overline{\overline{T}}^g_{s,f} \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s-1} \cdot \overline{\overline{R}}^g_{s,s+1}\right]^{-1} \cdot \overline{\overline{R}}^g_{s,s-1} U(f-s) \\
& + \overline{\overline{R}}^g_{f,f-1} \cdot \overline{\overline{T}}^g_{s,f} \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s+1} \cdot \overline{\overline{R}}^g_{s,s-1}\right]^{-1} U(s-f) \quad (44) \\
\overline{\overline{C}} = & \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s+1} \cdot \overline{\overline{R}}^g_{s,s-1}\right]^{-1} \cdot \overline{\overline{R}}^g_{s,s+1} \delta_{fs} \\
& + \overline{\overline{R}}^g_{f,f+1} \cdot \overline{\overline{T}}^g_{s,f} \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s-1} \cdot \overline{\overline{R}}^g_{s,s+1}\right]^{-1} U(f-s) \\
& + \overline{\overline{T}}^g_{s,f} \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s+1} \cdot \overline{\overline{R}}^g_{s,s-1}\right]^{-1} \cdot \overline{\overline{R}}^g_{s,s+1} U(s-f) \quad (45) \\
\overline{\overline{D}} = & \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s+1} \cdot \overline{\overline{R}}^g_{s,s-1}\right]^{-1} \cdot \overline{\overline{R}}^g_{s,s+1} \cdot \overline{\overline{R}}^g_{s,s-1} \delta_{fs} \\
& + \overline{\overline{R}}^g_{f,f+1} \cdot \overline{\overline{T}}^g_{s,f} \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s-1} \cdot \overline{\overline{R}}^g_{s,s+1}\right]^{-1} \cdot \overline{\overline{R}}^g_{s,s-1} U(f-s) \\
& + \overline{\overline{T}}^g_{s,f} \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}^g_{s,s+1} \cdot \overline{\overline{R}}^g_{s,s-1}\right]^{-1} U(s-f) \quad (46)
\end{aligned}$$

Hence, we have obtained the general expressions of the scattered dyadic Green's functions for multilayered biisotropic media stratified in $\hat{p}$-direction. Note that the utilization of the effective reflection and transmission concepts has avoided complicated formulation of the scattering coefficients and at the same time has provided good physical insights to the scattering mechanism. Furthermore, it is seen that the expressions (41) and (42) are very compact each containing 16 dyads at most and all the matrices to be dealt with are of size 2×2 only. In order to demonstrate the application of these expressions, let us consider explicitly the three canonical cases of planarly, cylindrically and spherically multilayered media. As mentioned above, only $\overline{\overline{E}}^{\gtrless}$ of each case is required to be determined. Using this $\overline{\overline{E}}^{\gtrless}$, all other matrices can be calculated readily by applying the same formulas.

4. CANONICAL MULTILAYERED BIISOTROPIC MEDIA

4.1 Planar

For planarly-stratified biisotropic media, we assume their planes of stratification are normal to $\hat{z}$ direction and let $\hat{p} = \hat{z}$. Using the commonly employed rectangular vector wave functions defined as [4]

$$\overline{M}(\overline{r}; k_x, k_y, k_z) = [\hat{x}ik_y - \hat{y}ik_x]e^{ik_x x + ik_y y + ik_z z} \tag{47}$$

$$\overline{N}(\overline{r}; k_x, k_y, k_z) = \frac{1}{k}[-\hat{x}k_z k_x - \hat{y}k_z k_y + \hat{z}(k_x^2 + k_y^2)]e^{ik_x x + ik_y y + ik_z z}$$

$$k^2 = k_x^2 + k_y^2 + k_z^2 \tag{48}$$

the dyadic Green's functions are expanded in (10), (14), (41) and (42) with ($q = s, f$)

$$\int \sum \equiv \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y, \qquad C_s = -\frac{\omega\mu_s}{8\pi^2(k_{vs} + k_{ws})(k_x^2 + k_y^2)},$$

$$C_s^{\substack{V \\ W}} = \frac{k_{\substack{v \\ w}s}}{k_{z\substack{v \\ w}s}} \tag{49}$$

$$k_{z\substack{v \\ w}q}^2 = k_{\substack{v \\ w}q}^2 - k_x^2 - k_y^2 \tag{50}$$

$$\overline{V}_q^{\gtrless} = \overline{M}(\overline{r}; k_x, k_y, \pm k_{zvq}) + \overline{N}(\overline{r}; k_x, k_y, \pm k_{zvq}) \tag{51}$$

$$\overline{V}'^{\gtrless}_q = \overline{M}(\overline{r}'; -k_x, -k_y, \mp k_{zvq}) + \overline{N}(\overline{r}'; -k_x, -k_y, \mp k_{zvq}) \tag{52}$$

$$\overline{W}_q^{\gtrless} = \overline{M}(\overline{r}; k_x, k_y, \pm k_{zwq}) - \overline{N}(\overline{r}; k_x, k_y, \pm k_{zwq}) \tag{53}$$

$$\overline{W}'^{\gtrless}_q = \overline{M}(\overline{r}'; -k_x, -k_y, \mp k_{zwq}) - \overline{N}(\overline{r}'; -k_x, -k_y, \mp k_{zwq}) \tag{54}$$

