Progress In Electromagnetics Research, PIER 20, 45–61, 1998

STABLE SOLUTION OF THE GMT-MoM METHOD BY TIKHONOV REGULARIZATION

L. Landesa, F. Obelleiro, J. L. Rodríguez, and M. R. Pino

Dpto. Tecnoloxías das Comunicacións
Universidade de Vigo
E-36200 Vigo, Spain

1. INTRODUCTION

The problem of electromagnetic scattering by obstacles has been subject to extensive research for many years due to its great importance in different application areas. These efforts have lead to a great variety of methods that are chosen according to the size and the properties of the object. It must be pointed, among others, the *Method of Moments* (MoM) [1] which entails the formulation of an integral equation for the induced surface current in the case of a perfectly conducting scatter, or the induced polarization current for a dielectric scatter. These unknowns are expanded using basis functions with unknown coefficients. The coefficients are then determined using suitable testing functions and finally, the scattered field is calculated from the knowledge of the induced surface or polarization currents.

On the other hand, the equivalent source methods, such as the *Generalized Multipole Technique* (GMT) [2–4] and the current filament method [5–8], have been found to be a very efficient alternative to the standard MoM formulation. The scattered field is represented as a summation of fields due to fictitious sources adequately placed. If the locations, number and values of the fictitious sources are appropriately chosen, the boundary conditions are satisfied on the scatter surface and the scattered field can be straightforward obtained without doing any integration over the surface currents.

In recent years other significant contributions were developed in order to reduce the matrix condition number and the amount of unknowns of such methods [9–13]. In these approaches, directive sources are employed in order to obtain generalized impedance matrices with banded structure. In [9–11], the directivity is achieved by positioning point sources into the complex space; while in [12, 13] the sources are arrays of isotropic elements.

When non-smooth bodies are involved, the previous methods may provide poor results, specially when near field quantities are of main interest. In order to overcome this problem, different solutions have been proposed. Some hybrid approaches combining fictitious sources with MoM are presented in [14–17]. Closely related is the use of specialized expansions [18–20]. Nevertheless, these hybrid methods inherently generate ill-conditioning and hence provide potentially unstable solutions.

Otherwise, another difficulty of the above mentioned methods, is the adequate selection of the number of sources and their locations, which highly influences the accuracy of the method. Some guidelines concerning these topics, may be found in [4, 5, 14, 21], while in [22, 23], the location of the MoM basis in the hybrid GMT-MoM method is addressed.

The impedance matrix ill-conditioning was addressed in [24] for the current filament method, using the *Singular Value Decomposition* (SVD). In other works, the SVD [26, 27] and a convergent minimum-norm solution [28] has been used to avoid matrix ill -conditioning produced by resonant frequencies in the EFIE integral equation.

In this work, the hybrid GMT-MoM method is investigated from the perspective of reducing its ill-posedness and consequently its high dependence on the choice of the number and location of the sources. The ill-conditioning problem is overcome by using a *Tikhonov regularization* [29] over the full GMT-MoM impedance matrix, by imposing a quadratic constraint over GMT and MoM coefficients. The proposed

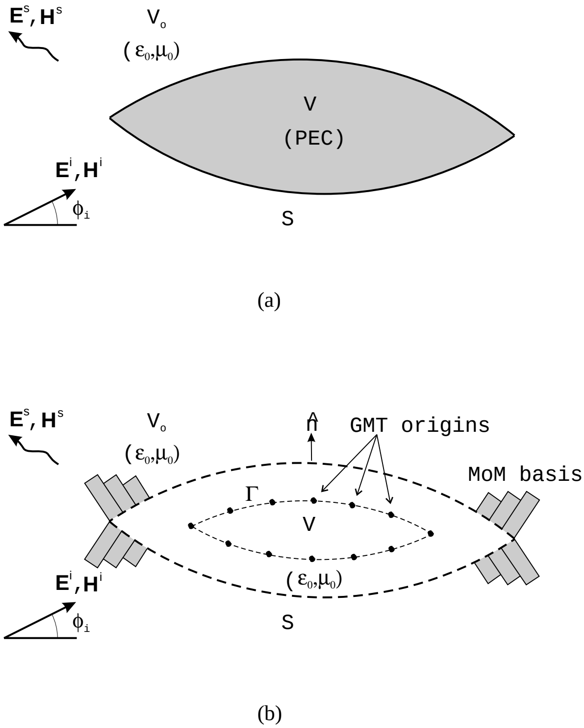

Figure 1. Scattering from a PEC. (a) General problem; (b) Equivalent problem.

technique has proved to provide significant improvements with respect to the original GMT-MoM solution. However, it should be mentioned that the results are highly dependent on the regularization parameters; a new technique, called *L-curve criterion*, is used in the present work in order to obtain a stable parameter for the *Tikhonov regularization*. This method provides a very good choice of the parameter, nevertheless there may be other efficient methods for selecting the regularization parameter, which is nowadays a subject under research [29].

