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ON THE EIGENFUNCTION EXPANSIONS OF THE DYADIC GREEN'S FUNCTIONS FOR BIANISOTROPIC MEDIA

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1. INTRODUCTION

The dyadic Green's function is a powerful and important tool in the study of electromagnetic wave propagation and scattering problems [1–4]. Since the past few decades, considerable amount of research work have been devoted to the dyadic Green's functions in various media. In [5], the unbounded dyadic Green's function for the most general bianisotropic media has been obtained in terms of complete three-dimensional Fourier integrals. In many applications such as multilayered structures, the dyadic Green's functions are expressed as two-dimensional spectral domain integrals. In [6], the dyadic Green's function for a layered uniaxial anisotropic medium with optic axis perpendicular to the plane of stratification has been determined. In [7], the case corresponding to a uniaxial anisotropic medium with tilted optic axis is considered where both unbounded and layered dyadic Green's functions have been obtained. In [8], the spectral domain dyadic Green's functions for a stratified arbitrarily magnetized linear plasma have been formulated using the concept of upgoing and downgoing waves. In

[9], the complete plane wave spectral vector wave function expansion of the dyadic Green's functions for multilayered symmetric gyroelectric media is derived utilizing the orthogonality of transverse modes together with Lorentz reciprocity theorem and multiple scattering method. In [10], the dyadic Green's functions pertaining to the surface electric currents in complex anisotropic layered media is obtained using the spectral domain 4×4 matrix approach. In [11], a systematic approach based on the equivalent boundary method is described to obtain the bidimensional spectral Green's dyad in multilayered complex bianisotropic media. In [12], a matrix exponential function approach based on Cayley-Hamilton theorem is applied to study the electromagnetic fields of elementary dipole antennas embedded in stratified general gyrotropic media. Recently, the complex spectral Green's dyadics for inhomogeneous anisotropic and bianisotropic media have been determined using the integral equation method in [13, 14].

In this paper, we present a novel approach for constructing the complete eigenfunction expansions of electric and magnetic dyadic Green's functions for general linear bianisotropic media. The eigenfunctions are expressed in terms of linear combinations of commonly employed vector wave functions $\overline{M}$, $\overline{N}$ and $\overline{L}$ [4]. There are certain advantages in writing the electromagnetic fields and dyadic Green's functions in terms of these wave functions. In particular, the $\overline{M}$ and $\overline{N}$ functions correspond to the TE and TM waves respectively, hence the representations attempt to provide some insight to the eigenwaves in bianisotropic media, although the interpretation may often be obscured by the complicated nature of the media. As demonstrated in [4, 7–9] and recently in [15], the dyadic Green's functions for multilayered media can be formulated in a fairly simple manner once the unbounded dyadic Green's functions in vector wave function representations have been determined. Therefore, the emphasis of this paper will be on the unbounded case. Moreover, by applying the appropriate plane wave expansion formulas [16], the vector wave functions (and hence the dyadic Green's functions) expressed in Cartesian coordinate system can be transformed readily into those in other coordinate systems, e.g., cylindrical and spherical. In addition to the eigenfunction representations, both electric and magnetic dyadic Green's functions require explicit dyadic delta function terms for complete expansions at the source point [17–20]. Following the approach described in [21], these source point singularities are derived based on the theory of distributions [22] directly from Maxwell equations cast in dyadic forms. This approach avoids any cumbersome extraction from three-dimensional spectral domain integrals [4, 23]. Moreover, no integration is required as compared to the principal volume method [19, 20]. Apart from the singularities, the discontinuities associated with the eigenfunction expansions across the source point [24, 25] are also obtained directly from Maxwell dyadic equations as by-products. These discontinuity relations constitute the fundamental equations from which the eigenfunction expansions outside the source point can be constructed. This application of discontinuity relations is parallel to the utilization of Lorentz reciprocity theorem [3, 25] for relating the source-free eigenwaves with sources. However, the theorem cannot be applied directly to nonreciprocal media without a complementary medium [26] whereas our approach deals with both reciprocal and nonreciprocal cases in the same manner. Furthermore, the discontinuity relations also yield directly the jump conditions for electric and magnetic fields across a current sheet. For generality, this current sheet is assumed to consist of both tangentially and normally directed components. To demonstrate the application of our approach, we will consider in detail the dyadic Green's functions for (nonreciprocal) biisotropic media in which their expansions are readily available for validation. To show the feasibility of the method for more complex media, we also present the expressions for general uniaxial bianisotropic media in Appendix A. Throughout the following analysis, $e^{-i\omega t}$ time dependence is assumed and suppressed.

2. EIGENFUNCTION EXPANSIONS OF ELECTRIC AND MAGNETIC FIELDS

A homogeneous linear bianisotropic medium can be characterized by the constitutive relations of the form [2]

$$\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} + \overline{\overline{\xi}} \cdot \overline{H} \tag{1}$$

$$\overline{B} = \overline{\overline{\zeta}} \cdot \overline{E} + \overline{\overline{\mu}} \cdot \overline{H} \tag{2}$$

where $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ are respectively the permittivity and permeability dyadics, while $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$ are the magneto-electric pseudodyadics. Applying the constitutive equations (1)–(2) into the source-free Maxwell equations, we have

$$\nabla \times \overline{E} = i\omega(\overline{\overline{\zeta}} \cdot \overline{E} + \overline{\overline{\mu}} \cdot \overline{H}) \tag{3}$$

$$\nabla \times \overline{H} = - i\omega(\overline{\overline{\epsilon}} \cdot \overline{E} + \overline{\overline{\xi}} \cdot \overline{H}). \tag{4}$$

Eliminating $\overline{H}$ in the above equations, we obtain the vector wave equation for the electric field as

$$\nabla \times \overline{\overline{\mu}}^{-1} \cdot \nabla \times \overline{E} - i\omega(\nabla \times \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}} \cdot \overline{E} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \nabla \times \overline{E}) - \omega^2(\overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}}) \cdot \overline{E} = \overline{0}. \tag{5}$$

Substituting the Fourier transformation

$$\overline{E}(\overline{r}) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \overline{E}(\overline{k}) e^{i\overline{k} \cdot \overline{r}}, \qquad \overline{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \tag{6}$$

into (5), the condition for nontrivial solutions of $\overline{E}$ reads

$$\det\left[\left(\frac{\overline{k}}{\omega}\times\overline{\overline{I}}+\overline{\overline{\xi}}\right)\cdot\overline{\overline{\mu}}^{-1}\cdot\left(\frac{\overline{k}}{\omega}\times\overline{\overline{I}}-\overline{\overline{\zeta}}\right)+\overline{\overline{\epsilon}}\right]=0 \tag{7}$$

where $\overline{\overline{I}}$ is the idemfactor. This is the dispersion relation relating the wave vector $\overline{k}$ and the angular frequency ω in compact form. In a Cartesian coordinate system described by unit vectors $(\hat{t}_1,\hat{t}_2,\hat{p})$, the wave vector can be decomposed into its components as $\overline{k}=k_{t1}\hat{t}_1+k_{t2}\hat{t}_2+k_p\hat{p}$. Assuming k_p is to be determined as a function of frequency, constitutive parameters and the transverse components (k_{t1} and k_{t2}), equation (7) then yields four roots which may be real, complex and/or multiples of each other. These roots may be solved analytically or numerically [27, 28].

