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FRACTIONAL SOLUTIONS FOR THE HELMHOLTZ'S EQUATION IN A MULTILAYERED GEOMETRY

Q. A. Naqvi and A. A. Rizvi

Communications Lab.,
Department of Electronics,
Quaid-i-Azam University,
Islamabad, Pakistan

1. INTRODUCTION

It is well known that solutions for the Helmholtz's equation are called one or two dimensional Green's functions when the source terms are one or two dimensional Dirac delta functions respectively. One dimensional Dirac delta function represents an infinite plate source while two dimensional Dirac delta function represents an infinite line source.

Engheta [1–2] has recently discussed the role of fractional calculus in electromagnetics. He has shown that in a homogeneous space one and two dimensional Green's functions are related via a fractional order integral operator. Relation between one and two dimensional Green's function in homogeneous space via fractional order integral operator D is given by the following expression

$$G_2(x=0,y) \approx \frac{1}{2\sqrt{\pi}} {}_0D_{k^2}^{-1/2} G_1. \tag{1}$$

The order of the operator is $-1/2$. The operational variable of the operator is k^2, where k is the propagation constant of the medium. Subscript i in G_i, $i = 1, 2$ represents the dimensions of the Green's function. It may be noted that expression (1) is valid for point of observation in the far-zone. The definition of the fractional order integral of a function $f(x)$, which is know as Riemann-Liouville definition [3], is written as

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-u)^{-\alpha-1} f(u)du \qquad \text{for } \alpha < 0, \text{ and } x > a $$

where $\Gamma(\cdot)$ is the Gamma function, α is the order of the operator and x is the variable of integration. Relation (1) shows that integral operator D with order $-1/2$ when operated on one dimensional Green's function yields two dimensional Green's function. However, integral operator D with order zero, i.e., no operation, yields one dimensional Green's function. It is shown in [1–2] that integral operator D with order α between zero and $-1/2$ yields solutions for the Helmholtz's equation that are an intermediate or fractional step between the two integer dimensional Green's functions. For fractional or intermediate solutions, expression (1) may be written in an appropriate form as

$$ G_f(x=0, y) \approx \frac{1}{2\sqrt{\pi}} {}_0D_{k^2}^{(1-f)/2} G_1, \qquad 1 < f < 2. \tag{2} $$

The fractional solutions for the Helmholtz's equation in homogeneous space can be obtained by substituting

$$ G_1 = \frac{\exp(iky)}{k}, \qquad y > 0 $$

in (2). The fractional solution G_f for the Helmholtz's equation in homogeneous space is [1]

$$ G_f(x, y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 - k_x^2}} \frac{\exp(ik_x x + i\sqrt{k^2 - k_x^2}\, y)}{(ik_x)^{2-f}} dk_x, \quad y > 0. \tag{3} $$

It is important to note that although expressions (1) and (2) yield results that are for the axis of symmetry with points of observations in the far-zone but these results can be very easily converted to a form which is valid for the whole region.

The analysis presented in [1–2] is for a homogeneous space, i.e., a geometry that is unbounded. The question arises whether it is possible to relate G_1 and G_2 via fractional order operator for a geometry which contains parallel plane interfaces. What is the nature of the operator and what are contributions of the new geometry? If such an operator exists, then what are the fractional solutions for the Helmholtz's equation? In order to answer these questions, different geometries with parallel plane interfaces will be considered. Efforts will be made to relate one and two dimensional Green's function via fractional order operator. General form of the fractional order operator which is valid for arbitrary geometry with parallel plane interfaces will be derived. The interfaces are assumed to be infinite in extent. Throughout the discussion it is assumed that each medium is lossless, homogeneous and isotropic and the point of observation lies in an unbounded region.

2. SOURCES

Two radiating sources carrying time harmonic current are considered one by one in different geometries. One of the two sources is a one dimensional current source $\vec{J}_1 = -i\omega\mu\delta(y)\hat{e}_z$, i.e., an infinite plate source, and is located at $y = 0$. The second source is a two dimensional source $\vec{J}_2 = -i\omega\mu\delta(x)\delta(y)\hat{e}_z$, i.e., an infinite line source, and is located at $x = y = 0$. The field radiated from line source is termed as two dimensional Green's function while from plate source is termed as one dimensional Green's function. The time-harmonic factor $\exp(-i\omega t)$ has been suppressed throughout the discussion.