This paper is organized as follows. In section 2, the hybrid GMT-MoM method is briefly outlined and the notation is established. Section 3 is devoted to describe the algorithm used to determine the location of the MoM basis. In section 4, the regularization process is presented. Section 5 contains some results illustrating the capabilities of the method, and finally, a summary and a few conclusions are presented in section 6.

2. HYBRID GMT-MoM FORMULATION

Consider an infinite perfectly electric conducting (PEC) cylinder of arbitrary cross section, S, depicted in Fig. 1.a. The region surrounding the cylinder, V_o, is free space and let us denote V as the region interior to S. The excitation is due to an external source with harmonic time dependence $e^{j\omega t}$. Our goal is to determine the current distribution over the surface S and the scattered field in V_0. Thus, the equivalent problem shown in Fig. 1.b is considered, in which the scattered field by the conducting body is simulated by a set of sources radiating in free space: multipolar expansions placed over the contour Γ within V and subdomain MoM basis functions placed near the geometrical discontinuities on S, in order to account for the singularities of the field in those regions [17]. An approximated solution for the field $\mathbf{E}^s$ scattered by the cylinder is setup by superposition of the fields due to the GMT and MoM sources.

The coefficients of the GMT and MoM sources are calculated by imposing the E–field boundary condition over S:

$$\hat{n} \times \mathbf{E}^s(\mathbf{r}) = -\hat{n} \times \mathbf{E}^{inc}(\mathbf{r}) \quad \text{at} \quad \mathbf{r} \in S \tag{1}$$

where $\hat{n}$ is a unit vector outward normal to S, $\mathbf{E}^{inc}$ is the incident field and $\mathbf{E}^s$ is the scattered field, which can be obtained as the sum of the fields due to GMT sources and MoM basis:

$$\mathbf{E}^s = \sum_{n=1}^{N} \mathbf{E}_n^{s,GMT} c_n^{GMT} + \sum_{m=1}^{M} \mathbf{E}_m^{s,MoM} c_m^{MoM}, \tag{2}$$

where $\mathbf{E}_n^{s,GMT}$ and $\mathbf{E}_m^{s,MoM}$ are the scattered fields due to GMT and MoM sources respectively, while c_n^{GMT} and c_m^{MoM} are the corresponding unknown coefficients, which are obtained by imposing (1) at a set of N_s uniformly distributed matching points on S. This can be expressed in a matrix form as:

$$\cdot = \begin{bmatrix} ^{GMT} & ^{MoM} \end{bmatrix} \cdot \begin{bmatrix} ^{GMT} \\ ^{MoM} \end{bmatrix} = \tag{3}$$

where is the full impedance matrix, is the hybrid coefficients vector, and $\mathbf{V}$ denotes the excitation vector due to the incident plane wave. In the same way as in the conventional GMT, the number of matching points must be greater than the number of unknowns ($N_s \geq N+M$) in order to obtain accurate results [4]. Thus, (3) is solved in a least-square sense:

$$\min_{c} \{\| \cdot\ - \mathbf{V}\|_2\} \tag{4}$$

Due to numerical considerations, the QR factorization [25] is preferably to the normal equations method [25] to obtain the solutions of the above matrix equation (see [4]). Once has been found, it is straightforward to calculate the scattered field and other related quantities of interest.

