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A QUEST FOR SYSTEMATIC CONSTITUTIVE FORMULATIONS FOR GENERAL FIELD AND WAVE SYSTEMS BASED ON THE VOLTERRA DIFFERENTIAL OPERATORS

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1. INTRODUCTION

Spatiotemporally dependent field theories, like Maxwell's electromagnetism (electrodynamics) [1, 2], are usually represented as a set of coupled, partial differential equations. The mathematical formalism modeling the physics is usually characterized by two properties that are addressed here: The mathematical formalism is indeterminate (meaning that the number of unknowns exceeds the number of available equations), and the formalism is indifferent to causation (meaning that solutions admit an inversion of the arrow of time).

In mechanical continuum theories, such as linear acoustics [3] and elastodynamics [4–6], one considers the equations of motion and adds the laws relating the geometry to forces, which are usually combined with material properties, e.g., mass density, to supplement missing parameters, e.g., to yield a definition of the phase velocity. In linear acoustics [3], for example, we have four scalar equations, one for the pressure (a scalar), and three for the particle velocity (a vector), altogether four equations and four unknowns constituting a determinate system. However, the original derivation of the equations from the fundamental laws of mechanics requires that we include the relation of pressure to volume, the so called compressibility relation, in order to define the phase velocity. In elastodynamics the corresponding step is to include Hooke's law of stress and strain. These relations are not part of the Newtonian laws of mechanics, but rather extraneous constitutive relations stipulating the material properties involved. The present fine distinction between the physical models or "laws" and the additional constitutive relations is a side issue in mechanical continuum fields, but in the context of electrodynamics this step is crucial. Electrodynamics is to-date part of our Einsteinian Special Relativistic conception of physical reality [7]. As such, it deals also with the formulation of physical models in various inertial systems in relative motion, and includes the new relativistic elements relevant to simultaneity and causality. For comments on causality and advanced and retarded solutions, see for example Morse and Feshbach [8]. The question of causality is common to physical models mathematically formalized in terms of differential equations (or an equivalent formulation such as a variational principle, integral equations, etc.) In the context of Special Relativity theory, questions of simultaneity and causality become even more acute, because they involve relatively moving observers [7, 9]. The essence of Einstein's theory [7] is the stipulation that Maxwell's equations are "covariant", or "form invariant", meaning that the functional structure (not the value of the individual independent variables!) of the equations is identical in all inertial systems. In each system the native spatiotemporal coordinates and the associated differential operators are used. Inasmuch as Einstein [7] addressed the free space (vacuum) case only, the question of constitutive relations did not arise, but as the theory evolved, it became clear that the covariance property above is actually attributed to the indeterminate set of equations [1, 2, 10].

Consequently, in material media, the covariance must be reexamined in view of the presence of new constitutive relations, initially defined for the comoving frame of reference, where the medium is at rest. An account of the history of this controversial problem and the methodology stipulated by Minkowski [11] is given by Sommerfeld [12]. Unfortunately this is not sufficiently general, as discussed below, in particular when one attempts to include bi-anisotropic, bi-isotropic, and nonlinear systems [13]. One generalized scheme, slightly different from Minkowski's original approach, is proposed below.

Three major aspects related to the constitutive properties are considered here. The first involves the dispersive properties of media, i.e., the spectral (frequency and propagation vector) dependence of the constitutive parameters. The second aspect is the spatiotemporal inhomogeneity of such parameters. The assumption of the existence of dispersion usually excludes inhomogeneity, and *vice-versa*, because it calls for a combined description in both the spatiotemporal and the spectral domains. A consistent discussion of constitutive parameters combining dispersion and inhomogeneity is only feasible in the context of the high frequency (eikonal) approximation. This difficulty is resolved below, by instituting the new Volterra differential operators. The third aspect is the consistent definition of nonlinear constitutive parameters in combined dispersive and inhomogeneous media. It is shown that by employing the new Volterra differential operators the analysis of linear fields is straightforwardly extended to nonlinear systems.

Usually, the statement of dispersive constitutive parameters involves spatiotemporal convolution integrals. Similarly, nonlinear systems involve the relevant Volterra functional series, which constitute generalized convolution integrals. In both cases, causality consideration related to the special-relativistic light cone concept are introduced via the integration limits. The extent to which this problem can be obviated when the new Volterra differential operators replace the convolution and functional series integrals, is discussed subsequently.