Corresponding to each of the roots, there exists one or more (for repeated k_p) eigenvectors representing the nontrivial solutions of (5). Depending on the root multiplicity which renders the dyadic in (7) planar or linear, these eigenvectors can be constructed using standard dyadic methods [29]. Alternatively, the eigenvectors can be expressed directly in terms of commonly employed vector wave functions, namely the $\overline{M}$, $\overline{N}$ and $\overline{L}$ functions (of $\overline{r}$ and $\overline{k}$) defined in [4]. Since these vector functions form a complete set [16], for each root designated as $k_{pj}\ (j=1,2,3,4)$, we let

$$\overline{E}_j(\overline{r})=\int_{k_t}\overline{E}_j(\overline{r};\overline{k})=\int_{k_t}a_{ej}\overline{M}_j+b_{ej}\overline{N}_j+c_{ej}\overline{L}_j \tag{8}$$

where $\int_{k_t}$ implies $\int_{-\infty}^{\infty}dk_{t1}\int_{-\infty}^{\infty}dk_{t2}$ and a_{ej}, b_{ej} and c_{ej} are the coefficients of expansion to be determined. Substituting (8) into (5) and using the relations

$$\nabla\times\overline{M}_j=k_j\overline{N}_j,\qquad \nabla\times\overline{N}_j=k_j\overline{M}_j \tag{9}$$

$$k_j^2=k_t^2+k_{pj}^2,\qquad k_t^2=k_{t1}^2+k_{t2}^2, \tag{10}$$

we obtain

$$\int_{k_t}\int_S\begin{bmatrix}\overline{v}_1\\\overline{v}_2\\\overline{v}_3\end{bmatrix}\cdot[\,\overline{V}_{aj}\quad\overline{V}_{bj}\quad\overline{V}_{cj}\,]\cdot\begin{bmatrix}a_{ej}\\b_{ej}\\c_{ej}\end{bmatrix}=\overline{0} \tag{11}$$

where

$$\begin{aligned}\overline{V}_{aj}=&ik_j\overline{k}_j\times\overline{\overline{I}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{N}_j+\omega\overline{k}_j\times\overline{\overline{I}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}}\cdot\overline{M}_j\\&+i\omega k_j\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{N}_j-\omega^2(\overline{\overline{\epsilon}}-\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}})\cdot\overline{M}_j\end{aligned} \tag{12}$$

$$\begin{aligned}\overline{V}_{bj} =& ik_j\overline{k}_j \times \overline{\overline{I}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{M}_j + \omega\overline{k}_j \times \overline{\overline{I}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}} \cdot \overline{N}_j \\ &+ i\omega k_j\overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{M}_j - \omega^2(\overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}}) \cdot \overline{N}_j \end{aligned} \tag{13}$$

$$\overline{V}_{cj} = \omega\overline{k}_j \times \overline{\overline{I}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}} \cdot \overline{L}_j - \omega^2(\overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}}) \cdot \overline{L}_j. \tag{14}$$

Here, $\overline{v}_1$, $\overline{v}_2$ and $\overline{v}_3$ are three linearly independent vectors which when dot-integrated with $\overline{V}_{aj}$, $\overline{V}_{bj}$ and $\overline{V}_{cj}$ over surface $\int_S \equiv \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2$ lead to extraction of a particular mode corresponding to a particular index in $\int_{k_t}$, e.g., k'_{t1} and k'_{t2}. Thus, (11) represents the homogeneous equation of the form $\overline{\overline{\mathrm{A}}} \cdot \overline{\mathrm{x}} = \overline{0}$ whose solutions express the bianisotropic eigenwaves directly as linear combinations of $\overline{M}$, $\overline{N}$ and $\overline{L}$. Since the homogeneous solutions can be scaled by any scalar factor, at least one of a_{ej}, b_{ej} or c_{ej} is left arbitrary. In practice, this arbitrary factor is to be determined from the impressed source or the external excitation. As an example, consider the isotropic medium where $\overline{\overline{\epsilon}} = \epsilon\overline{\overline{I}}$, $\overline{\overline{\mu}} = \mu\overline{\overline{I}}$ and $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = \overline{\overline{0}}$. The wavenumber in $\hat{p} = \hat{z}$ direction can be obtained easily as $k_z = \pm\sqrt{\omega^2\mu\epsilon - k_t^2}$ (each repeated once). From (11), we find that both a_{ej} and b_{ej} can be arbitrary, i.e. they are independent of each other. This deduction coincides with the fact that $\overline{M}$ and $\overline{N}$ are actually the electric eigenfunctions in isotropic media. Furthermore, we find $c_{ej} = 0$ indicating that only the solenoidal type vector wave functions are required for electric field expansion in source-free isotropic regions. For more complex media, the irrotational vector wave function would be present as well in the expansion.

Having determined the eigenfunction expansions for the electric field, we proceed to find those for the magnetic field. Corresponding to a particular set of a_{ej}, b_{ej} and c_{ej}, the magnetic field can be obtained readily from (3) by letting

$$\overline{H}_j(\overline{r}) = \int_{k_t} \overline{H}_j(\overline{r};\overline{k}) = \int_{k_t} a_{hj}\overline{M}_j + b_{hj}\overline{N}_j + c_{hj}\overline{L}_j. \tag{15}$$

The coefficients of expansion a_{hj}, b_{hj} and c_{hj} can then be determined from

$$\int_{k_t}\int_S \begin{bmatrix} \overline{v}_1 \\ \overline{v}_2 \\ \overline{v}_3 \end{bmatrix} \cdot [\,\overline{M}_j \quad \overline{N}_j \quad \overline{L}_j\,] \cdot \begin{bmatrix} a_{hj} \\ b_{hj} \\ c_{hj} \end{bmatrix} = \int_{k_t}\int_S \begin{bmatrix} \overline{v}_1 \\ \overline{v}_2 \\ \overline{v}_3 \end{bmatrix} \cdot [\,\overline{V}_{hj}\,] \tag{16}$$

where we have chosen the same set of $\overline{v}$'s as in (11) and

$$\overline{V}_{hj} = a_{ej}(\frac{k_j}{i\omega}\overline{\overline{\mu}}^{-1}\cdot\overline{N}_j - \overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}}\cdot\overline{M}_j) + b_{ej}(\frac{k_j}{i\omega}\overline{\overline{\mu}}^{-1}\cdot\overline{M}_j - \overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}}\cdot\overline{N}_j) - c_{ej}\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}}\cdot\overline{L}_j. \tag{17}$$

The solutions of (15)–(17) express the magnetic eigenwaves directly as linear combinations of $\overline{M}$, $\overline{N}$ and $\overline{L}$. Together with (8) and (11)–(14), they constitute the eigenfunction expansions of the electromagnetic fields in a source-free bianisotropic medium. These eigenfunctions will be used to construct the expansions for both electric and magnetic dyadic Green's functions. In addition to the source-free eigenfunction expansions, each of these dyadics requires an extra dyadic delta function term at the source point. This is discussed in the next section.

3. EIGENFUNCTION EXPANSIONS OF DYADIC GREEN'S FUNCTION

Assuming an electric current source $\overline{J}$ is impressed in a bianisotropic medium. (The case for magnetic sources can also be considered via duality principle.) Due to linearity of Maxwell equations, the electric and magnetic fields can be related directly to the current source as

$$\overline{E}(\overline{r}) = \iiint\limits_{V'} dv' \,\overline{\overline{G}}_e(\overline{r},\overline{r}') \cdot \overline{J}(\overline{r}') \tag{18}$$

$$\overline{H}(\overline{r}) = \iiint\limits_{V'} dv' \,\overline{\overline{G}}_m(\overline{r},\overline{r}') \cdot \overline{J}(\overline{r}') \tag{19}$$

where $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_m$ are respectively the electric and magnetic dyadic Green's functions. Substituting (18)–(19) into the Maxwell equations, we obtain the dyadic equations

$$\nabla \times \overline{\overline{G}}_e = i\omega(\overline{\overline{\zeta}} \cdot \overline{\overline{G}}_e + \overline{\overline{\mu}} \cdot \overline{\overline{G}}_m) \tag{20}$$

$$\nabla \times \overline{\overline{G}}_m = - i\omega(\overline{\overline{\epsilon}} \cdot \overline{\overline{G}}_e + \overline{\overline{\xi}} \cdot \overline{\overline{G}}_m) + \overline{\overline{I}}\delta(\overline{r}' - \overline{r}) \tag{21}$$

where $\delta(\overline{r}' - \overline{r})$ is the three-dimensional Dirac delta function defined as

$$\overline{J}(\overline{r}) = \iiint\limits_{V'} dv' \,\overline{\overline{I}}\delta(\overline{r}' - \overline{r}) \cdot \overline{J}(\overline{r}'). \tag{22}$$