3. AUTOMATIC LOCATION ALGORITHM

In order to find both the location and the extension of the regions covered with MoM basis functions, an automatic algorithm has been developed [22, 23]. For that purpose, an initial GMT solution is calculated, which leads to a residual of the boundary condition at the set of matching points. Instead of using this quantity for the following steps, is better to normalize it respect to the magnitude of the incident field, i.e.:

$$\Delta E^{bc} = \frac{\left|\mathbf{E}^{inc} + \mathbf{E}^{s,GMT}\right|}{\left|\mathbf{E}^{inc}\right|_{\max}} \tag{5}$$

where $\mathbf{E}^{s,GMT}$ is the scattered field due to the multipolar expansions. At a first glance, one can suppose that the local maxima of the residual could serve as an indication of the regions that are worst modelled by GMT sources. But, due to its rapidly varying behavior, it is convenient to define a related magnitude, which stands for the mean error along a small fraction of the surface:

$$\widehat{\Delta E}_q^{bc}(\mathbf{r}_m) = \frac{1}{2q+1} \sum_{j=m-q}^{m+q} \Delta E^{bc}(\mathbf{r}_j) \tag{6}$$

where $2q+1$ matching points are considered to find the *integrated error* at each point. In practice, it has been found that $(2q+1)$ must be chosen to cover an extension close to a wavelength to be a good trade-off for our purposes.

Those points whose error $\widehat{\Delta E}_q^{bc}$ exceeds a fixed threshold, are selected as the location of the MoM basis for the hybrid GMT-MoM approach. The threshold level can be chosen according to the maximum number of allowed unknowns. Once the MoM basis have been selected, the problem is solved as explained in section 2.

It is important to notice that the QR factorization can be updated [25], taking advantage of the initial GMT solution [23]; namely, GMT factorization does not need to be repeated. Nevertheless, this technique requires the Q matrix of the initial solution to be stored, which in some cases may be an important drawback. Otherwise, this automatic algorithm may lead to source distributions whose matrix in (3) has a high condition number, which inherently generates potential unstable results. In order to overcome this problem, a regularization algorithm, called *Tikhonov regularization*, has been applied as described in the following section.

4. REGULARIZATION OF ILL-POSED PROBLEMS

In this section we will present two methods for solving linear least squares systems of the form:

$$Ax = b \quad (A \in \mathcal{C}^{m\times n},\, x \in \mathcal{C}^{n\times 1},\, b \in \mathcal{C}^{m\times 1}) \tag{7}$$

when these present problems due to their ill–posedness. The main feature of such ill-posed problems is the fast decay to zero of the singular values of A and, consequently, the large condition number of A. Conventional methods for solving the linear least squares problem of eq. (7), such as normal equations or QR decomposition, usually lead to solutions dominated by perturbation errors (such as non-exact evaluation of A, errors in measurements for b or finite machine precision) when the problem becomes ill-posed [29].

So-called regularization methods are needed to obtain meaningful solution estimations for these ill-posed problems. The purpose of regularization is to incorporate further information about the desired solution in order to stabilize the problem and thus to find a more efficient and stable solution.

One of the most popular regularization technique is the TSVD *Truncated–SVD*), that uses the SVD to obtain a stable solution of the problem [29]. Thus, the SVD of the matrix A could be written as:

$$A = U\Sigma V^H = \sum_{i=1}^{n} u_i \sigma_i v_i^H \tag{8}$$

where $U = (u_1, \ldots, u_n) \in \mathcal{C}^{m\times n}$, $V = (v_1, \ldots, v_n) \in \mathcal{C}^{n\times n}$ are orthonormal matrices and Σ is a diagonal matrix $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_\text{n})$ where the σ_i are the singular values of A ordered in decreasing order. The superscript "H" denotes the conjugate transposition operator.

The solution of (7), according with (8), can be expressed as:

$$x = \sum_{i=1}^{n} \frac{\left(u_i^H b\right) v_i}{\sigma_i} \tag{9}$$

From last expression, it can be clearly seen that a small singular value σ_i, compared with σ_1, gives rise to difficulties, due to the amplification of errors in A and b. To overcome this problem, TSVD computes the rank$-k$ matrix

$$A_k \equiv \sum_{i=1}^{k} u_i \sigma_i v_i^H, \qquad k \leq n \tag{10}$$

which is a truncation in the finite series of (8). This matrix A_k, avoids the influence of the $n-k$ smaller singular values. So, the new equation system for the TSVD technique is

$$A_k x_k = b \tag{11}$$

This equation can be computed in the *minimum-norm least squares sense,* for example by using the previous SVD decomposition:

$$x_k = \sum_{i=1}^{k} \frac{\left(u_i^H b\right) v_i}{\sigma_i} \tag{12}$$

The TSVD method is useful when there is a well-determined gap between the large and small singular values of A; in this case the k parameter can be chosen in order to avoid these small singular values [29].

