

## **EXPLICIT MATRIX FORMULATION FOR THE ANALYSIS OF SYNTHETIC LINEARLY AND NON LINEARLY LOADED MATERIALS IN FDTD**

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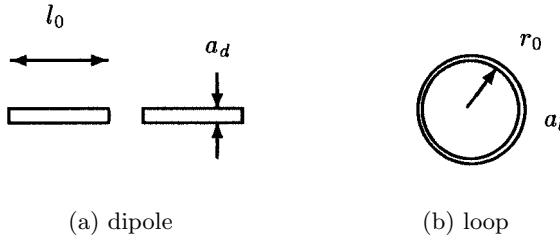
## 1. INTRODUCTION

A number of recent artificial constructs have indicated the possibility of designing the electric or magnetic properties of a medium. Several examples of these artificial materials have been shown in the past few years [1–6]. Most of them present a behavior that can be recovered using two basic material models: the Debye and Lorentz models.

We presented in some previous papers a new way to realize artificial dielectric, magnetic or bianisotropic materials, and how to derive their electromagnetic properties [7–10]. These materials are made of so called molecules (one or more small antennas connected to an electronic load circuit) embedded in a host medium. They can be linear or non linear, depending on the properties of the load circuit. The interaction of an EM wave with this kind of material has been studied, both theoretically and numerically. In this last case, the FDTD algorithm is very well suited. It has been used extensively for modeling effects in complex media including, for example dispersive dielectrics [11–18], ferrites [13, 16, 18–21] and nonlinear dielectrics [16, 22, 23]. It has been coupled very successfully with both linear and nonlinear circuit simulators [7–10, 16, 18, 24–32].

We recall in section 2 the main formulas and properties of these materials, in the case of linear molecules. Depending on the complexity of the load circuit, we recover the well known Debye and Lorentz models but also generalize them to more complicated models, such as the Time Derivative Debye model (TDD), One and Two Time Derivative Lorentz models (TDLM, 2TDLM). But if the load gets more complicated, the implementation of the differential equation describing the behavior of the molecule can become very hard. To overcome this problem, we derived a new matrix formulation method [33], based on a state variables approach developed in [34, 35] which for linear molecules maintains the explicit nature of the usual FDTD scheme. This method is explained in section 2, and the implementation is shown in section 3. Section 4 presents some validation cases, compared with results obtained by the usual FDTD scheme. We present in section 5 the generalization of our formulation to non linear molecules. Several numerical examples are given to illustrate the efficacy of the matrix formulation for both linear and nonlinear molecules.

All the computations use a 1D FDTD program. Because the circuit elements are assumed to be contained within one FDTD cell and result in a system of ordinary differential equations which simply augment the



**Figure 1.** Description of the antennas.

usual Maxwell's equations, this approach can easily be extended to 3D. We chose to work in 1D even though it neglects interesting polarization and directionality issues in order to simplify the presentation of our approach and the associated results.

A  $-j\omega t$  convention is understood and omitted throughout this paper.

## 2. LINEARLY LOADED MOLECULES

### 2.1 Previous Method

We consider here some electric or magnetic molecules, made of electrically small dipole or loop antennas (Fig. 1, i.e.,  $kl_0$ ,  $kr_0 \ll 1$ , where  $2l_0$  is the total length of the dipole,  $r_0$  is the radius of the loop, and  $k$  is the wavenumber in free space) connected to a linear electronic load circuit.

We derived in some earlier articles [7, 8] the following formulas for the equivalent susceptibility of the molecules

$$\chi_e = \frac{K_e}{-j\omega(Z_{in}^d + Z_L^d)} \quad K_e = \frac{l_0^2}{\epsilon_0 V} |\cos \psi_e| \sin \theta$$

$$\chi_m = \frac{K_h Z_L^l}{Z_{in}^l + Z_L^l} \quad K_h = \frac{\mu_0 S^2}{L_l V} |\cos \psi_h| \sin \theta$$

where  $\psi_e$  and  $\psi_h$  are the angles between the axis of the antenna and the incoming field,  $\theta$  is the spherical angle of radiation,  $S = \pi r_0^2$  is the surface of the loop,  $V$  is the small volume around the molecule where the EM properties can be considered as constant,  $Z_{in}^*$  and  $Z_L^*$  are respectively the input and load impedances of the molecules and

the superscripts are ‘ $d$ ’ for the dipole and ‘ $l$ ’ for the loop:

$$Z_{in}^d = \frac{1}{-j\omega C_d} \quad Z_{in}^l = -j\omega L_l$$

$C_d$  is the equivalent capacitance for the small dipole and  $L_l$  is the equivalent inductance for the small loop.  $\epsilon_0$  and  $\mu_0$  are the permittivity and permeability of free space.

This approach enabled us to show that the EM properties of those molecules recover some previously known behaviors (such as Debye and Lorentz) and generalize them to more complex behaviors (Multiple Time Derivative Debye and Lorentz) [9].

