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STATISTICAL NON-GAUSSIAN MODEL OF SEA SURFACE WITH ANISOTROPIC SPECTRUM FOR WAVE SCATTERING THEORY. PART I

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1. INTRODUCTION

Wave scattering from random surfaces depends on different parameters of surface depending on conditions. The important parameter of the scattering process is the Rayleigh parameter $Ra \equiv \nu_0\sigma$, where $\nu_0 = k\sin\theta_0$, θ_0 is the grazing angle of the incident wave, k is a wavenumber, ν_0 is a vertical component of the incident wave vector, and σ is the variance of the surface elevations.

If $Ra \ll 1$, the Bragg scattering mechanism works, and in this case the scattering cross section Σ depends only on the spectrum of the surface,

$$\Sigma \sim \Phi(\mathbf{q} - \mathbf{q}_0). \tag{1}$$

Here, Φ is the Fourier transform of the correlation function $B_\zeta(\mathbf{r})$ [1] of surface elevations $\zeta(\mathbf{r})$, i.e.,

$$B(\mathbf{r}) \equiv \left\langle \zeta\left(\mathbf{r}'\right) \zeta\left(\mathbf{r} + \mathbf{r}'\right)\right\rangle,$$

($\langle \cdots \rangle$ denotes the mean value) and

$$\begin{gathered} \Phi(\mathbf{q} - \mathbf{q}_0) = \frac{1}{4\pi^2} \iint \exp\left[-i(\mathbf{q} - \mathbf{q}_0)\,\mathbf{r}\right] B_\zeta(\mathbf{r})\, d^2 r, \\ B_\zeta(\mathbf{r}) = \iint \exp(i\mathbf{q}\mathbf{r})\, \Phi(\mathbf{q})\, d^2 q = \iint \cos(\mathbf{q}\mathbf{r})\, \Phi(\mathbf{q})\, d^2 q. \end{gathered} \tag{2}$$

In the case of $Ra \ll 1$, the scattering cross section does not depend on the probability distribution of elevations or surface slopes.

If $Ra \gg 1$, the Bragg scattering mechanism does not work, and several more complicated scattering theories must be applied. If the curvature radii of the surface are much larger than the wavelength, we can describe the scattering process using the Kirchhoff approximation. In this case, the scattering cross section depends on the characteristic function of *differences* in elevation at two arbitrary points, $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\left\langle \exp\left\{ i\alpha\left[\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)\right]\right\}\right\rangle, \; \alpha = \nu + \nu_0. \tag{3}$$

Here, ν is the vertical wavenumber of the scattered wave. In the case of very large k, only the linear term of expansion of $\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)$ in powers of $(\mathbf{r}_1 - \mathbf{r}_2)$ is important,

$$\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2) \approx (\mathbf{r}_1 - \mathbf{r}_2)\, \nabla\zeta(\mathbf{r}_2) + \cdots.$$

In this case the Kirchhoff approximation reduces to the geometric optics (GO) approximation and the scattering cross section depends

[1] In the following we also use the structure function of the surface elevations, the mean square of the difference of elevations in two points, $D_\zeta(\mathbf{r}' - \mathbf{r}'') \equiv \left\langle \left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right]^2 \right\rangle = 2\left[B_\zeta(0) - B_\zeta(\mathbf{r}' - \mathbf{r}'')\right]$. This function is related to the spectrum $\Phi(\mathbf{q})$ by the formula $D_\zeta(\mathbf{r}) = 2\iint \left[1 - \cos(\mathbf{q}\mathbf{r})\right] \Phi(\mathbf{q})\, d^2 q$. See more details in [1].

only on the probability distribution function (PDF) of surface slopes $\nabla\zeta(\mathbf{r})$. This result has a simple physical meaning: the scattering cross section is proportional to the number of surface facets having the appropriate slope, i.e., satisfying the condition of specular reflection. Thus, the scattering cross section in the Kirchhoff case depends on a quite different property of the surface: the PDF of differences in elevation (the PDF of slopes, in the GO case) rather than on the surface spectrum.

2. WHICH STATISTICAL CHARACTERISTICS OF THE SURFACE COMPLETELY DESCRIBE THE SCATTERING CROSS SECTION?

It is clear that, in general, the scattering cross section may depend not only on both of these parameters (spectrum and $\langle\exp\{i\alpha[\zeta(\mathbf{r}_1)-\zeta(\mathbf{r}_2)]\}\rangle$), but on some more complicated parameters of the surface.

It was shown by [2] that any solution of the scattering problem can be presented as a functional, depending only on the functions of the type

$$\mathcal{L}(\alpha,\mathbf{r})=\exp[i\alpha\zeta(\mathbf{r})] \tag{4}$$

with different α and $\mathbf{r}$. This means that the mean scattering cross section Σ can be presented as a functional Taylor series of the form

$$\begin{aligned}\Sigma=&A_0+\iint d^2r\int d\alpha A_1(\alpha,\mathbf{r})\langle\mathcal{L}(\alpha,\mathbf{r})\rangle\\&+\iint d^2r'\int d\alpha'\iint d^2r''\int d\alpha'' A_2(\alpha',\mathbf{r}';\alpha'',\mathbf{r}'')\left\langle\mathcal{L}(\alpha',\mathbf{r}')\mathcal{L}(\alpha'',\mathbf{r}'')\right\rangle\\&+\iint d^2r'\int d\alpha'\iint d^2r''\int d\alpha''\int d\alpha'''\iint d^2\mathbf{r}'''\\&\times A_3(\alpha',\mathbf{r}';\alpha'',\mathbf{r}'',\alpha''',\mathbf{r}''')\left\langle\mathcal{L}(\alpha',\mathbf{r}')\mathcal{L}(\alpha'',\mathbf{r}'')\mathcal{L}(\alpha''',\mathbf{r}''')\right\rangle+\cdots.\end{aligned} \tag{5}$$

If we take into account only the four beginning terms of this expansion (up to A_4), as was shown in [2], we obtain an approximate formula that includes in particular cases the Bragg scattering, the Kirchhoff approximation, the small-slope approximation [3, 4], the tilt-invariant approximation [5], and the double Kirchhoff approximation [6, 7]. The mean values, appearing in (5) are characteristic functions (CFs) of one-point, two-point, etc., joint PDF of surface elevations, i.e.,

$$\begin{aligned}\langle \mathcal{L}(\alpha,\mathbf{r})\rangle &= \chi_1(\alpha,\mathbf{r}) = \langle \exp[i\alpha\zeta(\mathbf{r})]\rangle \\ \left\langle \mathcal{L}\left(\alpha',\mathbf{r}'\right)\mathcal{L}\left(\alpha'',\mathbf{r}''\right)\right\rangle &= \chi_2\left(\alpha',\mathbf{r}';\alpha'',\mathbf{r}''\right) \\ &= \left\langle \exp\left[i\alpha'\zeta\left(\mathbf{r}'\right)+i\alpha''\zeta\left(\mathbf{r}''\right)\right]\right\rangle \\ &\dots \\ \langle \mathcal{L}(\alpha_1,\mathbf{r}_1)\cdots\mathcal{L}(\alpha_n,\mathbf{r}_n)\rangle &= \chi_n(\alpha_1,\mathbf{r}_1;\ldots;\alpha_n,\mathbf{r}_n) \\ &= \langle \exp[i\alpha_1\zeta(\mathbf{r}_1)+\cdots+i\alpha_n\zeta(\mathbf{r}_n)]\rangle .\end{aligned} \tag{6}$$

Thus, to calculate all these mean values it is enough to know the corresponding CF. The more orders of the scattering iterative term we consider, the more orders of CF are necessary.

The important property of scattering cross section Σ is its invariance with respect to translations of the scattering surface as a whole. If we denote the scattering cross section corresponding to the surface $z=\zeta(\mathbf{r})$ as $\Sigma[\zeta(\cdot)]$, this property is expressed by the formula

$$\Sigma[\zeta(\cdot)+h] = \Sigma[\zeta(\cdot)] . \tag{7}$$

It follows from this formula that Σ really depends only on such combinations of ζ that do not change during translations of the surface. In other words, Σ may depend only on the *differences* of the type $\zeta(\mathbf{r}')-\zeta(\mathbf{r}'')$.

In terms of CF (6) the invariance property (7) takes the form

$$\begin{aligned}\left\langle \exp\left\{ i\sum_{j=1}^{n}\alpha_j\left[\zeta(\mathbf{r}_j)+h\right]\right\}\right\rangle &= \left\langle \exp\left\{ i\sum_{j=1}^{n}\alpha_j\zeta(\mathbf{r}_j)+ih\sum\alpha_k\right\}\right\rangle \\ &= \left\langle \exp\left\{ i\sum_{j=1}^{n}\alpha_j\zeta(\mathbf{r}_j)\right\}\right\rangle .\end{aligned} \tag{8}$$

It follows from this formula that the only CF that may enter in the expression for Σ are those for which

$$\sum_{k=1}^{n}\alpha_k = 0. \tag{9}$$

The general formula for the characteristic function satisfying the property (9) is as follows (we use the special notation Θ for CF that satisfies the condition (9)):

$$\Theta\left(\alpha\right) \equiv \left\langle \exp\left[i\alpha_1\left(\zeta_1 - \zeta_1'\right) + \cdots + i\alpha_n\left(\zeta_n - \zeta_n'\right)\right]\right\rangle . \qquad (10)$$

Here, $\zeta_k \equiv \zeta\left(\mathbf{r}_k\right)$, $\zeta_k' \equiv \zeta\left(\mathbf{r}_k'\right)$, and $\alpha_k' = -\alpha_k$. In fact, (10) is the standard CF for *differences* $(\zeta_k - \zeta_k')$. Note that some of ζ_k may coincide with ζ_l or with ζ_l'; for instance, $\zeta_2 = \zeta_1'$. Because of this the total number of different points $\mathbf{r}_k$, $\mathbf{r}_j'$ in formula (10) may be either even or odd.

It follows from this analysis that the only statistical characteristics necessary to describe the scattering cross section are the joint PDF or the joint CF for *differences* in elevation at several points of the random surface.[2] Because of this the factor $A_1\left(\alpha, \mathbf{r}\right)$ in (5) must be zero.

If we consider the formulae for different terms of expansion of Σ obtained in [2], ((3) corresponds to the first term of this expansion), we can ascertain that all of them have the form (10).

Usually, in all theoretical (analytical and numerical) studies of the rough-surface scattering problem, the Gaussian PDF assumption is used. But many scattering surfaces have a PDF that differs from this basic law. As an example, we present in Figure 2 the PDF of water-surface elevations, obtained in [8] for wind-driven surface waves corresponding to frictional velocity $1.24\, m\cdot s^{-1}$. The significant deviation from the Gaussian PDF is evident. In [9] a statistical theory of gravity waves was developed in which it was shown that because of the nonlinearity of equations deviations from Gaussian PDF must appear. The method of cumulants was used in this paper to describe these deviations.