Another regularization technique, which is more suitable when the singular values decay gradually to zero, is the *Tikhonov regularization* [29, 30], which can be formulated as:

$$\min_x \left\{ \|Ax - b\|_2^2 + \lambda^2 \|Lx\|_2^2 \right\} \tag{13}$$

which is equivalent to apply a quadratic constraint of the form $\|Lx\|_2 \leq \alpha$ to the original least squares problem of (7). The α parameter depends on the regularization parameter λ in a nonlinear way. The matrix operator L uses to be a banded matrix, e.g., the identity matrix, a diagonal weighting matrix, or a derivative operator. Nevertheless, it should incorporate information concerning the physical problem, when available.

The Tikhonov formulation of (13) can be expressed as a new least squares problem:

$$\min_x \left\| \begin{pmatrix} A \\ \lambda L \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2 \tag{14}$$

that can be efficiently solved by doing a bidiagonalization of A and simple Givens rotations over λL matrix (supposing that this matrix has a banded structure). This leads to a bidiagonal system of equations that can be fast solved by back substitution. Similar to eq. (9), Tikhonov regularization solution can be also solved as:

$$x_\lambda = \sum_{i=1}^{n} f_i \frac{\left(u_i^H b\right) v_i}{\sigma_i}, \qquad f_i = \frac{\sigma_i^2}{\sigma_i^2 + \lambda} \tag{15}$$

for $L = I$ (notice that when $L \neq I$, the eq. (13) can be transformed to the so-called standard form with a $L' = I$. This expression shows that *Tikhonov regularization* considers all singular values, but each of them weighted by a filter factor f_i instead of truncating the matrix as done in TSVD.

It must be pointed out that the selection of the λ-parameter is not obvious, and sometimes it may require the computation of several λ -solutions, each of which can be easily obtained by back substitution in $\mathcal{O}(n)$ FLOPS. When the regularization parameter λ is too small the problem becomes underregularized and the solution is dominated by perturbation errors; on the other hand, too large values of λ provide solutions which are dominated by regularization errors. So, it is interesting to select the regularization parameter which balances both regularization and perturbation errors. There are different approaches to select the λ–parameter, from which we have chosen the *L-curve criterion* due to its stability and its illustrative notion [29, 31, 32].

The *L-curve criterion* is a relatively new method, based on a parametric plot of the regularized solution $\|Lx\|_2$ versus the residual norm $\|Ax - b\|_2$, in a *log-log* scale, with λ as the parameter. In ill-posed

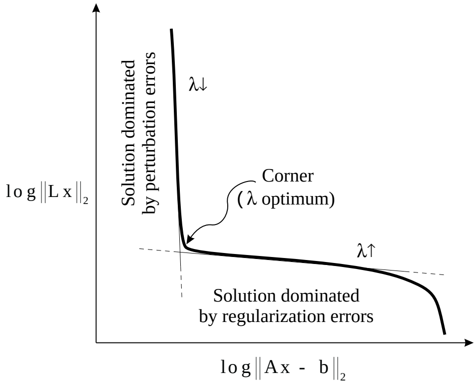

Figure 2. L-curve of an ill-posed problem.

problems, the L-curve plot has a characteristic *L-shape* appearance, with a sharp corner between vertical and horizontal parts. The vertical piece corresponds with a underregularized solution while horizontal part corresponds with a high regularized solution. The purpose of the *L-curve criterion* is to calculate the regularization parameter by choosing a point of this curve at the corner between the horizontal and vertical parts. By this way, a λ parameter that balances both the regularization and the perturbation errors is selected. Figure 2 shows these characteristics of a typical L-curve.

It is important to notice that a closed expression for the curvature of L-curve can be obtained when the *Tikhonov regularization* is used; so, the selection of the corner is performed by a simple maximum curvature search [29].

According with the application of the *Tikhonov regularization* to the electromagnetic problem stated in sections 2 and 3, it must be pointed that the hybrid GMT-MoM method tends to concentrate MoM sources near the edges, which inherently generates ill-conditioning and hence potentially unstable results [14]. When the problem becomes ill-conditioned, the singular values decay gradually to zero, so *Tikhonov regularization* is more suitable than TSVD.