However when the load circuit becomes complex, the differential equation describing the interaction of the EM wave with the molecule can become very complicated to derive and implement. For example, if the load is a series LC parallel to a series LR circuit, we have to deal with a fourth order model, characterized by a differential equation of the following type:

$$\sum_{i=0}^4 \alpha_i \frac{\partial^i P}{\partial t^i} = \sum_{j=0}^4 \beta_j \frac{\partial^j E}{\partial t^j}.$$

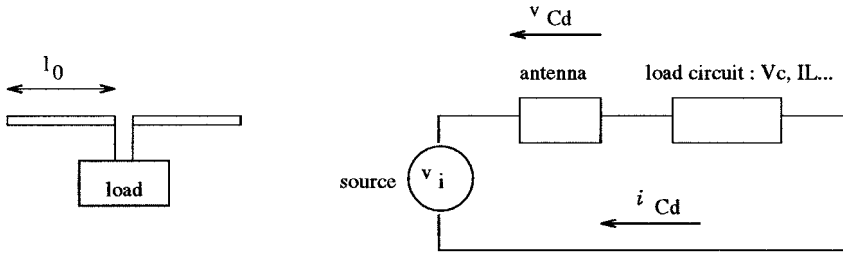
The method usually used for the FDTD implementation is the Auxiliary Differential Equation method (ADE) [16] which breaks this differential equation into an equivalent system of simultaneous differential equations of the first and second order. This implies that one needs to find a set of new local unknowns that do not appear in the final results. This can be very complicated.

The aim of our new approach is to overcome this difficulty, by the use of the state variables, which are the natural variables of the problem.

## 2.2 New Matrix Formulation

Our new method was developed in similarity to [34]. It is based on the decomposition of the problem in two separate problems:

- describe the molecule by its equivalent Thevenin’s or Norton’s circuit, analyze it and extract the corresponding linear matrix differential equation that most appropriately describes its behavior,
- consider the interaction of the wave and the molecule by adding Maxwell’s equations.



**Figure 2.** Dielectric molecule and equivalent Thevenin's circuit.

We will examine this approach in the case of molecules consisting of a single antenna: a dipole and a loop. Using network theory, one can write a set of first order differential equations describing the behavior of the electronic circuit equivalent to the molecule (normal form) [36]. The unknowns of this set of equations are the so called state variables of the circuit. We will show that these natural variables can be related to the intermediate variables one would use to simplify the ADE.

In the following, we will assume  $\psi_e = \psi_m = 0$  and  $\theta = \pi/2$  for simplicity.

### 2.2.1 Dipole antenna

In the case of the dipole antenna, we consider the unknown vector  $x$ , which is the vector (of dimension  $n$ , the total number of non resistor devices of the circuit) of the inductor currents and capacitance voltages of the molecule

$$x = \begin{bmatrix} \vdots \\ v_C \\ \vdots \\ i_L \\ \vdots \end{bmatrix},$$

where the very first unknown  $x_1$  is the voltage  $v_{Cd}$  at the equivalent capacitance of the antenna. From the equivalent Thevenin's circuit (Fig. 2), the set of Kirchoff's voltage and current equations can then be written in a matrix form as

$$\frac{\partial x}{\partial t} + ax = bx_i, \tag{1}$$

where  $a$  is a square matrix,  $b = [b_{*,1} \ b_{*,2}]$ ,  $b_{*,1}$  and  $b_{*,2}$  represent the elements of the first and second columns of  $b$ , and  $x_i$  is the excitation vector defined by:

$$x_i = \begin{bmatrix} \partial v_i \\ \frac{\partial v_i}{\partial t} \end{bmatrix},$$

where  $v_i$  is the source voltage.

We now need to describe the interaction of the equivalent circuit with the EM wave. For our one dimensional analysis, we can write Maxwell's electric curl equation (Faraday's law) as

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z} - \frac{1}{\epsilon_0} \frac{\partial P_x}{\partial t}. \quad (2)$$

We know from basic antenna theory that the polarizability can be related to the current  $i_{C_d}$  on the dipole with

$$-\frac{1}{\epsilon_0} \frac{\partial P_x}{\partial t} = \frac{l_0}{\epsilon_0 V} i_{C_d} = \frac{K_e}{l_0} i_{C_d},$$

and that  $v_i = -l_0 E_x$  [37]. Equations (2) and (1) lead to the following system

$$\begin{cases} \frac{\partial E_x}{\partial t} - \frac{K_e}{l_0} i_{C_d} = -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z}, \\ \frac{\partial x}{\partial t} + ax + l_0 b_{*,1} \frac{\partial E_x}{\partial t} + l_0 b_{*,2} E_x = 0, \end{cases} \quad (3)$$

which, using  $i_{C_d} = C_d \frac{\partial v_{C_d}}{\partial t} = C_d \frac{\partial x_1}{\partial t}$ , can be written as

$$\begin{cases} \frac{\partial E_x}{\partial t} - \frac{K_e}{l_0} C_d \frac{\partial x_1}{\partial t} = -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z}, \\ \frac{\partial x}{\partial t} + ax + K_e C_d b_{*,1} \frac{\partial x_1}{\partial t} + l_0 b_{*,2} E_x = \frac{l_0 b_{*,1}}{\epsilon_0} \frac{\partial H_y}{\partial z}. \end{cases}$$