3. GEOMETRY CONTAINING A DIELECTRIC INTERFACE

Consider a geometry shown in Fig. 1a. This geometry contains an infinite dielectric interface at $y = h$. The propagation constant of the medium above the dielectric interface is k_1 and it is called medium 1. The propagation constant of the medium below the dielectric interface is k_2 and it is called medium 2. Space is divided into three regions. Region I is above the dielectric interface, i.e., $y > b$ while region II is limited between $0 < y < b$. Region III is unbounded, i.e., $y < 0$.

Consider the situation shown in Fig. lb. An infinite plate source is buried in a dielectric interface geometry. Radiated field $G_1(x, y)$ from the plate source in region I is [4]

$$G_1 = \frac{-\omega\mu}{k_1 + k_2} \exp\{ik_2 b + ik_1(y - b)\}, \quad y > b.$$

Now consider the situation shown in Fig. 1c. In this geometry, an infinite line source is buried in a dielectric interface geometry. Radiated field $G_2(x, y)$ by the buried line source in region I along the axis of symmetry $(x = 0)$ is given below [4]

$$G_2(x = 0, y) = \frac{-\omega\mu}{2\pi} \int_{-\infty}^{\infty} A(k_{1x}) \exp\left(i\sqrt{k_1^2 - k_{1x}^2}\, y\right) dk_{1x}, \quad y > b$$

$$A(k_{1x}) = \frac{\exp\left(-i\sqrt{k_1^2 - k_{1x}^2}\, b + i\sqrt{k_2^2 - k_{1x}^2}\, b\right)}{\sqrt{k_1^2 - k_{1x}^2} + \sqrt{k_2^2 - k_{1x}^2}}.$$

Above radiated field expression is an even function of k_{1x}, therefore

$$G_2(x = 0, y) = -\frac{\omega\mu}{\pi} \int_{0}^{\infty} A(k_{1x}) \exp\left(i\sqrt{k_1^2 - k_{1x}^2}\, y\right) dk_{1x}.$$

Range of the integration is divided into two sub-ranges as

$$\begin{aligned} G_2 = &-\frac{\omega\mu}{\pi} \int_{0}^{k_1} A(k_{1x}) \exp\left(i\sqrt{k_1^2 - k_{1x}^2}\, y\right) dk_{1x} \\ &-\frac{\omega\mu}{\pi} \int_{k_1}^{\infty} A(k_{1x}) \exp\left(i\sqrt{k_{1x}^2 - k_1^2}\, y\right) dk_{1x}. \end{aligned}$$

In the far-zone, i.e., for large y, contribution of the second integral on the left hand side of above expression is negligibly small [4]. Under this approximation above expression reduces to

$$G_2 \approx \frac{-\omega\mu}{\pi} \int_{0}^{k_1} A(k_{1x}) \exp(i\sqrt{k_1^2 - k_{1x}^2}\, y) dk_{1x}.$$

Using change of variable $u = \sqrt{k_1^2 - k_{1x}^2}$ in the above equation yields the following

$$G_2 \approx \frac{-\omega\mu}{\pi} \int_{0}^{k_1} A(u) \frac{1}{\sqrt{k_1^2 - u^2}} \exp\{iu(y - b)\} du$$

$$A(u) = \frac{u \exp\{ib\sqrt{k_2^2 - k_1^2 + u^2}\}}{u + \sqrt{k_2^2 - k_1^2 + u^2}}.$$