2. LINEAR HOMOGENEOUS MEDIA

Physical field systems are modeled in terms of systems of coupled partial differential equations relating dependent variables which are physically measurable quantities (or the so called potentials, dependent variables from which such physically measurable quantities are derivable), involving space-time independent variables. An obvious example are Maxwell's equations for the electromagnetic fields, one of the most successful models of physics, especially in view of its compatibility with Einstein's Special Relativity theory. In the contemporary notation, in

MKS units, Maxwell's equations are presented [1, 2] in the form,

$$\partial_{\mathbf{x}} \times \mathbf{E} = -\partial_t \mathbf{B} - \mathbf{j}_m \tag{1a}$$

$$\partial_{\mathbf{x}} \times \mathbf{H} = \partial_t \mathbf{D} + \mathbf{j}_e \tag{1b}$$

$$\partial_{\mathbf{x}} \cdot \mathbf{D} = \rho_e \tag{1c}$$

$$\partial_{\mathbf{x}} \cdot \mathbf{B} = \rho_m \tag{1d}$$

where for example $\mathbf{E} = \mathbf{E}(\mathbf{X})$ is the electric vector field dependent on the spatiotemporal coordinates compacted here as a quadruplet $\mathbf{X} = \{\mathbf{x}, ict\}$, where $\mathbf{X}$ is a four-vector in Minkowski's space. In (1), space and time differential operators are indicated. The conventional current and charge densities (index e) are augmented by hypothetical corresponding magnetic (index m) densities. The current and charge densities can be sources, i.e., inhomogeneous terms of the differential equations, or dependent on other fields, or a combination thereof. This is not spelled out in the original representation (1), which is anyhow indeterminate (or undetermined, i.e., the number of unknowns exceeds the number of equations). In general, systems like (1) will be symbolized in a matric form by,

$$\mathcal{A}_{i,n}(\partial_{\mathbf{X}}) A_n(\mathbf{X}) + \mathcal{B}_{i,k}(\partial_{\mathbf{X}}) B_k(\mathbf{X}) = s_i(\mathbf{X}) \tag{2}$$

where

$$\partial_{\mathbf{X}} = (\partial_{\mathbf{x}}, -(i/c)\partial_t) \tag{3}$$

is the four-gradient, also a Minkowski four-vector, $\mathcal{A}_{i,n}(\partial_{\mathbf{X}})$, $\mathcal{B}_{i,k}(\partial_{\mathbf{X}})$ are operators involving those derivatives, and $s_i(\mathbf{X})$ are the inhomogeneous "source" terms of the system. The dependent variables $A_n(\mathbf{X})$, $B_k(\mathbf{X})$ have been divided arbitrarily into two groups, where $n = i$ groups dependent variables $A_n(\mathbf{X})$ whose number equals the number of equations, thus we need k additional relations for $B_k(\mathbf{X})$. For brevity, (2) is recast as

$$\mathcal{A}_{i,n} A_n + \mathcal{B}_{i,k} B_k = s_i \tag{4}$$

where the arguments are suppressed. Henceforth script font will denote operators.

Determinateness is achieved by invoking additional equations, usually suggested by associated coupled physical processes. But abstract models that are convenient for mathematical manipulation, or empirically suggested functions are also used. The (almost trivial) example for the unmagnetized cold plasma, (see for example [10]), based on the single electron equation of motion, demonstrated how elimination of the extraneous physical quantities leads to operator constitutive parameters. In this case, two constitutive equations are used: It is assumed that charge and current densities (dependent fields in the present example) are related by $\mathbf{j} = \rho\mathbf{v}$, with $\mathbf{v}$ being the velocity, and that the equation of motion of the electron is aptly represented by $q\mathbf{E} = m\partial_t\mathbf{v}$, with q, m, denoting the electron charge, mass, respectively. Consequently, after eliminating the extraneous mechanical quantity $\mathbf{v}$ and substituting back into Maxwell's equations, a (dielectric) constitutive operator $\varepsilon_r(\partial_t) = 1 + \omega_p^2\partial_t^{-1}\partial_t^{-1}$ is obtained, with ω_p denoting the plasma frequency. The Fourier transformed form is the more familiar $\varepsilon_r(-i\omega) = 1 - \omega_p^2/\omega^2$. See also [14] for more general cases and related differential operators techniques. In general, the differential operators are applied directly. Alternatively, to get rid of inverse operators like ∂_t^{-1}, the two sides of an equation are multiplied by the inverse operator. In more general terms, it is assumed that the k additional relations are given by