From the second relation of this system we can extract an explicit equation giving  $\partial x_1 / \partial t$  that we can plug into the first relation to obtain the final system of first order differential equations:

$$\begin{cases} \frac{\partial E_x}{\partial t} + \frac{K_e C_d b_{1,2}}{1 + K_e C_d b_{1,1}} E_x + a_{1,*} \frac{K_e C_d / l_0}{1 + K_e C_d b_{1,1}} x \\ \hspace{15em} = -\frac{1}{\epsilon_0 (1 + K_e C_d b_{1,1})} \frac{\partial H_y}{\partial z}, \\ \frac{\partial x}{\partial t} + ax + K_e C_d b_{*,1} \frac{\partial x_1}{\partial t} + l_0 b_{*,2} E_x = \frac{l_0 b_{*,1}}{\epsilon_0} \frac{\partial H_y}{\partial z}. \end{cases} \quad (4)$$

Considering these equations, we see that we can define a new unknown vector  $X$  as

$$X = \begin{bmatrix} E_x \\ x \end{bmatrix},$$

and rewrite system (4) as a new differential matrix equation of the following form

$$\frac{\partial X}{\partial t} + AX = B \quad (5)$$

where  $A$  is a square matrix, and  $B$  a one column matrix.

In many practical cases, we have  $b_{*,1} \equiv 0$ . The system (4) can then be simplified as following:

$$\begin{cases} \frac{\partial E_x}{\partial t} + \frac{K_e}{l_0} C_d (a_{1,*} x + b_{1,2} l_0 E_x) = -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z}, \\ \frac{\partial x}{\partial t} + ax + b_{*,2} l_0 E_x = 0. \end{cases}$$

The matrix  $A$  can then be written in simple form, where  $a$  appears as a submatrix.

$$A = \begin{bmatrix} b_{1,2} C_d K_e & a_{1,*} \frac{C_d K_e}{l_0} \\ b_{*,2} l_0 & a \end{bmatrix},$$

where  $a_{1,*}$  is the first line of  $a$ ,  $b_{*,2}$  is the second column of  $b$  and  $b_{1,2}$  its first element. The vector  $B$  is given by

$$B = \begin{bmatrix} -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z} \\ \mathcal{O} \end{bmatrix},$$

where  $\mathcal{O}$  is the null vector of dimension  $n$ . It is interesting to see that  $B$  is only related to the EM wave, and not to the molecule.

On the other hand, if  $b_{*,1} \neq 0$ , we can have the following property: every time derivative in  $\frac{\partial x}{\partial t}$  is independent of  $x_1 \equiv v_{C_d}$ . In this case, we can extract  $\frac{\partial x_1}{\partial t}$  from the first equation of the system (3) to obtain

$$\frac{\partial x_1}{\partial t} = -a_{1,*} x - l_0 b_{1,2} E_x - l_0 b_{1,1} \frac{\partial E_x}{\partial t},$$

which can be plugged into the second equation of the system (3). This process leads to the relation

$$\alpha \frac{\partial E_x}{\partial t} + K_e C_d b_{1,2} E_x + \frac{K_e C_d}{l_0} a_{1,2:*} x^* = -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z}, \quad (6)$$

where  $\alpha = 1 + K_e C_d b_{1,1}$  and  $x^*$  is  $x$  from which  $x_1$  is removed. Inserting this equality in the second equation of the system (3), we obtain the relation between  $\frac{\partial x^*}{\partial t}$  and  $(x^*, E_x)$

$$\begin{aligned} \frac{\partial x^*}{\partial t} + \left( a^* - \frac{K_e C_d}{\alpha} b_{*,1} a_{1,*} \right) x^* + \left( l_0 b_{*,2} - \frac{K_e C_d l_0}{\alpha} b_{1,2} b_{*,1} \right) E_x \\ = \frac{l_0}{\alpha \epsilon_0} b_{*,1} \frac{\partial H_y}{\partial z}, \end{aligned}$$

where,  $a^*$  is the matrix  $a$  without its first line and first row. Equation (6) and this last equation give us system (5) where we use

$$X = \begin{bmatrix} E_x \\ x^* \end{bmatrix}.$$

We thus reduced the size of our system. Solving the system (5) directly gives the new value of the electric field in the medium. Furthermore, this method leads to a fully explicit solution in the case of a linear molecule.

### 2.2.2 Loop antenna

Similar results can be derived for the loop case, by considering again the equivalent circuit of the magnetic molecule and deriving the matrix differential equations system. In this case the unknown state variable vector becomes

$$X' = \begin{bmatrix} H_y \\ x' \end{bmatrix},$$

where the vector  $x'$  is made of the inductor currents and capacitance voltages of the molecule. In a manne similar to the dipole case, we assume that the first unknown  $x'_1$  is the current at the inductor equivalent to the loop antenna.

## 3. IMPLEMENTATION

We can solve matrix differential systems in the form of equation (5) using two different techniques : standard or exponential discretization. We present the matrix expression of both cases in this section.