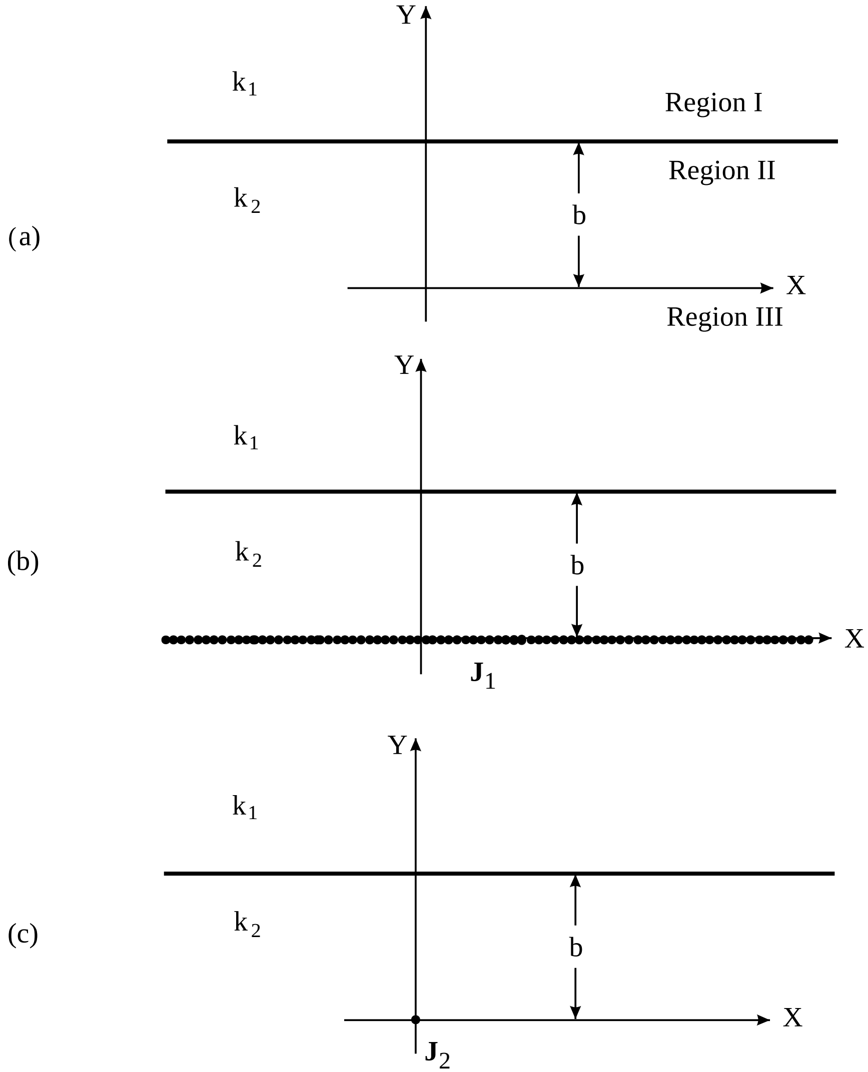

Figure 1. (a) Geometry containing a dielectric interface. (b) An infinite plate source is located at $\delta(y)$ in a dielectric interface geometry. (c) An infinite line source is located at $\delta(x)\delta(y)$ in a dielectric interface geometry.

One more change of variable $k_2^2+u^2=w$ in the above equation yields the following

$$G_2 \approx \frac{-\omega\mu}{\pi}\int_{k_2^2}^{k_1^2+k_2^2}\frac{1}{\sqrt{k_1^2+k_2^2-w}}A(w)\exp\left\{i\sqrt{w-k_2^2}(y-b)\right\}dw$$

$$A(w)=\frac{\exp\left\{ib(\sqrt{w-k_1^2})\right\}}{2(\sqrt{w-k_2^2}+\sqrt{w-k_1^2})}.$$

It is obvious from above expression that two dimensional Green's function can be expressed in terms of fractional order integral of G_1 as

$$\begin{aligned} G_2 &\approx \frac{1}{2\sqrt{\pi}}{}_{k_2^2}D^{-1/2}_{k_1^2+k_2^2}\frac{-\omega\mu}{k_1+k_2}\exp\{ik_2b+ik_1(y-b)\} \\ &= \frac{1}{2\sqrt{\pi}}{}_{k_2^2}D^{-1/2}_{k_1^2+k_2^2}\ G_1, \quad y>b \end{aligned} \tag{4}$$

where operator D represents fractional order integration [3] and the order of the fractional operator is $-1/2$.

The field radiated from the buried plate source in region $y<0$ and along the axis of symmetry is

$$\begin{aligned} G_1(x=0,y) &= \frac{-\omega\mu}{2k_2}\exp(-ik_2y) \\ &\quad + \frac{\omega\mu}{2k_2}\left(\frac{k_1-k_2}{k_1+k_2}\right)\exp\{-ik_2(y-2b)\}, \quad y<0. \end{aligned}$$

Field radiated in region III along the axis of symmetry by the line source is [4]

$$\begin{aligned} G_2(x=0,y) &= \frac{-\omega\mu}{4}H_0^{(1)}(k_2y) \\ &\quad + \frac{\omega\mu}{4\pi}\int_{-\infty}^{\infty}B(k_{2x})\exp\{-ik_{2y}(y-2b)\}dk_{1x}, \quad y<0 \\ B(k_{2x}) &= \frac{\sqrt{k_1^2-k_{2x}^2}-k_{2y}}{\sqrt{k_1^2-k_{2x}^2}+k_{2y}}\frac{1}{k_{2y}} \\ k_{2y} &= \sqrt{k_2^2-k_{2x}^2} \end{aligned}$$

where $H_0^{(1)}$ is the Hankel function of first kind and order zero. Adopting the procedure used in deriving (4) it can be shown that the fractional order relation between G_1 and G_2 is

$$G_2(x=0,y)\approx\frac{1}{2\sqrt{\pi}}{}_{k_1^2}D^{-1/2}_{k_1^2+k_2^2}\ G_1, \quad y<0. \tag{5}$$

