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UNIQUENESS IN THE SIMULTANEOUS RECONSTRUCTION OF MULTIPARAMETERS OF A TRANSMISSION LINE

D. Sheen

Department of Mathematics
Seoul National University
Seoul 151-742, Korea

D. Shepelsky

Institute for Low Temperature Physics
Kharkov 310164, Ukraine

1. INTRODUCTION

Methods for reconstructing the physical parameters of a nonuniform transmission line from the transient response are of interest in many applications such as design of nonuniform transmission line filters, analysis of a signal channel between a transmitter and receiver, optimal design of impedance matching sections, impedance–matched voltage transformers, etc.

During the last few decades, several approaches to inverse problems for transmission lines have been proposed in the both frequency and time domain, see [1–5] and references therein. In [4], the time–domain method based on coupled integral equations for the compact Green's functions has been developed to determine simultaneously two of the transmission line parameters. In [5], the simultaneous reconstruction of three parameters was presented that is based on the time–domain conjugate gradient search algorithm to minimize the difference between the target transient response and the calculated one. This method uses the transient responses with respect to incident signals from both sides. The examples of successful as well as unsuccessful reconstruction (when the different sets of the nonuniform line parameters give identical transient responses) were reported.

In the present paper, a frequency–domain approach to the inverse problem for a nonuniform transmission line is presented. It is shown that exactly two combinations of the physical parameters (shunt conductance, inductance, capacitance, and series resistance) as functions of the travel time, together with the attenuation factors, are determined uniquely by the transient response.

The paper is organized as follows. In Section 2, the model equations are given. The transformation of the split components and two propositions on the uniqueness in the direct problem, that is, the uniqueness of the transient response, are given in Section 3. Uniqueness in the inverse problem, that is, in the reconstruction of the line parameters, is discussed in Section 4. Details concerning the inverse scattering problem for the Zakharov–Shabat system are given in Appendix A, and the uniqueness results in the case of a hard reflection at the end of the line are presented in Appendix B.

2. BASIC EQUATIONS

A lossy nonuniform transmission line of a length l is described by the inductance $L(x)$, capacitance $C(x)$, series resistance $R(x)$, and shunt conductance $G(x)$, $0 < x < l$. The nonuniform line is joined to two lossless uniform lines $x < 0$ and $x > l$ characterized by the constant parameters $\{L_0, C_0\}$ and $\{L_l, C_l\}$, respectively ($R = G = 0$ outside $(0, l)$). The telegrapher's equations for the time–harmonic voltage $V(x, \omega)$ and current $I(x, \omega)$ are

$$\frac{d}{dx}\begin{pmatrix} V \\ I \end{pmatrix}(x,\omega) = -\,i\omega \begin{pmatrix} 0 & L(x) \\ C(x) & 0 \end{pmatrix}\begin{pmatrix} V \\ I \end{pmatrix}(x,\omega) \\ - \begin{pmatrix} 0 & R(x) \\ G(x) & 0 \end{pmatrix}\begin{pmatrix} V \\ I \end{pmatrix}(x,\omega), \quad x \in (-\infty,\infty). \tag{1}$$

According to the wave splitting approach to the scattering problems for continuously varying media (see, e.g., [6–9]), introduce the right– and left–moving components by diagonalizing the first term in the righthand side of (1):

$$Y(x,\omega) \equiv \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}(x,\omega) = \mathcal{T}(x)\begin{pmatrix} V \\ I \end{pmatrix}(x,\omega) \\ \equiv \begin{pmatrix} 1 & Z(x) \\ 1 & -Z(x) \end{pmatrix}\begin{pmatrix} V \\ I \end{pmatrix}(x,\omega),$$

where $Z(x) = \sqrt{\frac{L(x)}{C(x)}}$ is the characteristic impedance. Equation (1) becomes (cf. [5])

$$\frac{dY}{dx} = i\omega\frac{1}{c(x)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} Y + W(x)Y, \tag{2}$$

where $c(x) = \frac{1}{\sqrt{L(x)C(x)}}$ is the wavefront velocity,

$$W(x) \equiv \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix} \\ = \frac{1}{2}\begin{pmatrix} -ZG - \dfrac{R}{Z} + \dfrac{Z'}{Z} & -ZG + \dfrac{R}{Z} - \dfrac{Z'}{Z} \\ ZG - \dfrac{R}{Z} - \dfrac{Z'}{Z} & ZG + \dfrac{R}{Z} + \dfrac{Z'}{Z} \end{pmatrix}(x). \tag{3}$$

Note that $W(x) = 0$ for $x < 0$ and $x > l$.

It is assumed that $G(x), R(x)$, and $c(x)$ are piecewise continuous functions. The impedance $Z(x)$ is assumed to be absolutely continuous on $(0,l)$, with possible jump discontinuities at $x = 0$ and $x = l$, described by the parameters q_0 and q_l:

$$q_0 = \frac{1}{2}\left(1 - \frac{Z(-0)}{Z(+0)}\right), \qquad q_l = \frac{1}{2}\left(1 - \frac{Z(l-0)}{Z(l+0)}\right).$$

Let $Y^{\pm}(x,\omega)$ be two fundamental solutions of (2) for $x\in(-\infty,\infty)$ (so that $Y^{\pm}(x,\omega)$ are discontinuous at points where $Z(x)$ has jump discontinuities) determined by their behaviors on the uniform parts of the line:

$$Y^{-}(x,\omega)=\begin{pmatrix}\exp\left\{-i\omega\dfrac{x}{c(-0)}\right\} & 0\\ 0 & \exp\left\{i\omega\dfrac{x}{c(-0)}\right\}\end{pmatrix}\quad\text{for } x<0,$$
$$Y^{+}(x,\omega)=\begin{pmatrix}\exp\left\{-i\omega\dfrac{x}{c(l+0)}\right\} & 0\\ 0 & \exp\left\{i\omega\dfrac{x}{c(l+0)}\right\}\end{pmatrix}\quad\text{for } x>l. \tag{4}$$

The transient response of the line can be described in terms of a scattering matrix $S(\omega)$ that relates $Y^{-}(x,\omega)$ and $Y^{+}(x,\omega)$:

$$Y^{-}(x,\omega)S(\omega)=Y^{+}(x,\omega),\qquad x\in(-\infty,\infty). \tag{5}$$

The 2×2 scattering matrix $S(\omega)\equiv\begin{pmatrix}a_1(\omega) & b_2(\omega)\\ b_1(\omega) & a_2(\omega)\end{pmatrix}$ contains all informations about the transient responses from both left–sided and right–sided incidence of an input signal; namely, $a_1(\omega)$ and $b_1(\omega)$ describe the transient response to the left–sided incidence, whereas the second column of $S^{-1}(\omega)$, $\frac{1}{\det S(\omega)}\begin{pmatrix}-b_2(\omega)\\ a_1(\omega)\end{pmatrix}$, describes the response to an incident signal from the right.