### 3.1 Standard Discretization

We consider the matrix equation  $\frac{\partial X}{\partial t} + AX = B$ , and write the time derivative in the usual finite difference expression

$$\frac{\partial X}{\partial t} = \frac{X^{n+1} - X^n}{\Delta t},$$

where we wrote  $X^k = X(k\Delta t)$  for simplicity. Then equation (5) becomes

$$\left( Id + A\frac{\Delta t}{2} \right) X^{n+1} = \left( Id - A\frac{\Delta t}{2} \right) X^n + \Delta t B^{n+1/2}$$

where  $Id$  is the identity matrix. This can be solved as

$$X^{n+1} = \left( Id + A\frac{\Delta t}{2} \right)^{-1} \cdot \left( \left( Id - A\frac{\Delta t}{2} \right) X^n + \Delta t B^{n+1/2} \right). \quad (7)$$

This is the solution of our matrix differential equation, using a matrix version of the classic Yee algorithm.

In some cases, the elements of the matrix  $A\Delta t/2$  are very small compared to one, since  $\Delta t$  is very small. We can then use the approximation

$$\left( Id + A\frac{\Delta t}{2} \right)^{-1} = Id - A\frac{\Delta t}{2} + \mathcal{O}(\Delta t^2)$$

and simplify the update equation to obtain a simplified matrix relation

$$X^{n+1} = (Id - A\Delta t) X^n + \Delta t B^{n+1/2} + \mathcal{O}(\Delta t^2). \quad (8)$$

### 3.2 Exponential Discretization

If we set  $X = FY$  with the condition that  $F$  satisfies the differential equation

$$\frac{\partial F}{\partial t} + AF = 0, \quad (9)$$

we obtain, from the original ODE

$$F \frac{\partial Y}{\partial t} = B. \quad (10)$$

The solution of equation (9) can be written as  $F = e^{-At}$  where we define the exponential of a matrix as

$$e^{-At} = Id + \sum_{n=1}^{n=\infty} \frac{(-A)^n t^n}{n!}.$$

We can then show that  $(e^{-At})^{-1} = e^{At}$  which implies that the solution of equation (10) is

$$\frac{\partial Y}{\partial t} = F^{-1}B = e^{At}B.$$

Therefore, integrating between  $n\Delta t$  and  $(n+1)\Delta t$  and assuming the right hand side term is constant, we obtain

$$Y^{n+1} = Y^n + \Delta t e^{A(n+1/2)\Delta t} B^{n+1/2},$$

leading to the following update equation

$$X^{n+1} = e^{-A(n+1)\Delta t} e^{An\Delta t} X^n + \Delta t e^{-A(n+1)\Delta t} e^{A(n+1/2)\Delta t} B^{n+1/2}.$$

To compute  $e^{An\Delta t}$  one needs to know the eigenvalues and eigenvectors of  $A$ . If these eigenvalues are known, we can use the following formula :

$$e^{An\Delta t} = \Psi^{-1} E(n\Delta t) \Psi$$

where  $\Psi$  is the matrix of the eigenvectors and  $E(n\Delta t)$  is

$$E(n\Delta t) = \begin{bmatrix} \ddots & & 0 \\ & e^{\lambda_i n \Delta t} & \\ 0 & & \ddots \end{bmatrix}$$

where the  $\lambda_i$  are the eigenvalues of  $A$ .

This last method leads to a more complicated result, but both methods are consistent. The advantage of the matrix formulation is that all the matrices involved in the update equations can be computed only once, at the beginning of the computation.

In order to guarantee convergence of the numerical approach, the matrix differential equation, hence, the resulting matrix  $A$  must be constructed so that  $A$  will have no eigenvalues that lead to growing solutions. Through experimentation we have found that in practice it is always feasible to find such a matrix  $A$ .

## 4. FDTD RESULTS

### 4.1 Dielectric Lorentz Material

The dielectric Lorentz material can be obtained by connecting a simple series RLC load to the dipole antenna [7]. We combine the two capacitors as a single series capacitance  $C = C_L C_d / (C_L + C_d)$ ; our circuit unknown then takes the form

$$x = \begin{bmatrix} v_C \\ i_L \end{bmatrix}.$$

Applying Kirchhoff's voltage and current relations to the Thevenin's equivalent circuit of the molecule, we obtain the matrix differential equation

$$\frac{\partial x}{\partial t} + \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{R}{L} \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{L} \end{bmatrix} x_i.$$

Then, using  $v_i = -l_0 E_x$  and Maxwell's curl equation (2), we obtain the following differential equation

$$\frac{\partial X}{\partial t} + \begin{bmatrix} 0 & 0 & -\frac{K_e}{l_0} \\ 0 & 0 & \frac{-1}{C} \\ \frac{l_0}{L} & \frac{1}{L} & \frac{R}{L} \end{bmatrix} X = \begin{bmatrix} -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z} \\ 0 \\ 0 \end{bmatrix}, \text{ where } X = \begin{bmatrix} E_x \\ v_C \\ i_L \end{bmatrix}.$$