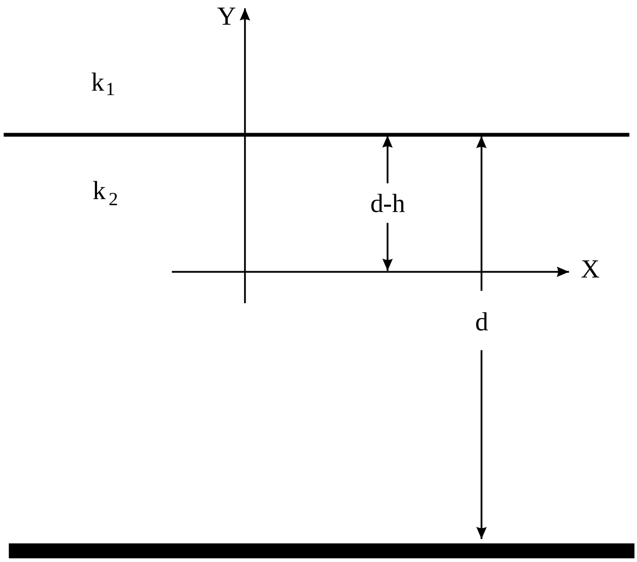

Figure 2. Geometry containing a grounded dielectric layer.

It is concluded from the above discussion that one dimensional and two dimensional Green's functions in dielectric interface geometry are also related via fractional order operator. The order of the operator is same as in the case of homogeneous space while the lower limit and variable of operation are different.

4. GROUNDED DIELECTRIC LAYER GEOMETRY

Consider a geometry shown in Fig. 2. In this geometry, a dielectric layer is placed on a perfectly conducting sheet. Thickness of the dielectric layer is d. Height of the sheet from the co-ordinate axis is h. Expression for fields radiated in region I due to a plate source embedded in the grounded dielectric layer is given below [4]

$$G_1(x,y) = i\omega\mu \frac{\sin(k_2 h)\exp\{-ik_1(d-h)\}}{k_2\cos(k_2 d) - ik_1\sin(k_2 d)} \exp(ik_1 y).$$

Expression for fields radiated in region I due to a line source embedded

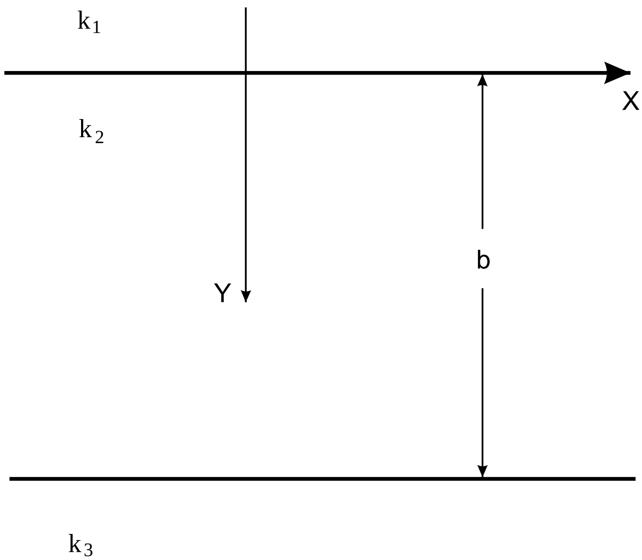

Figure 3. Geometry containing two dielectric interfaces.

in the grounded dielectric layer is given below [4–5]

$$G_2(x=0,y) = \frac{i\omega\mu}{2\pi}\int_{-\infty}^{\infty} \frac{\sin\left\{\sqrt{k_2^2-k_{1x}^2}h\right\}\exp\{ik_{1y}(y-d+h)\}}{\sqrt{k_2^2-k_{1x}^2}\cos\sqrt{k_2^2-k_{1x}^2}d - ik_{1y}\sin\sqrt{k_2^2-k_{1x}^2}d}\,dk_{1x}$$

where $k_{1y} = \sqrt{k_1^2 - k_{1x}^2}$. Using a similar process as in the previous section the following relation can be derived

$$G_2(x=0,y) \approx \frac{1}{2\sqrt{\pi}}{}_{k_2^2}D_{k_1^2+k_2^2}^{-1/2}\ G_1, \quad y > d-h. \tag{6}$$

It is observed that relationship between G_1 and G_2 in this geometry is exactly the same as the relation given by (4) in dielectric interface geometry. This means that perfectly conducting interface has not affected the operator, variable of operation and lower limit.