The matrix elements of the scattering matrix $S(\omega)$ are related directly to the time–domain reflection and transmission kernels, attenuation factors, and the travel time. For instance, in the case of $q_0=q_l=0$, that is, when the impedance is a continuous function on $(-\infty,\infty)$,

$$\frac{1}{a_1(\omega)}e^{i\omega\Delta}=b^{+}+\hat{T}^{+}(\omega),\qquad \frac{b_1(\omega)}{a_1(\omega)}=\hat{R}^{+}(\omega),$$
$$\hat{T}^{-}(\omega)=\frac{Z(0)}{Z(l)}\hat{T}^{+}(\omega),\qquad -\frac{b_2(\omega)}{a_1(\omega)}=\hat{R}^{-}(\omega)\exp\left\{2i\omega\frac{l}{c(l+0)}\right\},$$

where $\hat{R}^{\pm}(\omega)$ and $\hat{T}^{\pm}(\omega)$ are the Fourier transforms of the time–domain reflections and transmission kernels, respectively, for the left–side $(+)$ and right–side $(-)$ incidence, $b^{+}=\exp\{\int_0^l\alpha(s)ds\}$ is the attenuation factor, $\Delta = L - l/c(l+0)$, $L = \int_0^l \frac{ds}{c(s)}$ is the travel time of the wavefront from $x = 0$ to $x = l$ (see [5]).

For the response to the left–side incident signal in the mismatching case with $q_l \neq 0, q_0 = 0$ (see [4]),

$$\frac{1}{a_1(\omega)} \exp\{i\omega\Delta\} = \frac{1}{1-q_l} \exp\left\{\int_0^l \alpha(s)ds\right\} + \hat{T}^+(\omega),$$

$$\frac{b_1(\omega)}{a_1(\omega)} = \frac{q_l}{1-q_l} \exp\left\{\int_0^l (\alpha(s) - \delta(s))ds - 2i\omega L\right\} + \hat{R}^+(\omega).$$

3. TRANSFORMATION OF THE SPLIT COMPONENTS

The paper aims at determining the information about the characteristics of the nonuniform transmission line that can be obtained uniquely from the transient response. For this purpose, the inverse scattering method in the spectral domain for a 2×2 matrix differential equation, referred usually as a Zakharov–Shabat system, is applied (see, e.g., [10]).

First, if the line has mismatched impedance at the ends, the problem is reduced to that for an auxiliary impedance–matched line. The latter is characterized by the same characteristics inside the nonuniform part $0 < x < l$, whereas the uniform lines $x < 0$ and $x > l$ have the impedance $Z^{aux}(x) = Z(+0)$, $x < 0$, and $Z^{aux}(x) = Z(l-0)$, $x > l$, respectively. The split components $Y^{aux\pm}(x,\omega)$ are defined as continuous solutions of (2) for $x \in (-\infty, \infty)$ satisfying the same conditions (4) as $Y^{\pm}(x,\omega)$ do, and are related by the scattering matrix $S^{aux}(\omega)$

$$Y^{aux-}(x,\omega)S^{aux}(\omega) = Y^{aux+}(x,\omega), \qquad x \in (-\infty, \infty).$$

On the other hand, continuity of the voltage $V(x,\omega)$ and current $I(x,\omega)$ yields

$$\begin{aligned} Y^{\pm}(-0,\omega) &= P_0 Y^{\pm}(+0,\omega), \\ Y^{\pm}(l-0,\omega) &= P_l Y^{\pm}(l+0,\omega), \end{aligned} \tag{6}$$

where

$$P_a = \mathcal{T}(a-0)\mathcal{T}^{-1}(a+0) = \begin{pmatrix} 1-q_a & q_a \\ q_a & 1-q_a \end{pmatrix}, \quad a \in \{0, l\}.$$

It follows from (4) and (5) that the scattering matrices can be represented as

$$S(\omega) = Y^{+}(-0,\omega), \qquad S^{aux}(\omega) = Y^{aux+}(0,\omega). \tag{7}$$

Relations (6), (7), and the fact that the actual and auxiliary lines have the same characteristics inside the interval $(0,l)$ allow us to relate the actual and auxiliary scattering matrices. Indeed,

$$\begin{aligned} S(\omega) &= Y^{+}(-0,\omega) = P_0 Y^{+}(+0,\omega) = P_0\mathcal{R}(\omega)Y^{+}(l-0,\omega) \\ &= P_0\mathcal{R}(\omega)P_l Y^{+}(l+0,\omega), \\ S^{aux}(\omega) &= Y^{aux+}(0,\omega) = \mathcal{R}(\omega)Y^{aux+}(l,\omega), \end{aligned}$$

where $\mathcal{R}(\omega)$ is the transition matrix for equation (2) from $x = l-0$ to $x=+0$; hence,

$$S(\omega) = P_0 S^{aux}(\omega)D^{-1}(\omega)P_l D(\omega), \tag{8}$$

where

$$\begin{aligned} D(\omega) &= Y^{+}(l+0,\omega) = Y^{aux+}(l,\omega) \\ &= \begin{pmatrix} \exp\left\{-i\omega\dfrac{l}{c(l+0)}\right\} & 0 \\ 0 & \exp\left\{i\omega\dfrac{l}{c(l+0)}\right\} \end{pmatrix}. \end{aligned}$$

Further, the problem is transformed to make the "potential" part of equation (2), $W(x)$, off–diagonal. Setting $F(x,\omega) = \mathcal{E}(x)Y^{aux}(x,\omega)$, where

$$\mathcal{E}(x) = \begin{pmatrix} \exp\left\{-\displaystyle\int_0^x \alpha(s)ds\right\} & 0 \\ 0 & \exp\left\{-\displaystyle\int_0^x \delta(s)ds\right\} \end{pmatrix},$$

one gets

$$\begin{aligned} &\frac{dF}{dx}(x,\omega) = i\omega\frac{1}{c(x)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F(x,\omega) + W_1(x)F(x,\omega), \\ &x \in (-\infty,\infty), \end{aligned} \tag{9}$$

where $W_1(x) = \begin{pmatrix} 0 & Q_1(x) \\ Q_2(x) & 0 \end{pmatrix}$,

$$\begin{aligned} Q_1(x) &= \beta(x) \exp\left\{ \int_0^x (\delta - \alpha) ds \right\} \\ &= \frac{1}{2}\left(-ZG + \frac{R}{Z} - \frac{Z'}{Z} \right)(x) \exp\left\{ \int_0^x \left(ZG + \frac{R}{Z} \right) ds \right\}, \\ Q_2(x) &= \gamma(x) \exp\left\{ -\int_0^x (\delta - \alpha) ds \right\} \\ &= \frac{1}{2}\left(ZG - \frac{R}{Z} - \frac{Z'}{Z} \right)(x) \exp\left\{ -\int_0^x \left(ZG + \frac{R}{Z} \right) ds \right\}. \end{aligned} \tag{10}$$