$$\mathcal{C}_{l,k}B_k + \mathcal{D}_{l,i}A_i = u_l \tag{5}$$

where the elimination of extraneous fields was already performed, leaving us with the required $l = k$ new differential equations, generally but not necessarily including k new source terms u_k. Solving (5) for B_k yields

$$B_k = \mathcal{E}_{k,l}\mathcal{D}_{l,i}A_i - \mathcal{E}_{k,l}u_l \tag{6}$$

where $\mathcal{E}_{k,l}\mathcal{C}_{l,k} = -\mathcal{I}$, i.e., we have multiplied (5) by the negative inverse operator. Substituting back into (4) and redefining the fields B_k, we rearrange the system of equations as

$$\mathcal{A}_{i,n}A_n = -\mathcal{B}_{i,k}B_k + w_i \tag{7}$$

$$B_k = \mathcal{E}_{k,l}\mathcal{D}_{l,n}A_n = \mathcal{G}_{k,l}A_n \tag{8}$$

$$w_i = s_i + \mathcal{B}_{i,k}\mathcal{E}_{k,n}u_n \tag{9}$$

possessing the conventional structure of the Maxwell equations supplemented by additional constitutive relations, involving constitutive parameters that are in the present case differential operators.

For additional compactness of notation, the careful book-keeping of indices is now suppressed, using boldface characters to denote vectors and dyadics, the latter always appearing in script font. Thusly (7, 8) become

$$\boldsymbol{\mathcal{A}}(\partial_{\mathbf{X}}) \cdot \mathbf{A}(\mathbf{X}) = -\boldsymbol{\mathcal{B}}(\partial_{\mathbf{X}}) \cdot \mathbf{B}(\mathbf{X}) + \mathbf{w}(\mathbf{X}) \tag{10}$$

$$\mathbf{B}(\mathbf{X}) = \boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}}) \cdot \mathbf{A}(\mathbf{X}) \tag{11}$$

and once again, like in (4) and subsequent expressions, the arguments will be dropped where convenient, leaving us with the compacted determinate physical model

$$\boldsymbol{\mathcal{A}} \cdot \mathbf{A} = -\boldsymbol{\mathcal{B}} \cdot \mathbf{B} + \mathbf{w} \tag{12}$$

$$\mathbf{B} = \boldsymbol{\mathcal{G}} \cdot \mathbf{A} \tag{13}$$

Alternatively, one can start from the fundamental physical model (2) and represent the Spatiotemporally dependent functions in terms of the four-dimensional Fourier transform pair, e.g., for some function $f(\mathbf{X})$ we have

$$f(\mathbf{X}) = (2\pi)^{-4} \int (d^4\mathbf{K}) \overline{f}(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} \tag{14}$$

and its inverse

$$\overline{f}(\mathbf{K}) = \int (d^4\mathbf{X}) f(\mathbf{K}) e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{15}$$

where the bar denotes the transformed function, and $\mathbf{K}$ is a new quadruplet, once again constituting a Minkowski four-vector

$$\mathbf{K} = (\mathbf{k}, i\omega/c) \tag{16}$$

where k is the propagation vector and ω is the (angular) frequency, $\mathbf{K} \cdot \mathbf{X} = \mathbf{k} \cdot \mathbf{x} - \omega t$ is a plane wave appropriate phase, but care must be exercised, because thus far (14, 15) refer to arbitrary functions, without reference to any specific wave equation. In an obvious manner, derivatives affect the exponential, yielding

$$\partial_{\mathbf{X}} f(\mathbf{X}) = (2\pi)^{-4} \int (d^4\mathbf{K}) \overline{f}(\mathbf{K}) i\mathbf{K} e^{i\mathbf{K}\cdot\mathbf{X}} \tag{17}$$

