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NATURE OF ELECTROMAGNETIC FIELD AND ENERGY BEHAVIOR IN A PLANE RESONATOR WITH MOVING BOUNDARY

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1. INTRODUCTION

In many situations of convenient importance, engineers and physicists are confronted with effects begotten by the interaction of the electromagnetic waves with the mobile surface of electromagnetic systems. Many studies have been made in this area employing, in general, approximate mathematical methods [1–13]. Our main in this work is to analyze the nature of electromagnetic field and the energy behavior in a plane resonator constituted by two parallel, perfectly conducting plates, one of which is moving normally to itself in arbitrary fashion. Through a simple transformation of Maxwell's equations, the formulation of our problem is reduced to a one-dimensional hyperbolic initial-boundary-value problem with moving boundary conditions. For solving this problem, the analytical method given by the author in previous paper [14] is used. The exact expressions of field are found in the form of a modal nature. We remark that every mode has amplitude and a frequency depending on time. It is shown, also, that the energy grows when the space between the two walls is reduced.

2. FORMULATION OF PROBLEM

Consider the electromagnetic resonator constituted by two perfectly conducting plates of infinite extend. We take one plane to be in the yz axis plane and to cut the x-axis at $x = 0$. The second plane moves in arbitrary fashion, cutting the x-axis at $x = a(\tau)$, where $\tau = ct$ and c is the speed of light in vacuum. We set $a(0) = a_0$. Assume that $a(\tau)$ is twice continuously differentiable and the velocity of the moving plane is:

$$\left|a'(\tau)\right| < 1 \tag{1}$$

We suppose that before $\tau = 0$ the two planes were fixed and electromagnetic oscillations in the system were present.

The electromagnetic field is characterized by the electric vector $\vec{E}$ and magnetic vector $\vec{H}$ as follow:

$$\vec{E} = \{0, E, 0\}, \qquad \vec{H} = \{0, 0, H\} \tag{2}$$

Because of the geometry, the functions E and H depend only on x and τ

$$E = E(x, \tau), \qquad H = H(x, \tau) \tag{3}$$

From Maxwell's equations expressed in Gauss's system, we deduct the partial differential equations at which E and H must obey:

$$\frac{\partial H}{\partial x} + \frac{\partial E}{\partial \tau} = -\frac{4\pi}{c} J \tag{4}$$

$$\frac{\partial H}{\partial \tau} + \frac{\partial E}{\partial x} = 0 \tag{5}$$

where $J = J(x, \tau)$ is the current in the resonator flowing along the y axis The condition for the tangential part of electrical field $E_T = 0$ is expressed respectively for the fixed and moving wall by:

$$E(0, \tau) = 0 \tag{6}$$

$$E(a(\tau), \tau) - a'(\tau) H(a(\tau), \tau) = 0 \tag{7}$$

The condition (7) is the so-called "instantaneous rest-frame hypothesis" for a moving boundary [1, 15] .

Next for convenience, we introduce a new function $U(x,\tau)$ definite by the following relations:

$$E(x,\tau) = -\frac{\partial U(x,\tau)}{\partial \tau} \tag{8}$$

$$H(x,\tau) = \frac{\partial U(x,\tau)}{\partial x} \tag{9}$$

We assume that $U(x,\tau)$ is twice continually differentiable and $U(0,0) = 0$. By means of the function U, the system of equations (4, 5) is reduced in an alone differential equation to the partial derivatives of hyperbolic type. Indeed, in combining the relations (8) and (9) we obtain:

$$U_{\tau\tau}(x,\tau) - U_{xx}(x,\tau) = \theta(x,\tau) \tag{10}$$

$$\text{with } \theta(x,\tau) = -\frac{4\pi}{c} j(x,\tau) \tag{11}$$

The condition (6) expresses comfortably by:

$$U(0,\tau) = 0 \tag{12}$$

The condition (7) transforms as follow:

$$\left[\frac{\partial U}{\partial \tau} + \frac{\partial U}{\partial x} \cdot \frac{dx}{d\tau}\right]_{x=a(\tau)} = 0, \tag{13}$$

the expression (13) is the total derivative of the U function with regard to the variable τ. Consequently, we can write (13) under its original form, such that:

$$\left.\frac{dU(x,\tau)}{d\tau}\right|_{x=a(\tau)} = 0, \tag{14}$$

after integration, we obtain the condition at which the function U must satisfy on the mobile wall. It expresses by:

$$U(a(\tau),\tau) = 0 \tag{15}$$

Finally, the determination of electromagnetic field in the considered resonator amounts to solving the following one-dimensional hyperbolic initial-boundary-value problem:

$$U_{\tau\tau}(x,\tau) - U_{xx}(x,\tau) = \theta(x,\tau), \quad \tau \geq 0, \quad 0 \leq x \leq a(\tau) \tag{16}$$

with the initial conditions

$$U_\tau(x,0) = -E(x) \tag{17}$$
$$U_x(x,0) = H(x) \tag{18}$$

and the boundary conditions

$$U(0,\tau) = 0 \tag{19}$$
$$U(a(\tau),\tau) = 0 \tag{20}$$

$E(x)$ and $H(x)$ are the initial fields in the static resonator, $0 \leq x \leq a_0$, when the moving wall is stationary.