Referring to the expressions of $\overline{M}$ and $\overline{N}$ in (47) and (48), a convenient choice for $\overline{t}_1$ and $\overline{t}_2$ in (24) would be

$$\overline{t}_1 = \frac{1}{4\pi^2({k'_x}^2 + {k'_y}^2)}[\hat{x}k'_x + \hat{y}k'_y]e^{-ik'_x x - ik'_y y} \tag{55}$$

$$\overline{t}_2 = \frac{1}{4\pi^2({k'_x}^2 + {k'_y}^2)}[-\hat{x}k'_y + \hat{y}k'_x]e^{-ik'_x x - ik'_y y} \tag{56}$$

where $4\pi^2({k'_x}^2 + {k'_y}^2)$ represents the normalization factor. Note that $\overline{t}_1$ and $\overline{t}_2$ satisfy $\hat{z}\times\overline{t}_1 = \overline{t}_2$ and $\overline{t}_2\times\hat{z} = \overline{t}_1$, i.e., $\overline{t}_1\overline{t}_2\hat{z}$ forms a right-handed triad.Performing the surface integral of $\int dS \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy$ and dropping the primes associated with k'_x and k'_y, we obtain

$$\overline{\overline{E}}_f^{\gtrless} = \begin{bmatrix} \mp e^{\pm ik_{zvf}z}k_{zvf}/k_{vf} & \pm e^{\pm ik_{zwf}z}k_{zwf}/k_{wf} \\ -ie^{\pm ik_{zvf}z} & -ie^{\pm ik_{zwf}z} \end{bmatrix} \tag{57}$$

Having determined the above $\overline{\overline{E}}^{\gtrless}$, one can readily find all the local and global matrices for the scattering coefficients. For reference, the explicit expressions for the outgoing and incoming local reflection and transmission matrices at a planar biisotropic-biisotropic interface are given in Appendix A. Note that the exponential phase factors have been incorporated explicitly into the elements of these matrices. Furthermore, we see that the off-diagonal elements are nonzero implying that the eigenmodes of biisotropic media always couple with each other at the interfaces. From these matrices, the special case of chiral-isotropic, isotropic-chiral or isotropic-isotropic interface can be worked out readily. However, one must aware that there are different ways of defining the reflection and transmission matrices. In practice, one usually relates the most convenient eigenmodes in the corresponding medium, e.g., relating the linearly polarized modes in isotropic medium to the circularly polarized modes in chiral medium.

4.2 Cylindrical

For cylindrically multilayered biisotropic media, we assume the $\hat{z}$-axis of our coordinate system coincides with the core axis and let $\hat{p} = \hat{\rho}$. In order to obtain the compact expressions for the dyadic Green's functions as in (41) and (42), the exponential functions are chosen for the angular functions in the cylindrical vector wave functions [4]:

$$\overline{M}_n^{\overset{H}{J}}(\overline{r};\lambda,h) = \left[\hat{\rho}\frac{in}{\rho}Z_n^{\overset{H}{J}}(\lambda\rho) - \hat{\phi}\frac{\partial}{\partial\rho}Z_n^{\overset{H}{J}}(\lambda\rho)\right]e^{in\phi+ihz} \tag{58}$$

$$\begin{aligned}\overline{N}_n^{\overset{H}{J}}(\overline{r};\lambda,h) = \frac{1}{k}\Big[&\hat{\rho}ih\frac{\partial}{\partial\rho}Z_n^{\overset{H}{J}}(\lambda\rho) - \hat{\phi}\frac{hn}{\rho}Z_n^{\overset{H}{J}}(\lambda\rho) \\ &+ \hat{z}\lambda^2 Z_n^{\overset{H}{J}}(\lambda\rho)\Big]e^{in\phi+ihz}, \quad k^2 = \lambda^2 + h^2\end{aligned} \tag{59}$$

$$Z_n^{\substack{H \\ J}}(\lambda\rho) = \begin{cases} H_n^{(1)}(\lambda\rho) \\ J_n(\lambda\rho) \end{cases} \tag{60}$$

Using these functions, the dyadic Green's functions are expanded in (10), (14), (41) and (42) with ($q = s, f$)

$$\int\sum \equiv \int_{-\infty}^{\infty} dh \sum_{n=-\infty}^{\infty}, \quad C_s = -\frac{\omega\mu_s}{8\pi(k_{vs}+k_{ws})}$$

$$C_s^{\substack{V \\ W}} = \frac{k_{\substack{v \\ w}s}}{\lambda^2_{\substack{v \\ w}s}} \tag{61}$$

$$\lambda^2_{\substack{v \\ w}q} = k^2_{\substack{v \\ w}q} - h^2 \tag{62}$$

$$\overline{V}_q^{\gtrless} = \overline{M}_n^{\substack{H \\ J}}(\overline{r};\, \lambda_{vq}, h) + \overline{N}_n^{\substack{H \\ J}}(\overline{r};\, \lambda_{vq}, h) \tag{63}$$

$$\overline{V}'^{\gtrless}_q = \overline{M}_{-n}^{\substack{J \\ H}}(\overline{r}';\, -\lambda_{vq}, -h) + \overline{N}_{-n}^{\substack{J \\ H}}(\overline{r}';\, -\lambda_{vq}, -h) \tag{64}$$

$$\overline{W}_q^{\gtrless} = \overline{M}_n^{\substack{H \\ J}}(\overline{r};\, \lambda_{wq}, h) - \overline{N}_n^{\substack{H \\ J}}(\overline{r};\, \lambda_{wq}, h) \tag{65}$$

$$\overline{W}'^{\gtrless}_q = \overline{M}_{-n}^{\substack{J \\ H}}(\overline{r}';\, -\lambda_{wq}, -h) - \overline{N}_{-n}^{\substack{J \\ H}}(\overline{r}';\, -\lambda_{wq}, -h) \tag{66}$$