5. GEOMETRY CONTAINING TWO DIELECTRIC INTERFACES

Consider the third geometry shown in Fig. 3. This geometry contains two dielectric interfaces. The propagation constants of different dielectric media has been mentioned in the figure. Expression for fields radiated in region for $y < 0$ due to a plate source embedded in this geometry is given below [6]

$$G_1 = A\exp(-ik_1 y), \quad y < 0$$
$$A = \frac{\omega\mu\{(k_2 - k_3)\exp(ik_2 b) + (k_2 + k_3)\exp(-ik_2 b)\}}{(k_2 - k_1)(k_2 - k_3)\exp(ik_2 b) - (k_2 + k_1)(k_2 + k_3)\exp(-ik_2 b)}.$$

Expression for fields radiated in region I due to a line source embedded in this geometry is given below [6]

$$G_2(x = 0, y) = \int_{-\infty}^{\infty} A(k_{1x})\exp(-i\gamma_1 y)dk_{1x}, \quad y < 0$$
$$A(k_{1x}) =$$
$$\frac{\omega\mu}{2\pi}\frac{\{(\gamma_2 - \gamma_3)\exp(i\gamma_2 b) + (\gamma_2 + \gamma_3)\exp(-i\gamma_2 b)\}}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)\exp(i\gamma_2 b) - (\gamma_2 + \gamma_1)(\gamma_2 + \gamma_3)\exp(-i\gamma_2 b)}$$
$$\gamma_m = \sqrt{k_m^2 - k_{1x}^2}, \quad m = 1, 2, 3.$$

Using the similar procedure as in previous sections, it can be shown that the relation between Green's functions in the region $y < 0$ has two possible forms

$$G_2(x = 0, y) \approx \frac{1}{2\sqrt{\pi}}{}_{k_2^2}D_{k_1^2+k_2^2}^{-1/2}\ G_1 \qquad y < 0. \tag{7}$$

$$G_2(x = 0, y) \approx \frac{1}{2\sqrt{\pi}}{}_{k_3^2}D_{k_1^2+k_3^2}^{-1/2}\ G_1 \qquad y < 0. \tag{8}$$

It is obvious from (7) and (8) that in a given geometry, for same region of observation, there exists the possibility of different operational variables while the nature and order of the operator remains the same. It is also observed from (4), (6) and (7) that for different geometries there can exist a fractional order operator with same lower limit, order and operational variable. A common factor in all these cases is that

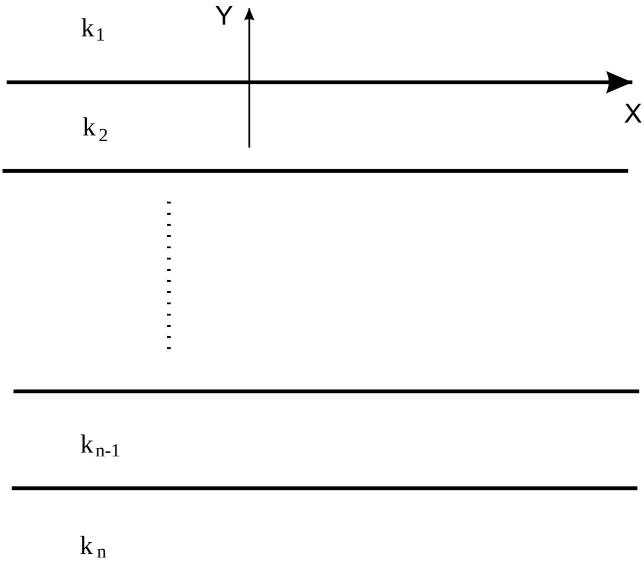

Figure 4. Geometry containing n dielectric interfaces.

region of observation has the same propagation constant, i.e., k_1 and also contains a region with propagation constant k_2 .

6. MULTILAYERED GEOMETRY

Consider a line source is placed in a geometry with parallel plane interfaces. The geometry is shown in Fig. 4. The field radiated from the line source, in an unbounded region along the axis of symmetry, can be written in form of spectrum as

$$G_2(x=0,y)=\int_{-\infty}^{\infty} A(\gamma_1,\gamma_2\gamma_3,\cdots,\gamma_n)\exp(i\gamma_1 y)dk_{1x}, \quad y>0 \quad (9)$$

where $\gamma_m=\sqrt{k_m^2-k_{1x}^2}$, $m=1,2\cdots,n$ and A is a spectrum function which can be evaluated using the boundary conditions of a given multilayered geometry. The propagation constant of the medium in which point of observation lies is k_1 . Above expression is an even function of k_{1x} . Therefore

$$G_2=2\int_0^{\infty} A(\gamma_1,\gamma_2,\gamma_3,\cdots,\gamma_n)\exp(i\gamma_1 y)dk_{1x}$$