At last, the travel time variable is introduced

$$\xi(x) = \int_0^x \frac{ds}{c(s)}. \tag{11}$$

Setting $\Phi(\xi, \omega) = F(x(\xi), \omega)$, one arrives at the equation

$$\frac{d\Phi}{d\xi}(\xi, \omega) = i\omega \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Phi(\xi, \omega) + U(\xi)\Phi(\xi, \omega), \quad \xi \in (-\infty, \infty), \tag{12}$$

where

$$U(\xi) = \begin{pmatrix} 0 & u_1(\xi) \\ u_2(\xi) & 0 \end{pmatrix}, \tag{13}$$

$$u_k(\xi) = Q_k(x(\xi))c(x(\xi)), \qquad k = 1, 2.$$

Define the scattering matrix $S^\Phi(\omega)$ and the related fundamental solutions $\Phi^\pm(\xi, \omega)$ of (12) by

$$\Phi^-(\xi, \omega) = \begin{pmatrix} e^{-i\omega\xi} & 0 \\ 0 & e^{i\omega\xi} \end{pmatrix} \qquad \text{for } \xi < 0, \tag{14}$$

$$\Phi^+(\xi, \omega) = \begin{pmatrix} e^{-i\omega\xi} & 0 \\ 0 & e^{i\omega\xi} \end{pmatrix} \qquad \text{for } \xi > L = \xi(l), \tag{15}$$

$$\Phi^-(\xi, \omega)S^\Phi(\omega) = \Phi^+(\xi, \omega) \qquad \text{for } -\infty < \xi < \infty. \tag{16}$$

(Also see (A1) in Appendix 1.) Comparing the large x–behavior of the split components yields

$$\begin{aligned} \Phi^-(\xi(x), \omega) &= \mathcal{E}(x) Y^{aux-}(x, \omega), \\ \Phi^+(\xi(x), \omega) &= \mathcal{E}(x) Y^{aux+}(x, \omega) \mathcal{E}^{-1}(l) \begin{pmatrix} e^{-i\omega\Delta} & 0 \\ 0 & e^{i\omega\Delta} \end{pmatrix}, \end{aligned}$$

so that the corresponding scattering matrices are related as

$$S^{aux}(\omega) = S^{\Phi}(\omega) \begin{pmatrix} \exp\left\{-\int_0^l \alpha(s)ds + i\omega\Delta\right\} & 0 \\ 0 & \exp\left\{-\int_0^l \delta(s)ds - i\omega\Delta\right\} \end{pmatrix}. \tag{17}$$

It follows from (8) and (17) that the actual scattering matrix $S(\omega)$ is determined completely by $S^{\Phi}(\omega)$ together with the constants $\int_0^l \alpha(s)ds$, $\int_0^l \delta(s)ds$, L, q_0, and q_l. On the other hand, $S^{\Phi}(\omega)$, being the scattering matrix of equation (12), is determined by $U(\xi)$, $\xi \in (0, L)$, that is, by $u_1(\xi)$ and $u_2(\xi)$. Therefore, we arrive at the following proposition.

Proposition 1 *Suppose that two nonuniform lines are characterized by the parameters* $\mathcal{P}^{(k)} = \{L^{(k)}(x), C^{(k)}(x), R^{(k)}(x), G^{(k)}(x)\}$, $x \in (0,l)$, $k = 1, 2$, *and joined to two lossless and uniform transmission lines* $x < 0$ *and* $x > l$. *Let their characteristics satisfy the following:*

1) $\int_0^l \alpha^{(1)}(s)ds = \int_0^l \alpha^{(2)}(s)ds,$ $\int_0^l \delta^{(1)}(s)ds = \int_0^l \delta^{(2)}(s)ds,$ $\int_0^l \frac{ds}{c^{(1)}(s)}ds = \int_0^l \frac{ds}{c^{(2)}(s)}ds \equiv L$,

where $c^{(k)}(x) = \left(L^{(k)}(x)C^{(k)}(x)\right)^{-\frac{1}{2}}$, $\alpha^{(k)}(x)$ *and* $\delta^{(k)}(x)$ *are constructed from* $\mathcal{P}^{(k)}$ *by (3);*

2) $u_j^{(1)}(\xi) = u_j^{(2)}(\xi)$, $\xi \in (0, L)$, $j = 1, 2$,

where $u_j^{(k)}(\xi)$ *relates to* $\mathcal{P}^{(k)}$ *by* (10) *and* (13), *with* $\xi(x) = \int_0^x \frac{ds}{c^{(k)}(s)}ds$;

3) $\frac{Z^{(1)}(-0)}{Z^{(1)}(+0)} = \frac{Z^{(2)}(-0)}{Z^{(2)}(+0)},$ $\frac{Z^{(1)}(l-0)}{Z^{(1)}(l+0)} = \frac{Z^{(2)}(l-0)}{Z^{(2)}(l+0)}$. *Then* $S^{(1)}(\omega) = S^{(2)}(\omega)$, *that is, the transient responses of the lines* $\mathcal{P}^{(1)}$ *and* $\mathcal{P}^{(2)}$ *are equivalent.*

The wavefront velocity $c(x)$ plays the special role in the problem transformation. Consider the particular case when $c(x)$ is fixed. As an immediate consequence of Proposition 1 we have

Proposition 2 *If two sets of parameters* $\{Z^{(k)}(x), R^{(k)}(x), G^{(k)}(x)\}$, $x \in (0,l)$, $k = 1, 2$, *are such that*

1) $\int_0^l \alpha^{(1)}(s)ds = \int_0^l \alpha^{(2)}(s)ds,$ $\int_0^l \delta^{(1)}(s)ds = \int_0^l \delta^{(2)}(s)ds$,

2) $Q_j^{(1)}(x) = Q_j^{(2)}(x)$, $x \in (0,l)$, $j = 1, 2$,

where $Q_j^{(k)}(x)$ *relates to* $\{Z^{(k)}(x), R^{(k)}(x), G^{(k)}(x)\}$ *by* (10),

3) $\frac{Z^{(1)}(-0)}{Z^{(1)}(+0)} = \frac{Z^{(2)}(-0)}{Z^{(2)}(+0)}, \qquad \frac{Z^{(1)}(l-0)}{Z^{(1)}(l+0)} = \frac{Z^{(2)}(l-0)}{Z^{(2)}(l+0)},$
then $S^{(1)}(\omega) = S^{(2)}(\omega)$ *.*

4. UNIQUENESS IN THE PARAMETER RECONSTRUCTION

In this section we address the problem which is inverse to that of Propositions 1 and 2. Suppose that two nonuniform lines produce the same transient response (or a part of it, related, for instance, to the one–sided incidence only). Can one conclude that the related combinations $u_j(\xi)$ of the line characteristics (as well as the constants involved in Propositions 1 and 2) are equal?