Note that the price to pay for the compact expressions is the larger (two-fold) extent for the summation index n. Considering $\overline{t}_1$ and $\overline{t}_2$ in (24), they are chosen to lie in the directions of the two transverse (to $\hat{\rho}$) vectors of the cylindrical coordinate system as

$$\overline{t}_1 = \frac{1}{4\pi^2} e^{-in'\phi - ih'z}\hat{\phi} \tag{67}$$

$$\overline{t}_2 = \frac{1}{4\pi^2} e^{-in'\phi - ih'z}\hat{z} \tag{68}$$

Carrying out the integration over the surface as $\int dS \equiv \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi$ and dropping the primes for n' and h', we obtain

$$\overline{\overline{E}}_f^{\gtrless} = \begin{bmatrix} -\frac{\partial}{\partial\rho} Z_n^{\substack{H \\ J}}(\lambda_{vf}\rho) - \frac{hn}{k_{vf}\rho} Z_n^{\substack{H \\ J}}(\lambda_{vf}\rho) & -\frac{\partial}{\partial\rho} Z_n^{\substack{H \\ J}}(\lambda_{wf}\rho) + \frac{hn}{k_{wf}\rho} Z_n^{\substack{H \\ J}}(\lambda_{wf}\rho) \\ Z_n^{\substack{H \\ J}}(\lambda_{vf}\rho)\lambda^2_{vf}/k_{vf} & -Z_n^{\substack{H \\ J}}(\lambda_{wf}\rho)\lambda^2_{wf}/k_{wf} \end{bmatrix} \tag{69}$$

With the above $\overline{\overline{E}}^{\gtrless}$, one can readily determine the local and global reflection and transmission matrices for the scattering coefficients.

Note that in these matrices, the propagating phase factors are buried in the Hankel $H_n^{(1)}$ and Bessel J_n functions which correspond to the outgoing and 'incoming' standing waves respectively. When a medium is lossy or the waves are highly evanescent, these Bessel/Hankel functions may become exponentially large. Therefore, care must be taken to prevent numerical overflows, e.g., renormalizing these functions in the definitions.

4.3 Spherical

For spherically multilayered biisotropic media, we assume their center point to be the origin of our spherical coordinate system and let $\hat{p} = \hat{r}$. As in the cylindrical case, the exponential functions are chosen for the angular (azimuthal) functions in the spherical vector wave functions [4]:

$$\overline{M}_{nm}^{\substack{h\\j}}(\overline{r};k) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\, z_n^{\substack{h\\j}}(kr) \left[\hat{\theta}\frac{im}{\sin\theta}P_n^m(\cos\theta) - \hat{\phi}\frac{\partial}{\partial\theta}P_n^m(\cos\theta)\right]e^{im\phi} \tag{70}$$

$$\overline{N}_{nm}^{\substack{h\\j}}(\overline{r};k) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\left[\hat{r}\frac{n(n+1)}{kr} z_n^{\substack{h\\j}}(kr)P_n^m(\cos\theta) + \frac{1}{kr}\frac{\partial}{\partial r}\left(r z_n^{\substack{h\\j}}(kr)\right)\left(\hat{\theta}\frac{\partial}{\partial\theta}P_n^m(\cos\theta) + \hat{\phi}\frac{im}{\sin\theta}P_n^m(\cos\theta)\right)\right]e^{im\phi} \tag{71}$$

$$z_n^{\substack{h\\j}}(kr) = \begin{cases} h_n^{(1)}(kr) \\ j_n(kr) \end{cases} \tag{72}$$

The dyadic Green's functions expanded using these functions are then (10), (14), (41) and (42) with ($q = s, f$)

$$\int\sum \equiv \sum_{n=1}^{\infty}\sum_{m=-n}^{n}, \qquad C_s = -\frac{\omega\mu_s}{n(n+1)(k_{vs}+k_{ws})},$$

$$C_s^{\substack{V\\W}} = k^2_{\substack{v\\w}s} \tag{73}$$

$$\overline{V}_q^{\substack{>\\<}} = \overline{M}_{nm}^{\substack{h\\j}}(\overline{r};k_{vq}) + \overline{N}_{nm}^{\substack{h\\j}}(\overline{r};k_{vq}) \tag{74}$$

$$\overline{V}'^{\substack{>\\<}}_{q} = \overline{M}^{\substack{j\\h}}_{n,-m}(\overline{r}'; k_{vq}) + \overline{N}^{\substack{j\\h}}_{n,-m}(\overline{r}'; k_{vq}) \tag{75}$$

$$\overline{W}^{\substack{>\\<}}_{q} = \overline{M}^{\substack{h\\j}}_{nm}(\overline{r}; k_{wq}) - \overline{N}^{\substack{h\\j}}_{nm}(\overline{r}; k_{wq}) \tag{76}$$

$$\overline{W}'^{\substack{>\\<}}_{q} = \overline{M}^{\substack{j\\h}}_{n,-m}(\overline{r}'; k_{wq}) - \overline{N}^{\substack{j\\h}}_{n,-m}(\overline{r}'; k_{wq}) \tag{77}$$

With reference to the expressions of $\overline{M}$ and $\overline{N}$ in (70) and (71), the convenient $\overline{t}_1$ and $\overline{t}_2$ for (24) are

$$\begin{aligned}\overline{t}_1 =& \frac{1}{n'(n'+1)}\sqrt{\frac{(2n'+1)(n'+m')!}{4\pi(n'-m')!}}\\ &\left[\hat{\theta}\frac{\partial}{\partial\theta}P_{n'}^{-m'}(\cos\theta) - \hat{\phi}\frac{im'}{\sin\theta}P_{n'}^{-m'}(\cos\theta)\right]e^{-im'\phi}\end{aligned} \tag{78}$$

$$\begin{aligned}\overline{t}_2 =& \frac{1}{n'(n'+1)}\sqrt{\frac{(2n'+1)(n'+m')!}{4\pi(n'-m')!}}\\ &\left[-\hat{\theta}\frac{im'}{\sin\theta}P_{n'}^{-m'}(\cos\theta) - \hat{\phi}\frac{\partial}{\partial\theta}P_{n'}^{-m'}(\cos\theta)\right]e^{-im'\phi}\end{aligned} \tag{79}$$