Since the dyadic Green's functions represent the field responses attributed to a point source, they can be written in general forms as

$$\overline{\overline{G}}_e = \overline{\overline{G}}{}_e^0 \delta(p' - p) + \overline{\overline{G}}{}_e^> U(p - p') + \overline{\overline{G}}{}_e^< U(p' - p) \tag{23}$$

$$\overline{\overline{G}}_m = \overline{\overline{G}}{}_m^0 \delta(p' - p) + \overline{\overline{G}}{}_m^> U(p - p') + \overline{\overline{G}}{}_m^< U(p' - p). \tag{24}$$

Here, we have expanded each dyadic Green's function into three parts weighted by different distributions. $\delta(p'-p)$ is the one-dimensional Dirac delta function and $U(\pm p \mp p')$ are the Heaviside unit step functions. The $\overline{\overline{G}}^0$ part together with $\delta(p'-p)$ gives the singular dyadic term required for complete expansion in the source region. The $\overline{\overline{G}}^{\gtrless}$ parts together with $U(\pm p \mp p')$ correspond to eigenfunction expansions for $p > p'$ and $p < p'$ respectively [25]. Substituting (23)–(24) into the Maxwell dyadic equations (20)–(21) and their divergences, we carry out the derivative operations in the sense of distributions using [30]

$$\nabla \times \left[\overline{\overline{G}}^0 \delta(p'-p)\right] = (\nabla \times \overline{\overline{G}}^0)\delta(p'-p) + \nabla\delta(p'-p) \times \overline{\overline{G}}^0 \tag{25}$$

$$\nabla \times \left[\overline{\overline{G}}^{\gtrless} U(\pm p \mp p')\right] = (\nabla \times \overline{\overline{G}}^{\gtrless})U(\pm p \mp p') \pm (\hat{p} \times \overline{\overline{G}}^{\gtrless})\delta(p-p') \tag{26}$$

$$\nabla \cdot \left[\overline{\overline{G}}^0 \delta(p'-p)\right] = (\nabla \cdot \overline{\overline{G}}^0)\delta(p'-p) + \nabla\delta(p'-p) \cdot \overline{\overline{G}}^0 \tag{27}$$

$$\nabla \cdot \left[\overline{\overline{G}}^{\gtrless} U(\pm p \mp p')\right] = (\nabla \cdot \overline{\overline{G}}^{\gtrless})U(\pm p \mp p') \pm (\hat{p} \cdot \overline{\overline{G}}^{\gtrless})\delta(p-p'). \tag{28}$$

Then, the singularities and discontinuities associated with the eigenfunction expansions of $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_m$ can be deduced as follows.

Corresponding to $\frac{\partial}{\partial p'}\delta(p'-p)$, we obtain the following equations for $p = p'$:

$$\hat{p} \times \overline{\overline{G}}_e^0 = \overline{\overline{0}} \tag{29}$$

$$\hat{p} \times \overline{\overline{G}}_m^0 = \overline{\overline{0}} \tag{30}$$

$$\hat{p} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{G}}_e^0 + \hat{p} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{G}}_m^0 = \frac{1}{i\omega}\delta_t \hat{p}' \tag{31}$$

$$\hat{p} \cdot \overline{\overline{\zeta}} \cdot \overline{\overline{G}}_e^0 + \hat{p} \cdot \overline{\overline{\mu}} \cdot \overline{\overline{G}}_m^0 = \overline{0} \tag{32}$$

where δ_t is the transverse delta function which together with $\delta(p'-p)$ forms the three-dimensional Dirac delta function, i.e. $\delta_t\delta(p'-p) = \delta(\overline{r}'-\overline{r})$. From these equations, it follows that

$$\begin{aligned}\overline{\overline{G}}_e^0 \delta(p'-p) &= \frac{1}{i\omega} g_e^0 \delta(\overline{r}'-\overline{r})\hat{p}\hat{p}', \\ g_e^0 &= \left.\frac{\overline{\overline{\mu}} : \hat{p}\hat{p}}{(\overline{\overline{\epsilon}} : \hat{p}\hat{p})(\overline{\overline{\mu}} : \hat{p}\hat{p}) - (\overline{\overline{\xi}} : \hat{p}\hat{p})(\overline{\overline{\zeta}} : \hat{p}\hat{p})}\right|_{\overline{r}=\overline{r}'}\end{aligned} \tag{33}$$

$$\overline{\overline{G}}{}^0_m \delta(p'-p) = \frac{1}{i\omega} g^0_m \delta(\overline{r}'-\overline{r})\hat{p}\hat{p}',$$
$$g^0_m = -\left.\frac{\overline{\overline{\zeta}} : \hat{p}\hat{p}}{(\overline{\overline{\epsilon}} : \hat{p}\hat{p})(\overline{\overline{\mu}} : \hat{p}\hat{p}) - (\overline{\overline{\xi}} : \hat{p}\hat{p})(\overline{\overline{\zeta}} : \hat{p}\hat{p})}\right|_{\overline{r}=\overline{r}'}. \quad (34)$$

Here, $\Big|_{\overline{r}=\overline{r}'}$ indicates that g^0_e and g^0_m are to be evaluated at the source point and $:$ is the double-dot product operator defined in [29]. Hence, we have obtained the explicit expressions of the source point dyadic delta function terms for $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_m$ as given by (33) and (34) respectively. These results are seen to be in accordance with those given in [14] for $\hat{p} = \hat{z}$. Note that the singular terms are obtained in a simple manner directly from Maxwell dyadic equations treated in the sense of distributions. This approach has avoided the somewhat cumbersome task of singularity extraction from three-dimensional spectral domain integrals [4, 23]. Moreover, there is no apparent need to specify the shape of an exclusion volume [31] and no integration is required as compared to the principal volume method [19, 20]. From (33)–(34), we see that for general bianisotropic media, both electric and magnetic dyadic Green's functions feature explicit dyadic delta function terms which depend on the constitutive parameters as well as on the preferred $\hat{p}$ direction of eigenfunction expansion.

Corresponding to $U(\pm p \mp p')$ in the Maxwell dyadic equations, we have

$$\nabla \times \overline{\overline{G}}{}^{\gtrless}_e = i\omega(\overline{\overline{\zeta}} \cdot \overline{\overline{G}}{}^{\gtrless}_e + \overline{\overline{\mu}} \cdot \overline{\overline{G}}{}^{\gtrless}_m) \quad (35)$$
$$\nabla \times \overline{\overline{G}}{}^{\gtrless}_m = -i\omega(\overline{\overline{\epsilon}} \cdot \overline{\overline{G}}{}^{\gtrless}_e + \overline{\overline{\xi}} \cdot \overline{\overline{G}}{}^{\gtrless}_m). \quad (36)$$

These results state that $\overline{\overline{G}}{}^{\gtrless}_e$ and $\overline{\overline{G}}{}^{\gtrless}_m$ actually correspond to the field responses in the source-free regions of $p > p'$ and $p < p'$. In view of these homogeneous equations, we anticipate that the dyadics can be expanded as

$$\overline{\overline{G}}{}^{\gtrless}_e = \int_{k_t} \overline{E}_{\substack{1\\3}} \overline{S}'_{\substack{1\\3}} + \overline{E}_{\substack{2\\4}} \overline{S}'_{\substack{2\\4}} \quad (37)$$
$$\overline{\overline{G}}{}^{\gtrless}_m = \int_{k_t} \overline{H}_{\substack{1\\3}} \overline{S}'_{\substack{1\\3}} + \overline{H}_{\substack{2\\4}} \overline{S}'_{\substack{2\\4}}. \quad (38)$$