Consider first the problem of determination of the constants. Due to the fact that $S^{\Phi}(\omega) \to I_2$ as $\omega \to \infty$, where I_2 is the 2×2 identity matrix (see Appendix A), the large ω–behavior of the one–sided response $a_1(\omega)$ and $b_1(\omega)$ allows reconstructing the propagation constants. Indeed, from (8) and (17) one has, as $\omega \to \infty$,

$$a_1(\omega) \exp\left\{ i\omega \frac{l}{c(l+0)} \right\} \simeq (1-q_0)(1-q_l) \exp\left\{ -\int_0^l \alpha ds + i\omega L \right\} + q_0 q_l \exp\left\{ -\int_0^l \delta ds - i\omega L \right\},$$

$$b_1(\omega) \exp\left\{ i\omega \frac{l}{c(l+0)} \right\} \simeq q_0(1-q_l) \exp\left\{ -\int_0^l \alpha ds + i\omega L \right\} + (1-q_0) q_l \exp\left\{ -\int_0^l \delta ds - i\omega L \right\},$$

which gives means to determine L, q_0, q_l, $\int_0^l \alpha ds$, and $\int_0^l \delta ds$.

Consider now the problem of reconstruction of $u_1(\xi)$ and $u_2(\xi)$. It is shown in Appendix A that $u_1(\xi)$ and $u_2(\xi)$, $\xi \in (0, L)$, are determined uniquely by $S^{\Phi}(\omega) \equiv \begin{pmatrix} a_1^{\Phi} & b_2^{\Phi} \\ b_1^{\Phi} & a_2^{\Phi} \end{pmatrix}(\omega)$, $\omega > 0$. In turn, as far as the constant parameters are determined, the transient response, that is, two reflection coefficients $r^{+}(\omega) = b_1(\omega)/a_1(\omega)$ and $r^{-}(\omega) = -(\det S(\omega))^{-1} b_2(\omega)/a_1(\omega)$, together with the transient coefficient $t(\omega) = a_1^{-1}(\omega)$, determine $S^{\Phi}(\omega)$ via (8) and (17) (notice that $\det S(\omega) = (1-2q_0)(1-2q_l)\exp\{-\int_0^l (\alpha + \delta) ds$). Therefore, we have the following proposition.

Proposition 3 *The two–sided transient response determines uniquely* $u_1(\xi)$ *and* $u_2(\xi)$ *.*

The possibility to reconstruct uniquely $u_1(\xi)$ and $u_2(\xi)$ simultaneously depends on whether the hard reflection at the back end of the line occurs or not. Consider first the case when there is no hard reflection at $x = l$, that is, $q_l = 0$. If $u_2(\xi) \equiv 0$, then, for any $u_1(\xi)$, one has $a_1^\Phi(\omega) \equiv 1$ and $b_1^\Phi(\omega) \equiv 0$, so that, in virtue of (8) and (17), $a_1(\omega) = (1 - q_0)\exp\{-\int_0^l \alpha ds + i\omega\Delta\}$ and $b_1(\omega) = q_0 \exp\{-\int_0^l \alpha ds + i\omega\Delta\}$. Therefore, in this case, a family of lines with different $u_1(\xi)$ produces the same left–sided transient response.

The one–sided response becomes more informative in the presence of the hard reflection, cf. [4]. The following proposition holds.

Proposition 4 *If* $q_l \neq 0$ *then the left–side incidence response, that is,* $a_1(\omega)$ *and* $b_1(\omega)$*, determines* $u_1(\xi)$ *and* $u_2(\xi)$ *uniquely.*

The reason for this is that in the case of hard reflection, an expression for $a_1(\omega)$ in terms of the elements of $S^\Phi(\omega)$ involves $b_2^\Phi(\omega)$ as well. The proof of Proposition 4 is given in Appendix B.

5. CONCLUSION

In this paper we have presented an analysis of the uniqueness question in the multiparameter reconstruction of a nonuniform transmission line from the transient response. By using the transformation of the problem (in the frequency domain) to a canonical form of the Zakharov–Shabat equation and the solution of the related inverse scattering problem, we have shown that only two travel time dependent combinations of the line characteristics are determined uniquely by the transient response (or, in the case of hard back reflection, by the one–sided transient response). In particular, in the case of known wavefront velocity, all three other parameters enter these combinations, so that the characteristic impedance can varies (together with the dissipative parameters like series resistance and shunt conductance) without affecting the transient response of the line (cf. [5]).

APPENDIX A. INVERSE SCATTERING PROBLEM FOR THE ZAKHAROV–SHABAT SYSTEM

The inverse scattering problem for equation (12) consists of determining the potential matrix $U(\xi)$ from the scattering matrix $S^{\Phi}(\omega)$ (see, e.g., [10]). In this paper we are interested in the case when supp $U(\xi) \subset [0, L]$, and $U(\xi)$ has the structure (13), with real–valued $u_1(\xi)$ and $u_2(\xi)$.

The solutions $\Phi^{\pm}(\xi, \omega)$ can be represented as

$$\begin{aligned} \Phi^-(\xi,\omega) &= e^{i\omega\xi J} + \int_{-\xi}^{\xi} K^-(\xi,t)e^{i\omega tJ}dt, \qquad \xi \ge 0, \\ \Phi^+(\xi,\omega) &= e^{i\omega\xi J} + \int_{\xi}^{2L-\xi} K^+(\xi,t)e^{i\omega tJ}dt, \qquad \xi \le L. \end{aligned} \tag{A1}$$

(See (14) and (15) for their representations in the complementary regions.) Here, $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and the 2×2 matrix functions $K^{\pm}(\xi, t)$ solve (in weak sense, if needed) the Goursat problems for the hyperbolic systems of equations

$$\begin{aligned} &\frac{\partial K^-_{11}(\xi,t)}{\partial \xi} + \frac{\partial K^-_{11}(\xi,t)}{\partial t} = u_1(\xi)K^-_{21}(\xi,t) \quad \text{in } D^-, \\ &\frac{\partial K^-_{21}(\xi,t)}{\partial \xi} - \frac{\partial K^-_{21}(\xi,t)}{\partial t} = u_2(\xi)K^-_{11}(\xi,t) \quad \text{in } D^-, \\ &K^-_{11}(\xi,-\xi) = 0, \qquad K^-_{21}(\xi,\xi) = \frac{1}{2}u_2(\xi) \quad \text{for } \xi \ge 0; \end{aligned} \tag{A2}$$