Note that $\hat{r} \times \overline{t}_1 = -\overline{t}_2$ and $\hat{r} \times \overline{t}_2 = \overline{t}_1$. Moreover, these vector functions can be shown to be mutually orthonormal with respect to $\int dS \equiv \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta$. Carrying out the integration and dropping the primes for n' and m', we obtain

$$\overline{\overline{E}}^{\substack{>\\<}}_{f} = \begin{bmatrix} \frac{1}{k_{vf}r}\frac{\partial}{\partial r}\left(r z_n^{\substack{h\\j}}(k_{vf}r)\right) & -\frac{1}{k_{wf}r}\frac{\partial}{\partial r}\left(r z_n^{\substack{h\\j}}(k_{wf}r)\right) \\ z_n^{\substack{h\\j}}(k_{vf}r) & z_n^{\substack{h\\j}}(k_{wf}r) \end{bmatrix} \tag{80}$$

Notice that $\overline{\overline{E}}^{\substack{>\\<}}$ are functions of n but not m, hence the resultant local and global matrices for the scattering coefficients will be independent of m as well.

5. OTHER BIISOTROPIC CONTSTITUTIVE RELATIONS

The above formulation of the dyadic Green's functions has been carried out based on Post-Jaggard constitutive relations. However, for chiral or biisotropic media, it is well-known that there are many other sets of constitutive relations available due to the various means of connecting the electric and magnetic quantities. Among them, the constitutive relations of Drude-Born-Federov (DBF) are used in common [24]:

$$\overline{D}_F = \epsilon_F[\overline{E}_F + (\beta - i\alpha)\nabla \times \overline{E}_F] \tag{81}$$
$$\overline{B}_F = \mu_F[\overline{H}_F + (\beta + i\alpha)\nabla \times \overline{H}_F] \tag{82}$$

Here, chirality is incorporated explicitly through β while nonreciprocity through α. Due to the possible distinctions between DBF and Post-Jaggard relations, the permittivity, permeability, vectors and dyadics are subscripted with F denoting DBF in addition to the layer number indices.

Substituting the constitutive equations (81) and (82) into the Maxwell equations and consider first the unbounded dyadic Green's functions defined analogous to (3) and (4), we find

$$\begin{aligned}\overline{\overline{G}}_{e0F}\delta_{fs} = &\frac{1}{i\omega\epsilon_{Fs}}\delta(\overline{r}' - \overline{r})\hat{p}\hat{p}'\\ &+ \int\sum C_s\Big\{\Big[F_s^V C_s^V \overline{V}_s^{>}\overline{V}'^{>}_s + F_s^W C_s^W \overline{W}_s^{>}\overline{W}'^{>}_s\Big]U^{>}\\ &+ \Big[F_s^V C_s^V \overline{V}_s^{<}\overline{V}'^{<}_s + F_s^W C_s^W \overline{W}_s^{<}\overline{W}'^{<}_s\Big]U^{<}\Big\}\end{aligned} \tag{83}$$

$$\begin{aligned}\overline{\overline{G}}_{m0F}\delta_{fs} = &\int\sum C_s\Big\{\Big[F_s^V C_s^V \eta_s^V \overline{V}_s^{>}\overline{V}'^{>}_s + F_s^W C_s^W \eta_s^W \overline{W}_s^{>}\overline{W}'^{>}_s\Big]U^{>}\\ &+ \Big[F_s^V C_s^V \eta_s^V \overline{V}_s^{<}\overline{V}'^{<}_s + F_s^W C_s^W \eta_s^W \overline{W}_s^{<}\overline{W}'^{<}_s\Big]U^{<}\Big\}\end{aligned} \tag{84}$$

where

$$F_s^{\substack{V\\W}} = 1 \pm ik_{\substack{v\\w}s}(\alpha_s - i\beta_s) \tag{85}$$
$$\eta_s^{\substack{V\\W}} = \frac{k_{Fs}^2}{i\omega\mu_{Fs}k_{os}^2}\Big(\pm\frac{k_{vs} + k_{ws}}{2} - ik_{os}^2\alpha_s\Big) \tag{86}$$
$$k_{\substack{v\\w}s} = \pm k_{os}^2\beta_s + k_{os}\sqrt{1 + k_{os}^2\beta_s^2} \tag{87}$$
$$k_{os}^2 = \frac{k_{Fs}^2}{1 - k_{Fs}^2(\alpha_s^2 + \beta_s^2)} \tag{88}$$
$$k_{Fs}^2 = \omega^2\mu_{Fs}\epsilon_{Fs} \tag{89}$$

Notice that subscript F is not attached to some of the above notations because they are equal to those of Post-Jaggard if their constitutive parameters are mapped as [24]. Similar arguments apply to the $\overline{V}$, $\overline{W}$ and the C coefficients in the expansions. Referring to (83) and (84), it is noteworthy that for the DBF magnetic dyadic Green's function, there is no singular term like that of Post-Jaggard, although the source point term for their electric dyadic Green's functions coincides with one another. This can be accounted for by the divergenceless property of the DBF magnetic field, i.e., $\nabla \cdot \overline{H}_F = 0$, hence the solenoidal eigenfunction expansion is sufficient for the corresponding dyadic Green's function.