The question arises: How significant is this deviation for wave scattering from the sea surface? This problem was discussed in [10–12].

To approach an answer to this question we developed in [13] a statistical model of the surface that possesses the following properties: (1) It has the given PDF of elevations in any fixed point of the surface. (2) It has the given anisotropic spectrum. (3) It is possible to find explicit analytical formulae for any characteristic function of the type (6) for any n. As was shown in [2], using this information we are able

[2] If we are interested in the reflection coefficient, including its phase, we also need information about the PDF of elevations.

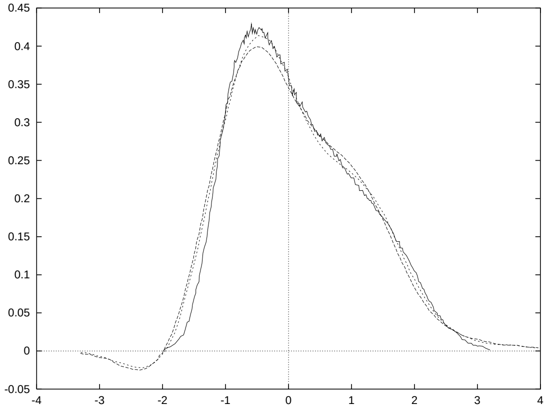

Figure 1. Measured PDF of elevations corresponding to the frictional velocity $U = 1.24\,\mathrm{m/s}$ appears as fluctuated line. Expansion suggested by Longuet-Higgins [26] appears as dashed line; the eight-term Edgeworth expansion, limited by four cumulants appears as dotted line. This Figure is taken from [8]).

to find, in an analytical form, the mean values entering into scattering theories.

The problem of how to construct a statistical model that satisfies conditions 1 to 3 has an infinite number of solutions because there are no restrictions on the highest (two-point, three-point, etc.) PDF. The model developed in [13] is only one of many possibilities. Some related problems were considered earlier in [14–17].

It is clear from the preceding analysis that statistics on *differences* in elevation are more important for the scattering theory than statistics on elevations. On the other hand, it is clear that if the spectrum of the surface is anisotropic, that is, if it depends on the angle between the wind direction and the wave-vector direction, we can expect that the PDF of differences also may be anisotropic. The experimental data for the PDF of slopes [18, 19] support this conclusion. Because of this, we try in this paper to construct a statistical model of the surface that satisfies the following conditions: (a) It has the given anisotropic spectrum (or correlation or structure functions). (b) It has the given joint PDF of slopes in two principal directions (for the wind-driven surface waves, these directions are upwind and cross-wind.) (c) It is possible to find the explicit analytical formulae for any characteristic function for differences of the type (10) for any n.

In most publications devoted to non-Gaussian surfaces, the cumulant expansions (Edgeworth or Gram-Charlier series) were used. This method is standard for describing non-Gaussian distributions. However, it is known that the PDF with the final number of cumulants (except the Gaussian PDF) does not exist (see, e.g., [20]). Because of this, the truncation of Edgeworth or Gram-Charlier expansions *necessarily* leads to the appearance of negative probabilities (see example on Figure 1, taken from the paper [8]):

These negative probabilities may affect the results of calculations of scattering cross sections and violate the energy conservation law.

In describing the non-Gaussian multivariate PDF, we will use decomposition of an arbitrary PDF in the sum of an auxiliary multivariate Gaussian PDF (for a single random variable this method is sometimes used in the Monte Carlo simulation of a non-Gaussian PDF). This approach replaces the conventional cumulant expansion. The method suggested in this paper does not lead to negative probabilities (see Figure 2) and, because of its simplicity, it successfully replaces the cumulant expansion. The solution obtained is simple enough to (1) perform all necessary calculations, and (2) obtain the analytical formulae for joint CF of differences in elevation. It allows us to obtain the scattering cross section in the Kirchhoff and other approximations for non-Gaussian surfaces with the realistic anisotropic spectrum and the PDF of the principal slopes.

The results obtained show that deviations from the Gaussian PDF may be important and may cause differences in the scattering cross section in several times.

To aid the reader's understanding of this paper, we will first describe its logical structure. We then consider the following problems, each of which can be solved once the preceding problem has been solved. They are:

***A**. Joint PDF of differences in elevation taken in two principal directions*

We start with an examination of a joint non-Gaussian PDF $W(\Delta_1, \Delta_2)$ for two differences in elevation:

$$\Delta_1(l_1) = \zeta(\mathbf{r} + l_1\mathbf{m}_1) - \zeta(\mathbf{r}), \quad \Delta_2(l_2) = \zeta(\mathbf{r} + l_2\mathbf{m}_2) - \zeta(\mathbf{r}),$$

taken in two principal directions: upwind, described by the unit vector $\mathbf{m}_1$, and cross-wind, described by the unit vector $\mathbf{m}_2$. The function $W(\Delta_1, \Delta_2)$ is approximated by the sum of the two-dimensional

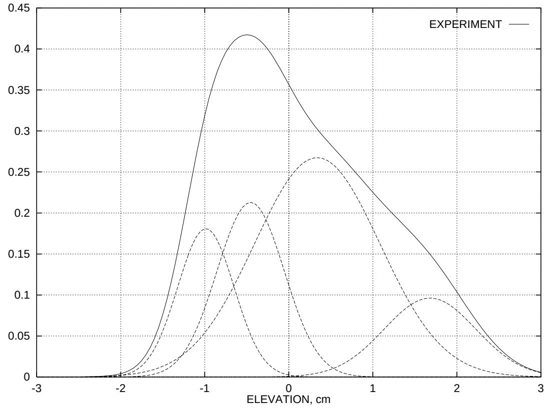

Figure 2. Non-Gaussian probability density function of elevations for the frictional velocity 1.24 m/s taken from the paper [8] and its approximation by the sum of four Gaussian terms. In contrast to the cumulant expansion, no negative probabilities appear in this representation. The sum of four Gaussian components does not exceed experimental thresholds.

Gaussian PDF (12) having different positions and different matrixes of second moments. The parameters of these auxiliary Gaussian PDFs are expressed in terms of the (anisotropic) correlation (or structure) function of the surface, which is assumed to be known from experimental data. The formula (49) for characteristic function (CF) of this non-Gaussian PDF thus obtained contains several uncertain numerical parameters: P_μ, λ_μ, $\kappa_{1\mu}$, and $\kappa_{2\mu}$. (The index μ denotes the different Gaussian terms of decomposition).

***B**. Joint PDF of two principal slopes*

From consideration of the particular case $l_1, l_2 \to 0$, it is possible to obtain the joint PDF (71) of two principal slopes, γ_1 and γ_2, in terms of the same uncertain parameters: P_μ, λ_μ, $\kappa_{1\mu}$, and $\kappa_{2\mu}$. All these parameters can be determined from a comparison of the approximate formula (71) with the experimental data. After finding these parameters we can substitute them in the formula for the joint PDF or the joint CF of two differences in elevation Δ_1 and Δ_2. The result is a formula that agrees with the correlation properties of the surface and with the joint PDF of two principal slopes.

***C**. PDF of a single, arbitrarily directed, difference in elevations*

The next step is to find the PDF and CF of a single, arbitrarily directed, difference in elevation,

$$\Delta\left(\mathbf{r}',\mathbf{r}''\right)=\zeta\left(\mathbf{r}'\right)-\zeta\left(\mathbf{r}''\right)$$

that can be easily expressed in terms of the joint CF of two differences in elevation, taken in the principal directions. This CF is a superposition of the corresponding Gaussian CFs with the same parameters, P_μ, λ_μ, $\kappa_{1\mu}$, and $\kappa_{2\mu}$. A particular case of this CF for $l_1, l_2 \to 0$ provides a CF for a slope in an arbitrary direction.

D. *Joint PDF of an arbitrary number of arbitrarily directed differences in elevation*

The joint PDF $\mathcal{P}$ for M arbitrarily directed differences in elevation also can be presented in the form of a superposition of M-dimensional Gaussian PDF. The coefficients of this superposition do not depend on M. Therefore, we can use the same parameters P_μ, λ_μ, $\kappa_{1\mu}$, and $\kappa_{2\mu}$ to construct $\mathcal{P}$. The function $\mathcal{P}$, or the corresponding CF (107) obtained in this way, describes the random surface with the given anisotropic spectrum (or given structure function) and the given joint PDF of two slopes (derivatives of elevation in two principal directions).

E. *Scattering cross sections*

Scattering cross sections from the absolutely reflecting interface can be obtained for different approximations in terms of the obtained CF. Numerical evaluation of the corresponding integrals in the Kirchhoff approximation shows that deviation from the Gaussian distribution can be very important and can cause a significant difference in scattering cross sections, especially in the range of low grazing angles.

F. *Universal angular dependence of the variance of slope*

We show in Appendix A, Part II that only from the symmetry of the spectrum of surface with respect to wind direction does it follow the universal dependence of slope variance $\left\langle\gamma^2\left(\psi\right)\right\rangle=\left\langle\gamma_1^2\right\rangle\cos^2\psi+\left\langle\gamma_2^2\right\rangle\sin^2\psi$ on the angle ψ with wind direction.

3. JOINT PDF FOR UPWIND AND CROSS-WIND DIFFERENCES IN ELEVATION

Let us consider the joint PDF for two finite differences in elevation, Δ_1 and Δ_2, taken in upwind and cross-wind directions:

$$\Delta_1\left(l_1\right)\equiv\zeta\left(\mathbf{r}+l_1\mathbf{m}_1\right)-\zeta\left(\mathbf{r}\right),\quad \Delta_2\left(l_2\right)\equiv\zeta\left(\mathbf{r}+l_2\mathbf{m}_2\right)-\zeta\left(\mathbf{r}\right), \quad (11)$$

for arbitrary values of l_1 and l_2. We assume that the joint PDF for Δ_1 and Δ_2, the function $W(\Delta_1, \Delta_2)$, can be approximated by the sum of two-dimensional Gaussian surfaces of the general type, $W_\mu(\Delta_1, \Delta_2)$:

$$W_\mu(\Delta_1, \Delta_2) = \frac{1}{2\pi\sigma_{1\mu}\sigma_{2\mu}\sqrt{1-\rho_\mu^2}} \times \exp\left\{ -\frac{\left(\Delta_1 - \overline{\Delta}_{1,\mu}\right)^2}{2\sigma_{1,\mu}^2\left(1-\rho_\mu^2\right)} - \frac{\left(\Delta_2 - \overline{\Delta}_{2,\mu}\right)^2}{2\sigma_{2,\mu}^2\left(1-\rho_\mu^2\right)} + \frac{2\rho_\mu\left(\Delta_1 - \overline{\Delta}_{1,\mu}\right)\left(\Delta_2 - \overline{\Delta}_{2,\mu}\right)}{2\sigma_{1,\mu}\sigma_{2,\mu}\left(1-\rho_\mu^2\right)} \right\}. \tag{12}$$