Here, $\overline{S}'_j$ are the source vectors to be determined as functions of source (primed) coordinates. $\overline{E}_j$ and $\overline{H}_j$ are given by (8) and (15) satisfying (3)–(4) and hence (35)–(36). Furthermore, k_{p1} and k_{p2} are assumed to correspond to $p > p'$ while k_{p3} and k_{p4} to $p < p'$. In reality, the association of each k_{pj} with the respective region is rather intricate [32]. As k_{t1} and k_{t2} vary from $-\infty$ to ∞, each of these roots traces a certain path in the complex k_p plane. Based on the Jordan's lemma requirement, one must select for $p > p'$ those roots with $\mathrm{Im}\, k_{pj} > 0$ and for $p < p'$ those with $\mathrm{Im}\, k_{pj} < 0$. For lossless medium, all roots may be located right on the real k_p axis, i.e. $\mathrm{Im}\, k_{pj} = 0$. In this case, one can introduce a small loss and examine the root behavior in the complex plane as the loss goes to zero. Considering the real part of k_{pj} which indicates the propagating direction, we may expect that incoming waves existing at large distance from the source [32]. This conclusion is seen to contradict the well-known Sommerfeld radiation condition for isotropic media which requires waves to propagate outward at infinity. Indeed, for general bianisotropic media, the radiation condition should be based on the concept of bounded solutions and both incoming as well as (decaying) evanescent waves may be present in addition to the outgoing waves.

Corresponding to $\delta(p' - p)$ in the Maxwell dyadic equations, we obtain the following by-products for $p = p'$:

$$\hat{p} \times (\overline{\overline{G}}_e^{>} - \overline{\overline{G}}_e^{<}) = -\frac{1}{i\omega} g_e^0 \nabla_t \delta_t \times \hat{p}\hat{p}' + \overline{\overline{I}}_t \cdot (g_e^0 \overline{\overline{\zeta}} + g_m^0 \overline{\overline{\mu}}) \cdot \hat{p}\hat{p}' \delta_t \tag{39}$$

$$\hat{p} \times (\overline{\overline{G}}_m^{>} - \overline{\overline{G}}_m^{<}) = \overline{\overline{I}}_t \delta_t - \frac{1}{i\omega} g_m^0 \nabla_t \delta_t \times \hat{p}\hat{p}' - \overline{\overline{I}}_t \cdot (g_e^0 \overline{\overline{\epsilon}} + g_m^0 \overline{\overline{\xi}}) \cdot \hat{p}\hat{p}' \delta_t \tag{40}$$

where $\overline{\overline{I}}_t$ is the transverse (to $\hat{p}$) part of idemfactor and ∇_t is the gradient operator taken with respect to transverse coordinates. Equations (39)–(40) describe the discontinuities present in the eigenfunction expansions of the dyadic Green's functions for general linear bianisotropic media. These discontinuities denote the changes undergone in the tangential components ($\hat{p}\times$) of the dyadics across the source point. They have emerged naturally due to the representations of $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_m$ in the forms of (23)–(24) where a point singularity at $\overline{r} = \overline{r}'$ has been modeled by an equivalent layer of surface singularity at $p = p'$. For isotropic media, the discontinuity relations (39)–(40) can be verified readily as demonstrated in [24, 25]. In actuality, these relations constitute the fundamental equations from which the dyadics $\overline{\overline{G}}_e^{\gtrless}$ and $\overline{\overline{G}}_m^{\gtrless}$ can be determined explicitly. Specifically, substituting (37)–(38) into (39)–(40) and applying some orthogonal relationships via two linearly independent transverse vector functions ($\overline{v}_{t1}$, $\overline{v}_{t2}$), we obtain

$$\int_{k_t}\int_S \begin{bmatrix} \overline{v}_{t1}\cdot\hat{p}\times\overline{E}_1 & \overline{v}_{t1}\cdot\hat{p}\times\overline{E}_2 & -\overline{v}_{t1}\cdot\hat{p}\times\overline{E}_3 & -\overline{v}_{t1}\cdot\hat{p}\times\overline{E}_4 \\ \overline{v}_{t2}\cdot\hat{p}\times\overline{E}_1 & \overline{v}_{t2}\cdot\hat{p}\times\overline{E}_2 & -\overline{v}_{t2}\cdot\hat{p}\times\overline{E}_3 & -\overline{v}_{t2}\cdot\hat{p}\times\overline{E}_4 \\ \overline{v}_{t1}\cdot\hat{p}\times\overline{H}_1 & \overline{v}_{t1}\cdot\hat{p}\times\overline{H}_2 & -\overline{v}_{t1}\cdot\hat{p}\times\overline{H}_3 & -\overline{v}_{t1}\cdot\hat{p}\times\overline{H}_4 \\ \overline{v}_{t2}\cdot\hat{p}\times\overline{H}_1 & \overline{v}_{t2}\cdot\hat{p}\times\overline{H}_2 & -\overline{v}_{t2}\cdot\hat{p}\times\overline{H}_3 & -\overline{v}_{t2}\cdot\hat{p}\times\overline{H}_4 \end{bmatrix}_{p=p'}$$

$$\cdot\begin{bmatrix}\overline{S}'_1\\ \overline{S}'_2\\ \overline{S}'_3\\ \overline{S}'_4\end{bmatrix} = \int_S \begin{bmatrix} \left[\frac{1}{i\omega}g_e^0\nabla_t\cdot(\hat{p}\times\overline{v}_{t1}) + \overline{v}_{t1}\cdot(g_e^0\overline{\overline{\zeta}} + g_m^0\overline{\overline{\mu}})\cdot\hat{p}\right]\hat{p}'\delta_t \\ \left[\frac{1}{i\omega}g_e^0\nabla_t\cdot(\hat{p}\times\overline{v}_{t2}) + \overline{v}_{t2}\cdot(g_e^0\overline{\overline{\zeta}} + g_m^0\overline{\overline{\mu}})\cdot\hat{p}\right]\hat{p}'\delta_t \\ \overline{v}_{t1}\delta_t + \left[\frac{1}{i\omega}g_m^0\nabla_t\cdot(\hat{p}\times\overline{v}_{t1}) - \overline{v}_{t1}\cdot(g_e^0\overline{\overline{\epsilon}} + g_m^0\overline{\overline{\xi}})\cdot\hat{p}\right]\hat{p}'\delta_t \\ \overline{v}_{t2}\delta_t + \left[\frac{1}{i\omega}g_m^0\nabla_t\cdot(\hat{p}\times\overline{v}_{t2}) - \overline{v}_{t2}\cdot(g_e^0\overline{\overline{\epsilon}} + g_m^0\overline{\overline{\xi}})\cdot\hat{p}\right]\hat{p}'\delta_t \end{bmatrix}_{p=p'} \tag{41}$$

Solving the above 4×4 matrix, we can then cast the resultant $\overline{S}'_j$'s into linear combinations of $\overline{M}$, $\overline{N}$ and $\overline{L}$ following the similar procedure as described in the previous section. Having determined these source functions, we have all the dyadics in the right sides of (23)–(24) known and hence the complete expansions of the dyadic Green's functions have been obtained.