It has been found that both GMT and MoM coefficients grow without bound when the problem becomes ill–posed. In order to overcome this drawback, the *Tikhonov regularization* (13) is applied to eq. (4), which can be written in a general form as:

$$\min_c \left\{ \| \cdot \; - \; \|_2^2 + \lambda^2 \, \| \cdot \|_2^2 \right\} \tag{16}$$

Matrix operator is selected to be the identity matrix the identity matrix = , in order to provide a physical constraint for the GMT and MoM coefficients. But, due to the different order of magnitude between MoM and GMT coefficients, a new parameter γ must be included in :

$$= \begin{bmatrix} {}_{N\times N} & \mathbf{0} \\ \mathbf{0} & \gamma \cdot {}_{M\times M} \end{bmatrix} \tag{17}$$

that can be interpreted as a combination of two independent operators (one for the MoM sources and the other for GMT sources) by a *Sobolev norm* [29].

An interesting consequence of the application of the quadratic constraint to the coefficients is that the radiated power of each source alone is controlled, and therefore the coupling between them is reduced in such a way. Furthermore, the GMT-MoM method becomes less dependent on the location of the sources as the source coupling becomes weaker, which has been found to be a very important factor to provide a significant improvement in the accuracy of the solution.

5. NUMERICAL RESULTS

Some numerical results are presented, concerning the convergence and stability of the proposed hybrid solution, together with its dependence on the number and location of the MoM sources. For illustrative purposes we consider the case of a 2D PEC body illuminated by TM and TE plane waves, as shown in Fig. 3, which also contains a description of the location of the GMT and MoM basis.

The results for TM polarization are shown in Figs. 4 and 5. Fig. 4 shows the average error in the induced current density as a function of the number of MoM sources (M) located nearby the corner of the geometry. This error is defined as

$$\xi = mean \left\{ \frac{|\mathbf{J}_{ref}(\mathbf{r}) - \mathbf{J}(\mathbf{r})|}{2 \cdot |\hat{n} \times \mathbf{H}^{inc}|_{\max}} \right\} \times 100 \quad \text{at} \quad \mathbf{r} \in S \tag{18}$$

where $\mathbf{J}_{ref}$ is the induced current density obtained with the MoM reference solution, $\mathbf{H}^{inc}$ is the incident magnetic field, and $\mathbf{J}$ is the

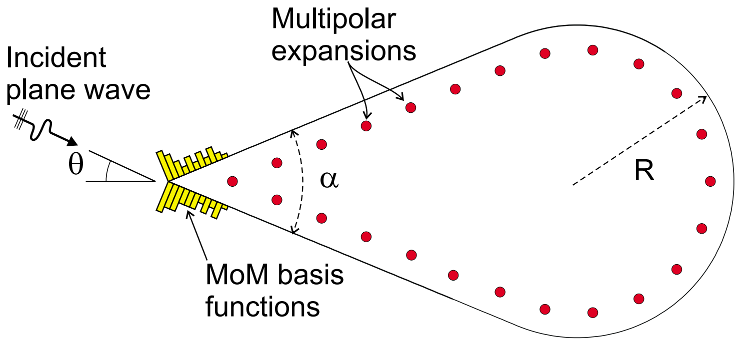

Figure 3. Test geometry. $R = 2.92\ \lambda$, $\alpha = 45°$, $\theta = 0°$, $N = 72$ multipolar sources located in 24 origins, $M = 0 \dots 40$ pulse basis functions.

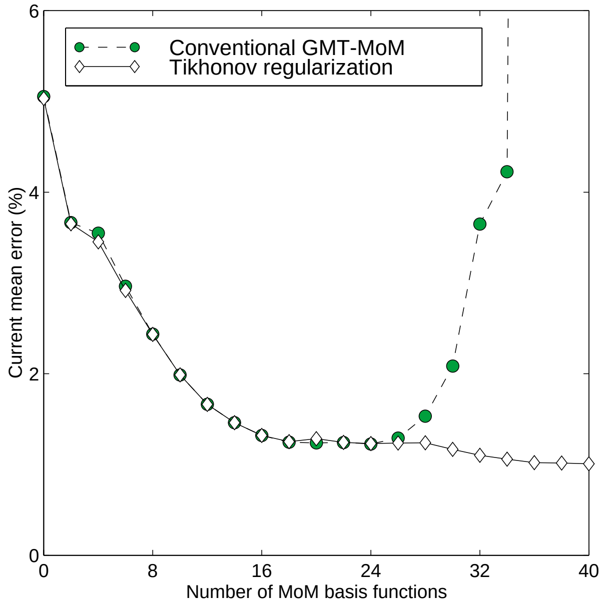

Figure 4. Plot of average current error as a function of number of MoM sources (M). TM polarization.

induced current density obtained by the hybrid method using the H–field boundary condition over S. From Fig. 4, it can be seen that the GMT-MoM method improves the solution obtained by conventional GMT $(M = 0)$ for a range of values of M. Nevertheless, for larger values of M the problem becomes ill-posed, and therefore the solution is not accurate. Otherwise, the *Tikhonov regularization* provides an uniformly convergent solution, achieving better results as the number of MoM basis increases.