Each Gaussian surface over the plane (Δ_1, Δ_2) described by the function (12) is centered in the point

$$\left(\overline{\Delta}_{1,\mu}, \overline{\Delta}_{2,\mu}\right)$$

and is characterized by the parameters $\sigma_{1,\mu}$, $\sigma_{2,\mu}$, and ρ_μ. These parameters are expressed in terms of the mean values calculated with the PDF $W_\mu(\Delta_1, \Delta_2)$. We call them conditional mean values:

$$\overline{\Delta}_{1,\mu} \equiv \iint W_\mu(\Delta_1, \Delta_2)\,\Delta_1 d\Delta_1 d\Delta_2 = \langle \Delta_1 | \, \mu \rangle, \tag{13}$$

$$\overline{\Delta}_{2,\mu} = \iint W_\mu(\Delta_1, \Delta_2)\,\Delta_2 d\Delta_1 d\Delta_2 = \langle \Delta_2 | \, \mu \rangle, \tag{14}$$

$$\sigma_{1,\mu}^2 = \iint W_\mu(\Delta_1, \Delta_2)\left(\Delta_1 - \overline{\Delta}_{1,\mu}\right)^2 d\Delta_1 d\Delta_2 = \left\langle \Delta_1^2 \right| \mu \rangle - \langle \Delta_1 | \, \mu \rangle^2 = \iint W_\mu(\Delta_1, \Delta_2)\,\Delta_1^2 d\Delta_1 d\Delta_2 - \overline{\Delta}_{1,\mu}^2, \tag{15}$$

$$\sigma_{2,\mu}^2 = \iint W_\mu(\Delta_1, \Delta_2)\left(\Delta_2 - \overline{\Delta}_{2,\mu}\right)^2 d\Delta_1 d\Delta_2 = \left\langle \Delta_2^2 \right| \mu \rangle - \langle \Delta_2 | \, \mu \rangle^2 = \iint W_\mu(\Delta_1, \Delta_2)\,\Delta_2^2 d\Delta_1 d\Delta_2 - \overline{\Delta}_{2,\mu}^2, \tag{16}$$

$$\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu = \iint W_\mu(\Delta_1, \Delta_2)\left(\Delta_1 - \overline{\Delta}_{1,\mu}\right)\left(\Delta_2 - \overline{\Delta}_{2,\mu}\right) d\Delta_1 d\Delta_2 = \langle \Delta_1\Delta_2 | \, \mu \rangle - \langle \Delta_1 | \, \mu \rangle \langle \Delta_2 | \, \mu \rangle. \tag{17}$$

We seek an approximation of the joint non-Gaussian PDF of two differences in elevation Δ_1 and Δ_2, the function $W(\Delta_1, \Delta_2)$, in the form[3]

$$W(\Delta_1, \Delta_2) \approx \sum_{\mu} P_\mu W_\mu (\Delta_1, \Delta_2), \tag{18}$$

where $P_\mu > 0$. Because each function W_μ is normalized, the normalization condition for W_μ takes the form

$$\sum_{\mu} P_\mu = 1. \tag{19}$$

Thus, we can consider the positive numbers P_μ as probabilities and the functions $W_\mu (\Delta_1, \Delta_2)$ as conditional PDF for fixed μ.

Let us consider the joint CF for Δ_1 and Δ_2:

$$\Theta_\Delta (\alpha_1, l_1; \alpha_2, l_2) \equiv \langle \exp i [\alpha_1 \Delta_1 (l_1) + i\alpha_2 \Delta_2 (l_2)] \rangle . \tag{20}$$

If we use the approximation (18) for $W(\Delta_1, \Delta_2)$, we obtain the corresponding approximation for the CF:

$$\begin{aligned} \Theta_\Delta (\alpha_1, l_1; \alpha_2, l_2) &\approx \sum_{\mu} P_\mu \Theta_{\Delta,\mu} (\alpha_1, l_1; \alpha_2, l_2) \\ &= \sum_{\mu} P_\mu \exp \Big[i\alpha_1 \overline{\Delta}_{1,\mu} + i\alpha_2 \overline{\Delta}_{2,\mu} \\ &\quad - \frac{1}{2} \left(\alpha_1^2 \sigma_{1,\mu}^2 + 2\alpha_1\alpha_2 \sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu + \alpha_2^2 \sigma_{2,\mu}^2 \right) \Big] . \end{aligned} \tag{21}$$

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[3] It was shown by D. DeWolf (Private communication, 1984 and [21]) for the one-dimensional case that any smooth function can be approximated by the sum of Gaussian functions. In the case of a two-dimensional PDF a similar consideration can be performed. We start from the identity $W(x, y) = \iint \delta (x - x') \delta (y - y') W(x', y') dx'dy'$. If we replace the product $\delta (x - x') \delta (y - y')$ by the Gaussian function of the type (12) with small σ_1, σ_2, we obtain the approximate formula that becomes precise if $\sigma_1, \sigma_2 \to 0$. If we replace the integral by the finite sum, we obtain the approximate formula of the type (18) which becomes precise if $\sigma_1, \sigma_2 \to 0$ and $N \to \infty$. In practice, very often we do not need to use very small σ_1 and σ_2 and very large N.

Here,

$$\Theta_{\Delta,\mu}(\alpha_1, l_1; \alpha_2, l_2) = \exp\left[i\alpha_1\overline{\Delta}_{1,\mu} + i\alpha_2\overline{\Delta}_{2,\mu} - \frac{1}{2}\left(\alpha_1^2\sigma_{1,\mu}^2 + 2\alpha_1\alpha_2\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu + \alpha_2^2\sigma_{2,\mu}^2\right)\right] \tag{22}$$

is a CF corresponding to the conditional Gaussian PDF (12).

To determine the unknown coefficients and functions, entering in (21) and (22), we compare the expansions of $\Theta(\alpha_1, l_1; \alpha_2, l_2)$ that follow from the definition (20) and from the representation (21). The expansion of (20) in powers of α_1 and α_2 has the form:

$$\Theta(\alpha_1, l_1; \alpha_2, l_2) = 1 + i\alpha_1\langle\Delta_1\rangle + i\alpha_2\langle\Delta_2\rangle - \frac{1}{2}\left[\alpha_1^2\left\langle\Delta_1^2\right\rangle + 2\alpha_1\alpha_2\langle\Delta_1\Delta_2\rangle + \alpha_2^2\left\langle\Delta_2^2\right\rangle\right] + \cdots. \tag{23}$$

Because $\langle\zeta\rangle = 0$, we obtain $\langle\Delta_1\rangle = \langle\Delta_2\rangle = 0$ and

$$\Theta(\alpha_1, l_1; \alpha_2, l_2) = 1 - \frac{1}{2}\left[\alpha_1^2\left\langle\Delta_1^2\right\rangle + 2\alpha_1\alpha_2\langle\Delta_1\Delta_2\rangle + \alpha_2^2\left\langle\Delta_2^2\right\rangle\right] + \cdots. \tag{24}$$

The expansion of (21) in powers of α_1, α_2 has the form

$$\Theta_\Delta(\alpha_1, l_1; \alpha_2, l_2) \approx \sum_\mu P_\mu\Big\{1 + i\alpha_1\overline{\Delta}_{1,\mu} + i\alpha_2\overline{\Delta}_{2,\mu} - \frac{1}{2}\left(\alpha_1^2\sigma_{1,\mu}^2 + 2\alpha_1\alpha_2\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu + \alpha_2^2\sigma_{2,\mu}^2\right) - \frac{1}{2}\left(\alpha_1^2\overline{\Delta}_{1,\mu}^2 + 2\alpha_1\alpha_2\overline{\Delta}_{1,\mu}\overline{\Delta}_{2,\mu} + \alpha_2^2\overline{\Delta}_{2,\mu}^2\right) + \cdots\Big\}. \tag{25}$$

From comparison of the zero-order in α-s terms of expansions (24) and (25) we obtain the same relation (19). From comparison of the linear in α_1 and α_2 terms we obtain

$$\sum_\mu P_\mu\overline{\Delta}_{1,\mu} = 0, \ \sum_\mu P_\mu\overline{\Delta}_{2,\mu} = 0. \tag{26}$$

From comparison of the coefficients in front of α_1^2, α_2^2, and $\alpha_1\alpha_2$ we obtain:

$$\sum_\mu P_\mu\left[\sigma_{1,\mu}^2 + \overline{\Delta}_{1,\mu}^2\right] = \left\langle\Delta_1^2\right\rangle, \tag{27}$$

$$\sum_{\mu} P_{\mu}\left[\sigma_{2,\mu}^{2}+\overline{\Delta}_{2,\mu}^{2}\right]=\left\langle\Delta_{2}^{2}\right\rangle, \tag{28}$$

$$\sum_{\mu} P_{\mu}\left[\sigma_{1,\mu}\sigma_{2,\mu}\rho_{\mu}+\overline{\Delta}_{1,\mu}\overline{\Delta}_{2,\mu}\right]=\left\langle\Delta_{1}\Delta_{2}\right\rangle. \tag{29}$$

Note that all of the values $\Delta_1,\ \Delta_2,\ \overline{\Delta}_{1,\mu},\ \overline{\Delta}_{2,\mu},\ \sigma_{1,\mu}^2,\ \sigma_{2,\mu}^2$, and ρ_μ depend on l_1 or l_2. If we substitute $\overline{\Delta}_{1,\mu},\ \overline{\Delta}_{2,\mu},\ \sigma_{1,\mu}^2,\ \sigma_{2,\mu}^2$, and ρ_μ in (27) to (29) in terms of conditional mean values (13) to (17), we obtain:

$$\sum_{\mu} P_{\mu}\left\langle \Delta_{1}^{2}\left(l_{1}\right)\right|\mu\left.\right\rangle=\left\langle\Delta_{1}^{2}\left(l_{1}\right)\right\rangle, \tag{30}$$

$$\sum_{\mu} P_{\mu}\left\langle \Delta_{2}^{2}\left(l_{2}\right)\right|\mu\left.\right\rangle=\left\langle\Delta_{2}^{2}\left(l_{2}\right)\right\rangle, \tag{31}$$

$$\sum_{\mu} P_{\mu}\left\langle \Delta_{1}\left(l_{1}\right)\Delta_{2}\left(l_{2}\right)\right|\mu\left.\right\rangle=\left\langle\Delta_{1}\left(l_{1}\right)\Delta_{2}\left(l_{2}\right)\right\rangle. \tag{32}$$

We can satisfy all of the equations (30) to (32) if we set

$$\left\langle \Delta_{1}^{2}\left(l_{1}\right)\right|\mu\left.\right\rangle=\lambda_{\mu}\left\langle\Delta_{1}^{2}\left(l_{1}\right)\right\rangle, \tag{33}$$

$$\left\langle \Delta_{2}^{2}\left(l_{2}\right)\right|\mu\left.\right\rangle=\lambda_{\mu}\left\langle\Delta_{2}^{2}\left(l_{2}\right)\right\rangle, \tag{34}$$

$$\left\langle \Delta_{1}\left(l_{1}\right)\Delta_{2}\left(l_{2}\right)\right|\mu\left.\right\rangle=\lambda_{\mu}\left\langle\Delta_{1}\left(l_{1}\right)\Delta_{2}\left(l_{2}\right)\right\rangle. \tag{35}$$