Dividing range of integration into two subranges as

$$G_2 = 2\int_0^{k_1} A(\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_n) \exp(i\gamma_1 y) dk_{1x} + 2\int_{k_1}^{\infty} A(\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_n) \exp(i\gamma_1 y) dk_{1x}.$$

The second term in the above expression may be neglected for the points, i.e., when $k_1 y \gg 1$. This approximation reduces the above expression to following

$$G_2 \approx 2 \int_0^{k_1} A(\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_n) \exp(i\gamma_1 y) dk_{1x}$$

Using the substitution $u = \sqrt{k_1^2 - k_{1x}^2}$, following is obtained

$$G_2 \approx 2 \int_0^{k_1} A(\Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_n) \frac{u}{\sqrt{k_1^2 - u^2}} \exp(iuy) du$$

where $\Gamma_m = \sqrt{k_m^2 - k_1^2 + u^2}$, $m = 1, 2 \cdots, n$. Change of variable $k_j^2 + u^2 = w$, $j \neq 1$ in the above expression takes the following form

$$G_2 \approx \int_{k_j^2}^{k_j^2 + k_1^2} A(\xi_1, \xi_2, \xi_3, \cdots, \xi_n) \frac{1}{\sqrt{k_j^2 + k_1^2 - w}} \exp(i\sqrt{w - k_j^2} y) dw$$

where $\xi_m = \sqrt{k_m^2 - k_1^2 + w - k_j^2}$, $m = 1, 2 \cdots, n$. Comparing the above expression with the definition of fractional integral of a function, one can easily write the above expression in terms of fractional integral operator as

$$G_2(x = 0, y) \approx {}_{k_j^2}D_{k_j^2 + k_1^2}^{-1/2} \sqrt{\pi} A(k_1, k_2, k_3, \cdots, k_n) \exp(ik_1 y). \tag{10}$$

The corresponding plate source solution can be obtained by integrating the line source solution (9) from $-\infty$ to ∞ along x-axis and using the relation

$$2\pi\delta(p) = \int_{-\infty}^{\infty} \exp(ipx) dx.$$

After this process following result will be obtained

$$G_1 = 2\pi A(k_1, k_2, k_3, \cdots, k_n) \exp(ik_1 y).$$

This means that expression (9) can be written as

$$G_2(x=0,y) \approx \frac{1}{2\sqrt{\pi}} {}_{k_j^2}D_{k_j^2+k_1^2}^{-1/2} G_1. \tag{11}$$

It is concluded that regardless of the number of interfaces in a geometry one and two dimensional Green's functions along the axis of symmetry can be related via fractional order integral operator. The order of the operator is independent of the geometry while operational variable and lower limit are functions of the geometry. It is also concluded that operational variable is always sum of squares of two propagation constants. One of these corresponds to the propagation constant of the medium in which point of observation lies and the second propagation constant can be selected randomly from any medium other than the medium in which point of observation lies and square of same propagation constant will be the lower limit.

7. FRACTIONAL SOLUTIONS

It is noted from the above discussion that in a parallel plane interface geometry integral operator D with order $\alpha = -1/2$ when operated on G_1 yields G_2 while with $\alpha = 0$ yields G_1 itself. Question arises that if the order α of the operator varies between zero and $-1/2$, what will be the resulting expression for radiated fields? What will be the corresponding source distribution that yields these radiated fields? Is it possible to propose some solutions that can be regarded as an intermediate or fractional step between one and two dimensional Green's functions as are proposed for case of homogeneous space [1–2]? Last question can be restated as; Would the fractional solutions in a parallel plane interface geometry also contain both plane and cylindrical waves as for the case of homogeneous space geometry?

For this purpose a geometry with one dielectric interface is selected for discussion. Efforts will be made to generalize the results for other geometries with parallel plane interfaces. A relation between G_1 and G_2 in this geometry is given by the expression (4). Variable $f = (1-2\alpha)$ is introduced such that, when the order α varies between zero and $-1/2$ the variable f sweeps a range between one and two. Introduction of variable f modifies expression (4) to following form

$$G_f(x=0,y) \approx \frac{1}{2\sqrt{\pi}} {}_{k_2^2}D_{k_1^2+k_2^2}^{(1-f)/2}\, G_1, \quad y > b \quad \text{and} \quad 1 < f < 2.$$