The above mentioned behavior is also illustrated in Fig. 5, where the condition number is plotted as a function of M. It can be seen that the *Tikhonov regularization* stabilizes the problem, maintaining its condition number into an acceptable range of values.

The same results for TE polarization are plotted in Figs. 6 and 7, showing the same behavior previously commented for the TM polarization, although the condition number presents weak oscillations for large values of M, that anyway does not provide any significant inconvenience as can be shown in the mean error plot.

Finally, the L-curve for the *Tikhonov regularization* (TM case, $M = 32$), is shown in Fig. 8. This L-curve illustrates the trade-off between minimizing the two quantities involved in the regularization problem, namely, the residual norm and the solution norm, showing how this quantities depend on the regularization parameter λ. For the particular problem considered here, the L-curve has a clearly sharp corner, whose position (marked in Fig. 8) corresponds to a regularized solution in which the perturbation error ant the regularization error are balanced.

6. CONCLUSION

A regularized solution of the hybrid GMT-MoM method has been presented in this paper. The study has been focused on the convergence and accuracy aspects of the solution, examining their dependence on the location and the number of the GMT and MoM sources. As was expected, it has been found that the regularization highly improves the conventional GMT-MoM solution and besides reduces the condition number associated with the problem. Otherwise the use of this regularization tool allows to overcome one of the main drawback for the users of the GMT-MoM method, namely, its great dependence on the sources location. *Tikhonov regularization* has been selected instead of TSVD because of the following reasons: (a) It is more stable with

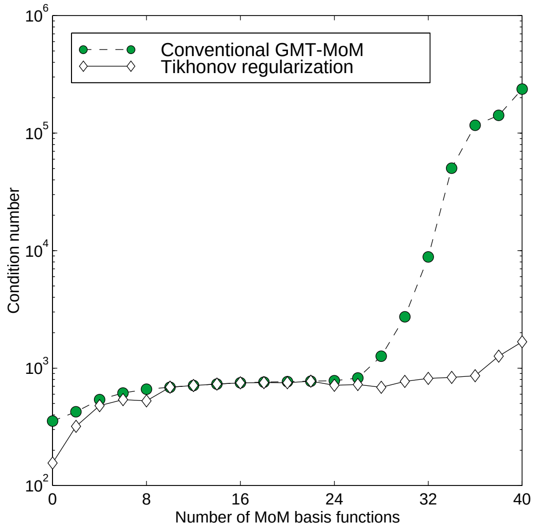

Figure 5. Plot of the condition number as a function of number of MoM basis (M). TM polarization.

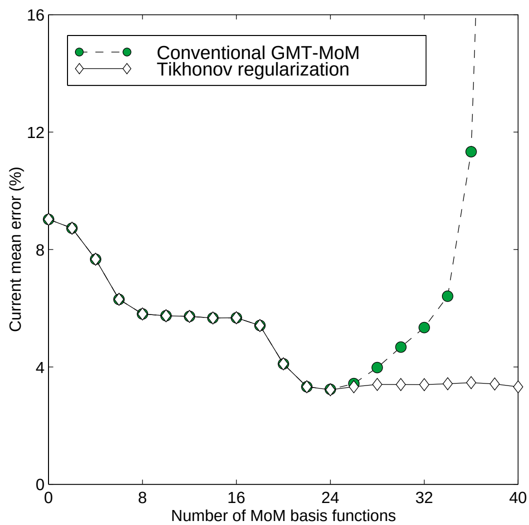

Figure 6. Plot of average current error as a function of number of MoM sources (M). TE polarization.

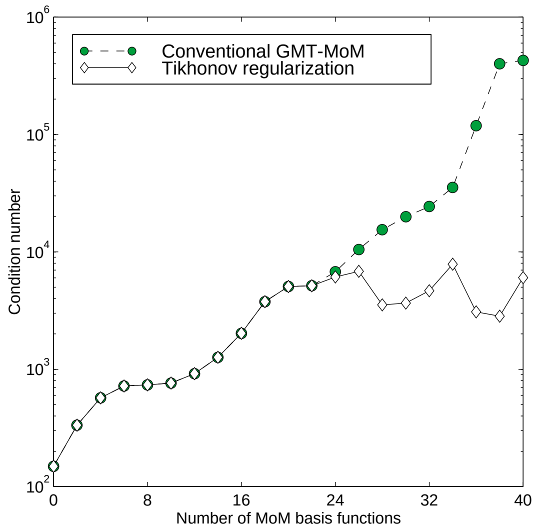

Figure 7. Plot of the condition number as a function of number of MoM basis (M). TE polarization.