In other words, all of the conditional second moments of differences are proportional to corresponding known unconditional second moments with the same coefficient λ_μ. In this case, all of the equations (30) to (32) formulated in terms of functions of l_1, l_2 reduce to a single equation with respect to the numbers λ_μ:

$$\sum_{\mu} P_{\mu}\lambda_{\mu}=1. \tag{36}$$

Note that in terms of the structure function of the surface,

$$D\left(\mathbf{r}'-\mathbf{r}''\right)=\left\langle\left[\zeta\left(\mathbf{r}'\right)-\zeta\left(\mathbf{r}''\right)\right]^{2}\right\rangle, \tag{37}$$

the functions (33) and (34) take the form

$$\left\langle \Delta_{1}^{2}\left(l_{1}\right)\right|\mu\left.\right\rangle=\lambda_{\mu}D\left(l_{1}\mathbf{m}_{1}\right), \tag{38}$$

$$\left\langle \Delta_{2}^{2}\left(l_{2}\right)\right|\mu\left.\right\rangle=\lambda_{\mu}D\left(l_{2}\mathbf{m}_{2}\right). \tag{39}$$

The expression

$$\langle \Delta_1(l_1)\,\Delta_2(l_2)\rangle = \langle [\zeta(\mathbf{r}+l_1\mathbf{m}_1)-\zeta(\mathbf{r})]\,[\zeta(\mathbf{r}+l_2\mathbf{m}_2)-\zeta(\mathbf{r})]\rangle$$

can be transformed using the Yaglom identity [22]:

$$(A-B)(C-D) = \frac{1}{2}\left[(A-D)^2+(B-C)^2-(A-C)^2-(B-D)^2\right] \tag{40}$$

as follows:

$$\langle \Delta_1(l_1)\,\Delta_2(l_2)\rangle = \frac{1}{2}\left[D(l_1\mathbf{m}_1)+D(l_2\mathbf{m}_2)-D(l_1\mathbf{m}_1-l_2\mathbf{m}_2)\right]. \tag{41}$$

Thus, we can rewrite (35) in the form

$$\langle\, \Delta_1(l_1)\,\Delta_2(l_2)|\,\mu\rangle = \frac{\lambda_\mu}{2}\left[D(l_1\mathbf{m}_1)+D(l_2\mathbf{m}_2)-D(l_1\mathbf{m}_1-l_2\mathbf{m}_2)\right]. \tag{42}$$

Let us consider now the equations (26). The derivatives of $\overline{\Delta}_{1,\mu}(l_1)$ and $\overline{\Delta}_{2,\mu}$ with respect to l_1 or l_2 in the points $l_1=0$ or $l_2=0$ are equal to the conditional mean values of slopes. We will find a bit later that these values must be nonzero. Because of this we cannot set these functions $\overline{\Delta}_{1,\mu}(l_1)$ and $\overline{\Delta}_{2,\mu}(l_2)$ to be zero, despite that such a choice is consistent with (26).

The typical value of the difference

$$\overline{\Delta}_{1,\mu} = \langle\, \zeta(\mathbf{r}+l_1\mathbf{m}_1)-\zeta(\mathbf{r})|\,\mu\rangle$$

is on the order of $\sqrt{\left\langle \Delta_1^2(l_1)\right\rangle}$. On the other hand, $\overline{\Delta}_{1,\mu}$ as a function of l_1 must be an odd function, that is,

$$\overline{\Delta}_{1,\mu}(-l_1) = -\overline{\Delta}_{1,\mu}(l_1). \tag{43}$$

Because of this we can seek $\overline{\Delta}_{1,\mu}$ and $\overline{\Delta}_{2,\mu}$ in the form [4]

$$\begin{aligned}\overline{\Delta}_{1,\mu}(l_1) &= \kappa_{1,\mu}\frac{l_1}{|l_1|}\sqrt{\left\langle \Delta_1^2(l_1)\right\rangle} = \kappa_{1,\mu}l_1\sqrt{\frac{\left\langle \Delta_1^2(l_1)\right\rangle}{l_1^2}},\\ \overline{\Delta}_{2,\mu}(l_2) &= \kappa_{2,\mu}\frac{l_2}{|l_2|}\sqrt{\left\langle \Delta_2^2(l_2)\right\rangle} = \kappa_{2,\mu}l_2\sqrt{\frac{\left\langle \Delta_2^2(l_2)\right\rangle}{l_2^2}}.\end{aligned} \tag{44}$$

[4] We assumed the coefficients $\kappa_{1,\mu}$ and $\kappa_{2,\mu}$ to be different because of the different type of symmetry of slopes in the upwind and cross-wind directions.

Note that if $l_1 \to 0$, the function $\left\langle \Delta_1^2 (l_1) \right\rangle$ is proportional to l_1^2 and

$$\sqrt{\left\langle \Delta_1^2 (l_1) \right\rangle / l_1^2} \sim \text{Constant.}$$

Thus, the function $\overline{\Delta}_{1,\mu} (l_1)$ is proportional to l_1 for small l_1, i.e., it has a continuous derivative in the point $l_1 = 0$.

After substituting (44) in the equations (26) they reduce to the equations with respect to the numbers $\kappa_{1,\mu}$ and $\kappa_{2,\mu}$:

$$\sum_\mu P_\mu \kappa_{1,\mu} = 0, \ \sum_\mu P_\mu \kappa_{2,\mu} = 0. \tag{45}$$

We expressed all the functions

$$\overline{\Delta}_{1,\mu}, \overline{\Delta}_{2,\mu}, \left\langle \Delta_1^2 (l_1) \middle| \mu \right\rangle, \left\langle \Delta_2^2 (l_2) \middle| \mu \right\rangle, \text{ and } \left\langle \Delta_1 (l_1) \Delta_2 (l_2) \middle| \mu \right\rangle$$

in terms of the known structure function $D(\mathbf{r}' - \mathbf{r}'')$ of the surface and the unknown numbers λ_μ, $\kappa_{1,\mu}$, and $\kappa_{2,\mu}$. If we substitute the formulae obtained in expressions (15) to (17), we obtain

$$\sigma_{1,\mu}^2 = \left\langle \Delta_1^2 \middle| \mu \right\rangle - \left\langle \Delta_1 \middle| \mu \right\rangle^2 = \left(\lambda_\mu - \kappa_{1,\mu}^2\right) D(l_1 \mathbf{m}_1), \tag{46}$$

$$\sigma_{2,\mu}^2 = \left\langle \Delta_2^2 \middle| \mu \right\rangle - \left\langle \Delta_2 \middle| \mu \right\rangle^2 = \left(\lambda_\mu - \kappa_{2,\mu}^2\right) D(l_2 \mathbf{m}_2), \tag{47}$$

$$\begin{aligned} \sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu &= \left\langle \Delta_1 \Delta_2 \middle| \mu \right\rangle - \left\langle \Delta_1 \middle| \mu \right\rangle \left\langle \Delta_2 \middle| \mu \right\rangle \\ &= \frac{\lambda_\mu}{2} \left[D(l_1 \mathbf{m}_1) + D(l_2 \mathbf{m}_2) - D(l_1 \mathbf{m}_1 - l_2 \mathbf{m}_2) \right] \\ &\quad - \kappa_{1,\mu}\kappa_{2,\mu} l_1 l_2 \sqrt{\frac{D(l_1 \mathbf{m}_1) D(l_2 \mathbf{m}_2)}{l_1^2 l_2^2}}. \end{aligned} \tag{48}$$

For the joint CF of the differences in the elevation of the surface, substituting (46) to (48) in (21) we obtain:

$$\begin{aligned} &\Theta_\Delta (\alpha_1, l_1; \alpha_2, l_2) = \left\langle \exp\left\{ i\alpha_1 \Delta_1 (l_1) + i\alpha_2 \Delta_2 (l_2) \right\} \right\rangle \\ &\approx \sum_\mu P_\mu \exp\left\{ i \left[\alpha_1 \kappa_{1,\mu} l_1 \sqrt{\frac{D(\mathbf{m}_1 l_1)}{l_1^2}} + \alpha_2 \kappa_{2,\mu} l_2 \sqrt{\frac{D(\mathbf{m}_2 l_2)}{l_2^2}} \right] \right. \\ &\quad - \frac{1}{2} \left(\lambda_\mu - \kappa_{1,\mu}^2\right) \alpha_1^2 D(\mathbf{m}_1 l_1) - \frac{1}{2} \left(\lambda_\mu - \kappa_{2,\mu}^2\right) \alpha_2^2 D(\mathbf{m}_2 l_2) \\ &\quad - \alpha_1 \alpha_2 \left[\frac{\lambda_\mu}{2} \left[D(l_1 \mathbf{m}_1) + D(l_2 \mathbf{m}_2) - D(l_1 \mathbf{m}_1 - l_2 \mathbf{m}_2) \right] \right. \\ &\quad \left. \left. - \kappa_{1,\mu}\kappa_{2,\mu} l_1 l_2 \sqrt{\frac{D(l_1 \mathbf{m}_1) D(l_2 \mathbf{m}_2)}{l_1^2 l_2^2}} \right] \right\}. \end{aligned} \tag{49}$$

Formula (49) does not contain any unknown functions, but only unknown numerical parameters P_μ, λ_μ, $\kappa_{1,\mu}$, and $\kappa_{2,\mu}$. To find these parameters, we consider the particular case of (49) while l_1, $l_2 \to 0$. In this case we obtain from CF for differences in elevation the CF for *derivatives* of the surface, i.e., for the slopes of the surface.