3. GENERAL SOLUTION FOR AN ARBITRARY MOVING BOUNDARY

By using the analytical method given in previous paper [14], we obtain the exact solution of the above homogeneous problem (i.e., $J = 0$). This solution expresses by the following relations:

$$U(\tau,x) = \sum_{-\infty}^{+\infty} A_n \left\{ \exp\left[\frac{i\pi n}{\eta_0}\psi(\tau+x)\right] - \exp\left[\frac{i\pi n}{\eta_0}\psi(\tau-x)\right]\right\} \tag{21}$$

$$A_n = \frac{1}{4i\pi n}\int_{-a_0}^{+a_0} \chi(x)\exp\left[\frac{-i\pi n}{\eta_0}\psi(x)\right] dx \tag{22}$$

with

$$\chi(x) = H(x) - E(x) \qquad \text{if } x \geq 0 \tag{23}$$

$$\chi(x) = H(x) + E(x) \qquad \text{if } x \leq 0 \tag{24}$$

From (8, 9) we deduct the exact expressions of electric and magnetic field

$$E(\tau,x) = -\sum_{-\infty}^{+\infty} \frac{i\pi n}{\eta_0} A_n \left\{\psi'(\tau+x)\exp\left[\frac{i\pi n}{\eta_0}\psi(\tau+x)\right] - \psi'(\tau-x)\exp\left[\frac{i\pi n}{\eta_0}\psi(\tau-x)\right]\right\} \tag{25}$$

$$H(\tau,x)=\sum_{-\infty}^{+\infty}\frac{i\pi n}{\eta_0}A_n$$
$$\left\{\psi'(\tau+x)\exp\left[\frac{i\pi n}{\eta_0}\psi(\tau+x)\right]+\psi'(\tau-x)\exp\left[\frac{i\pi n}{\eta_0}\psi(\tau-x)\right]\right\}\quad(26)$$

Remark

We remind that Ψ is the inverse of an analytical function F permitting a conformal mapping between the time varying domain $0\leq x\leq a(\tau)$ of (τ,x) plane and a band $(\eta=0,\ \eta=\eta_0)$ of (ξ,η) plane. The functions Ψ and F depend on the moving boundary law. The following relation gives the width of the band (see [14]):

$$\eta_0=\frac{1}{2}\left[\psi(\tau+a(\tau))-\psi(\tau-a(\tau))\right]\quad(27)$$

Commentaries

The expressions (25) and (26) give the analytic solution of electromagnetic field in the resonator with a moving boundary of arbitrary manner.

By taking into account (27) we can show easily that the expressions of the fields verify the boundary conditions (6) and (7) Furthermore we remark, that the electromagnetic field has a modal nature. Every mode is constituted by two waves progressing in opposite directions. According to η_0, the modes are dynamics. We note also the presence of a double generalized Doppler shift.

It is completely apparent that if the speed of the moving boundary becomes null, then the functions F and Ψ become identity functions. Thus, we find again all the features of stationary resonator.

4. COMPUTATION OF RATE OF CHANGE OF ENERGY

In the goal of knowing the influence of the mobile wall on the energy, the best is of calculating its rate of change.

Because of the moving boundary the energy is not preserved and its expression for every solution is given by the relation [1]

$$e(\tau)=\int_0^{a(\tau)}\left[E^2(x,\tau)+H^2(x,\tau)\right]dx\quad(28)$$

The following relation express the rate of change:

$$\frac{de(\tau)}{d\tau}=\frac{\partial}{\partial\tau}\int_0^{a(\tau)}\left[E^2(x,\tau)+H^2(x,\tau)\right]dx\quad(29)$$

According to the rules of integration with variable limits, expression (29) becomes:

$$\begin{aligned}\frac{de(\tau)}{d\tau} = &\int_0^{a(\tau)} \frac{\partial}{\partial\tau}\left[E^2(x,\tau)+H^2(x,\tau)\right]dx \\ &+ a'(\tau)\left[E^2(a(\tau),\tau)+H^2(a(\tau),\tau)\right]\end{aligned} \tag{30}$$