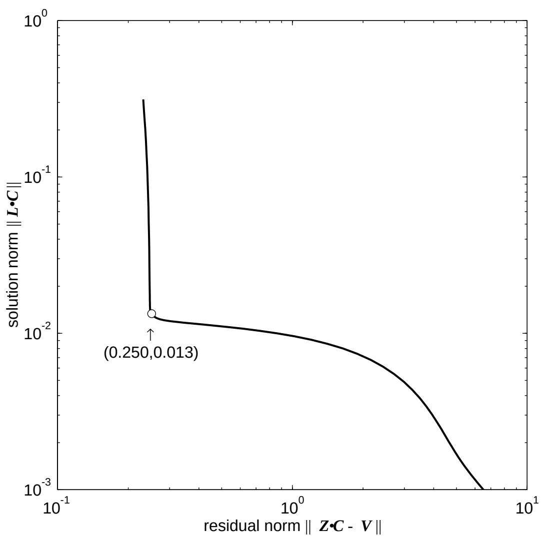

Figure 8. *L-curve* corresponding to the previous example with $M = 32$ MoM basis functions and TM polarization.

respect to the selection of the regularization parameter; (b) it obtains better results (a few tenth percent), due to the gradually decay of the singular values; and (c) its implies a minor number of FLOPS.

The main drawback of the proposed regularization method is the selection of the regularization parameter λ which is not obvious. The adequate selection of this parameter is a very important task, for which we have chosen the *L-curve criterion* that has been found to be very efficient and straightforward to apply.

ACKNOWLEDGEMENTS

This work has been partially supported by the Comisión Interministerial de Ciencia y Tecnología (CICYT), Project Ref. TIC97-0821-C02-01, and by the European INTAS Program Proj. Ref. INTAS-96-2139.

REFERENCES

1. Harrington, R. F., *Field Computation by Moment Method*, IEEE Press, 1993.
2. Ludwig, A. C., "A Comparison of spherical wave boundary value matching versus integral equation scattering solutions for a perfectly conducting body," *IEEE Trans. Ant. Prop.*, Vol. AP-34, No. 7, 857–865, July, 1986.
3. Ludwig, A., "A new technique for numerical electromagnetics," *IEEE Antennas Propagat. Newsletter*, Vol. 3, 40–41, Feb., 1889.
4. Hafner, C., *The Generalized Multipole Technique for Computational Electromagnetics*, Artech House, London, 1990.
5. Leviatan, Y., and A. Boag, "Analysis of electromagnetic scattering from dielectric cylinders using a multifilament current model," *IEEE Trans. Ant. Prop.*, Vol. 35, No. 10, 1119–1126, Oct., 1987.
6. Leviatan, Y., A. Boag, and A. Boag, "Analysis of TE scattering from dielectric cylinders using a multifilament magnetic current model," *IEEE Trans. Ant. Prop.*, Vol. 36, No. 7, 1026–1031, Jul., 1988.
7. Leviatan, Y., A. Boag, and A. Boag, "Generalized formulations for electromagnetic scattering from perfectly conducting and homogeneous material bodies – theory and numerical solution," *IEEE Trans. Ant. Prop.*, Vol. 36, No. 12, 1722–1734, Dec., 1988.
8. Leviatan, Y., A. Boag, and A. Boag, "Analysis of electromagnetic scattering using a current model method," *Computer Physics Communications*, Vol. 68, 331–345, 1991.