We then use the scheme described in equation (7) to find the following update matrix equation :

$$X^{n+1} = \frac{1}{D} \mathcal{U} X^n - \frac{\Delta t}{D \epsilon_0} \frac{\partial H_y}{\partial z} \begin{bmatrix} 4LC + 2RC\Delta t + \Delta t^2 \\ -l_0 \Delta t^2 \\ -2l_0 C \Delta t \end{bmatrix} \quad (11)$$

where  $D = 4LC + 2RC\Delta t + (1 + K_e C)\Delta t^2$  and the matrix  $\mathcal{U}$  is given by

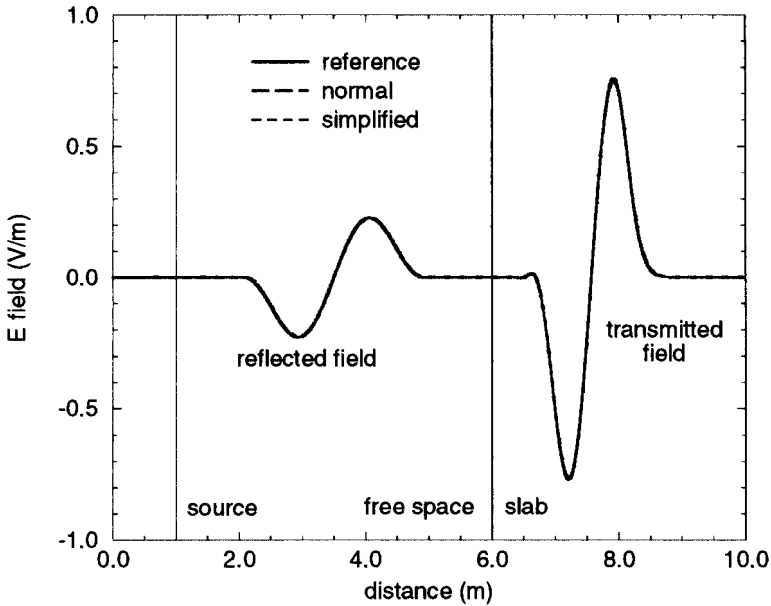
$$\mathcal{U} = \begin{bmatrix} 4LC + 2RC\Delta t + (1 - K_e C)\Delta t^2 & -2K_e C\Delta t^2/l_0 & & & \\ -2l_0\Delta t^2 & 4LC + 2RC\Delta t + (-1 + K_e C)\Delta t^2 & & & \\ -4l_0C\Delta t & & -4C\Delta t & & \\ & 4K_e LC\Delta t/l_0 & & & \\ & 4L\Delta t & & & \\ & 4LC - 2RC\Delta t - (1 + K_e C)\Delta t^2 & & & \end{bmatrix}.$$

If the elements of  $A\Delta t$  are negligible, we can use the simplified update equation (8), leading to

$$X^{n+1} = \begin{bmatrix} 1 & 0 & \frac{K_e\Delta t}{l_0} \\ 0 & 1 & \frac{\Delta t}{C} \\ \frac{-l_0\Delta t}{L} & \frac{-\Delta t}{L} & 1 - \frac{R\Delta t}{L} \end{bmatrix} X^n + \begin{bmatrix} -\frac{\Delta t}{\epsilon_0} \frac{\partial H_y}{\partial z} \\ 0 \\ 0 \end{bmatrix}. \quad (12)$$

We implemented both of these schemes into a standard staggered grid FDTD method. This example uses a one dimensional TE mode Yee algorithm, where the  $E_x$  field component is located at integer values of the space discretization step and the  $H_y$  field component at half integer values. Therefore, we put the capacitor voltage  $v_C$  at the same location as the electric field, and the inductor current  $i_L$  at the same location as the magnetic field. A similar leapfrog scheme is arranged in time, where  $E_x$  and  $v_C$  are at integer time steps, and  $H_y$  and  $i_L$  are taken at half integer time steps. Of course, this approach can be easily extended to 3D.

In order to test the matrix approach, we have run comparisons among several methods for a canonical problem dealing with the scattering of a pulsed plane wave from a Lorentz dispersive, dielectric slab. Figure 3 shows the values of the reflected and transmitted fields for a dielectric Lorentz material slab, for an incoming field described by a bipolar pulse which has the form  $E^i \sim (1 - t^2)^3$  over a finite time period. The figure compares a reference case computed with the usual FDTD scheme, the normal matrix formulation from equation (11), and the simplified formulation using equation (12). All space and time discretization are the same; they have values  $\Delta z = 1$  mm and  $\Delta T = 0.33 \times 10^{-11}$  s. The slab is 0.4 m thick. The total field/scattered



**Figure 3.** Dielectric Lorentz material : Electric field (V/m) vs. distance (m) for  $l_0 = 1.8$  cm,  $R = 100\Omega$ ,  $L = 470$  nH,  $C_d = 0.17$  pF,  $C = C_d/2$ .