At this point, it is clear from above the role played by each distribution in the Maxwell dyadic equations. In particular, the derivative of delta function $\frac{\partial}{\partial p'}\delta(p'-p)$ leads to straightforward derivation of the source point dyadic delta function terms $\overline{\overline{G}}_e^0$ and $\overline{\overline{G}}_m^0$ in (23) and (24). The Heaviside unit step functions $U(\pm p \mp p')$ correspond to the (source-free) eigenfunction expansions of field antecedents in $\overline{\overline{G}}_e^{\gtrless}$ and $\overline{\overline{G}}_m^{\gtrless}$ of (35) and (36). Finally, the correspondence of delta function $\delta(p'-p)$ provides a direct mean to determine the expansion coefficients and source consequents in $\overline{\overline{G}}_e^{\gtrless}$ and $\overline{\overline{G}}_m^{\gtrless}$, thus asserting the importance of (39)–(40). In fact, the significance of these discontinuities has been emphasized in [1] when (40) is utilized in the method of $\overline{\overline{G}}_m$ to derive the singular term associated with the electric dyadic Green's function for isotropic media. Moreover, from the discontinuity relations, we obtain the jump conditions for electric and magnetic fields across a current sheet $\overline{J}_s$ as

$$\hat{p}\times(\overline{E}^> - \overline{E}^<) = -\frac{1}{i\omega}g_e^0\nabla\times\overline{J}_{ps} + \overline{\overline{I}}_t\cdot(g_e^0\overline{\overline{\zeta}} + g_m^0\overline{\overline{\mu}})\cdot\overline{J}_{ps} \tag{42}$$

$$\hat{p}\times(\overline{H}^> - \overline{H}^<) = \overline{J}_{ts} - \frac{1}{i\omega}g_m^0\nabla\times\overline{J}_{ps} - \overline{\overline{I}}_t\cdot(g_e^0\overline{\overline{\epsilon}} + g_m^0\overline{\overline{\xi}})\cdot\overline{J}_{ps}. \tag{43}$$

Here, it is assumed that the current sheet may consist of both tangentially and normally directed components denoted as $\overline{J}_{ts} = \overline{\overline{I}}_t \cdot \overline{J}_s$ and $\overline{J}_{ps} = \hat{p}\hat{p} \cdot \overline{J}_s$ respectively. From (42)–(43), it is interesting to observe the manner $\overline{J}_{ps}$, via certain components of the constitutive dyadics, gives rise to the discontinuities in the tangential components of electromagnetic fields. Note that these jump conditions have resulted as direct consequences of interpreting the Maxwell equations in the distribution sense.

4. APPLICATION EXAMPLE

In this section, the procedures developed above is applied to obtain the explicit eigenfunction expansion of the dyadic Green's functions for unbounded bianisotropic media. These eigenfunctions are assumed to 'propagate' along $\hat{p} = \hat{z}$ transverse to $\hat{t}_1 = \hat{x}$ and $\hat{t}_2 = \hat{y}$. For simplicity, we will consider in detail the dyadic Green's functions for biisotropic media in which their expansions are readily available for validation. To show the feasibility of the method for more complex media, we also present the expressions for general uniaxial bianisotropic in Appendix A. These materials have received much attention recently due to their potential applications and they can be fabricated easily by inserting metal helices in an isotropic host medium [33, 34].

For biisotropic media, the constitutive dyadics are multiples of idemfactor and hence they can be characterized by four scalar coefficients as

$$\overline{\overline{\epsilon}} = \epsilon \overline{\overline{I}}, \qquad \overline{\overline{\mu}} = \mu \overline{\overline{I}}, \qquad \overline{\overline{\xi}} = \xi \overline{\overline{I}}, \qquad \overline{\overline{\zeta}} = \zeta \overline{\overline{I}}. \tag{44}$$

Using (44), we obtain the dispersion equation in terms of k_z as

$$\begin{aligned} k_z^4 &+ \left[\omega^2(\xi^2 + \zeta^2 - 2\epsilon\mu) + 2k_t^2\right] k_z^2 \\ &+ \left[\omega^4(\epsilon\mu - \xi\zeta)^2 + \omega^2(\xi^2 + \zeta^2 - 2\epsilon\mu)k_t^2 + k_t^4\right] = 0 \end{aligned} \tag{45}$$

where $k_t^2 = k_x^2 + k_y^2$. This biquadratic equation yields four roots designated as k_{z1}, k_{z2}, $k_{z3} = -k_{z1}$ and $k_{z4} = -k_{z2}$ ($\mathrm{Im}\, k_{z1}, k_{z2} > 0$). Corresponding to each root, we can determine from (11) the electric eigenfunctions in terms of [4]

$$\overline{M}(\overline{r}; k_x, k_y, k_z) = [\hat{x}ik_y - \hat{y}ik_x]\, e^{ik_x x + ik_y y + ik_z z} \tag{46}$$

$$\overline{N}(\overline{r}; k_x, k_y, k_z) = \frac{1}{k}\left[-\hat{x}k_z k_x - \hat{y}k_z k_y + \hat{z}(k_x^2 + k_y^2)\right] e^{ik_x x + ik_y y + ik_z z}, \quad k^2 = k_x^2 + k_y^2 + k_z^2 \tag{47}$$

$$\overline{L}(\overline{r}; k_x, k_y, k_z) = [\hat{x}ik_x + \hat{y}ik_y + \hat{z}ik_z]\, e^{ik_x x + ik_y y + ik_z z}. \tag{48}$$

By choosing some convenient $\overline{v}$'s in the surface integral, for instance,

$$\overline{v}_1 = \frac{1}{4\pi^2 k_t'^2}\left[\hat{x}k_x' + \hat{y}k_y'\right] e^{-ik_x'x - ik_y'y} \tag{49}$$

$$\overline{v}_2 = \frac{1}{4\pi^2 k_t'^2}\left[-\hat{x}k_y' + \hat{y}k_x'\right] e^{-ik_x'x - ik_y'y}, \tag{50}$$

we find (dropping the primes associated with k_x' and k_y')

$$b_{ej} = \frac{i\left[k_j^2 - \omega^2(\epsilon\mu - \xi\zeta)\right]}{\omega k_j(\xi - \zeta)} a_{ej} \tag{51}$$

$$c_{ej} = -\frac{k_{zj}\left[k_j^4 + \omega^2(\xi^2 + \zeta^2 - 2\epsilon\mu)k_j^2 + \omega^4(\epsilon\mu - \xi\zeta)^2\right]}{\omega^3 k_j^2(\xi - \zeta)(\epsilon\mu - \xi\zeta)} a_{ej}. \tag{52}$$

Inserting (51)–(52) into the third equation of (11) and choosing say, $\overline{v}_3 = \hat{z}e^{-ik_x'x - ik_y'y}$, we obtain the dispersion equation in the form of (45) in an alternative approach. In other words, (11) can be treated as a nonhomogeneous equation with a_{ej} assumed known while b_{ej}, c_{ej} and k_{zj} stay as unknowns. Furthermore, substituting each k_{zj} explicitly into (51)–(52), we find $b_{ej} = \pm a_{ej}$ for

$$k_{\frac{1}{2}}^2 = \pm\frac{\omega(\xi - \zeta)}{2}\sqrt{(\xi + \zeta)^2 - 4\epsilon\mu} + \omega^2\left(\epsilon\mu - \frac{\xi^2 + \zeta^2}{2}\right) \tag{53}$$

respectively. These eigenwaves are precisely the well-known left- and right-circularly polarized modes which have been obtained in the literature mostly by employing the Bohren transformation [35]. For both of these modes, we have $c_{ej} = 0$ reasserting the solenoidal property of the electric field in source-free regions. Corresponding to each a_{ej} and b_{ej}, the magnetic eigenfunctions can be determined from (16) as

$$a_{hj} = \frac{1}{i\omega\mu}(k_j b_{ej} - i\omega\zeta a_{ej}) \tag{54}$$

$$b_{hj} = \frac{1}{i\omega\mu}(k_j a_{ej} - i\omega\zeta b_{ej}) \tag{55}$$

with the biisotropic admittances given by

$$\eta_{\frac{1}{2}} = \frac{1}{i\omega\mu}(\pm k_{\frac{1}{2}} - i\omega\zeta). \tag{56}$$

Again, $c_{hj} = 0$ implying that the magnetic field is also divergenceless in source-free regions. Normalizing a_{ej} as unity, we can write the electric and magnetic eigenfunctions as

$$\overline{E}_{\frac{1}{2}} = \overline{M}(k_x, k_y, k_{z\frac{1}{2}}) \pm \overline{N}(k_x, k_y, k_{z\frac{1}{2}}), \qquad \overline{H}_{\frac{1}{2}} = \eta_{\frac{1}{2}} \overline{E}_{\frac{1}{2}} \tag{57}$$

$$\overline{E}_{\frac{3}{4}} = \overline{M}(k_x, k_y, -k_{z\frac{1}{2}}) \pm \overline{N}(k_x, k_y, -k_{z\frac{1}{2}}), \qquad \overline{H}_{\frac{3}{4}} = \eta_{\frac{1}{2}} \overline{E}_{\frac{3}{4}}. \tag{58}$$

Next, we proceed to derive the dyadic Green's functions in terms of these eigenfunctions.