Or

$$\begin{aligned}\frac{de(\tau)}{d\tau} = &\int_0^{a(\tau)} \left[2E\frac{\partial}{\partial\tau}E(x,\tau)+2H\frac{\partial}{\partial\tau}H(x,\tau)\right]dx \\ &+ a'(\tau)\left[E^2(a(\tau),\tau)+H^2(a(\tau),\tau)\right]\end{aligned} \tag{31}$$

By taking into account equations (4) and (5) for $J = 0$, the relation (31) writes:

$$\begin{aligned}\frac{de(\tau)}{d\tau} = &\int_0^{a(\tau)} \left[-2E\frac{\partial}{\partial x}H(x,\tau)-2H\frac{\partial}{\partial x}E(x,\tau)\right]dx \\ &+ a'(\tau)\left[E^2(a(\tau),\tau)+H^2(a(\tau),\tau)\right]\end{aligned} \tag{32}$$

Or

$$\begin{aligned}\frac{de(\tau)}{d\tau} &= -2\int_0^{a(\tau)} \left[\frac{\partial}{\partial x}E(x,\tau)\cdot H(x,\tau)\right]dx \\ &\quad + a'(\tau)\left[E^2(a(\tau),\tau)+H^2(a(\tau),\tau)\right] \\ &= -2E(a(\tau),\tau)\cdot H(a(\tau),\tau) \\ &\quad + a'(\tau)\left[E^2(a(\tau),\tau)+H^2(a(\tau),\tau)\right]\end{aligned} \tag{33}$$

The condition (7) allows us to express the rate of change of the energy by the more explicit following relation:

$$\frac{de(\tau)}{d\tau} = a'(\tau)\left[a'^2(\tau)-1\right]\cdot H^2(a(\tau),\tau) \tag{34}$$

The above result is in perfect agreement with the one found by J. Cooper in 1993 [1].

Commentary

Since $|a'(\tau)| < 1$, we note, then, that the sign of rate of change of energy depends of the one of the speed of the moving wall. Consequently, the energy increases every time that the wave is in contact with the mobile wall and that the speed of this is negative (the space between the two planes reduces). The mechanical energy of the moving wall is thus converted in the energy of the wave. Conversely, the energy decreases when the speed is positive.

5. APPLICATION FOR A LINEAR BOUNDARY MOTION

Suppose that the moving wall moves following the linear law:

$$a(\tau) = a_0 + \beta\tau \tag{35}$$

where $\beta = a'(\tau) = \nu/c$, $\tau = ct$, ν is the speed of the moving wall and c the one of light in vacuum.

According to (25) and (26), the electromagnetic field depend on Ψ, Ψ' and η_0. With the help of [14], we have:

$$\begin{aligned} &\psi(\tau \pm x) = \ln\left(1 + \frac{\beta}{a_0}(\tau \pm x)\right), \\ &\psi'(\tau \pm x) = \frac{\beta}{a_0 + \beta(\tau \pm x)}, \qquad \eta_0 = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \end{aligned} \tag{36}$$

The A_n coefficients are determining from the initial conditions deducted from stationary resonator that is when $0 \leq x \leq a_0$. It is obvious that anyone of the eigen functions of static resonator can be chosen as initial condition. So for a chosen m state of stationary resonator, the expressions of fields write as follow:

$$E(x) = -\frac{i\pi m}{a_0} A_m \left[\exp\left(\frac{i\pi m}{a_0}x\right) - \exp\left(-\frac{i\pi m}{a_0}x\right)\right] \tag{37}$$

$$H(x) = \frac{i\pi m}{a_0} A_m \left[\exp\left(\frac{i\pi m}{a_0}x\right) + \exp\left(-\frac{i\pi m}{a_0}x\right)\right] \tag{38}$$

We deduct easily the coefficients A_n by:

$$A_n = \frac{1}{4i\pi n} \cdot \frac{2i\pi m}{a_0} A_m \int_{-a_0}^{+a_0} \exp\left(\frac{i\pi m}{a_0}x\right) \cdot \exp\left[\frac{-i\pi n}{\eta_0}\psi(x)\right] dx \tag{39}$$

Remark

The coefficient A_n can be chosen real or imaginary. Thus, there are two types of initial conditions namely of electric type $E(x)$ and magnetic type $H(x)$:

For $A_m = \frac{\hat{A}}{2}$ real

$$\hat{E}(x) = \frac{\pi m}{a_0}\hat{A}\sin\left(\frac{\pi m x}{a_0}\right), \qquad \hat{H}(x) = 0 \tag{40}$$

For $A_m = -i\frac{\hat{A}}{2}$ imaginary

$$\hat{H}(x) = \frac{\pi m}{a_0}\hat{A}\cos\left(\frac{\pi m x}{a_0}\right), \qquad \hat{E}(x) = 0 \tag{41}$$

Where $\hat{A}$ is real.