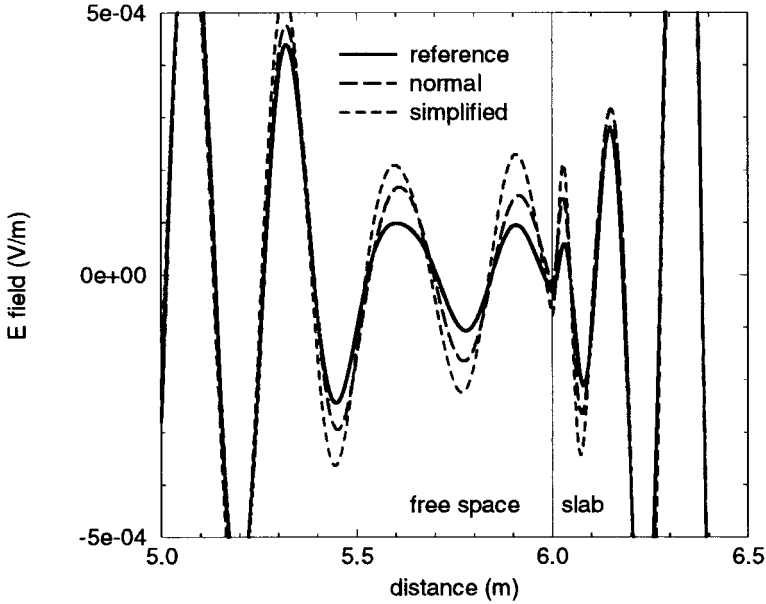
field source is located 5.0 m from the slab. The three curves seem to be perfectly superimposed.

Figure 4 displays a closer look at the electric fields at the free space - medium interface. Even if the standard scheme is more precise, the error in the matrix formulation is less than 0.02% of the amplitude of the input field.

For the dielectric Lorentz material, one can easily show that the inductor current is related to the current density  $J_x$  and the capacitance voltage to the polarization  $P_x$  with :

$$J_x = -\frac{\epsilon_0 K_e}{l_0} i_L \quad \text{and} \quad P_x = -\frac{\epsilon_0 K_e}{l_0} C v_C.$$

The term  $\frac{\epsilon_0 K_e}{l_0}$  has the units of inverse area.



**Figure 4.** Dielectric Lorentz material : difference between the 3 formulations.

## 4.2 Dielectric Two Time Derivative Lorentz Material

The two time derivative Lorentz material (2TDLM) model is obtained with a parallel RLC load connected to the dipole antenna. This 2TDLM case is slightly more difficult because we have a different  $x$  vector in addition to some new terms that appear in the expression of  $\frac{\partial v_i}{\partial t}$ :

$$\frac{\partial x}{\partial t} + \begin{bmatrix} 0 & \frac{-1}{R(C_L + C_d)} & \frac{-1}{C_L + C_d} \\ 0 & \frac{1}{R(C_L + C_d)} & \frac{1}{C_L + C_d} \\ 0 & \frac{-1}{L} & 0 \end{bmatrix} x = \begin{bmatrix} \frac{C_L}{C_L + C_d} & 0 \\ \frac{C_d}{C_L + C_d} & 0 \\ 0 & 0 \end{bmatrix} x_i$$

where  $x = \begin{bmatrix} v_{C_d} \\ v_{C_L} \\ i_L \end{bmatrix}$ .

The voltage  $v_{C_d}$  is measured across the capacitance equivalent to the antenna,  $v_{C_L}$  is the voltage across the capacitance of the load and

$i_L$  is the current through the inductor of the load. We have non zero terms in the first row of the  $b$  matrix, so we can remove  $v_{C_d}$  and set

$$X = \begin{bmatrix} E_x \\ v_{C_L} \\ i_L \end{bmatrix}$$

to obtain, using the simplified update equation (8) :

$$\frac{\partial X}{\partial t} + \begin{bmatrix} 0 & -\frac{\beta}{R} & -\beta \\ 0 & \frac{\gamma}{R} & \gamma \\ 0 & -\frac{1}{L} & 0 \end{bmatrix} X = \begin{bmatrix} -\frac{l_0}{\epsilon_0 \alpha} \frac{\partial H_y}{\partial z} \\ l_0^2 C_d \frac{\partial H_y}{\partial z} \\ \epsilon_0 \alpha (C_L + C_d) \frac{\partial H_y}{\partial z} \end{bmatrix},$$

where

$$\alpha = 1 + K_e \frac{C_L C_d}{C_L + C_d}, \quad \beta = \frac{K_e}{l_0 \alpha} \frac{C_d}{C_L + C_d} \quad \text{and} \quad \gamma = \frac{1 + \beta l_0 C_d}{C_L + C_d}.$$

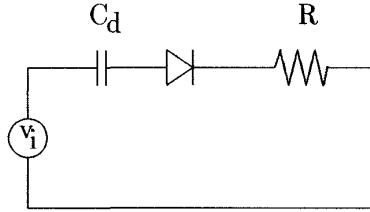
This can be solved using equations (7) or (8).

## 5. NON LINEARLY LOADED MOLECULES

Nonlinear materials result from the introduction of active circuit elements into the circuit which loads the antennas. The actual usefulness of the resulting active materials as “smart skins” (i.e., a surface that could actively respond to variations in the incident field) can also be assessed using FDTD calculations. As was done before [38], the generalized polarization field equation is coupled with Maxwell’s equations in a natural way and solved numerically with the FDTD approach. We still use a method derived from [34]; but in the case of a nonlinear load, we can not decompose our problem into independent circuit and field components. We now have to consider as a whole the circuit and the antenna illuminated by the incoming field.