Noting that $\overline{\overline{I}} : \hat{z}\hat{z} = 1$, we can determine the singularities required in the source region directly from (33)–(34) as

$$g_e^0 = \frac{\mu}{\epsilon\mu - \xi\zeta} \tag{59}$$

$$g_m^0 = -\frac{\zeta}{\epsilon\mu - \xi\zeta}. \tag{60}$$

The eigenfunction expansions of the dyadic Green's functions for $z > z'$ and $z < z'$ will be now determined. In conventional methods, one usually employs the Lorentz reciprocity theorem [3, 25] to relate the source-free eigenwaves with sources in a reciprocal medium. For nonreciprocal media, the theorem cannot be applied directly although a complementary (adjoint) medium may be introduced in the modified reciprocity theorem [26]. Here, we provide an alternative approach which is applicable to both reciprocal and nonreciprocal media to derive $\overline{\overline{G}}_e^{\gtrless}$ and $\overline{\overline{G}}_m^{\gtrless}$ directly based on equations (39)–(40). Using the explicit expressions of (57)–(60), along with $\overline{v}_{t1} = \overline{v}_1$ and $\overline{v}_{t2} = \overline{v}_2$, we can write (41) as

$$\begin{bmatrix} ie^{ik_{z1}z'} & ie^{ik_{z2}z'} & -ie^{-ik_{z1}z'} & -ie^{-ik_{z2}z'} \\ -\frac{k_{z1}}{k_1} e^{ik_{z1}z'} & \frac{k_{z2}}{k_2} e^{ik_{z2}z'} & -\frac{k_{z1}}{k_1} e^{-ik_{z1}z'} & \frac{k_{z2}}{k_2} e^{-ik_{z2}z'} \\ ie^{ik_{z1}z'}\eta_1 & ie^{ik_{z2}z'}\eta_2 & -ie^{-ik_{z1}z'}\eta_1 & -ie^{-ik_{z2}z'}\eta_2 \\ -\frac{k_{z1}}{k_1} e^{ik_{z1}z'}\eta_1 & \frac{k_{z2}}{k_2} e^{ik_{z2}z'}\eta_2 & -\frac{k_{z1}}{k_1} e^{-ik_{z1}z'}\eta_1 & \frac{k_{z2}}{k_2} e^{-ik_{z2}z'}\eta_2 \end{bmatrix} \cdot \begin{bmatrix} \overline{S}_1' \\ \overline{S}_2' \\ \overline{S}_3' \\ \overline{S}_4' \end{bmatrix}$$
$$= \frac{e^{-ik_x x' - ik_y y'}}{4\pi^2 k_t^2} \begin{bmatrix} \overline{0} \\ \hat{z}\frac{\mu k_t^2}{\omega(\epsilon\mu - \xi\zeta)} \\ \hat{x}k_x + \hat{y}k_y \\ -\hat{x}k_y + \hat{y}k_x - \hat{z}\frac{\zeta k_t^2}{\omega(\epsilon\mu - \xi\zeta)} \end{bmatrix}. \tag{61}$$

Solving (61), we obtain

$$\overline{S}'_{\frac{1}{2}} = \frac{ik_{\frac{1}{2}}}{8\pi^2(\eta_1-\eta_2)k_t^2 k_{z\frac{1}{2}}}\left[\overline{M}'(-k_x,-k_y,-k_{z\frac{1}{2}}) \pm \overline{N}'(-k_x,-k_y,-k_{z\frac{1}{2}})\right] \quad (62)$$

$$\overline{S}'_{\frac{3}{4}} = \frac{ik_{\frac{1}{2}}}{8\pi^2(\eta_1-\eta_2)k_t^2 k_{z\frac{1}{2}}}\left[\overline{M}'(-k_x,-k_y,k_{z\frac{1}{2}}) \pm \overline{N}'(-k_x,-k_y,k_{z\frac{1}{2}})\right]. \quad (63)$$

Hence, the complete eigenfunction expansions of the dyadic Green's functions take the forms

$$\begin{aligned}\overline{\overline{G}}_e =& \frac{\mu}{i\omega(\epsilon\mu-\xi\zeta)}\delta(\overline{r}'-\overline{r})\hat{z}\hat{z} + \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \frac{i}{8\pi^2(\eta_1-\eta_2)k_t^2} \\ &\cdot\left\{\left[\frac{k_1}{k_{z1}}\overline{V}(k_x,k_y,k_{z1})\overline{V}'(-k_x,-k_y,-k_{z1})\right.\right. \\ &\left. + \frac{k_2}{k_{z2}}\overline{W}(k_x,k_y,k_{z2})\overline{W}'(-k_x,-k_y,-k_{z2})\right]U(z-z') \\ &+\left[\frac{k_1}{k_{z1}}\overline{V}(k_x,k_y,-k_{z1})\overline{V}'(-k_x,-k_y,k_{z1})\right. \\ &\left.\left. + \frac{k_2}{k_{z2}}\overline{W}(k_x,k_y,-k_{z2})\overline{W}'(-k_x,-k_y,k_{z2})\right]U(z'-z)\right\} \quad (64)\end{aligned}$$

$$\begin{aligned}\overline{\overline{G}}_m =& -\frac{\zeta}{i\omega(\epsilon\mu-\xi\zeta)}\delta(\overline{r}'-\overline{r})\hat{z}\hat{z} + \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \frac{i}{8\pi^2(\eta_1-\eta_2)k_t^2} \\ &\cdot\left\{\left[\frac{\eta_1 k_1}{k_{z1}}\overline{V}(k_x,k_y,k_{z1})\overline{V}'(-k_x,-k_y,-k_{z1})\right.\right. \\ &\left. + \frac{\eta_2 k_2}{k_{z2}}\overline{W}(k_x,k_y,k_{z2})\overline{W}'(-k_x,-k_y,-k_{z2})\right]U(z-z') \\ &+\left[\frac{\eta_1 k_1}{k_{z1}}\overline{V}(k_x,k_y,-k_{z1})\overline{V}'(-k_x,-k_y,k_{z1})\right. \\ &\left.\left. + \frac{\eta_2 k_2}{k_{z2}}\overline{W}(k_x,k_y,-k_{z2})\overline{W}'(-k_x,-k_y,k_{z2})\right]U(z'-z)\right\} \quad (65)\end{aligned}$$

where we have employed the notations of $\overline{V} = \overline{M} + \overline{N}$ and $\overline{W} = \overline{M} - \overline{N}$. These results obviously agree with those found in the literature, where it is known that the dyadic Green's function for an unbounded biisotropic medium can be expressed as combinations of two equivalent isotropic ones [36].