Having determined the DBF unbounded dyadic Green's functions, the DBF scattered dyadic Green's functions can be obtained by referring to their expansion forms. From above, we observe that except for the source point singular term which does not affect the scattered dyadics, $\overline{\overline{G}}_{e0F}$ and $\overline{\overline{G}}_{m0F}$ are given by (10) and (14) with C_s^V and C_s^W replaced by $F_s^V C_s^V$ and $F_s^W C_s^W$ respectively. It then follows that $\overline{\overline{G}}_{eSF}^{(fs)}$ and $\overline{\overline{G}}_{mSF}^{(fs)}$ are given by (41) and (42) provided we replace their corresponding C_s^V and C_s^W. Upon making these changes, the dyadic Green's functions for the canonical cases of multilayered DBF media can be expanded with their respective C_s as

$$\text{Planar}: \quad C_s = -\frac{\omega \mu_{Fs} k_{os}^2}{8\pi^2 k_{Fs}^2 (k_{vs} + k_{ws})(k_x^2 + k_y^2)} \tag{90}$$

$$\text{Cylindrical}: \quad C_s = -\frac{\omega \mu_{Fs} k_{os}^2}{8\pi k_{Fs}^2 (k_{vs} + k_{ws})} \tag{91}$$

$$\text{Spherical}: \quad C_s = -\frac{\omega \mu_{Fs} k_{os}^2}{k_{Fs}^2 n(n+1)(k_{vs} + k_{ws})} \tag{92}$$

Another set of biisotropic constitutive relations which has been in much use is that of Condon-Tellegen (denoted by subscript T) [24]:

$$\overline{D}_T = \epsilon_T \overline{E}_T + (\gamma + i\omega\chi)\overline{H}_T \tag{93}$$

$$\overline{B}_T = \mu_T \overline{H}_T + (\gamma - i\omega\chi)\overline{E}_T \tag{94}$$

where chirality is introduced through χ and nonreciprocity through γ. Applying these constitutive equations into the Maxwell equations, we obtain the same set of equations as those of Post-Jaggard assuming their constitutive parameters are properly mapped. Consequently, the Condon-Tellegen unbounded and scattered dyadic Green's functions are similar to (10), (14), (41) and (42), including the extra source point dyadic delta function term which is required for the magnetic dyadic Green's function due to $\nabla \cdot \overline{\overline{G}}_{m0T} \neq \overline{0}$.

6. MULTILAYERED ISOTROPIC MEDIA

Thus far, the general expressions of the layered-media dyadic Green's functions have been obtained for general biisotropic media. Their reductions to chiral, Tellegen or isotropic media are certainly straightforward and easy. However, in usual practice for the isotropic case, one prefers the dyadic Green's functions to be written directly in terms of linearly polarized modes $\overline{N}$ and $\overline{M}$ (or TM and TE), rather than the circularly polarized modes $\overline{V}$ and $\overline{W}$. For this purpose, the definitions for (24) and (25) should read

$$\overline{\overline{E}}_f^{0\gtrless} = \int \sum \int dS \begin{bmatrix} \overline{t}_1 \\ \overline{t}_2 \end{bmatrix} \begin{bmatrix} \cdot \overline{N}_f^{\gtrless} & \overline{M}_f^{\gtrless} \end{bmatrix} \tag{95}$$

$$\overline{\overline{H}}_f^{0\gtrless} = \overline{\overline{E}}_f^{0\gtrless} \cdot \begin{bmatrix} 0 & \eta_f^0 \\ \eta_f^0 & 0 \end{bmatrix} \tag{96}$$

where

$$\eta_f^0 = \frac{k_{0f}}{i\omega\mu_{0f}}, \qquad k_{0f}^2 = \omega^2 \mu_{0f}\epsilon_{0f} \tag{97}$$

Note that $\overline{\overline{E}}_f^{0\gtrless}$ can also be derived from $\overline{\overline{E}}_f^{\gtrless}$ by

$$\overline{\overline{E}}_f^{0\gtrless} = \frac{1}{2} \overline{\overline{E}}_f^{\gtrless} \Big|_{\xi=\psi=0} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{98}$$

Using this $\overline{\overline{E}}_f^{0\gtrless}$ in the reflection and transmission matrix formulas, we obtain the unbounded and scattered dyadic Green's functions for $\hat{p}$-stratified multilayered isotropic media as

$$\begin{aligned} \overline{\overline{G}}_{e0}^0 \delta_{fs} = & \frac{1}{i\omega\epsilon_{0s}} \delta(\overline{r}' - \overline{r}) \hat{p}\hat{p}' + \int \sum C_s^0 \Big\{ \Big[\overline{N}_s^> \overline{N}'^>_s + \overline{M}_s^> \overline{M}'^>_s \Big] U^> \\ & + \Big[\overline{N}_s^< \overline{N}'^<_s + \overline{M}_s^< \overline{M}'^<_s \Big] U^< \Big\} \end{aligned} \tag{99}$$

$$\overline{\overline{G}}_{m0}^0 \delta_{fs} = \int \sum C_s^0 \eta_s^0 \Big\{ \Big[\overline{M}_s^> \overline{N}'^>_s + \overline{N}_s^> \overline{M}'^>_s \Big] U^>$$

$$+\left[\overline{M}_s^{<}\overline{N}'^{<}_s+\overline{N}_s^{<}\overline{M}'^{<}_s\right]U^{<}\Bigg\} \tag{100}$$

$$\begin{aligned}\overline{\overline{G}}{}^{0(fs)}_{eS}=&\int\sum C_s^0\Bigg\{\left[\overline{N}_f^{>},\overline{M}_f^{>}\right]\cdot\overline{\overline{A}}{}^0\cdot\left[\overline{N}'^{>}_s,\overline{M}'^{>}_s\right]^T\\&+\left[\overline{N}_f^{>},\overline{M}_f^{>}\right]\cdot\overline{\overline{B}}{}^0\cdot\left[\overline{N}'^{<}_s,\overline{M}'^{<}_s\right]^T\\&+\left[\overline{N}_f^{<},\overline{M}_f^{<}\right]\cdot\overline{\overline{C}}{}^0\cdot\left[\overline{N}'^{>}_s,\overline{M}'^{>}_s\right]^T\\&+\left[\overline{N}_f^{<},\overline{M}_f^{<}\right]\cdot\overline{\overline{D}}{}^0\cdot\left[\overline{N}'^{<}_s,\overline{M}'^{<}_s\right]^T\Bigg\}\end{aligned} \tag{101}$$