4. MATCHING WITH THE PDF FOR SLOPES

The slope of a surface in a point $\mathbf{r}$ taken in a direction determined by the unit vector $\mathbf{n}$ is given by the formula

$$\gamma(\mathbf{n}, \mathbf{r}) \equiv \mathbf{n}\nabla\zeta(\mathbf{r}). \tag{50}$$

We assume that the spectrum of surface $\Phi(\mathbf{q})$ is symmetrical with respect to the wind direction determined by the unit vector $\mathbf{m}_1$. If we choose the x-axis along the vector $\mathbf{m}_1$, we obtain

$$\mathbf{m}_1 = (1, 0). \tag{51}$$

The vector $\mathbf{q}$ can be presented in the form

$$\mathbf{q} = (q\cos\varphi, q\sin\varphi), \tag{52}$$

where φ is the angle between $\mathbf{q}$ and the wind direction. The symmetry of the spectrum with respect to the wind direction means that

$$\Phi(q, \varphi) = \Phi(q, -\varphi). \tag{53}$$

The structure function of the surface,

$$D(\mathbf{r}' - \mathbf{r}'') \equiv \left\langle \left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right]^2 \right\rangle, \tag{54}$$

in terms of the spectrum Φ, has the form (compare with (2)):

$$D(r, \psi) = 2\iint [1 - \cos(\mathbf{qr})]\,\Phi(q, \varphi)\, q\,dq\,d\varphi. \tag{55}$$

Let us consider in (49) the case l_1, $l_2 \to 0$ and substitute

$$\begin{aligned} \Delta_1 &= \zeta(\mathbf{r} + l_1\mathbf{m}_1) - \zeta(\mathbf{r}) \to l_1\mathbf{m}_1\nabla\zeta(\mathbf{r}) = l_1\gamma_1(\mathbf{r}) \\ \Delta_2 &= \zeta(\mathbf{r} + l_2\mathbf{m}_2) - \zeta(\mathbf{r}) \to l_2\mathbf{m}_2\nabla\zeta(\mathbf{r}) = l_2\gamma_2(\mathbf{r}). \end{aligned} \tag{56}$$

Here,

$$\gamma_1(\mathbf{r}) \equiv \mathbf{m}_1 \nabla\zeta(\mathbf{r}), \;\; \gamma_2(\mathbf{r}) \equiv \mathbf{m}_2 \nabla\zeta(\mathbf{r}) \tag{57}$$

are the slopes of the surface at the point $\mathbf{r}$, taken in the upwind direction $\mathbf{m}_1$ and in the cross-wind direction $\mathbf{m}_2$. We obtain:

$$\begin{aligned}\Theta_\Delta(\alpha_1, l_1; \alpha_2, l_2) &\to \langle \exp\{i(\alpha_1 l_1)\gamma_1(\mathbf{r}) + i(\alpha_2 l_2)\gamma_2(\mathbf{r})\}\rangle \\ &\equiv \Theta_\gamma(\alpha_1 l_1, \alpha_2 l_2).\end{aligned} \tag{58}$$

If we denote

$$\beta_1 = \alpha_1 l_1, \;\; \beta_2 = \alpha_2 l_2, \tag{59}$$

and consider the case $l_1,\ l_2 \to 0,\ \beta_1,\ \beta_2 =$ Constant, we obtain the relation between Θ_Δ and Θ_γ:

$$\Theta_\gamma(\beta_1, \beta_2) = \lim_{l_1, l_2 \to 0} \Theta_\Delta\left(\frac{\beta_1}{l_1}, l_1; \frac{\beta_2}{l_2}, l_2\right). \tag{60}$$

According to the definitions of slopes,

$$\lim_{l_1 \to 0} \frac{D(l_1 \mathbf{m}_1)}{l_1^2} = \langle \gamma_1^2 \rangle, \quad \lim_{l_2 \to 0} \frac{D(l_2 \mathbf{m}_2)}{l_2^2} = \langle \gamma_2^2 \rangle. \tag{61}$$

Therefore, the following limiting formulae are true for the values entering in (49):

$$\begin{aligned}\frac{\beta_1}{l_1}\kappa_{1,\mu} l_1 \sqrt{\frac{D(l_1 \mathbf{m}_1)}{l_1^2}} &\to \beta_1 \kappa_{1,\mu} \sqrt{\langle \gamma_1^2 \rangle}, \\ \frac{\beta_2}{l_2}\kappa_{2,\mu} l_2 \sqrt{\frac{D(l_2 \mathbf{m}_2)}{l_2^2}} &\to \beta_2 \kappa_{2,\mu} \sqrt{\langle \gamma_2^2 \rangle}.\end{aligned} \tag{62}$$

The term

$$D(l_1 \mathbf{m}_1) + D(l_2 \mathbf{m}_2) - D(\mathbf{m}_1 l_1 - \mathbf{m}_2 l_2) = 2\langle \Delta_1(l_1)\Delta_2(l_2)\rangle,$$

following the product $\alpha_1 \alpha_2$ in (49), needs more attention. Using spectral representation (55), we find

$$\langle \Delta_1 \Delta_2 \rangle = \iint \Phi(\mathbf{q})\, d^2 q \{1 - \cos(\mathbf{q}\mathbf{m}_1 l_1) - \cos(\mathbf{q}\mathbf{m}_2 l_2) + \cos(\mathbf{q}\mathbf{m}_1 l_1 - \mathbf{q}\mathbf{m}_2 l_2)\} \tag{63}$$

But

$$\{\cdots\} = 4\sin\frac{\mathbf{qm}_1 l_1}{2}\sin\frac{\mathbf{qm}_2 l_2}{2}\cos\frac{\mathbf{qm}_1 l_1 - \mathbf{qm}_2 l_2}{2},$$

and

$$\langle\Delta_1\Delta_2\rangle = 4\iint \Phi(\mathbf{q})\sin\frac{\mathbf{qm}_1 l_1}{2}\sin\frac{\mathbf{qm}_2 l_2}{2}\cos\frac{\mathbf{qm}_1 l_1 - \mathbf{qm}_2 l_2}{2} d^2 q. \tag{64}$$

For $l_1, l_2 \to 0$ we obtain

$$4\sin\frac{\mathbf{qm}_1 l_1}{2}\sin\frac{\mathbf{qm}_2 l_2}{2}\cos\frac{\mathbf{qm}_1 l_1 - \mathbf{qm}_2 l_2}{2} \to l_1 l_2 (\mathbf{qm}_1)(\mathbf{qm}_2)$$

and

$$\begin{aligned}\lim_{l_1, l_2 \to 0}\frac{\langle\Delta_1\Delta_2\rangle}{l_1 l_2} &= \langle\gamma_1\gamma_2\rangle \\ &= \iint \Phi(\mathbf{q})(\mathbf{qm}_1)(\mathbf{qm}_2)\, d^2 q \\ &= \int_0^\infty q^3 dq \int_{-\pi}^{\pi} \Phi(q,\varphi)\sin\varphi\cos\varphi d\varphi = 0\end{aligned}$$

because of $\Phi(q,\varphi) = \Phi(q,-\varphi)$ (the integrand is an odd function with respect to φ). Thus, we proved that the term

$$\frac{\langle\Delta_1\Delta_2\rangle}{l_1 l_2} \to 0 \text{ while } l_1, l_2 \to 0,$$

or

$$\lim_{l_1, l_2 \to 0}\frac{D(l_1\mathbf{m}_1) + D(l_2\mathbf{m}_2) - D(\mathbf{m}_1 l_1 - \mathbf{m}_2 l_2)}{l_1 l_2} = 0, \tag{65}$$

vanishes while $l_1, l_2 \to 0$. This relation also can be written in the form

$$\langle\gamma_1\gamma_2\rangle = 0. \tag{66}$$

It follows from (66) that two principal slopes in the same point on a surface are uncorrelated. (It is shown in Appendix A, Part II that this relation follows only from the symmetry of the spectrum with respect to wind direction and is independent of the PDF of slopes; therefore, the relation (66) can be derived without using any approximate formulae for the PDF).

Thus, using (60), (61), (62), and (65), we obtain from (49):

$$\begin{aligned}\Theta_\gamma(\beta_1,\beta_2) &= \langle \exp\{i\beta_1\gamma_1(\mathbf{r}) + i\beta_2\gamma_2(\mathbf{r})\}\rangle \\ &\approx \sum_\mu P_\mu \exp\Bigg\{ i\beta_1\kappa_{1,\mu}\sqrt{\langle\gamma_1^2\rangle} + i\beta_2\kappa_{2,\mu}\sqrt{\langle\gamma_2^2\rangle} \\ &\quad -\frac{\beta_1^2}{2}\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\langle\gamma_1^2\rangle - \frac{\beta_2^2}{2}\left(\lambda_\mu - \kappa_{2,\mu}^2\right)\langle\gamma_2^2\rangle \\ &\quad +\beta_1\beta_2\kappa_{1,\mu}\kappa_{2,\mu}\sqrt{\langle\gamma_1^2\rangle\langle\gamma_2^2\rangle}\Bigg\}.\end{aligned} \tag{67}$$

It is easy to verify by direct differentiation of the right-hand side of (67) that

$$-\left.\frac{\partial^2\Theta_\gamma(\beta_1,\beta_2)}{\partial\beta_1\partial\beta_2}\right|_{\beta_1=\beta_2=0} = \langle\gamma_1\gamma_2\rangle = 0 \tag{68}$$

for any values of the parameters. The mean value of the slope $\langle\gamma_1\rangle$,

$$\langle\gamma_1\rangle = \frac{1}{i}\left.\frac{\partial\Theta_\gamma(\beta_1,\beta_2)}{\partial\beta_1}\right|_{\beta_1=\beta_2=0} = \sqrt{\langle\gamma_1^2\rangle}\sum_\mu P_\mu\kappa_{1,\mu} = 0 \tag{69}$$

because of (45). A similar formula is true for γ_2. Thus, the principal slopes γ_1 and γ_2 are statistically dependent, but uncorrelated.[5]

It follows from (67) that the conditional Gaussian distribution, marked by subscript μ, has the following parameters:

$$\begin{aligned}\langle\gamma_1|\,\mu\rangle &= \kappa_{1,\mu}\sqrt{\langle\gamma_1^2\rangle}, \qquad \langle\gamma_2|\,\mu\rangle = \kappa_{2,\mu}\sqrt{\langle\gamma_2^2\rangle} \\ \sigma_{\gamma_1,\mu}^2 &= \left(\lambda_\mu - \kappa_{1,\mu}^2\right)\langle\gamma_1^2\rangle, \quad \sigma_{\gamma_2,\mu}^2 = \left(\lambda_\mu - \kappa_{2,\mu}^2\right)\langle\gamma_2^2\rangle, \\ \sigma_{\gamma_1,\mu}\sigma_{\gamma_2,\mu}\rho_\mu &= -\kappa_{1,\mu}\kappa_{2,\mu}\sqrt{\langle\gamma_1^2\rangle\langle\gamma_2^2\rangle} \\ \rho_\mu &= -\frac{\kappa_{1,\mu}\kappa_{2,\mu}}{\sqrt{\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\left(\lambda_\mu - \kappa_{2,\mu}^2\right)}}, \\ 1-\rho_\mu^2 &= \frac{\lambda_\mu\left(\lambda_\mu - \kappa_{1,\mu}^2 - \kappa_{2,\mu}^2\right)}{\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\left(\lambda_\mu - \kappa_{2,\mu}^2\right)}\end{aligned} \tag{70}$$

[5] This is an interesting example of statistically dependent, but uncorrelated, random values. Only in the case of the Gaussian PDF are two random variables statistically independent if they are uncorrelated.

$$\sigma_{1,\mu}^2\sigma_{2,\mu}^2\left(1-\rho_\mu^2\right)=\left\langle\gamma_1^2\right\rangle\left\langle\gamma_2^2\right\rangle\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right).$$