5. OTHER COORDINATE SYSTEMS

So far, the dyadic Green's functions have been expanded in Cartesian coordinate system in terms of the Cartesian vector wave functions. Occasionally, one may wish to express the expansions in other coordinate systems, e.g., cylindrical and spherical. By using the plane wave expansion in cylindrical and spherical wave functions, the Cartesian vector wave functions can be transformed readily to those of cylindrical and spherical types. For instance, the commonly employed cylindrical vector wave functions are defined as [4]

$$\overline{M}_m(\overline{r}; k_\rho, k_z) = \left[\hat{\rho}\frac{im}{\rho}J_m(k_\rho\rho) - \hat{\phi}\frac{\partial}{\partial\rho}J_m(k_\rho\rho)\right] e^{im\phi+ik_z z} \tag{66}$$

$$\begin{aligned}\overline{N}_m(\overline{r}; k_\rho, k_z) =& \frac{1}{k}\left[\hat{\rho}ik_z\frac{\partial}{\partial\rho}J_m(k_\rho\rho) - \hat{\phi}\frac{mk_z}{\rho}J_m(k_\rho\rho) + \hat{z}k_\rho^2 J_m(k_\rho\rho)\right] \\ &\cdot e^{im\phi+ik_z z}, \qquad\qquad k^2 = k_\rho^2 + k_z^2\end{aligned} \tag{67}$$

$$\overline{L}_m(\overline{r}; k_\rho, k_z) = \left[\hat{\rho}\frac{\partial}{\partial\rho}J_m(k_\rho\rho) + \hat{\phi}\frac{im}{\rho}J_m(k_\rho\rho) + \hat{z}ik_z J_m(k_\rho\rho)\right] e^{im\phi+ik_z z} \tag{68}$$

Using the rectangular-cylindrical coordinate transformation of the wave numbers together with the plane wave identity [16]

$$k_x = k_\rho\cos\phi_k, \qquad k_y = k_\rho\sin\phi_k \tag{69}$$

$$e^{ik_\rho\rho\cos(\phi-\phi_k)} = \sum_{m=-\infty}^{\infty} i^m J_m(k_\rho\rho)e^{im(\phi-\phi_k)}, \tag{70}$$

the Cartesian vector wave functions (46)–(48) can be related to the cylindrical vector wave functions (66)–(68) as

$$\overline{M}(k_x, k_y, k_z) = \sum_{m=-\infty}^{\infty} i^m e^{-im\phi_k}\overline{M}_m(k_\rho, k_z) \tag{71}$$

$$\overline{N}(k_x, k_y, k_z) = \sum_{m=-\infty}^{\infty} i^m e^{-im\phi_k}\overline{N}_m(k_\rho, k_z) \tag{72}$$

$$\overline{L}(k_x, k_y, k_z) = \sum_{m=-\infty}^{\infty} i^m e^{-im\phi_k}\overline{L}_m(k_\rho, k_z). \tag{73}$$

Returning to the dyadic Green's functions for biisotropic media (64)–(65), their integrals are then transformed as

$$\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y = \int_0^{\infty} k_\rho dk_\rho \int_0^{2\pi} d\phi_k \tag{74}$$

$$\overline{V}(k_x,k_y,\pm k_{z1})\overline{V}'(-k_x,-k_y,\mp k_{z1}) = \sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} i^{m-m'}e^{-i(m-m')\phi_k}\overline{V}_m(k_\rho,\pm k_{z1})\overline{V}'_{-m'}(-k_\rho,\mp k_{z1}) \quad (75)$$

$$\overline{W}(k_x,k_y,\pm k_{z2})\overline{W}'(-k_x,-k_y,\mp k_{z2}) = \sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} i^{m-m'}e^{-i(m-m')\phi_k}\overline{W}_m(k_\rho,\pm k_{z2})\overline{W}'_{-m'}(-k_\rho,\mp k_{z2}). \quad (76)$$

Since the integrand does not depend on ϕ_k except in the exponential terms, the integral $d\phi_k$ can be evaluated easily by noting

$$\int_0^{2\pi} d\phi_k e^{-i(m-m')\phi_k} = 2\pi\delta_{mm'} \quad (77)$$

where $\delta_{mm'}$ is the Kronecker delta symbol. Thus, the resultant dyadic Green's functions become

$$\begin{aligned}\overline{\overline{G}}_e =& \frac{\mu}{i\omega(\epsilon\mu-\xi\zeta)}\delta(\overline{r}'-\overline{r})\hat{z}\hat{z} + \int_0^\infty k_\rho dk_\rho \sum_{m=-\infty}^{\infty}\frac{i}{4\pi(\eta_1-\eta_2)k_\rho^2}\\ &\cdot\Bigg\{\bigg[\frac{k_1}{k_{z1}}\overline{V}_m(k_\rho,k_{z1})\overline{V}'_{-m}(-k_\rho,-k_{z1})\\ &\qquad + \frac{k_2}{k_{z2}}\overline{W}_m(k_\rho,k_{z2})\overline{W}'_{-m}(-k_\rho,-k_{z2})\bigg]U(z-z')\\ &\quad + \bigg[\frac{k_1}{k_{z1}}\overline{V}_m(k_\rho,-k_{z1})\overline{V}'_{-m}(-k_\rho,k_{z1})\\ &\qquad + \frac{k_2}{k_{z2}}\overline{W}_m(k_\rho,-k_{z2})\overline{W}'_{-m}(-k_\rho,k_{z2})\bigg]U(z'-z)\Bigg\} \end{aligned} \quad (78)$$

$$\begin{aligned}\overline{\overline{G}}_m =& -\frac{\zeta}{i\omega(\epsilon\mu-\xi\zeta)}\delta(\overline{r}'-\overline{r})\hat{z}\hat{z} + \int_0^\infty k_\rho dk_\rho \sum_{m=-\infty}^{\infty}\frac{i}{4\pi(\eta_1-\eta_2)k_\rho^2}\\ &\cdot\Bigg\{\bigg[\frac{\eta_1 k_1}{k_{z1}}\overline{V}_m(k_\rho,k_{z1})\overline{V}'_{-m}(-k_\rho,-k_{z1})\\ &\qquad + \frac{\eta_2 k_2}{k_{z2}}\overline{W}_m(k_\rho,k_{z2})\overline{W}'_{-m}(-k_\rho,-k_{z2})\bigg]U(z-z')\\ &\quad + \bigg[\frac{\eta_1 k_1}{k_{z1}}\overline{V}_m(k_\rho,-k_{z1})\overline{V}'_{-m}(-k_\rho,k_{z1})\\ &\qquad + \frac{\eta_2 k_2}{k_{z2}}\overline{W}_m(k_\rho,-k_{z2})\overline{W}'_{-m}(-k_\rho,k_{z2})\bigg]U(z'-z)\Bigg\}. \end{aligned} \quad (79)$$

By the same token, the dyadic Green's functions can also be cast into spherical wave representation using the rectangular-spherical or cylindrical-spherical coordinate transformation together with the appropriate identity [16, 37].

6. CONCLUSIONS

This paper has presented a novel approach for obtaining the complete eigenfunction expansions of electric and magnetic dyadic Green's functions for general linear bianisotropic media. The eigenfunctions have been expressed in terms of linear combinations of commonly employed solenoidal and irrotational vector wave functions. Based on the theory of distributions, the singularities and discontinuities associated with the eigenfunction expansions of the dyadic Green's functions have been derived directly from Maxwell equations cast in dyadic forms. It is seen that both electric and magnetic dyadic Green's functions feature explicit source point dyadic delta function terms which depend on the constitutive parameters as well as on the preferred direction of eigenfunction expansion. The discontinuity relations describe the changes undergone in the tangential components of the dyadics across the source point. These relations constitute the fundamental equations from which the eigenfunction expansions outside the source point can be constructed. This approach is parallel to the utilization of Lorentz reciprocity theorem for relating eigenwaves with sources and it is applicable directly to both nonreciprocal as well as reciprocal media. Furthermore, the discontinuity relations have been used to obtain the jump conditions for electric and magnetic fields across a current sheet. As an illustration of our approach, the dyadic Green's functions for biisotropic media have been considered in detail. The expressions for general uniaxial bianisotropic media have also been presented in Appendix A showing the feasibility of the method for more complex media. Although the dyadic Green's functions have been expanded mainly in Cartesian coordinate system, they can be transformed readily to other coordinate systems, e.g., cylindrical and spherical, by employing the corresponding vector wave functions in these systems. Furthermore, the approach described above may be extended to bounded medium provided the corresponding eigenfunctions can be determined readily (via some orthogonality relationships) and the infinite integrals are replaced with finite and/or discrete ones. With the availability of vector wave function representations of eigenfunctions, the dyadic Green's functions for multilayered bianisotropic media can be solved in a manner resembling those of isotropic media, e.g., by making use of the well-developed recursive algorithm [4, 15]. This work will be reported in the near future.