$$\begin{aligned} &\frac{\partial K^-_{22}(\xi,t)}{\partial \xi} + \frac{\partial K^-_{22}(\xi,t)}{\partial t} = u_2(\xi)K^-_{12}(\xi,t) \quad \text{in } D^-, \\ &\frac{\partial K^-_{12}(\xi,t)}{\partial \xi} - \frac{\partial K^-_{12}(\xi,t)}{\partial t} = u_1(\xi)K^-_{11}(\xi,t) \quad \text{in } D^-, \\ &K^-_{22}(\xi,-\xi) = 0, \qquad K^-_{12}(\xi,\xi) = \frac{1}{2}u_1(\xi) \quad \text{for } \xi \ge 0, \end{aligned} \tag{A3}$$

where $D^- = \{(\xi, t) :\ -\xi < t < \xi, \xi > 0\}$;

$$\begin{aligned} &\frac{\partial K^+_{11}(\xi,t)}{\partial \xi} + \frac{\partial K^+_{11}(\xi,t)}{\partial t} = u_1(\xi)K^+_{21}(\xi,t) \quad \text{in } D^+, \\ &\frac{\partial K^+_{21}(\xi,t)}{\partial \xi} - \frac{\partial K^+_{21}(\xi,t)}{\partial t} = u_2(\xi)K^+_{11}(\xi,t) \quad \text{in } D^+, \\ &K^+_{11}(\xi,2L-\xi) = 0, \qquad K^+_{21}(\xi,\xi) = -\frac{1}{2}u_2(\xi) \quad \text{for } \xi \le L; \end{aligned} \tag{A4}$$

$$\frac{\partial K_{22}^{+}(\xi,t)}{\partial \xi} + \frac{\partial K_{22}^{+}(\xi,t)}{\partial t} = u_2(\xi) K_{12}^{+}(\xi,t) \quad \text{in } D^{+},$$
$$\frac{\partial K_{12}^{+}(\xi,t)}{\partial \xi} - \frac{\partial K_{12}^{+}(\xi,t)}{\partial t} = u_1(\xi) K_{11}^{+}(\xi,t) \quad \text{in } D^{+},$$
$$K_{22}^{+}(\xi, 2L-\xi) = 0, \qquad K_{12}^{+}(\xi,\xi) = -\frac{1}{2} u_1(\xi) \quad \text{for } \xi \le L;, \tag{A5}$$

where $D^{+} = \{(\xi,t) : \xi < t < 2L-\xi, \xi < L\}$. Because of the finite support of $u_1(\xi)$ and $u_2(\xi)$, the kernels K^{-} and K^{+} vanish outside the domains $\{(\xi,t) : 0 < \xi - t < 2L, 0 < \xi + t < 2L\}$ and $\{(\xi,t) : 0 < t - \xi < 2L, 0 < \xi + t < 2L\}$, respectively. The kernels $K^{\pm}(\xi,t)$ can be obtained as unique solutions of Volterra integral equations of the second kind which are equivalent to the problems (A2–A5). Since $u_1(\xi)$ and $u_2(\xi)$ are real–valued, the components $K_{jk}^{\pm}(\xi,t), j,k = 1,2,$ are real–valued also, and

$$\Phi^{\pm}(\xi,-\omega) = \overline{\Phi^{\pm}(\xi,\omega)}.$$

Notice that $K^{\pm}(\xi,t)$ are closely related to the compact Green's functions $G^{c\pm}(x,t)$ used in the time domain reconstruction methods, see, e.g., [5]. These relations are particularly simple for an impedance–matched line ($q_0 = q_l = 0$). Comparing the expressions for the split components in the time and frequency domain gives

$$G^{c+}(x,t) = \exp\left\{-\int_x^l \alpha ds\right\} K_{11}^{+}(\xi(x), t+\xi(x)),$$
$$G^{c-}(x,t) = \exp\left\{\int_0^x \delta ds - \int_0^l \alpha ds\right\} K_{21}^{+}(\xi(x), t+\xi(x)).$$

In the mismatching case, G^{c+} relates to both K_{11}^{+} and K_{12}^{+}, whereas the expression for G^{c-} involves K_{21}^{+} and K_{22}^{+}.

Off–diagonal structure of $U(\xi)$ implies

$$\det \Phi^{\pm}(\xi,\omega) = \det S^{\Phi}(\omega) \equiv 1.$$

The elements of the scattering matrix $S^{\Phi} = \begin{pmatrix} a_1^{\Phi} & b_2^{\Phi} \\ b_1^{\Phi} & a_2^{\Phi} \end{pmatrix}$ can be expressed in terms of the kernels $K^{\pm}(\xi,t)$. From $S^{\Phi}(\omega) = \Phi^{+}(0,\omega)$ it follows that

$$a_1^\Phi(\omega) = 1 + \int_0^{2L} K_{11}^+(0,t)e^{-i\omega t}dt,$$
$$b_1^\Phi(\omega) = \int_0^{2L} K_{21}^+(0,t)e^{-i\omega t}dt, \tag{A6}$$

$$a_2^\Phi(\omega) = 1 + \int_0^{2L} K_{22}^+(0,t)e^{i\omega t}dt,$$
$$b_2^\Phi(\omega) = \int_0^{2L} K_{12}^+(0,t)e^{i\omega t}dt. \tag{A7}$$

The integral representations (A1), (A6), and (A7) imply that:

1) for a fixed ξ, $\Phi_{jk}^{\pm}(\xi,\omega)$ and $S_{ij}^\Phi(\omega), j,k = 1,2,$ are entire functions in the ω–complex plane;
2) as $|\omega| \to \infty$,
$a_1^\Phi(\omega) \to 1$, $b_1^\Phi(\omega) \to 0$, $\quad$ Im $\omega \le 0$,
$a_2^\Phi(\omega) \to 1$, $b_2^\Phi(\omega) \to 0$, $\quad$ Im $\omega \ge 0$;
$(\Phi_{(1)}^-,\ \Phi_{(2)}^+) \simeq e^{i\omega\xi J}$, $\quad$ Im $\omega \ge 0$,
$(\Phi_{(1)}^+,\ \Phi_{(2)}^-) \simeq e^{i\omega\xi J}$, $\quad$ Im $\omega \le 0$;
(here $\Phi_{(j)}$ denotes the j-th column of Φ).