The PDF that corresponds to CF (67) is given by the formula

$$W_\gamma\left(\gamma_1,\gamma_2\right)=\sum_\mu\frac{P_\mu}{2\pi\sqrt{\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)\left\langle\gamma_1^2\right\rangle\left\langle\gamma_2^2\right\rangle}}$$
$$\times\exp\left\{-\frac{\left(\lambda_\mu-\kappa_{2,\mu}^2\right)\left[\gamma_1-\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle}\right]^2}{2\left\langle\gamma_1^2\right\rangle\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}\right.$$
$$-\frac{\left(\lambda_\mu-\kappa_{1,\mu}^2\right)\left[\gamma_2-\kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}\right]^2}{2\left\langle\gamma_2^2\right\rangle\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}$$
$$\left.-\frac{\kappa_{1,\mu}\kappa_{2,\mu}\left[\gamma_1-\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle}\right]\left[\gamma_2-\kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}\right]}{\sqrt{\left\langle\gamma_1^2\right\rangle\left\langle\gamma_2^2\right\rangle}\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}\right\}. \tag{71}$$

Because the values $\sigma_{1,\mu}^2$, $\sigma_{2,\mu}^2$, and $1-\rho_\mu^2$ must be non-negative, we obtain the following restrictions for the parameters λ_μ, $\kappa_{1,\mu}$, and $\kappa_{2,\mu}$:

$$\lambda_\mu-\kappa_{\alpha,\mu}^2\geq 0,\quad \alpha=1,2;\quad \lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\geq 0. \tag{72}$$

5. FINDING THE PARAMETERS λ_μ, κ_μ, AND P_μ

The next step is to find parameters λ_μ, κ_μ, and P_μ. If the function $W_\gamma\left(\gamma_1,\gamma_2\right)$ is known (for example, from the experimental data), we can approximate this function by the formula (71). (See footnote[3] after the formula (17)). It follows from the statistical symmetry of slopes with respect to wind direction that distribution in the cross-wind direction must be symmetrical, i.e.,

$$W_\gamma\left(\gamma_1,\gamma_2\right)=W_\gamma\left(\gamma_1,-\gamma_2\right). \tag{73}$$

The top of each conditional Gaussian PDF numbered by the subscript μ has on the plane $\left(\gamma_1,\gamma_2\right)$ the coordinates

$$\left(\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle},\,\kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}\right). \tag{74}$$

It follows from the formula (71) and the symmetry condition (73) that each point numbered by μ and having coordinates (74), must be accompanied by the dissymmetric point, numbered by some μ', and having the coordinates

$$\kappa_{1,\mu'}\sqrt{\langle\gamma_1^2\rangle} = \kappa_{1,\mu}\sqrt{\langle\gamma_1^2\rangle}, \quad \kappa_{2,\mu'}\sqrt{\langle\gamma_2^2\rangle} = -\kappa_{2,\mu}\sqrt{\langle\gamma_2^2\rangle}, \qquad (75)$$

and the same values of $P_{\mu'} = P_\mu$ and $\lambda_{\mu'} = \lambda_\mu$. It is convenient to numerate this point by $\mu' = -\mu$. In this case,[6]

$$\kappa_{1,-\mu} = \kappa_{1,\mu}; \quad \kappa_{2,-\mu} = -\kappa_{2,\mu}; \quad P_\mu = P_{-\mu},\ \lambda_\mu = \lambda_{-\mu}. \qquad (76)$$

Therefore, we must approximate the experimental joint PDF of two principal slopes by the formula (71) with the additional conditions

$$\begin{gathered}\sum_\mu P_\mu = 1,\ P_\mu > 0, \quad \sum_\mu P_\mu \kappa_{1,\mu} = \sum_\mu P_\mu \kappa_{2,\mu} = 0, \quad \sum_\mu P_\mu \lambda_\mu = 1 \\ P_{-\mu} = P_\mu, \quad \lambda_{-\mu} = \lambda_\mu, \quad \kappa_{1,-\mu} = \kappa_{1,\mu}; \\ \kappa_{2,-\mu} = -\kappa_{2,\mu}, \quad \lambda_\mu \geq \kappa_{1,\mu}^2 + \kappa_{2,\mu}^2. \end{gathered} \qquad (77)$$

The quantities $\langle\gamma_1^2\rangle$ and $\langle\gamma_2^2\rangle$ can be determined from the joint experimental PDF $W_\gamma(\gamma_1, \gamma_2)$. Therefore, only the numbers $\kappa_{1,\mu}, \kappa_{2,\mu}, \lambda_\mu$, and P_μ need to be found. Note that, in general, the conditional 2-D Gaussian PDF is characterized by five independent parameters: two shifts and three coefficients of the quadratic form. In our case, only three independent parameters $\kappa_{1,\mu}, \kappa_{2,\mu}, \lambda_\mu$ remain; the two other coefficients of the quadratic form are some functions of $\kappa_{1,\mu}, \kappa_{2,\mu}, \lambda_\mu$.

The procedure of approximation can be performed by minimization of the integrated squared difference between the given joint PDF and its approximation by the formula (71). In the process of approximation we find all the numerical parameters $P_\mu, \kappa_{1,\mu}, \kappa_{2,\mu}$, and λ_μ (see the example in the section "Numerical Results for the Radar Cross Section for Cox-Munk PDF and 2-D Anisotropic Spectra" in Part II). [7]

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[6] The index of summation runs from $-N'$ to N' instead of $1, N$.

[7] We must emphasize that, as with other methods of finding PDF approximations by using a finite number of moments, the procedure for approximating PDF by the sum of Gaussian components is unstable. If we approximate some experimental PDF $W(\gamma)$, we always can

6. THE CF FOR THE ARBITRARILT DIRECTED DIFFERENCE $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$

We have already obtained the formula (49) for the joint CF of two differences, taken in the two perpendicular principal directions. In this section we generalize this formula for the arbitrarily directed difference of the type

$$\Delta\left(\mathbf{r}', \mathbf{r}''\right) = \zeta\left(\mathbf{r}'\right) - \zeta\left(\mathbf{r}''\right). \tag{78}$$

It is easy to formulate the problem of finding the CF for such differences in terms of the solved problem. Let us draw through the point $\mathbf{r}'$ a straight line in the direction of the vector $\mathbf{m}_1$ (the upwind direction) and draw through the point $\mathbf{r}''$ a straight line in the direction of the vector $\mathbf{m}_2$ (the cross-wind direction). These two lines intersect at some point $\mathbf{r}^*$, depending on $\mathbf{r}'$ and $\mathbf{r}''$. This point is determined by the equations

$$\begin{aligned} &\mathbf{r}' = \left(x', y'\right), \quad \mathbf{r}'' = \left(x'', y''\right), \quad \mathbf{r}^* = \left(x'', y'\right) \\ &l_1 = l_1\left(\mathbf{r}', \mathbf{r}''\right) = \mathbf{m}_1\left(\mathbf{r}'' - \mathbf{r}'\right) = x'' - x', \\ &l_2 = l_2\left(\mathbf{r}', \mathbf{r}''\right) = \mathbf{m}_2\left(\mathbf{r}'' - \mathbf{r}'\right) = y'' - y'. \end{aligned} \tag{79}$$

We can present the difference $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$ as follows:

$$\begin{aligned} \zeta\left(\mathbf{r}'\right) - \zeta\left(\mathbf{r}''\right) &= \Delta\left(\mathbf{r}', \mathbf{r}''\right) \\ &= \left[\zeta\left(\mathbf{r}'\right) - \zeta\left(\mathbf{r}^*\right)\right] - \left[\zeta\left(\mathbf{r}''\right) - \zeta\left(\mathbf{r}^*\right)\right] \\ &= \Delta_1\left(l_1\right) - \Delta_2\left(l_2\right). \end{aligned} \tag{80}$$

Thus, for the CF of $\Delta(\mathbf{r}', \mathbf{r}'')$ we obtain

$$\begin{aligned} \Theta_\Delta(\alpha) &= \left\langle \exp\left\{i\alpha\Delta\left(\mathbf{r}', \mathbf{r}''\right)\right\}\right\rangle \\ &= \left\langle \exp\left\{i\alpha\Delta_1\left(l_1\right) - i\alpha\Delta_2\left(l_2\right)\right\}\right\rangle \\ &= \Theta_\Delta\left(\alpha, l_1; -\alpha, l_2\right), \end{aligned} \tag{81}$$

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add some additional Gaussian component with a very small coefficient, centered far beyond the experimental range. such an additional term will not affect the accuracy of approximation in the considered range but can significantly change the highest moments. Because of this, in the process of approximation we must restrict ourselves to a level of accuracy that is consistent with the accuracy of measured the measured PDF.

where $\Theta_\Delta(\alpha, l_1; -\alpha, l_2)$ is given by the formula (49). Substitution of (49) in (81) leads to the formula

$$\Theta_\Delta(\alpha) = \left\langle \exp\left\{i\alpha\left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right]\right\}\right\rangle \approx \exp\left\{i\alpha\mathcal{L} - \alpha^2\mathcal{Q}\right\}, \qquad (82)$$

where

$$\mathcal{L} = \kappa_{1,\mu} l_1 \sqrt{\frac{D(\mathbf{m}_1 l_1)}{l_1^2}} - \kappa_{2,\mu} l_2 \sqrt{\frac{D(\mathbf{m}_2 l_2)}{l_2^2}} \qquad (83)$$

and

$$\begin{aligned}\mathcal{Q} = &\frac{1}{2}\left(\lambda_\mu - \kappa_{1,\mu}^2\right) D(\mathbf{m}_1 l_1) + \frac{1}{2}\left(\lambda_\mu - \kappa_{2,\mu}^2\right) D(\mathbf{m}_2 l_2)\\ &- \left[\frac{\lambda_\mu}{2}\Big[D(\mathbf{m}_1 l_1) + D(\mathbf{m}_2 l_2) + D(\mathbf{m}_1 l_1 - \mathbf{m}_2 l_2)\Big]\right.\\ &\left. + \kappa_{1,\mu}\kappa_{2,\mu} l_1 l_2 \sqrt{\frac{D(\mathbf{m}_1 l_1) D(\mathbf{m}_2 l_2)}{l_1^2 l_2^2}}\right].\end{aligned} \qquad (84)$$

After cancellation of several terms following the factor $\alpha^2\lambda_\mu$, we obtain

$$\begin{aligned}\Theta_\Delta(\alpha) &= \left\langle \exp\left\{i\alpha\left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right]\right\}\right\rangle\\ &\approx \sum_\mu P_\mu \exp\left\{i\alpha\mathcal{L} - \frac{1}{2}\alpha^2\lambda_\mu D(\mathbf{m}_1 l_1 - \mathbf{m}_2 l_2) + \frac{\alpha^2\mathcal{L}^2}{2}\right\}.\end{aligned} \qquad (85)$$

It is easy to show that the vector

$$\mathbf{m}_1 l_1 - \mathbf{m}_2 l_2 = \left(\mathbf{r}^* - \mathbf{r}'\right) + \left(\mathbf{r}^* - \mathbf{r}''\right) = \left(x'' - x', y' - y''\right)$$

is dissymmetric to the vector $\mathbf{r}''-\mathbf{r}'$ with respect to the direction of $\mathbf{m}_1$ (i.e., to the wind direction). Because we assumed the symmetry of the spectrum (and the structure function) with respect to wind direction, we obtain from this symmetry:

$$D\left(\mathbf{m}_1 l_1(\mathbf{r}', \mathbf{r}'') - \mathbf{m}_2 l_2(\mathbf{r}', \mathbf{r}'')\right) = D\left(\mathbf{r}'' - \mathbf{r}'\right). \qquad (86)$$