### 5.1 Dipole Antenna

We again introduce an unknown vector  $X$  defined as the vector of the electric field, the capacitor voltages, including the antenna, and the inductor currents. However we must take into account a new vector



**Figure 5.** Equivalent circuit to the clamping molecule.

$W$  for the nonlinear devices, composed of their driving voltages and currents. Consequently, our matrix differential system becomes more complicated, and can be written as

$$\begin{cases} \dot{X} + AX = BW + C \\ f(W) = D^T X + E^T W, \end{cases}$$

where the multidimensional function  $f$  describes the behavior of the nonlinear components. The time derivative of the second equation gives

$$\left( \frac{\partial f}{\partial W} - E^T \right) \frac{\partial W}{\partial t} = D^T (BW + C - AX)$$

which can be solved with a differential equation solver for  $W$  (4<sup>th</sup> order Runge Kutta for example). On the other hand, the first part of the system can be solved, using equation (7) as usual in the FDTD scheme

$$X^{n+1} = \left( Id + A \frac{\Delta t}{2} \right)^{-1} \cdot \left( \left( Id - A \frac{\Delta t}{2} \right) X^n + \Delta t (BW + C)^{n+1/2} \right)$$

leading to  $X$  and the electric field using the value of  $W$  previously calculated.

### 5.1.1 Application: Clamping circuit

A load composed of a resistor  $R$  and a diode is connected to the dipole antenna (Figure 5). The diode's behavior is modeled using the nonlinear equation  $i = I_s(e^{\alpha v_d} - 1)$  where  $v_d$  is the diode voltage. Using Kirchhoff's laws and equation (2) we have the following differential system



$$\frac{\partial X}{\partial t} + \begin{bmatrix} \frac{K_e}{R} & \frac{K_e}{l_0 R} \\ \frac{l_0}{RC_d} & \frac{1}{RC_d} \end{bmatrix} X = \begin{bmatrix} -\frac{K_e}{l_0 R} \\ -1 \\ \frac{1}{RC_d} \end{bmatrix} W + \begin{bmatrix} -1 & \frac{\partial H_y}{\partial z} \\ \epsilon_0 & 0 \end{bmatrix} \quad (13)$$

where  $X = \begin{bmatrix} E_x \\ v_{C_d} \end{bmatrix}$  and  $W = [v_d]$ ,

and the nonlinear equation

$$i = f(v_d) = I_s(e^{\alpha v_d} - 1) = \frac{-1}{R} (l_0 E_x + v_{C_d}) - \frac{v_d}{R}. \quad (14)$$

The time derivative of equation (14) gives us our nonlinear differential equation, solved with a 4<sup>th</sup> order Runge Kutta algorithm

$$\begin{aligned} & \frac{\partial v_d}{\partial t} (1 + \alpha(RI_s - l_0 E_x - v_{C_d} - v_d)) \\ &= \frac{l_0}{\epsilon_0} \frac{\partial H_y}{\partial z} + (l_0 E_x + v_{C_d} + v_d) \left( \frac{K_e + 1/C_d}{R} \right). \end{aligned}$$

Using relation (7) to solve equation (13) we obtain

$$X^{n+1} = \frac{1}{D} \mathcal{U} X^n + \frac{\Delta t}{D} \begin{bmatrix} -\frac{\Delta t + 2RC_d}{\epsilon_0} \frac{\partial H}{\partial z} - 2\frac{K_e C_d}{l_0} v_d \\ \frac{l_0 \Delta t}{\epsilon_0} \frac{\partial H}{\partial z} - 2v_d \end{bmatrix}. \quad (15)$$

where

$$\mathcal{U} = \begin{bmatrix} 2RC_d + \Delta t(1 - K_e C_d) & -2\frac{K_e C_d}{l_0} \Delta t \\ -2l_0 \Delta t & 2RC_d - \Delta t(1 - K_e C_d) \end{bmatrix},$$

$$D = 2RC_d + \Delta t(1 + K_e C_d).$$

After finding  $v_d$  we use equation (15) to compute and update the electric field  $E_x$ .

This approach actually leads to unstable (infinite) results. The  $\mathcal{U}$  matrix coefficient of the  $\mathcal{U}$  matrix is usually close to  $-1$  because the term  $K_e C_d \Delta t$  is greater than the other terms. If we increase the resistor value, in order to change the sign of the field term, the clamping

effect disappears, as the molecule becomes closer to a resistive Debye molecule.