Denote the contour M in the ω-plane by $M = \mathbf{R} \cup \{\omega : |\omega| = R\}$ where R is large enough such that all zeros of $a_1^\Phi(\omega)$ and $a_2^\Phi(\omega)$ are contained in the disk $\{\omega : |\omega| < R\}$. Define a 2×2 matrix function $G(\xi,\omega)$ holomorphic in $\mathbf{C} \setminus M$ (ξ being a parameter) as follows:

$$G(\xi,\omega) = \begin{cases} (\Phi_{(1)}^-,\ \Phi_{(2)}^+)(\xi,\omega)e^{-i\omega\xi J}, & \text{Im } \omega > 0, |\omega| > R, \\ (\Phi_{(1)}^+,\ \Phi_{(2)}^-)(\xi,\omega)\dfrac{1}{a_1(\omega)}e^{-i\omega\xi J}, & \text{Im } \omega < 0, |\omega| > R, \\ \Phi^-(\xi,\omega)e^{-i\omega\xi J}, & |\omega| < R. \end{cases} \tag{A8}$$

Since

$$\det G(\xi,\omega) = \begin{cases} a_2(\omega), & \text{Im } \omega > 0, |\omega| > R, \\ \dfrac{1}{a_1(\omega)}, & \text{Im } \omega < 0, |\omega| > R, \\ 1, & |\omega| < R, \end{cases}$$

the matrix $G(\xi,\omega)$ is nondegenerate for any ξ and ω. Besides, $G(\xi,\omega) \to I_2$ as $\omega \to \infty$.

The contour M is the boundary for the open set $\Omega^+ = \{\omega : \mathrm{Im}\,\omega > 0, |\omega| > R\} \ \cup \{\omega : \mathrm{Im}\,\omega < 0, |\omega| < R\}$, as well as for $\Omega^- = \mathbf{C} \setminus (M \cup \Omega^+)$. Let

$$G^\pm(\xi,\nu) = \lim_{\substack{\omega\to\nu \\ \omega\in\Omega^\pm}} G^\pm(\xi,\omega), \qquad \nu \in M.$$

Then the scattering relation (16) is transformed to

$$G^+(\xi,\nu) = C^-(\xi,\nu)\cdot g(\xi,\nu), \qquad \nu \in M, \tag{A9}$$

where $g(\xi,\nu) = e^{i\omega\xi J} g_0(\nu) e^{-i\omega\xi J}$,

$$g_0(\nu) = \begin{cases} \begin{pmatrix} 1 & b_2^\Phi \\ -b_1^\Phi & 1 \end{pmatrix}(\nu), & \nu \in (-\infty,-R)\cup(R,\infty), \\ \begin{pmatrix} 1 & 0 \\ -b_1^\Phi & a_1^\Phi \end{pmatrix}(\nu), & |\nu| = R,\ \mathrm{Im}\,\nu < 0, \\ \begin{pmatrix} 1 & b_2^\Phi \\ 0 & a_2^\Phi \end{pmatrix}(\nu), & |\nu| = R,\ \mathrm{Im}\,\nu > 0, \\ I_2, & \nu \in (-R,R). \end{cases}$$

In the inverse problem, $S^\Phi(\omega) = \begin{pmatrix} a_1^\Phi & b_2^\Phi \\ b_1^\Phi & a_2^\Phi \end{pmatrix}(\omega)$ is given on the real axis $\omega \in \mathbf{R}$ (notice that $S^\Phi(-\omega) = \overline{S^\Phi(\omega)}$ and $\det S^\Phi(\omega) = 1$). The conjugation matrix $g_0(\nu)$, $\nu \in M$, is determined uniquely by using the analytic continuation of the corresponding matrix elements of $S^\Phi(\omega)$.

Therefore, we arrive at the following problem: given $g(\xi,\nu)$, $\xi \in [0,L]$, $\nu \in M$, find a matrix function $G(\xi,\omega)$, which is piecewise holomorphic and nondegenerate in $\mathbf{C}\setminus M$, such that $G(\xi,\omega) \to I_2$ as $\omega\to\infty$, and its limiting values $G^\pm(\xi,\nu)$, $\nu \in M$, are related by $g(\xi,\nu)$ through (A9).

The solution of this problem is unique. Indeed, if there are two solutions, $G^{(1)}(\xi,\omega)$ and $G^{(2)}(\xi,\omega)$, then $G^{(1)}(\xi,\omega)\left[G^{(2)}(\xi,\omega)\right]^{-1}$ is continuous across M, tends to I_2 as $\omega\to\infty$, hence it is I_2 identically. Therefore, $G(\xi,\omega)$ (and, as a consequence, $u_1(\xi)$ and $u_2(\xi)$) is uniquely determined by the scattering matrix $S^\Phi(\omega)$, $\omega > 0$.

APPENDIX B. UNIQUENESS IN THE CASE OF HARD BACK REFLECTION

Let us consider the case when $q_l \neq 0$, $q_0 = 0$. The input data are taken to be $a_1(\omega)$ and $b_1(\omega)$. As it was shown above, the constants

L, q_l, $\int_0^l \alpha ds$, and $\int_0^l \delta ds$ are determined from the large ω–behavior of $a_1(\omega)$ and $b_1(\omega)$. From (8) and (17) we have

$$\begin{aligned} a_1(\omega) =& (1-q_l)\exp\left\{-\int_0^l \alpha ds + i\omega\Delta\right\} a_1^{\Phi}(\omega) \\ &+ q_l \exp\left\{-\int_0^l \delta ds - i\omega\Delta'\right\} b_2^{\Phi}(\omega), \\ b_1(\omega) =& (1-q_l)\exp\left\{-\int_0^l \alpha ds + i\omega\Delta\right\} b_1^{\Phi}(\omega) \\ &+ q_l \exp\left\{-\int_0^l \delta ds - i\omega\Delta'\right\} a_2^{\Phi}(\omega), \end{aligned} \tag{B1}$$

where $\Delta' = \Delta + \frac{2l}{c(l+0)}$.

Proposition 5 *Let two sets of the line parameters (with* $q_l \neq 0$, $q_0 = 0$*) related to* $U^{(1)}(\xi) = \begin{pmatrix} 0 & u_1^{(1)}(\xi) \\ u_2^{(1)}(\xi) & 0 \end{pmatrix}$ *and* $U^{(2)}(\xi) = \begin{pmatrix} 0 & u_1^{(2)}(\xi) \\ u_2^{(2)}(\xi) & 0 \end{pmatrix}$*, respectively, produce the same* $a_1(\omega)$ *and* $b_1(\omega)$.

Then $u_1^{(1)}(\xi) = u_1^{(2)}(\xi)$ *and* $u_2^{(1)}(\xi) = u_2^{(2)}(\xi)$.