Thus, we can simplify formula (85) and write

$$\begin{aligned}\Theta_\Delta(\alpha) &= \left\langle \exp\left\{i\alpha\left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right]\right\}\right\rangle\\ &\approx \sum_\mu P_\mu \exp\left\{i\alpha\mathcal{L} - \frac{1}{2}\alpha^2\left[\lambda_\mu D\left(\mathbf{r}'' - \mathbf{r}'\right) - \mathcal{L}^2\right]\right\}.\end{aligned} \qquad (87)$$

It follows from (87) that the coefficient following the factor α in the exponent presents the conditional mean value of $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$:

$$\left\langle \left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right] \middle| \mu \right\rangle = \mathcal{L} = \kappa_{1,\mu} l_1 \sqrt{\frac{D(\mathbf{m}_1 l_1)}{l_1^2}} - \kappa_{2,\mu} l_2 \sqrt{\frac{D(\mathbf{m}_2 l_2)}{l_2^2}}, \tag{88}$$

and the coefficient following the factor $\alpha^2/2$ presents the conditional variance of the same difference:

$$\left\langle \left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right]^2 \middle| \mu \right\rangle - \left\langle \left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right] \middle| \mu \right\rangle^2 = \lambda_\mu D(\mathbf{r}'' - \mathbf{r}') - \mathcal{L}^2 \geq 0. \tag{89}$$

From comparison of (89) and (88) it follows that

$$\left\langle \left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right]^2 \middle| \mu \right\rangle = \lambda_\mu D(\mathbf{r}'' - \mathbf{r}'). \tag{90}$$

This result extends (38), (39) to an arbitrarily directed argument of the conditional structure function.

6.1 The CF for an Arbitrarily Directed Slope

Let us set in (87)

$$\mathbf{r}' = \mathbf{r} + \frac{\rho}{2}, \quad \mathbf{r}'' = \mathbf{r} - \frac{\rho}{2}, \tag{91}$$

and consider the case $|\rho| \to 0$. For the difference $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$ we obtain:

$$\zeta(\mathbf{r}') - \zeta(\mathbf{r}'') \approx \rho \nabla \zeta(\mathbf{r}) + \cdots = \rho \gamma(\mathbf{r}) + \cdots. \tag{92}$$

Because in the chosen coordinate system we have

$$\rho = (l_1, l_2) \text{ and } \gamma = (\gamma_1, \gamma_2),$$

we can also write

$$\zeta(\mathbf{r}') - \zeta(\mathbf{r}'') = l_1 \gamma_1 + l_2 \gamma_2 + \cdots. \tag{93}$$

For the values entering in (87) for $l_1 \to 0$, $l_2 \to 0$ we obtain

$$\frac{D(\mathbf{m}_1 l_1(\mathbf{r}', \mathbf{r}''))}{l_1^2(\mathbf{r}', \mathbf{r}'')} \to \left\langle \gamma_1^2 \right\rangle, \quad \frac{D(\mathbf{m}_2 l_2(\mathbf{r}', \mathbf{r}''))}{l_2^2(\mathbf{r}', \mathbf{r}'')} \to \left\langle \gamma_2^2 \right\rangle \tag{94}$$

and (because $\langle \gamma_1 \gamma_2 \rangle = 0$),

$$D\left(\mathbf{r}'' - \mathbf{r}'\right) \to \left\langle \left[l_1\gamma_1 + l_2\gamma_2\right]^2 \right\rangle = l_1^2 \left\langle \gamma_1^2 \right\rangle + l_2^2 \left\langle \gamma_2^2 \right\rangle. \tag{95}$$

Thus, for $l_1 \to 0$, $l_2 \to 0$ we can write denoting

$$\mathcal{A}\left(\alpha, \beta\right) \equiv \alpha \kappa_{1,\mu} \sqrt{\left\langle \gamma_1^2 \right\rangle} - \beta \kappa_{2,\mu} \sqrt{\left\langle \gamma_2^2 \right\rangle}, \tag{96}$$

$$\begin{aligned} &\left\langle \exp\left[i\alpha\left(l_1\gamma_1 + l_2\gamma_2\right)\right]\right\rangle = \Theta_\gamma\left(\alpha\rho\right) \\ &\approx \sum_\mu P_\mu \exp\left\{ i\alpha\mathcal{A}\left(l_1, l_2\right) - \frac{1}{2}\alpha^2 \left\{ \lambda_\mu \left[l_1^2 \left\langle \gamma_1^2 \right\rangle + l_2^2 \left\langle \gamma_2^2 \right\rangle \right] - \mathcal{A}^2\left(l_1, l_2\right) \right\} \right\}. \end{aligned} \tag{97}$$

The function $\Theta_\gamma\left(\alpha\rho\right)$ really depends on the product $\alpha\rho = \beta = (\alpha l_1, \alpha l_2)$:

$$\begin{aligned} &\left\langle \exp\left[i\beta\gamma\left(\mathbf{r}\right)\right]\right\rangle \equiv \Theta_\gamma\left(\beta\right) \\ &\approx \sum_\mu P_\mu \exp\left\{ i\mathcal{A}\left(\beta_1, \beta_2\right) - \frac{1}{2}\left\{ \lambda_\mu \left[\beta_1^2 \left\langle \gamma_1^2 \right\rangle + \beta_2^2 \left\langle \gamma_2^2 \right\rangle \right] - \mathcal{A}^2(\beta_1, \beta_2) \right\} \right\}. \end{aligned} \tag{98}$$

We can present (98) in another form, if we set

$$\beta_1 = \beta \cos\psi, \quad \beta_2 = \beta \sin\psi, \tag{99}$$

where ψ is the angle with respect to wind direction. In this case,

$$\gamma_1 \cos\psi + \gamma_2 \sin\psi \equiv \gamma\left(\psi\right) \tag{100}$$

is the slope in ψ-direction and

$$\beta_1^2 \left\langle \gamma_1^2 \right\rangle + \beta_2^2 \left\langle \gamma_2^2 \right\rangle = \beta^2 \left[\cos^2\psi \left\langle \gamma_1^2 \right\rangle + \sin^2\psi \left\langle \gamma_2^2 \right\rangle\right] = \beta^2 \left\langle \gamma^2\left(\psi\right) \right\rangle. \tag{101}$$

The last equality is true because $\langle \gamma_1 \gamma_2 \rangle = 0$. We emphasize that formula (101) describes the dependence of the rms of slope on the direction. This dependence is universal; it follows only from the symmetry of the spectrum with respect to wind direction and does not depend on the PDF. (See Appendix A, Part II for a derivation of (101) that is based only on the symmetry of the spectrum).

The scalar product $\beta\gamma(\mathbf{r})$ takes the form

$$\beta\gamma(\mathbf{r}) = \beta_1\gamma_1 + \beta_2\gamma_2 = \beta[\gamma_1\cos\psi + \gamma_2\sin\psi] = \beta\gamma(\psi). \tag{102}$$

Substituting (99) to (102) in (98), we obtain:

$$\begin{aligned}\langle\exp[i\beta\gamma(\psi)]\rangle &\equiv \Theta_{\gamma(\psi)}(\beta)\\ &\approx \sum_\mu P_\mu \exp\left\{i\beta\mathcal{A}(\psi) - \frac{\beta^2}{2}\left\{\lambda_\mu\left\langle\gamma^2(\psi)\right\rangle - \mathcal{A}^2(\psi)\right\}\right\},\end{aligned} \tag{103}$$

where

$$\mathcal{A}(\psi) \equiv \mathcal{A}(\cos\psi, \sin\psi) = \kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle}\cos\psi - \kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle} \tag{104}$$

7. MULTIVARIATE PDF FOR DIFFERENCES $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$

For many problems it is necessary to know the joint CF for several differences of the type

$$\Delta_1\left(\mathbf{r}_1', \mathbf{r}_1''\right) = \zeta\left(\mathbf{r}_1'\right) - \zeta\left(\mathbf{r}_1''\right), ..., \Delta_n\left(\mathbf{r}_n', \mathbf{r}_n''\right) = \zeta\left(\mathbf{r}_n'\right) - \zeta\left(\mathbf{r}_n''\right). \tag{105}$$

For example, such CFs appear in the theory of wave scattering from rough surfaces; they contain all the information necessary to calculate the scattering cross sections.

We seek this CF,

$$\Theta_\Delta(\alpha_1, ..., \alpha_n) \equiv \left\langle\exp\left\{i\sum\alpha_i\Delta_i\left(\mathbf{r}_i', \mathbf{r}_i''\right)\right\}\right\rangle, \tag{106}$$

in the form

$$\begin{aligned}\Theta_\Delta(\alpha_1, ..., \alpha_n) \approx \sum_\mu P_\mu \exp\Bigg\{ & i\sum_{i=1}^n \alpha_i\left\langle \Delta_i\left(\mathbf{r}_i', \mathbf{r}_i''\right)\middle|\,\mu\right\rangle\\ & - \frac{1}{2}\sum_{i,k=1}^n B_{ik}\left(\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_k', \mathbf{r}_k''|\mu\right)\alpha_i\alpha_k\Bigg\},\end{aligned} \tag{107}$$

with the same values P_μ, $\kappa_{1,\mu}$, $\kappa_{2,\mu}$, and λ_μ that have already been determined. Here,

$$\begin{aligned}& B_{ik}\left(\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_k', \mathbf{r}_k''|\mu\right)\\ &= \left\langle \Delta_i\left(\mathbf{r}_i', \mathbf{r}_i''\right)\Delta_k\left(\mathbf{r}_k', \mathbf{r}_k''\right)\middle|\,\mu\right\rangle - \left\langle \Delta_i\left(\mathbf{r}_i', \mathbf{r}_i''\right)\middle|\,\mu\right\rangle\left\langle \Delta_k\left(\mathbf{r}_k', \mathbf{r}_k''\right)\middle|\,\mu\right\rangle.\end{aligned} \tag{108}$$