Until present, we have supposed that the movement of the wall was relativistic, that is when $\beta = a'(\tau) = \nu/c < 1$. As the wall of resonator moves only with non-relativistic speeds $(\beta \ll 1)$, then the following approximations hold extremely well.

$$\psi(x) = \ln\left(1 + \frac{\beta}{a_0}\right) \approx \frac{\beta}{a_0}x \tag{42}$$

$$\eta_0 = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \approx \beta \tag{43}$$

With the above approximations the relation (39) giving the A_n coefficients becomes:

$$\begin{aligned} A_n &= \frac{1}{4i\pi n}\cdot\frac{2i\pi m}{a_0}A_m\int_{-a_0}^{+a_0}\exp\left(\frac{i\pi m}{a_0}x\right)\cdot\exp\left[\frac{-i\pi n}{\beta}\cdot\frac{\beta}{a_0}x\right]dx \\ &= \frac{1}{4i\pi n}\cdot\frac{2i\pi m}{a_0}A_m\int_{-a_0}^{+a_0}\exp\left[\frac{i\pi}{a_0}(m-n)x\right]dx \end{aligned} \tag{44}$$

According to the orthogonality relations, we have:

$$\int_{-a_0}^{+a_0}\exp\left[\frac{i\pi}{a_0}(m-n)x\right]dx = 2a_0\delta_{mn} \tag{45}$$

Finally, we obtain for A_n the following result:

$$A_n = A_m\delta_{mn}$$

With $\delta_{mn} = 1$ if $m = n$ and 0 if $m \neq n$

The result (46) shows that for a non-relativistic moving boundary, the electromagnetic field in the resonator presents in form of a functional Fourier series with only one term $n = m$. Thus, the non-relativistic solution is given by:

$$U(x,\tau) = A_m\left[e^{\frac{i\pi m}{\beta}\psi(\tau+x)} - e^{\frac{i\pi m}{\beta}\psi(\tau-x)}\right] \tag{47}$$

$$E(x,\tau) = -\frac{im\pi}{\beta}A_m\left[\psi'(\tau+x)e^{\frac{i\pi m}{\beta}\psi(\tau+x)} - \psi'(\tau-x)e^{\frac{i\pi m}{\beta}\psi(\tau-x)}\right] \tag{48}$$

$$H(x,\tau) = \frac{im\pi}{\beta}A_m\left[\psi'(\tau+x)e^{\frac{i\pi m}{\beta}\psi(\tau+x)} + \psi'(\tau-x)e^{\frac{i\pi m}{\beta}\psi(\tau-x)}\right] \tag{49}$$

Where $\psi(\tau \pm x)$ and $\psi'(\tau \pm x)$ are given by (36).

The above expressions are expressed in their complex form. It is sometimes useful of expressing the electromagnetic field under the real form. By taking into account that A_m can be chosen real or imaginary answering to the two fundamental types of initial conditions, and after transforming the exponential to sinus and cosine functions, we obtain:

a) For $A_m = \frac{\hat{A}}{2}$

$$\begin{aligned}\hat{E}(x,\tau) = \frac{\pi}{2}m\hat{A}\left\{\frac{1}{a_0+\beta(\tau+x)}\cdot\sin\left[\frac{m\pi}{\beta}\psi(\tau+x)\right]\right.\\ \left.-\frac{1}{a_0+\beta(\tau-x)}\cdot\sin\left[\frac{m\pi}{\beta}\psi(\tau-x)\right]\right\}\end{aligned} \tag{50}$$

$$\begin{aligned}\hat{H}(x,\tau) = -\frac{\pi}{2}m\hat{A}\left\{\frac{1}{a_0+\beta(\tau+x)}\cdot\sin\left[\frac{m\pi}{\beta}\psi(\tau+x)\right]\right.\\ \left.+\frac{1}{a_0+\beta(\tau-x)}\cdot\sin\left[\frac{m\pi}{\beta}\psi(\tau-x)\right]\right\}\end{aligned} \tag{51}$$