The proof is based on using the transformation operator that relates the solutions $\Phi^{(k)-}(\xi,\omega)$, $k = 1,2$, of equation (12) with $U(\xi) = U^{(k)}(\xi)$, respectively:

$$\Phi^{(1)-}(\xi,\omega) = \Phi^{(2)-}(\xi,\omega) + \int_{-\xi}^{\xi} N(\xi,t)\Phi^{(2)-}(t,\omega)dt. \tag{B2}$$

Substituting (B2) into (12) implies that the kernel 2×2 matrix function $N(\xi,t)$ solves the Goursat problem for the hyperbolic system (cf. (A2, A3))

$$\begin{aligned} &\frac{\partial N(\xi,t)}{\partial \xi} + J\frac{\partial N(\xi,t)}{\partial t}J = -JN(\xi,t)JU^{(2)}(t) + U^{(1)}(\xi)N(\xi,t), \\ &\qquad -\xi < t < \xi,\ 0 < \xi < L, \\ &N(\xi,-\xi) + JN(\xi,-\xi)J = 0, \qquad 0 \le \xi \le L, \\ &N(\xi,\xi) - JN(\xi,\xi)J = U^{(1)}(\xi) - U^{(2)}(\xi), \qquad 0 \le \xi \le L, \end{aligned} \tag{B3}$$

or, in terms of the matrix elements,

$$\begin{aligned}
\frac{\partial N_{11}(\xi,t)}{\partial \xi} + \frac{\partial N_{11}(\xi,t)}{\partial t} &= u_2^{(2)}(t)N_{12}(\xi,t) + u_1^{(1)}(\xi)N_{21}(\xi,t),\\
\frac{\partial N_{21}(\xi,t)}{\partial \xi} - \frac{\partial N_{21}(\xi,t)}{\partial t} &= -u_2^{(2)}(t)N_{22}(\xi,t) + u_2^{(1)}(\xi)N_{11}(\xi,t),\\
\frac{\partial N_{22}(\xi,t)}{\partial \xi} + \frac{\partial N_{22}(\xi,t)}{\partial t} &= u_1^{(2)}(t)N_{21}(\xi,t) + u_2^{(1)}(\xi)N_{12}(\xi,t),\\
\frac{\partial N_{12}(\xi,t)}{\partial \xi} - \frac{\partial N_{12}(\xi,t)}{\partial t} &= -u_1^{(2)}(t)N_{11}(\xi,t) + u_1^{(1)}(\xi)N_{22}(\xi,t),
\end{aligned} \tag{B4}$$

in the domain $D = \{(\xi,t) : -\xi < t < \xi, 0 < \xi < L\}$, and

$$N_{11}(\xi,-\xi) = N_{22}(\xi,-\xi) = 0, \qquad 0 \le \xi \le L, \tag{B5}$$

$$\begin{aligned}
N_{12}(\xi,\xi) &= u_1^{(1)}(\xi) - u_1^{(2)}(\xi),\\
N_{21}(\xi,\xi) &= u_2^{(1)}(\xi) - u_2^{(2)}(\xi), \qquad 0 \le \xi \le L.
\end{aligned} \tag{B6}$$

Since $(S^\Phi)^{-1}(\omega) \equiv \begin{pmatrix} a_2^\Phi & -b_2^\Phi \\ -b_1^\Phi & a_1^\Phi \end{pmatrix}(\omega) = \exp\{-i\omega LJ\}\Phi^-(L,\omega)$, relations (B1) yield

$$(-1, \quad \rho e^{2i\omega L})e^{-i\omega LJ}\Phi^{(1)-}(L,\omega) = (-1, \quad \rho e^{2i\omega L})e^{-i\omega LJ}\Phi^{(2)-}(L,\omega), \tag{B7}$$

where $\rho = \frac{1-q_l}{q_l}\exp\left\{\int_0^l(\delta-\alpha)ds\right\}$.

Substituting (B7) into (B2) gives

$$\hat{h}\int_{-L}^{L} N(L,t)\Phi^{(2)-}(t,\omega)dt = 0, \tag{B8}$$

where $\hat{h} = (-1, \ \rho)$. Since $\Phi^{(2)-}(t,\omega)$ is also related to $\exp\{i\omega tJ\}$ by the transformation (A1), equation (B8) can be written, by a suitable changes of variables, as

$$\begin{aligned}
0 &= \hat{h}\left\{\int_{-L}^{0} N(L,t)e^{i\omega tJ}dt + \int_0^L N(L,t)e^{i\omega tJ}dt + \int_0^L N(L,t)\right.\\
&\qquad \left.\int_{-t}^{t} K^-(t,s)e^{i\omega sJ}dsdt\right\}\\
&= \hat{h}\left\{\int_{-L}^{0}\left[N(L,t) + \int_{-t}^{L} N(L,s)K^-(s,t)ds\right]e^{i\omega tJ}dt\right.\\
&\qquad \left. + \int_0^L\left[N(L,t) + \int_t^L N(L,s)K^-(s,t)ds\right]e^{i\omega tJ}dt\right\}.
\end{aligned} \tag{B9}$$

Since (B9) holds for any $\omega \in \mathbf{R}$, one gets

$$\hat{h}\left\{N(L,t)+\int_{-t}^{L}N(L,s)K^{-}(s,t)ds\right\}=0, \qquad -L\le t\le 0, \quad \text{(B10)}$$

$$\hat{h}\left\{N(L,t)+\int_{t}^{L}N(L,s)K^{-}(s,t)ds\right\}=0, \qquad 0\le t\le L. \quad \text{(B11)}$$

The integral equation (B11) is the uniform Volterra equation of the second kind (with respect to $\hat{h}N(L,t)$, $t\in[0,L]$), the only solution of which is the trivial one:

$$\hat{h}N(L,t)=0, \qquad 0\le t\le L. \quad \text{(B12)}$$

Then, (B10) gives

$$\hat{h}N(L,t)=0, \qquad -L\le t\le 0. \quad \text{(B13)}$$

In terms of the matrix elements of $N(L,t)$, relations (B12) and (B13) read

$$\begin{aligned} N_{12}(L,t)&=\rho N_{22}(L,t),\\ N_{21}(L,t)&=\frac{1}{\rho}N_{11}(L,t). \end{aligned} \quad \text{(B14)}$$

Now consider the boundary value problem (B4) + (B5) + (B14) for $N(\xi,t)$ in D. If we show that the solution of this problem vanishes identically, then, in virtue of (B6), $u_1^{(1)}(\xi)-u_1^{(2)}(\xi)=N_{12}(\xi,\xi)=0$ and $u_2^{(1)}(\xi)-u_2^{(2)}(\xi)=N_{21}(\xi,\xi)=0$.