Using the Yaglom identity (40), we obtain:

$$\begin{aligned}
&\left\langle \Delta_i\left(\mathbf{r}_i', \mathbf{r}_i''\right) \Delta_j\left(\mathbf{r}_j', \mathbf{r}_j''\right) \middle|\, \mu \right\rangle \\
&\quad \equiv \left\langle \left[\zeta\left(\mathbf{r}_i'\right) - \zeta\left(\mathbf{r}_i''\right)\right]\left[\zeta\left(\mathbf{r}_j'\right) - \zeta\left(\mathbf{r}_j''\right)\right] \middle|\, \mu \right\rangle \\
&\quad = \frac{1}{2}\left\{ \left\langle \left[\zeta\left(\mathbf{r}_i'\right) - \zeta\left(\mathbf{r}_j''\right)\right]^2 \middle|\, \mu \right\rangle + \left\langle \left[\zeta\left(\mathbf{r}_j'\right) - \zeta\left(\mathbf{r}_i''\right)\right]^2 \middle|\, \mu \right\rangle \right. \\
&\qquad \left. - \left\langle \left[\zeta\left(\mathbf{r}_i'\right) - \zeta\left(\mathbf{r}_j'\right)\right]^2 \middle|\, \mu \right\rangle - \left\langle \left[\zeta\left(\mathbf{r}_i''\right) - \zeta\left(\mathbf{r}_j''\right)\right]^2 \middle|\, \mu \right\rangle \right\}.
\end{aligned} \tag{109}$$

But for the arbitrarily directed, conditional, mean value and structure function we have already obtained formulae (88) and (90):

$$\left\langle \left[\zeta\left(\mathbf{r}'\right) - \zeta\left(\mathbf{r}''\right)\right] \middle|\, \mu \right\rangle = \left\langle \Delta\left(\mathbf{r}', \mathbf{r}''\right) \middle|\, \mu \right\rangle = \mathcal{L}\left(\nabla', \mathbf{r}''\right), \tag{110}$$

$$\left\langle \left[\zeta\left(\mathbf{r}'\right) - \zeta\left(\mathbf{r}''\right)\right]^2 \middle|\, \mu \right\rangle = \lambda_\mu D\left(\mathbf{r}'' - \mathbf{r}'\right), \tag{111}$$

where, according to (79),

$$l_1 = \left(\mathbf{r}'' - \mathbf{r}'\right)\mathbf{m}_1 = x'' - x', \quad l_2 = \left(\mathbf{r}'' - \mathbf{r}'\right)\mathbf{m}_2 = y'' - y'. \tag{112}$$

(i.e., the arguments of the anisotropic structure functions in (110) are the upwind and the cross-wind components of the vector $\mathbf{r}'' - \mathbf{r}'$).

Substituting (111) in (109), we obtain

$$\begin{aligned}
&\left\langle \Delta_i\left(\mathbf{r}_i', \mathbf{r}_i''\right) \Delta_j\left(\mathbf{r}_j', \mathbf{r}_j''\right) \middle|\, \mu \right\rangle \\
&\equiv \frac{\lambda_\mu}{2}\left\{ D\left(\mathbf{r}_i' - \mathbf{r}_j''\right) + D\left(\mathbf{r}_j' - \mathbf{r}_i''\right) - D\left(\mathbf{r}_i' - \mathbf{r}_j'\right) - D\left(\mathbf{r}_i'' - \mathbf{r}_i''\right) \right\},
\end{aligned} \tag{113}$$

and the formula (108) for $B_{ij}\left(\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_j', \mathbf{r}_j'' | \mu\right)$ takes the form

$$\begin{aligned}
&B_{ij}\left(\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_j', \mathbf{r}_j'' | \mu\right) \\
&\quad = \frac{\lambda_\mu}{2}\left[D\left(\mathbf{r}_i' - \mathbf{r}_j''\right) + D\left(\mathbf{r}_j' - \mathbf{r}_i''\right) - D\left(\mathbf{r}_i' - \mathbf{r}_j'\right) - D\left(\mathbf{r}_i'' - \mathbf{r}_i''\right)\right] \\
&\qquad - \mathcal{L}\left(\mathbf{r}_i', \mathbf{r}_i''\right)\mathbf{L}\left(\mathbf{r}_j', \mathbf{r}_j''\right).
\end{aligned} \tag{114}$$

Formula (107), where $\left\langle \Delta_i\left(\mathbf{r}_i', \mathbf{r}_i''\right) \middle|\, \mu \right\rangle$ is determined by (110), B_{ij} is determined by (114), and $l_1\left(\mathbf{r}_j', \mathbf{r}_j''\right), l_2\left(\mathbf{r}_j', \mathbf{r}_j''\right)$ are determined by

(112), presents the joint multivariate CF for several arbitrarily directed differences in elevation.

8. CONCLUSIONS. COMPARISON WITH OTHER METHODS OF STATISTICAL DESCRIPTION OF SEA SURFACES

There are several different approaches to the problem of the statistical description of sea surfaces. All of these approaches are based on the general theory of random functions (see, e.g., [1, 22–24]. The paper of [9], devoted to random surfaces, served as a starting point for works describing the statistics of nonlinear surface waves. [9] is based on a special model of the rough surface. This model is equivalent to the following representation of a random 2-D field:

$$\zeta(\mathbf{r}) = \iint \xi(\mathbf{q}) \exp(i\mathbf{q}\mathbf{r})\, d^2q. \tag{115}$$

Here, the random spectral density $\xi(\mathbf{q})$ is determined by the following relations:

$$\begin{aligned} &\xi(\mathbf{q}) \quad \text{is Gaussian random function} \\ &\langle \xi(\mathbf{q}) \rangle = 0 \\ &\left\langle \xi(\mathbf{q}')\, \xi^*(\mathbf{q}'') \right\rangle = E(\mathbf{q}')\, \delta(\mathbf{q}' - \mathbf{q}'') \\ &\left\langle \xi(\mathbf{q}')\, \xi(\mathbf{q}'') \right\rangle = \left\langle \xi^*(\mathbf{q}')\, \xi^*(\mathbf{q}'') \right\rangle = 0. \end{aligned} \tag{116}$$

Representation (116) is widely used in the theory of turbulence [25] and wave propagation in random media [1, 24].[8] Thus, the random surfaces considered in [9] are Gaussian.

In a subsequent paper of [26], the model of random functions developed in [9], was applied to nonlinear surface gravity waves. In this case, the surface is non-Gaussian and the following decomposition (in terms of (116)) was used:

[8] In paper [9] the more cumbersome representation that includes the finite sum $\zeta(\mathbf{r}) = \operatorname{Re} \sum_{n=1}^{N} c_n \exp(i\mathbf{k}_n\mathbf{r})$ with the random coefficients c_n and the following limiting process $N \to \infty$, was used. But all of the results of this approach are obtainable from a more compact representation (116).

$$
\begin{aligned}
\zeta(\mathbf{r}) = & \iint \xi(\mathbf{q}) \exp(i\mathbf{q}\mathbf{r})\, d^2 q \\
& + \iint d^2 q_1 \iint d^2 q_2 \exp\left[i\left(\mathbf{q}_1 + \mathbf{q}_2\right)\mathbf{r}\right] C_2\left(\mathbf{q}_1, \mathbf{q}_2\right) \xi\left(\mathbf{q}_1\right) \xi\left(\mathbf{q}_2\right) \\
& + \iint d^2 q_1 \iint d^2 q_2 \iint d^2 q_3 \exp\left[i\left(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3\right)\mathbf{r}\right] \\
& \times C_3\left(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\right) \xi\left(\mathbf{q}_1\right) \xi\left(\mathbf{q}_2\right) \xi\left(\mathbf{q}_3\right) + \cdots .
\end{aligned}
\tag{117}
$$

The coefficients $C_k(\mathbf{q}_1, ..., \mathbf{q}_k)$ were determined by the substitution of (117) in the hydrodynamic equations, expanded in the perturbation series in powers of ζ. As a result, the expansion of the non-Gaussian PDF in the Gram-Charlier series was obtained. This method describes only small deviations from the Gaussian distribution, because it uses the perturbation expansion. The random Gaussian field $\xi(\mathbf{q})$, entering in (117), is completely auxiliary and has no direct meaning.

The model of [26] was used in [27] for description of radar impulses reflection from the sea surface in GO approximation. This model was extended in [28] for the joint PDF of elevation and two slopes, and applied to radar altimetry.

In [29] the method of [9] was generalized for random Stokes waves. This work also starts from the auxiliary Gaussian field, but the field undergoes some nonlinear transforming, induced by the shape of the Stokes wave. As a result, an explicit formula for the PDF of elevations was obtained. In paper [30] the same method was applied to the joint PDF of elevation and slope for the random Stokes waves. This was possible because of the dynamic relationship between the elevation and slope for the Stokes waves. In [31] the restrictions related to the appearance of breaking waves were included in the consideration.[9]

[9] The assumption that the first-order solution is Gaussian, used in [26] and in the many subsequent papers, is an additional assumption. It is rather difficult to ground this assumption, because in the presence of nonlinear effects, the first-order component by itself has no physical meaning. It is applied to a nonexistent physical object. If the waves are really linear (amplitudes are very small), the Gaussian PDF seems to be natural, but this fact has no relation to the first-order component of the nonlinear waves. Thus, this assumption can be considered

The main goal of the present paper is to develop a model of the sea surface that will allow us to calculate the CF of the surface of an arbitrary order, and at the same time satisfy necessary conditions for the second-order PDF of slopes and for the spectrum. Such CFs appear in the modern theories of rough surface scattering (see section 2 of this paper). The above-mentioned sea-surface models do not allow us to achieve this goal. For instance, the approach of papers [26] and [28] leads to the truncated Gram-Charlier series that necessarily entail negative probabilities. The method of papers [29, 30] and [31] is free from this disadvantage, but it does not allow us to find the high-order CFs, nor does the method of [26] and [28]. For instance, none of the above-mentioned methods allows us to calculate the scattering cross sections even in the Kirchhoff approximation; only the simplest GO approximation can be considered. The method developed in this paper allows us to calculate a scattering cross section in any scattering theory (see section 2).

Another difference in the method presented in this paper is its phenomenological nature. We did not try to utilize dynamical equations of motion, but used the experimental data instead. However, we could have used not only experimental data, but also any results of theoretical consideration.

Usage of the decomposition of the multivariate non-Gaussian PDF in the sum of a Gaussian PDF allows us to describe such a non-Gaussian PDF without the difficulties related to the truncation of the Gram-Charlier series. This method can be applied to various problems dealing with non-Gaussian distributions.

We should emphasize that the model of the random surface developed is *not ergodic*. This means that it is impossible to create a single surface large enough that the averaging over this single surface leads to the same mean values as statistical averaging. Each realization of the surface has the (conditional) Gaussian PDF. If we want to use this mathematical model for some numerical method of calculation of the scattering field, and apply the Monte Carlo simulation method (instead of analytical averaging) we must first prepare the set of models of Gaussian random surfaces. Each of these surfaces must have its specific values of λ_μ, $\kappa_{\mu,1}$, and $\kappa_{\mu,2}$, and the total number of sur-

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as a convenient and effective working hypothesis, but only successful comparison with the experimental data can serve as a justification for its use.

faces having these parameters and included in the ensemble must be proportional to P_μ.[10]

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[10] If we put all of these samples on a single joint surface, we obtain a nonhomogeneous random surface with the *Gaussian* PDF and the correlation function, depending not only on the differences of coordinates, but on both coordinates. The sum of the partial PDF of the form $W = \sum_\mu P_\mu W_\mu$ corresponds to the random choice of the surface numbered by μ with the probability P_μ.

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