APPENDIX A

In this Appendix, we give the explicit expressions for the eigenfunction expansions of the dyadic Green's functions for a general uniaxial bianisotropic medium, which can be characterized by the following constitutive relations:

$$\begin{aligned}\overline{\overline{\epsilon}} &= \epsilon_t \overline{\overline{I}}_t + \epsilon_z \hat{z}\hat{z}, \qquad \overline{\overline{\mu}} = \mu_t \overline{\overline{I}}_t + \mu_z \hat{z}\hat{z}\\ \overline{\overline{\xi}} &= \xi_t \overline{\overline{I}}_t + \xi_z \hat{z}\hat{z}, \qquad \overline{\overline{\zeta}} = \zeta_t \overline{\overline{I}}_t + \zeta_z \hat{z}\hat{z}.\end{aligned} \tag{A1}$$

Choosing $\hat{p} = \hat{z}$, we find that the dispersion equation is biquadratic in k_z, i.e. $k_{z3} = -k_{z1}$ and $k_{z4} = -k_{z2}$ ($\operatorname{Im} k_{z1}, k_{z2} > 0$). The electric ($\overline{E}_j$) and magnetic ($\overline{H}_j$) eigenfunctions associated with each root can be determined as $(j = 1, 2, 3, 4)$

$$\begin{aligned} b_{ej} =& i\omega a_{ej}\Big\{ [\mu_t(\epsilon_t\mu_z - \xi_t\zeta_z) + \zeta_t(\mu_t\zeta_z - \mu_z\zeta_t)]\, k_{zj}^2 \\ &+ \mu_t(\epsilon_t\mu_t - \xi_t\zeta_t)k_t^2 - \omega^2\mu_z(\epsilon_t\mu_t - \xi_t\zeta_t)^2\Big\}/(k_j D) \end{aligned} \tag{A2}$$

$$\begin{aligned} c_{ej} =& -k_{zj}a_{ej}\Big\{k_j^2(\mu_t k_t^2 + \mu_z k_{zj}^2) + \omega^4\mu_z(\epsilon_t\mu_t - \xi_t\zeta_t)^2 \\ &+ \omega^2[\mu_z(\xi_t^2 - \epsilon_t\mu_t)k_j^2 + \zeta_t(\mu_t\zeta_z k_t^2 + \mu_z\zeta_t k_{zj}^2) \\ &- \epsilon_t\mu_t(\mu_t k_t^2 + \mu_z k_{zj}^2) - \mu_t\xi_t(\zeta_z - \zeta_t)k_t^2]\Big\}/(\omega k_j^2 D) \end{aligned} \tag{A3}$$

$$D = \omega^2(\epsilon_t\mu_t - \xi_t\zeta_t)(\mu_z\xi_t - \mu_t\zeta_z) + (\mu_t\zeta_z - \mu_z\zeta_t)k_{zj}^2 \tag{A4}$$

$$a_{hj} = \frac{1}{i\omega\mu_t}\left[k_j b_{ej} - i\omega\zeta_t a_{ej}\right] \tag{A5}$$

$$\begin{aligned} b_{hj} =& \frac{1}{i\omega\mu_t\mu_z k_j^2}[(\mu_t k_t^2 + \mu_z k_{zj}^2)k_j a_{ej} - i\omega(\mu_t\zeta_z k_t^2 + \mu_z\zeta_t k_{zj}^2)b_{ej} \\ &- \omega(\mu_z\zeta_t - \mu_t\zeta_z)k_j k_{zj} c_{ej}] \end{aligned} \tag{A6}$$

$$\begin{aligned} c_{hj} =& \frac{1}{\omega\mu_t\mu_z k_j^3}[(\mu_z - \mu_t)k_j k_t^2 k_{zj} a_{ej} + i\omega(\mu_t\zeta_z - \mu_z\zeta_t)k_t^2 k_{zj} b_{ej} \\ &- \omega(\mu_z\zeta_t k_t^2 + \mu_t\zeta_z k_{zj}^2)k_j c_{ej}]. \end{aligned} \tag{A7}$$

Notice that $b_{e_{\substack{3\\4}}} = b_{e_{\substack{1\\2}}}$, $c_{e_{\substack{3\\4}}} = -c_{e_{\substack{1\\2}}}$, $a_{h_{\substack{3\\4}}} = a_{h_{\substack{1\\2}}}$, $b_{h_{\substack{3\\4}}} = b_{h_{\substack{1\\2}}}$ and $c_{h_{\substack{3\\4}}} = -c_{h_{\substack{1\\2}}}$. The source functions $\overline{S}'_j$ associated with each eigenfunction can also be determined as

$$\begin{aligned} \overline{S}'_{\substack{1\\2}} = -\frac{1}{8\pi^2(a_{h1} - a_{h2})\Delta k_t^2}\Big[&\mp ik_{\substack{1\\2}}\gamma_{e_{\substack{2\\1}}}(a_{h1} - a_{h2})\overline{M}'(-k_x, -k_y, -k_{z_{\substack{1\\2}}}) \\ &+ \frac{\alpha_{\substack{1\\2}} \mp ik_{z_{\substack{1\\2}}}\Delta}{k_{\substack{1\\2}}}\overline{N}'(-k_x, -k_y, -k_{z_{\substack{1\\2}}}) \end{aligned}$$

$$+\frac{ik_{z_{\frac{1}{2}}}\alpha_{\frac{1}{2}} \mp k_t^2\Delta}{k_{\frac{1}{2}}^2}\overline{L}'(-k_x,-k_y,-k_{z_{\frac{1}{2}}})\bigg] \quad \text{(A8)}$$

$$\overline{S}'_{\frac{3}{4}} = -\frac{1}{8\pi^2(a_{h1}-a_{h2})\Delta k_t^2}\bigg[\mp ik_{\frac{1}{2}}\gamma_{e_{\frac{2}{1}}}(a_{h1}-a_{h2})\overline{M}'(-k_x,-k_y,k_{z_{\frac{1}{2}}})$$
$$+\frac{\alpha_{\frac{1}{2}} \mp ik_{z_{\frac{1}{2}}}\Delta}{k_{\frac{1}{2}}}\overline{N}'(-k_x,-k_y,k_{z_{\frac{1}{2}}})$$
$$-\frac{ik_{z_{\frac{1}{2}}}\alpha_{\frac{1}{2}} \mp k_t^2\Delta}{k_{\frac{1}{2}}^2}\overline{L}'(-k_x,-k_y,k_{z_{\frac{1}{2}}})\bigg] \quad \text{(A9)}$$

where

$$\gamma_{ej} = -b_{ej}k_{zj} + ic_{ej}k_j, \qquad \gamma_{hj} = -b_{hj}k_{zj} + ic_{hj}k_j \quad \text{(A10)}$$
$$\Delta = \gamma_{e1}\gamma_{h2} - \gamma_{h1}\gamma_{e2} \quad \text{(A11)}$$
$$\alpha_{\frac{1}{2}} = \mp\frac{(\mu_z\gamma_{h_{\frac{2}{1}}} + \zeta_z\gamma_{e_{\frac{2}{1}}})(a_{h1}-a_{h2})k_{\frac{1}{2}}k_t^2}{\omega(\epsilon_z\mu_z - \xi_z\zeta_z)}. \quad \text{(A12)}$$

With $\overline{E}_j$, $\overline{H}_j$ and $\overline{S}'_j$ solved explicitly, we thus obtain the complete eigenfunction expansions of the dyadic Green's functions taking into account the corresponding source point dyadic delta function terms.

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