Substituting the value of G_1 in the above expression yields following expression for $G_f(x,y)$ in a spectrum form

$$G_f(x,y) = \frac{-\omega\mu}{2\pi} \int_{-\infty}^{\infty} A(k_{1x}) \frac{1}{(ik_{1x})^{2-f}} \exp(ik_{1x}x + i\sqrt{k_1^2 - k_{1x}^2}y) dk_{1x} \tag{12}$$

$$A(k_{1x}) = \frac{\exp\left(-i\sqrt{k_1^2 - k_{1x}^2}b + i\sqrt{k_2^2 - k_{1x}^2}b\right)}{\sqrt{k_1^2 - k_{1x}^2} + \sqrt{k_2^2 - k_{1x}^2}}.$$

It is desired to calculate corresponding source distribution that gives above solution for the Helmholtz's equation in a dielectric interface geometry. The Helmholtz's equation for a line source J_2

$$J_2 = -i\omega\mu\delta(x)\delta(y) = \frac{-i\omega\mu\delta(y)}{2\pi} \int_{-\infty}^{\infty} \exp(ik_{2x}x) dk_{2x}$$

placed in the medium 2 of the dielectric interface geometry yields following solution in region I of the geometry [4]

$$G_2(x,y) = \frac{-\omega\mu}{2\pi} \int_{-\infty}^{\infty} A(k_{1x}) \exp(ik_{1x}x + i\sqrt{k_1^2 - k_{1x}^2}y) dk_{1x} \tag{13}$$

$$A(k_{1x}) = \frac{\exp\left\{-i\sqrt{k_1^2 - k_{1x}^2}b + i\sqrt{k_2^2 - k_{1x}^2}b\right\}}{(\sqrt{k_1^2 - k_{1x}^2} + \sqrt{k_2^2 - k_{1x}^2})}.$$

This means that for each term $\frac{-i\omega\mu\delta(y)}{2\pi}\exp(ik_{2x}x)$ as a source there corresponds a solution

$$\frac{-\omega\mu}{2\pi} A(k_{1x}) \exp\left(ik_{1x}x + i\sqrt{k_1^2 - k_{1x}^2}y\right).$$

Using the above argument one can write the source distribution that radiates the fields given in (12) and is given as

$$J_f(x,y) = \frac{-i\omega\mu\delta(y)}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(ik_{2x}x)}{(ik_{2x})^{2-f}} dk_{2x} \quad \text{for} \quad 1 < f < 2. \tag{14}$$

Comparison of (12) and (13) shows that solutions for the sources J_f in a dielectric interface geometry is approximately the same as corresponding case for a line source J_2. Sources J_f have introduced a factor in the form of a branch point at $k_{1x} = 0$.

From the above observation following general expression for radiated fields is proposed for a geometry with parallel plane interfaces

$$G_f(x,y) = \frac{-\omega\mu}{2\pi}\int_{-\infty}^{\infty}\zeta(k_{ix})\frac{1}{(ik_{ix})^{(2-f)}}\exp(ik_{ix}x + ik_{iy}y)dk_{ix}, \quad y > 0 \tag{15a}$$

where $\zeta(k_{ix})$ is corresponding spectrum function for the case of a line source J_2 and $k_{iy} = \sqrt{k_i^2 - k_{ix}^2}$ and k_i is propagation constant of an unbounded region in which point of observation lies. General expression for the corresponding source distribution is

$$J_f(x,y) = \frac{-i\omega\mu\delta(y)}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(ik_{jx}x)}{(ik_{jx})^{2-f}}dk_{jx} \tag{15b}$$

$k_{jy} = \sqrt{k_j^2 - k_{jx}^2}$ and k_j is propagation constant of a medium that contains the source.

Source distribution J_f can be expressed in terms of fractional order integral of two dimensional Dirac delta function $-i\omega\mu\delta(x)\delta(y)$ as [1–2]

$$J_f(x,y) = {}_{-\infty}D_x^{f-2}[-i\omega\mu\delta(x)\delta(y)] = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{-i\omega\mu\delta(y)x^{1-f}}{\Gamma(2-f)} & \text{for } x > 0 \end{cases} \quad \text{for} \quad 1 < f < 2. \tag{16}$$

It is obvious that above expression yields the two dimensional Dirac delta function $-i\omega\mu\delta(x)\delta(y)$ for the limit of $f = 2$, i.e., a source of two dimensional Green's function. While yields function $-i\omega\mu\delta(y)U(x)$ for the limit of $f = 1$, which is not a source of one dimensional Green's function. $U(x)$ is the unit step function in x. Even part ${}_eJ_f(x,y)$ of the source distribution $J_f(x,y)$ is considered so that (16) can yield the desired source distribution, i.e., one and two dimensional Dirac delta functions for the limiting cases for $f = 1$ and $f = 2$ respectively. Even part of the source distribution is given by [1]

$${}_eJ_f(x,y) = \frac{-i\omega\mu\delta(y)|x|^{1-f}}{2\Gamma(2-f)} \quad \text{for} \quad 1 < f < 2.$$

Above source distribution can be regarded as an intermediate step between two integer dimensional Dirac delta functions.