b) For $A_m = -i\frac{\hat{A}}{2}$

$$\begin{aligned}\hat{E}(x,\tau) = -\frac{\pi}{2}m\hat{A}\left\{\frac{1}{a_0+\beta(\tau+x)}\cdot\cos\left[\frac{m\pi}{\beta}\psi(\tau+x)\right]\right.\\ \left.-\frac{1}{a_0+\beta(\tau-x)}\cdot\cos\left[\frac{m\pi}{\beta}\psi(\tau-x)\right]\right\}\end{aligned} \tag{52}$$

$$\begin{aligned}\hat{H}(x,\tau) = \frac{\pi}{2}m\hat{A}\left\{\frac{1}{a_0+\beta(\tau+x)}\cdot\cos\left[\frac{m\pi}{\beta}\psi(\tau+x)\right]\right.\\ \left.+\frac{1}{a_0+\beta(\tau-x)}\cdot\cos\left[\frac{m\pi}{\beta}\psi(\tau-x)\right]\right\}\end{aligned} \tag{53}$$

Note that in the two cases, the boundary conditions (6) and (7) and the initial conditions are satisfied.

c) Computation of rate of change of energy

The substitution of (50) and (51) into the relation (30) giving the rate of variation of the energy duct to:

$$\begin{aligned}\frac{de(\tau)}{d\tau} = 2\left(\frac{m\pi\hat{A}}{\beta}\right)^2 \int_0^{a(\tau)} \frac{\partial}{\partial\tau}\Bigg\{&\psi'^2(\tau+x)\sin^2\left[\frac{m\pi}{\beta}(\tau+x)\right]\\ &+\psi'^2(\tau-x)\sin^2\left[\frac{m\pi}{\beta}(\tau-x)\right]\Bigg\}dx\\ +\beta 2\left(\frac{m\pi\hat{A}}{\beta}\right)^2 \Bigg\{&\psi'^2(\tau+a(\tau))\sin^2\left[\frac{m\pi}{\beta}\psi(\tau+a(\tau))\right]\\ &+\psi'^2(\tau-a(\tau))\sin^2\left[\frac{m\pi}{\beta}\psi(\tau-a(\tau))\right]\Bigg\}\end{aligned} \tag{54}$$

The expression (54) can also be expressed by:

$$\begin{aligned}\frac{de(\tau)}{d\tau} = 2\left(\frac{m\pi\hat{A}}{\beta}\right)^2 \int_0^{a(\tau)} \frac{\partial}{\partial\tau}\Bigg\{&\psi'^2(\tau+x)\sin^2\left[\frac{m\pi}{\beta}(\tau+x)\right]\\ &-\psi'^2(\tau-x)\sin^2\left[\frac{m\pi}{\beta}(\tau-x)\right]\Bigg\}dx\\ +\beta 2\left(\frac{m\pi\hat{A}}{\beta}\right)^2 \Bigg\{&\psi'^2(\tau+a(\tau))\sin^2\left[\frac{m\pi}{\beta}\psi(\tau+a(\tau))\right]\\ &+\psi'^2(\tau-a(\tau))\sin^2\left[\frac{m\pi}{\beta}\psi(\tau-a(\tau))\right]\Bigg\}\end{aligned} \tag{55}$$

Or

$$\begin{aligned}\frac{de(\tau)}{d\tau} &= -2\int_0^{a(\tau)}\left[\frac{\partial}{\partial x}E(x,\tau)\cdot H(x,\tau)\right]dx\\ &\quad+\beta\left[E^2(a(\tau),\tau)+H^2(a(\tau),\tau)\right]\\ &= -2E(a(\tau),\tau)\cdot H(a(\tau),\tau)\\ &\quad+\beta\left[E^2(a(\tau),\tau)+H^2(a(\tau),\tau)\right]\end{aligned} \tag{56}$$

Finally, amount held of (7), we obtain the following result for the rate of change of energy:

$$\frac{de(\tau)}{d\tau} = \beta\left[\beta^2 - 1\right] \cdot H^2(a(\tau), \tau) \tag{57}$$

Note that (57) is in good agreement with the general results (34) and duct therefore to the same conclusion.

6. CONCLUSION

By means of a simple transformation, Maxwell's equations in the resonator with moving wall are reduced to an initial-boundary-value-problem of the wave equation with a moving boundary. By using the analytical method developed by the author, the exact expressions of electromagnetic field are determined. The analysis of the results shows that:

–The electromagnetic field has a modal nature.

–The modes are dynamic; everyone has an amplitude and a frequency depending on time.

–The presence of a generalized double Doppler shift.

–The transfer of the energy between the moving wall and the wave.

The author believes that this study can be extending to a laser cavity and to possible application to devises for constructing ultra-short pulses and optical modulation.

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