In the characteristic variables

$$\mu=\xi+t, \quad \eta=\xi-t, \qquad \tilde{N}(\mu,\eta)=N\left(\frac{\mu+\eta}{2},\frac{\mu-\eta}{2}\right),$$

the problem (B4) + (B5) + (B14) becomes

$$\begin{aligned}
\frac{\partial\tilde{N}_{11}(\mu,\eta)}{\partial\mu}&=\frac{1}{2}u_2^{(2)}\left(\frac{\mu-\eta}{2}\right)\tilde{N}_{12}(\mu,\eta)+\frac{1}{2}u_1^{(1)}\left(\frac{\mu+\eta}{2}\right)\tilde{N}_{21}(\mu,\eta),\\
\frac{\partial\tilde{N}_{21}(\mu,\eta)}{\partial\eta}&=-\frac{1}{2}u_2^{(2)}\left(\frac{\mu-\eta}{2}\right)\tilde{N}_{22}(\mu,\eta)+\frac{1}{2}u_2^{(1)}\left(\frac{\mu+\eta}{2}\right)\tilde{N}_{11}(\mu,\eta),\\
\frac{\partial\tilde{N}_{22}(\mu,\eta)}{\partial\mu}&=\frac{1}{2}u_1^{(2)}\left(\frac{\mu-\eta}{2}\right)\tilde{N}_{21}(\mu,\eta)+\frac{1}{2}u_2^{(1)}\left(\frac{\mu+\eta}{2}\right)\tilde{N}_{12}(\mu,\eta),\\
\frac{\partial\tilde{N}_{12}(\mu,\eta)}{\partial\eta}&=-\frac{1}{2}u_1^{(2)}\left(\frac{\mu-\eta}{2}\right)\tilde{N}_{11}(\mu,\eta)+\frac{1}{2}u_1^{(1)}\left(\frac{\mu+\eta}{2}\right)\tilde{N}_{22}(\mu,\eta)
\end{aligned} \quad \text{(B15)}$$

in the domain $\tilde{D} = \{(\mu, \eta) : \mu > 0, \eta > 0, \mu + \eta < 2L\}$, with the boundary conditions

$$\begin{aligned} &\tilde{N}_{11}(0,\eta) = \tilde{N}_{22}(0,\eta) = 0, \qquad 0 \le \eta \le 2L, \\ &\tilde{N}_{12}(\mu, 2L-\mu) = \rho \tilde{N}_{22}(\mu, 2L-\mu), \qquad 0 \le \mu \le 2L, \\ &\tilde{N}_{21}(\mu, 2L-\mu) = \frac{1}{\rho}\tilde{N}_{11}(\mu, 2L-\mu), \qquad 0 \le \mu \le 2L. \end{aligned} \tag{B16}$$

Integrating the equations in (B15) with respect to the corresponding variables, μ or η, using the boundary conditions (B16), yields

$$\begin{aligned} \tilde{N}_{11}(\mu,\eta) &= \frac{1}{2}\int_0^\mu \left[u_2^{(2)}\left(\frac{s-\eta}{2}\right)\tilde{N}_{12}(s,\eta) \right. \\ &\qquad \left. + u_1^{(1)}\left(\frac{s+\eta}{2}\right)\tilde{N}_{21}(s,\eta)\right] ds, \\ \tilde{N}_{21}(\mu,\eta) &= \frac{1}{2}\int_\eta^{2L-\mu} \left[u_2^{(2)}\left(\frac{\mu-\tau}{2}\right)\tilde{N}_{22}(\mu,\tau) \right. \\ &\qquad \left. - u_2^{(1)}\left(\frac{\mu-\tau}{2}\right)\tilde{N}_{11}(\mu,\tau)\right] d\tau + \tilde{N}_{21}(\mu, 2L-\mu) \\ &= \frac{1}{2}\int_\eta^{2L-\mu} \left[u_2^{(2)}\left(\frac{\mu-\tau}{2}\right)\tilde{N}_{22}(\mu,\tau) \right. \\ &\qquad \left. - u_2^{(1)}\left(\frac{\mu-\tau}{2}\right)\tilde{N}_{11}(\mu,\tau)\right] d\tau \\ &\qquad + \frac{1}{2\rho}\int_0^\mu \left[u_2^{(2)}\left(\frac{s-2L+\mu}{2}\right)\tilde{N}_{12}(s, 2L-\mu) \right. \\ &\qquad \left. + u_1^{(1)}\left(\frac{s+2L-\mu}{2}\right)\tilde{N}_{21}(s, 2L-\mu)\right] ds, \\ \tilde{N}_{22}(\mu,\eta) &= \frac{1}{2}\int_0^\mu \left[u_1^{(2)}\left(\frac{s-\eta}{2}\right)\tilde{N}_{21}(s,\eta) \right. \\ &\qquad \left. + u_2^{(1)}\left(\frac{s+\eta}{2}\right)\tilde{N}_{12}(s,\eta)\right] ds, \\ \tilde{N}_{12}(\mu,\eta) &= \frac{1}{2}\int_\eta^{2L-\mu} \left[u_1^{(2)}\left(\frac{\mu-\tau}{2}\right)\tilde{N}_{11}(\mu,\tau) \right. \\ &\qquad \left. - u_1^{(1)}\left(\frac{\mu-\tau}{2}\right)\tilde{N}_{22}(\mu,\tau)\right] d\tau \\ &\qquad + \frac{\rho}{2}\int_0^\mu \left[u_1^{(2)}\left(\frac{s-2L+\mu}{2}\right)\tilde{N}_{21}(s, 2L-\mu) \right. \end{aligned}$$

$$+u_2^{(1)}\left(\frac{s+2L-\mu}{2}\right)\tilde{N}_{12}(s,2L-\mu)\Big]\,ds. \tag{B17}$$

The system of the Volterra equations (B17) can be written as

$$\mathbf{N}(\mu,\eta)=\mathcal{A}\mathbf{N}(\mu,\eta), \tag{B18}$$

where $\mathbf{N}=(\tilde{N}_{11},\tilde{N}_{21},\tilde{N}_{12},\tilde{N}_{22})^{\top}$, and $\mathcal{A}$ is a 4×4 matrix integral operator.

Define the norm $\|\mathbf{N}\|=\sup_{\substack{(\mu,\eta)\in\tilde{D}\\ 1\le j,k\le 2}}|\tilde{N}_{jk}(\mu,\eta)|$. By induction, it is seen that

$$|(\mathcal{A}^n\mathbf{N})_j(\mu,\eta)|\le\|\mathbf{N}\|C^n\frac{(\mu+2L-\eta)^n}{n!},$$

for all $(\mu,\eta)\in\tilde{D}$, $1\le j\le 4$, and $n\in\mathbf{N}$, where

$$C=\max\left\{C_1,\rho C_1,\frac{C_1}{\rho}\right\},\qquad C_1=\sup_{\substack{0\le\xi\le L\\ 1\le j,k\le 2}}|u_j^{(k)}(\xi)|;$$

thus, $\|\mathcal{A}^n\mathbf{N}\|\le\|\mathbf{N}\|\frac{(2LC)^n}{n!}$. Since $\|\mathcal{A}^n\|\to 0$ as $n\to\infty$, the system of equations (B18) has only the trivial solution.

The case $q_0\neq 0$ is reduced to the case considered above by using $P_0^{-1}\begin{pmatrix}a_1(\omega)\\ b_1(\omega)\end{pmatrix}$ instead of $\begin{pmatrix}a_1(\omega)\\ b_1(\omega)\end{pmatrix}$.

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