For the above symmetric source distribution, which can be considered as an intermediate step between two integer dimensional Dirac delta functions, solution to the Helmholtz's equation for medium above the dielectric interface can be written as

$$\begin{aligned} {}_eG_f(x,y) &= -\frac{\omega\mu}{4\pi}\int_{-\infty}^{\infty} A(k_{1x})\exp\left\{ik_{1x}x + i\sqrt{k_1^2-k_{1x}^2}y\right\}dk_{1x} \\ &\quad -\frac{\omega\mu}{4\pi}\int_{-\infty}^{\infty} A(k_{1x})\exp\left\{-ik_{1x}x + i\sqrt{k_1^2-k_{1x}^2}y\right\}dk_{1x} \\ &= I_1 + I_2 \qquad (17) \\ A(k_{1x}) &= \frac{\exp\left\{-i\sqrt{k_1^2-k_{1x}^2}b + i\sqrt{k_2^2-k_{1x}^2}b\right\}}{\left(\sqrt{k_1^2-k_{1x}^2}+\sqrt{k_2^2-k_{1x}^2}\right)(ik_{1x})^{2-f}} \end{aligned}$$

and the pre-subscript "e" in ${}_eG_f(x,y)$ indicates that this solution of Helmholtz's equation is even symmetric with respect to y-z plane. Using the change of variables, from Cartesian coordinate system to cylindrical coordinate system, by the following transformations

$$\begin{aligned} x &= \rho\cos\theta, \qquad & k_{1x} &= k_1\cos\alpha \\ y &= \rho\sin\theta, \qquad & k_{2x} &= k_2\cos\alpha \end{aligned}$$

expression (17) reduces to the following form

$$\begin{aligned} I_{\substack{1\\2}} &= \frac{-\omega\mu k_1}{4\pi}\int_C A(\alpha)\exp\{\pm ik_1\rho\cos(\theta\mp\alpha)\}d\alpha \qquad (18) \\ A(\alpha) &= \frac{k_1\sin^2\alpha - \sin\alpha\sqrt{k_2^2-k_1^2\cos^2\alpha}}{(k_1^2-k_2^2)(ik_1\cos\alpha)^{2-f}} \\ &\quad \cdot\exp\left\{-ik_1b\sin\alpha + ib\sqrt{k_2^2-k_1^2\cos^2\alpha}\right\} \end{aligned}$$

where C is the contour in the complex α-plane. Integrals in (18) can be calculated using asymptotic technique [7]. First consider the integral I_1. If the point of observation lies in range $x > 0$, i.e., $0 < \theta < \pi/2$, the deformed path, i.e., steepest decent path, will not intersect the branch cut at $\alpha = \pi/2$. Treatment will be the approximately same as if a line source buried in a dielectric half-space and will yield cylindrical waves. When the observation point lies in range $\pi/2 < \theta < \pi$ the deformed path will intersect the branch cut at $\alpha = \pi/2$. Therefore the contribution along the branch cut around the branch point is required to note the additional contributions due to the fractional source [1]. The asymptotic contribution due to integral I_2 can be calculated by replacing θ with $\theta+\pi$ and x with $-x$ in the asymptotic expression for I_1. The far-zone radiated fields when the observation point is not too close to $\theta = \pi/2$ is given by the following expression

$$\begin{aligned} {}_eG_f(x,y) \sim & -\cos(\pi f/2)(k_1|\cos\theta|)^{f-2}G_2 \\ & + \frac{1}{2\Gamma(2-f)k_1^{1-f}}\frac{G_1}{(k_1|x|)^{f-1}}, \quad y > b \end{aligned} \tag{19}$$

where G_1 and G_2 represents the one and two dimensional Green's functions in the far-zone. k_1 is the propagation constant of the medium in which point of observation lies. This means that in a dielectric interface geometry fractional sources yields solutions that are also combination of both cylindrical and plane waves as for the case of a homogeneous space geometry [1]. It is expected that expression for other geometry with parallel plane interfaces can be obtained by placing in (19) the values of G_1, G_2 and k_1 corresponding to that geometry.

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