$$\begin{aligned}\overline{\overline{G}}{}^{0(fs)}_{mS}=&\int\sum C_s^0\eta_f^0\Bigg\{\left[\overline{M}_f^{>},\overline{N}_f^{>}\right]\cdot\overline{\overline{A}}{}^0\cdot\left[\overline{N}'^{>}_s,\overline{M}'^{>}_s\right]^T\\&+\left[\overline{M}_f^{>},\overline{N}_f^{>}\right]\cdot\overline{\overline{B}}{}^0\cdot\left[\overline{N}'^{<}_s,\overline{M}'^{<}_s\right]^T\\&+\left[\overline{M}_f^{<},\overline{N}_f^{<}\right]\cdot\overline{\overline{C}}{}^0\cdot\left[\overline{N}'^{>}_s,\overline{M}'^{>}_s\right]^T\\&+\left[\overline{M}_f^{<},\overline{N}_f^{<}\right]\cdot\overline{\overline{D}}{}^0\cdot\left[\overline{N}'^{<}_s,\overline{M}'^{<}_s\right]^T\Bigg\}\end{aligned} \tag{102}$$

For the canonical cases of planarly, cylindrically and spherically multilayered isotropic media, their C_s^0 coefficients in (99)–(102) are respectively

$$\text{Planar}: \quad C_s^0=-\frac{\omega\mu_{0s}}{8\pi^2k_{z0s}(k_x^2+k_y^2)},\quad k_{z0s}^2=k_{0s}^2-k_x^2-k_y^2 \tag{103}$$

$$\text{Cylindrical}: \quad C_s^0=-\frac{\omega\mu_{0s}}{8\pi\lambda_{0s}^2},\quad \lambda_{0s}^2=k_{0s}^2-h^2 \tag{104}$$

$$\text{Spherical}: \quad C_s^0=-\frac{\omega\mu_{0s}k_{0s}}{n(n+1)} \tag{105}$$

and the arguments for the unprimed ($\overline{M}^{\gtrless},\overline{N}^{\gtrless}$) and primed ($\overline{M}'^{\gtrless},\overline{N}'^{\gtrless}$) vector wave functions follow those of $\overline{V}^{\gtrless}$ and $\overline{W}^{\gtrless}$ but with k_0 as the isotropic wave number.

Using the $\bar{t}_1$ and $\bar{t}_2$ vectors of Section 4 in the above $\overline{\overline{E}}{}^{0\gtrless}$, the explicit expressions for the outgoing and incoming local reflection and transmission matrices at a planar isotropic-isotropic interface can be found easily as in Appendix B. Notice that these matrices are diagonal implying that the isotropic eigenmodes are decoupled at the planar interface. (Similar arguments apply to the spherical but not the cylindrical interface). Except for the exponential phase factors which account for the propagating directions, the diagonal elements coincide with the well-known TM and TE Fresnel reflection and transmission coefficients. However, it should be noted that all our reflection and transmission matrices are defined in terms of electric fields, hence some of these expressions may differ from those relating magnetic fields by certain impedance ratio. Using these local reflection and transmission matrices in the formulas of Section 3, all the global matrices for the scattering coefficients of (101) and (102) can be determined readily.

7. CONCLUSION

This paper has presented a systematic and unified formulation of the dyadic Green's functions for multilayered biisotropic media stratified in $\hat{p}$ direction. Based on the principle of scattering superposition, both electric and magnetic dyadic Green's functions are expressed in terms of unbounded and scattered parts. For the unbounded dyadic Green's functions, their complete eigenfunction expansions have been obtained in terms of commonly employed vector wave functions. It is seen that the magnetic dyadic Green's function may feature an extra source point dyadic delta function term in addition to its solenoidal eigenfunction expansion. For the scattered dyadic Green's functions, their scattering coefficients have been determined by applying the effective reflection and transmission concepts. This approach has avoided cumbersome operations and has also provided good physical insights to the scattering mechanism. Throughout the formulation, all the matrices are seen to be of size 2×2 only. The resulting general expressions have been written in very compact forms each containing 16 dyads only. As an application for these expressions, the canonical cases of planarly, cylindrically and spherically multilayered biisotropic media have been considered explicitly. It is noted that for each of these cases, one needs to determine the particular $\overline{\overline{E}}^{\gtrless}$ only and all other matrices can be calculated readily using the same formulas. The dyadic Green's functions have been derived based on Post-Jaggard constitutive relations while those for Drude-Born-Federov and Condon-Tellegen relations have also been discussed for comparisons. For the special case of multilayered isotropic media, the general expressions for their dyadic Green's functions have been given in the most common and compact forms. As a final note, the above formulation is seen to be very general and its applications to more complex media are currently under investigation.

APPENDIX

Appendix A

In this Appendix, we give the explicit expressions for the outgoing and incoming local reflection and transmission matrices at a planar biisotropic-biisotropic interface, Fig. 2a. Using the simplified notations of $a = q$, $b = q+1$ and $z = Z_q$, the elements of these matrices defined in the manner of (19) are
outgoing:

$$\begin{aligned} R_{ab}^{VV} &= [+\eta_1\eta_2(k_1-k_2)+\eta_3\eta_4(k_3-k_4) \\ &\quad -\eta_5\eta_6(k_5-k_6)]e^{2ik_{zva}z}/\Delta & (A1) \\ R_{ab}^{VW} &= -2\eta_2\eta_4(k_2+k_4)e^{i(k_{zva}+k_{zwa})z}/\Delta & (A2) \\ R_{ab}^{WV} &= -2\eta_1\eta_3(k_1+k_3)e^{i(k_{zva}+k_{zwa})z}/\Delta & (A3) \\ R_{ab}^{WW} &= [-\eta_1\eta_2(k_1-k_2)-\eta_3\eta_4(k_3-k_4) \\ &\quad -\eta_5\eta_6(k_5-k_6)]e^{2ik_{zwa}z}/\Delta & (A4) \\ T_{ab}^{VV} &= -2\eta_3\eta_5(k_3+k_5)e^{i(k_{zva}-k_{zvb})z}/\Delta & (A5) \\ T_{ab}^{VW} &= 2\eta_2\eta_5(k_2-k_5)e^{i(k_{zwa}-k_{zvb})z}/\Delta & (A6) \\ T_{ab}^{WV} &= -2\eta_1\eta_5(k_1-k_5)e^{i(k_{zva}-k_{zwb})z}/\Delta & (A7) \\ T_{ab}^{WW} &= 2\eta_4\eta_5(k_4+k_5)e^{i(k_{zwa}-k_{zwb})z}/\Delta & (A8) \end{aligned}$$

incoming:

$$\begin{aligned} R_{ba}^{VV} &= [+\eta_1\eta_2(k_1-k_2)-\eta_3\eta_4(k_3-k_4) \\ &\quad +\eta_5\eta_6(k_5-k_6)]e^{-2ik_{zvb}z}/\Delta & (A9) \\ R_{ba}^{VW} &= -2\eta_2\eta_3(k_2+k_3)e^{-i(k_{zvb}+k_{zwb})z}/\Delta & (A10) \\ R_{ba}^{WV} &= -2\eta_1\eta_4(k_1+k_4)e^{-i(k_{zvb}+k_{zwb})z}/\Delta & (A11) \\ R_{ba}^{WW} &= [-\eta_1\eta_2(k_1-k_2)+\eta_3\eta_4(k_3-k_4) \\ &\quad +\eta_5\eta_6(k_5-k_6)]e^{-2ik_{zwb}z}/\Delta & (A12) \\ T_{ba}^{VV} &= 2\eta_4\eta_6(k_4+k_6)e^{i(k_{zva}-k_{zvb})z}/\Delta & (A13) \\ T_{ba}^{VW} &= -2\eta_2\eta_6(k_2-k_6)e^{i(k_{zva}-k_{zwb})z}/\Delta & (A14) \\ T_{ba}^{WV} &= 2\eta_1\eta_6(k_1-k_6)e^{i(k_{zwa}-k_{zvb})z}/\Delta & (A15) \\ T_{ba}^{WW} &= -2\eta_3\eta_6(k_3+k_6)e^{i(k_{zwa}-k_{zwb})z}/\Delta & (A16) \end{aligned}$$

where the notations are introduced as

$$
\begin{aligned}
&k_1 = k_{zva}k_{zvb}k_{wa}k_{wb}, \qquad k_2 = k_{zwa}k_{zwb}k_{va}k_{vb}\\
&k_3 = k_{zva}k_{zwb}k_{wa}k_{vb}, \qquad k_4 = k_{zwa}k_{zvb}k_{va}k_{wb}\\
&k_5 = k_{zva}k_{zwa}k_{vb}k_{wb}, \qquad k_6 = k_{zvb}k_{zwb}k_{va}k_{wa}\\
&\eta_1 = \eta_a^V - \eta_b^V, \qquad \eta_2 = \eta_a^W - \eta_b^W\\
&\eta_3 = \eta_a^V - \eta_b^W, \qquad \eta_4 = \eta_a^W - \eta_b^V\\
&\eta_5 = \eta_a^V - \eta_a^W, \qquad \eta_6 = \eta_b^V - \eta_b^W\\
&\Delta = \eta_1\eta_2(k_1+k_2) + \eta_3\eta_4(k_3+k_4) - \eta_5\eta_6(k_5+k_6)
\end{aligned}
$$

Appendix B

For a planar isotropic-isotropic interface (refer to Fig. 2a with $a = q$, $b = q+1$, $z = Z_q$), the explicit expressions for the outgoing and incoming local reflection and transmission matrices are given by
outgoing:

$$\overline{\overline{R}}_{ab}^{0l} = \begin{bmatrix} R_{ab}^{NN} & 0 \\ 0 & R_{ab}^{MM} \end{bmatrix}, \qquad \overline{\overline{T}}_{ab}^{0l} = \begin{bmatrix} T_{ab}^{NN} & 0 \\ 0 & T_{ab}^{MM} \end{bmatrix} \tag{B1}$$

$$R_{ab}^{NN} = \frac{\mu_a k_{za} k_b^2 - \mu_b k_{zb} k_a^2}{\mu_a k_{za} k_b^2 + \mu_b k_{zb} k_a^2} e^{2ik_{za}z} \tag{B2}$$

$$R_{ab}^{MM} = \frac{\mu_b k_{za} - \mu_a k_{zb}}{\mu_b k_{za} + \mu_a k_{zb}} e^{2ik_{za}z} \tag{B3}$$

$$T_{ab}^{NN} = \frac{2\mu_b k_b k_a k_{za}}{\mu_a k_{za} k_b^2 + \mu_b k_{zb} k_a^2} e^{i(k_{za}-k_{zb})z} \tag{B4}$$

$$T_{ab}^{MM} = \frac{2\mu_b k_{za}}{\mu_b k_{za} + \mu_a k_{zb}} e^{i(k_{za}-k_{zb})z} \tag{B5}$$

incoming:

$$\overline{\overline{R}}_{ba}^{0l} = \begin{bmatrix} R_{ba}^{NN} & 0 \\ 0 & R_{ba}^{MM} \end{bmatrix}, \qquad \overline{\overline{T}}_{ba}^{0l} = \begin{bmatrix} T_{ba}^{NN} & 0 \\ 0 & T_{ba}^{MM} \end{bmatrix} \tag{B6}$$

$$R_{ba}^{NN} = \frac{\mu_b k_{zb} k_a^2 - \mu_a k_{za} k_b^2}{\mu_b k_{zb} k_a^2 + \mu_a k_{za} k_b^2} e^{-2ik_{zb}z} \tag{B7}$$

$$R_{ba}^{MM} = \frac{\mu_a k_{zb} - \mu_b k_{za}}{\mu_a k_{zb} + \mu_b k_{za}} e^{-2ik_{zb}z} \tag{B8}$$

$$T_{ba}^{NN} = \frac{2\mu_a k_a k_b k_{zb}}{\mu_b k_{zb} k_a^2 + \mu_a k_{za} k_b^2} e^{i(k_{za}-k_{zb})z} \tag{B9}$$

$$T_{ba}^{MM} = \frac{2\mu_a k_{zb}}{\mu_a k_{zb} + \mu_b k_{za}} e^{i(k_{za} - k_{zb})z} \qquad (B10)$$