Another choice for  $W$  is more appropriate. Setting  $W = [i]$  leads to

$$\frac{\partial X}{\partial t} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} \frac{K_e}{l_0} \\ \frac{1}{C_d} \end{bmatrix} W + \begin{bmatrix} \frac{-1}{\epsilon_0} \frac{\partial H_y}{\partial z} \\ 0 \end{bmatrix}$$

Differentiating the equation of the diode with respect to time gives us

$$\frac{\partial i}{\partial t} = \alpha(i + I_s) \frac{\partial v_d}{\partial t}$$

where we can plug the time derivative of the voltage relation  $v_d = v_i - v_{C_d} - Ri$  to obtain our nonlinear differential equation

$$\frac{\partial i}{\partial t} = \frac{\alpha(i + I_s)}{1 + R\alpha(i + I_s)} \left[ \frac{l_0}{\epsilon_0} \frac{\partial H_y}{\partial z} - \left( K_e + \frac{1}{C_d} \right) i \right].$$

Solving this equation will allow us to find the current  $i$  which can be plugged into the update equation, obtained with (8) :

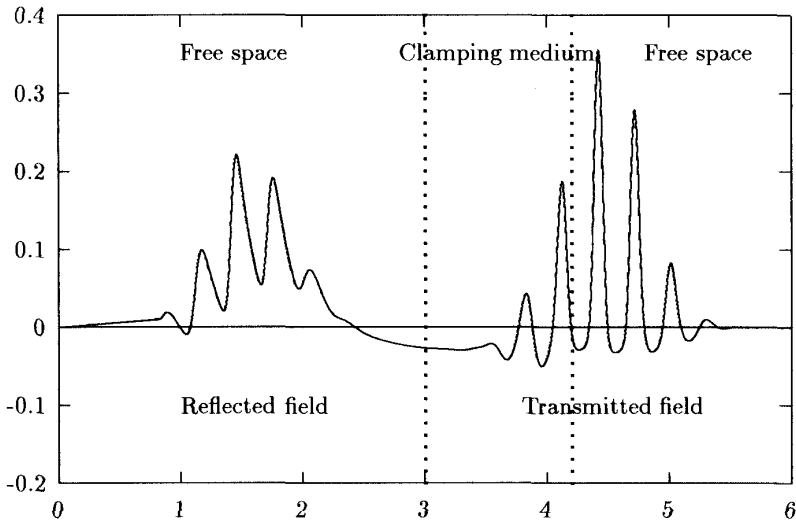
$$X^{n+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X^n + \begin{bmatrix} \frac{K_e \Delta t}{l_0} \\ \frac{\Delta t}{C_d} \end{bmatrix} W^{n+1/2} + \begin{bmatrix} \frac{-\Delta t}{\epsilon_0} \frac{\partial H_y}{\partial z} \\ 0 \end{bmatrix}^{n+1/2}$$

This approach is simpler, and shows that the results are greatly dependent of the choice of the unknowns.

Figure 6 shows the value of the reflected and transmitted electric fields, in the case of a clamping molecule with :  $l_0 = 5 \text{ cm}$ ,  $V = 3.10^{-8} \text{ m}^3$ , the diode coefficients  $\alpha = 40 \text{ V}^{-1}$  and  $I_s = 2.10^{-9} \text{ A}$ , and a  $10 \Omega$  resistor. The incident field is a sinusoidal pulse of 1 GHz frequency. The results are identical to [38].

## 5.2 Loop antenna

We have shown in an earlier work [38] that, due to the fact that most of the nonlinear devices available today are voltage driven, no application for a loop antenna. Consequently, we will not detail the matrix formulation for a loop-based load. Nonetheless, it would be straightforward to derive it from the discussions given above.



**Figure 6.** Clamping material : Electric field (V/m) vs. distance (m) for  $\alpha = 40 \text{ V}^{-1}$ ,  $R = 10 \Omega$ ,  $l_0 = 5 \text{ cm}$ .

## 6. CONCLUSION

We presented a new matrix method for the computation, with an FDTD method, of the field diffracted by a slab of artificial material. This approach uses a state variable method and is fully explicit in the case of linear materials. The ADE method [16], for example, would lead to a high order differential equation, that would have to be decomposed into a set of simultaneous first and second order differential equations which would then require determining some new and adequate intermediate set of variables to solve the problem. This process is naturally handled in our method. The intermediate variables are immediately found by writing the normal form of the network Kirchhoff's equations. A matrix formulation was introduced that enables us to solve this system in a direct way. A simplification is possible, in many cases, that leads to a fully explicit solution, i.e., without any matrix inversion.

Considering the decomposition of the physical permittivity and/or permeability functions presented in [39], it appears possible to apply our artificial material models and matrix method formulation to any kind of linear dielectric and/or magnetic material. In particular, a method was given in [39] to model any kind of linear material as

an electronic circuit composed of several RLC stages in parallel. We could use our matrix method to find the proper state variables and easily decompose the associated set of differential equations. The explicit matrix formulation would then enable us to simulate quickly the interaction of an electromagnetic wave with such a material.

It was shown that the matrix method can be generalized to nonlinear loads and leads to the resolution of a system of linear differential equations describing the wave propagation effects coupled with the nonlinear equations describing the nonlinear load elements. We presented an example, a diode clamped material, which shows that the choice of the intermediate variables is very important and that the choice of the natural state variables leads to a stable system of equations. Previously predicted results for scattering from this material were recovered efficiently and accurately with the matrix method.

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