9. Boag, A., and R. Mittra, "Complex multipole beam approach to electromagnetic scattering problems," *IEEE Trans. Ant. Prop.*, Vol. 42, No. 3, 366–372, Mar., 1994.
10. Boag, A., and R. Mittra, "Complex multipole-beam approach to three-dimensional electromagnetic scattering problems," *J. of the Opt. Soc. of Am.*, Vol. 11, No. 4, 1505–1512, Apr., 1994.
11. Erez, E., and Y. Leviatan, 'Electromagnetic scattering analysis using a model of dipoles located in complex space," *IEEE Trans. Ant. Prop.*, Vol. 42, No. 12, 1620-1624, Dec., 1994
12. Leviatan, Y., Z. Baharav, and E. Heyman, "Analysis of electromagnetic scattering using arrays of fictitious sources," *IEEE Trans. Ant. Prop.*, Vol. 43, No. 10, 1091–1098, Oct., 1995
13. Baharav, Z., and Y. Leviatan, "Scattering analysis using fititious wavelet array sources," *Journal of Electromagnetic Waves and Applications*, Vol. 10, 1683–1697, 1996.
14. Eisler, S., and Y. Leviatan, "Analysis of electromagnetic scattering from metallic and penetrable cylinders with edges using a multifilament current model," *IEE Proceedings, Pt. H*, Vol. 136, No. 6, 431–438, Dec., 1989.
15. Hafner, C., J. Waldvogel, J. Mosig, J. Zheng, and Y. Brand, "On the Combination of MMP with MoM," *Appl. Comp. Elec. Soc. J.*, Vol. 9, No. 3, 18–27, 1994.
16. Boag, A., and E. Michelssen, and R. Mittra, "Hybrid multipole-Beam approach to electromagnetic scattering problems," *Appl. Comp. Elec. Soc. J.*, Vol. 9, No. 3, 7–17, 1994.
17. Rodríguez, J. L., F. Obelleiro, and A. G. Pino, "A hybrid multipolar-expansion-moment-method approach for electromagnetic scattering problems," *Micr. Opt. Tech. Lett.*, Vol. 11, No. 6, 304–308, Apr., 1996.
18. Leuchtmann, P., "MMP modeling technique with curved line multipoles," *Appl. Comp. Elec. Soc. J.*, Vol. 9, No. 3, 69–78, 1994
19. Leutchmann, P., "The construction of practically useful fast converging expansions for the GMT," *1989 IEEE AP-S Intl. Symp.*, 176–179, 1989.
20. Shifman, Y., M. Friedmann, and Y. Leviatan, "Analysis of electromagnetic scattering by cylinders with edges using a hybrid moment method, *IEE Proceedings Microwave, Antennas and Propagation*, Vol. 144, No. 4, 235–240, Aug., 1997.
21. Beshir, K. L., and J. E. Richie, "On the location and number of expansion centers for the generalized multipole technique," *IEEE Trans. on Elec. Compat.*, Vol. 38, No. 2, 177–180, May, 1996.

22. Rodríguez, J. L., and A. G. Pino, "An automatic location algorithm of MoM basis in the hybrid GMT-MoM method," *Micr. Opt. Tech. Lett.*, Vol. 13, No. 6, 327–329, Dec, 1996.
23. Rodríguez, J. L., "Desarrollo de métodos eficientes para el estudio de problemas de dispersión electromagnética," Ph. D. Thesis *E. T. S. I. Telecomunicación. Universidad de Vigo*, 1997.
24. Boag, A., Y. Leviatan, and A. Boag, "On the use of SVD-improved point matching in the current-model method," *IEEE Trans. Ant. Prop.* , Vol. 41, No. 7, 926–933, Jul., 1993.
25. Golub, G. H., and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press,1989.
26. Canning, F. X., "Singular value decomposition of integral equations of EM and applications to the cavity resonance problem," *IEEE Trans. Ant. Prop.*, Vol. 37, No. 9, 1156–1163, Sep., 1989.
27. Canning, F. X., "Protecting EFIE-based scattering computations from effects of interior resonances," *IEEE Trans. Ant. Prop.*, Vol. 39, No. 11, 1545–1552, Nov., 1991.
28. Sarkar, T. K., and S. M. Rao, "A simple technique for solving E-field integral equation for conducting bodies at internal resonances," *IEEE Trans. Ant. Prop.*, Vol. 30, No. 6, 1250–1254, Nov., 1982.
29. Hansen, P. C., *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion*, SIAM, Philadelphia, 1998.
30. Tikhonov, A. N., and V. Y. Arsenin, *Solutions of Ill-posed Problems*, Winston, Washington, D.C., 1997.
31. Hansen, P. C., and D. P. O'Leary, "The use of the L-curve in the regularization of discrete ill-posed problems," *SIAM Journal of Scientific Computing*, Vol. 14, 1487–1503, 1993.
32. Hanke, M., "Limitations of the L-curve method in ill-posed problems," *BIT*, Vol. 36, 287–301, 1996.