REFERENCES

1. Tai, C. T., *Dyadic Green's Functions in Electromagnetic Theory.* New York: IEEE Press, 1993.
2. Kong, J. A., *Electromagnetic Wave Theory.* New York: Wiley, 1990.
3. Collin, R. E., *Field Theory of Guided Waves.* New York: IEEE Press, 1991.
4. Chew, W. C., *Waves and Fields in Inhomogeneous Media.* New York: Van Nostrand, 1990.
5. Barkeshli, S., and P. H. Pathak, "On the dyadic Green's function for a planar multilayered dielectric/magnetic media," *IEEE Trans. Microwave Theory Tech.*, Vol. 40, No. 1, 128–142, 1992.
6. Li, L. W., P. S. Kooi, M. S. Leong, and T. S. Yeo, "On the eigenfunction expansion of dyadic Green's function in planarly stratified media," *J. Electrom. Waves Applicat.*, Vol. 8, No. 6, 663–678, 1994.
7. Li, L. W., P. S. Kooi, M. S. Leong, and T. S. Yeo, "Electromagnetic dyadic Green's function in spherically multilayered media," *IEEE Trans. Microwave Theory Tech.*, Vol. 42, No. 12, 2302–2310, 1994.
8. Xiang, Z., and Y. Lu, "Electromagnetic dyadic Green's function in cylindrically multilayered media," *IEEE Trans. Microwave Theory Tech.*, Vol. 44, No. 4, 614–621, 1996.
9. Lee, J. K., and J. A. Kong, "Dyadic Green's functions for layered anisotropic medium," *Electromag.*, Vol. 3, 111–130, 1983.
10. Habashy, T. M., S. M. Ali, J. A. Kong and M. D. Grossi, "Dyadic Green's functions in a planar stratified arbitrary magnetized linear plasma," *Radio Sci.*, Vol. 26, 701–715, 1991.
11. Bassiri, S., N. Engheta, and C. H. Papas, "Dyadic Green's function and dipole radiation in chiral media," *Alta Freq.*, Vol. 55, 83-88, 1986.
12. Jaggard, D. L., X. Sun, and N. Engheta, "Canonical Sources and Duality in Chiral Media," *IEEE Trans. Antennas Propagat.*, Vol. 36, No. 7, 1007–1013, 1988.
13. Lindell, I. V., A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-Isotropic Media.* London: Artech House, 1994.
14. Lakhtakia, A., *Beltrami Fields in Chiral Media.* Singapore: World Scientific, 1994.

15. Ali, S. M., T. M. Habashy, and J. A. Kong, "Spectral-domain dyadic Green's function in layered chiral media," *J. Opt. Soc. Am. A*, Vol. 9, No. 3, 413–423, 1992.
16. Toscano, A., and L. Vegni, "Spectral dyadic Green's function formulation for planar integrated structures with a grounded chiral slab," *J. Electrom. Waves Applicat.*, Vol. 6, 751–769, 1992.
17 Yin, W., P. Li., and W. Wang, "The theory of dyadic Green's function and their radiation characteristics of sources in stratified bi-isotropic madia," *Progress in Electromagnetics Research*, Vol. 9, 117–136, 1994.
18. Yin, W., and W. Wang, "Dyadic Green's function of cylindrical multilayered chiral media and its applications," *J. Electrom. Waves Applicat.*, Vol. 7, No. 7, 1005–1027, 1993.
19. Li, L. W., P. S. Kooi, M. S. Leong, and T. S. Yeo, "Analytic representation of scattering dyadic Green's functions' coefficients for cylindrically multilayered chiral media," *J. Electrom. Waves Applicat.*, Vol. 9, No. 9, 1207–1221, 1993.
20. Engheta, N., and M. W. Kowarz, "Antenna radiation in the presence of a chiral sphere," *J. Appl. Phys.*, Vol. 67, No. 2, 639–647, 1990.
21. Yin, W., and P. Li, "Theory of multilayered spherical chirodome," *Microwave Opt. Technol. Lett.*, Vol. 6, No. 3, 179–182, 1993.
22. Yin, W., and P. Li,"The focal points of radially stratified spherical chirolenses," *Microwave Opt. Technol. Lett.*, Vol. 6, No. 9, 561–564, 1993.
23 Li, L. W., P. S. Kooi, M. S. Leong, and T. S. Yeo, "A general expression of dyadic Green's functions in radially multilayered chiral media," *IEEE Trans. Antennas Propagat.*, Vol. 43, No. 3, 232–238, 1995.
24. Sihvola, A. H., and I. V. Lindell, "Bi-Isotropic Constitutive Relations," *Microwave Opt. Technol. Lett.*, Vol. 4, No. 8, 295–297, 1991.
25. Pathak, P. H., "On the eigenfunction expansion of electromagnetic dyadic Green's functions," *IEEE Trans. Antennas Propagat.*, Vol. 31, No. 6, 837–846, 1983.
26. Tai, C. T., *Generalized Vector and Dyadic Analysis.* New York: IEEE Press, 1991.