Consequently (10, 11) become a set of algebraic equations

$$\boldsymbol{\mathcal{A}}(i\mathbf{K}) \cdot \overline{\mathbf{A}}(\mathbf{K}) = -\boldsymbol{\mathcal{B}}(i\mathbf{K}) \cdot \overline{\mathbf{B}}(\mathbf{K}) + \overline{\mathbf{w}}(\mathbf{K}) \tag{18}$$

$$\overline{\mathbf{B}}(\mathbf{K}) = \boldsymbol{\mathcal{G}}(i\mathbf{K}) \cdot \overline{\mathbf{A}}(\mathbf{K}) \tag{19}$$

Note carefully that in the transition from $\boldsymbol{\mathcal{A}}(\partial_{\mathbf{X}})$ to $\boldsymbol{\mathcal{A}}(i\mathbf{K})$, etc., we have simply replaced each component of the four-gradient operator by the corresponding component of $i\mathbf{K}$. The functional structure of $\boldsymbol{\mathcal{A}}$ itself remains unchanged. On the other hand $\overline{\mathbf{A}}(\mathbf{K})$ etc., are the Fourier transformed functions of $\mathbf{A}(\mathbf{X})$ etc.

Augmentation of the original undeterminate set of equations (4) by a set of differential equations in **X**-space (5), as done above, is only one alternative for dealing with the problem of determinateness. One could start with an heuristic model for the dispersive properties of the medium in question, based on some physical model or on some empirical results. That would lead to a statement similar in its appearance to (19), but not quite the same, because at the outset we allow an arbitrary constitutive function $\overline{\overset{*}{\boldsymbol{\mathcal{G}}}}(\mathbf{K})$

$$\overline{\mathbf{B}}(\mathbf{K}) = \overline{\overset{*}{\boldsymbol{\mathcal{G}}}}(\mathbf{K}) \cdot \overline{\mathbf{A}}(\mathbf{K}) \tag{20}$$

Here we start in **K**-space, and therefore we put a bar on all symbols defining independent variables. The inverse Fourier transform of a product of functions as in (20) becomes a convolution integral in **X**-space in terms of the inverse transforms of the **K**-space functions in question, i.e.,

$$\mathbf{B}(\mathbf{X}) = \int (d^4\mathbf{X}_1) \overset{*}{\boldsymbol{\mathcal{G}}}(\mathbf{X}_1) \cdot \mathbf{A}(\mathbf{X} - \mathbf{X}_1) \tag{21}$$

The two functions $\overset{*}{\boldsymbol{\mathcal{G}}}(\mathbf{X})$ in (21) and $\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}})$ in (11) should not be confused: Note that $\overset{*}{\boldsymbol{\mathcal{G}}}(\mathbf{X})$ denotes the inverse Fourier transform of the corresponding function $\overline{\overset{*}{\boldsymbol{\mathcal{G}}}}(\mathbf{K})$ in (20). Inasmuch as in (21) the contributions to the value of the integral are obtained by scanning the whole of $\mathbf{X}_1$-space, in order to obtain the value for some specific **X**, (21) describes a non-local interaction. On the other hand (11), by virtue of the differential operators, describes a local interaction. In the integral form (21), and in view of the finite speed of propagation assumed in Special Relativity theory, the limits of the integral must be restricted to ensure that all interactions are within the relativistic light cone, i.e., that all events affecting the outcome at **X** are within a distance that can be traversed at a speed $\leq c$.

The above analysis clearly shows that the manner in which the constitutive relations are stated has far-reaching implications. Is dispersion a local phenomenon, as derived by augmenting the physical laws (e.g., Maxwell's equations) by means of additional differential equations (e.g., the equation of motion for charge as performed for the one electron model for a cold plasma, or the one fluid model [14]), or should we always allow for non-local interactions?

From the mathematical point of view, one can safely assume that if $\overset{*}{\bar{\boldsymbol{\mathcal{G}}}}(\mathbf{K})$ in (20) can be exactly recast or approximated as a rational function (ratio of polynomials) in the components of $\mathbf{K}$, then the structure

$$\overset{*}{\bar{\boldsymbol{\mathcal{G}}}}(\mathbf{K}) = \boldsymbol{\mathcal{G}}(i\mathbf{K}) \tag{22}$$

of (19) can be recovered, and we can work our way backwards to (11). Conversely, one could start from (21) as the given constitutive relation and arrive at (11), subject to the proviso that the interaction is local. We start from the basic definition of the one-dimensional Dirac impulse- (or δ-) function

$$\int_a^b dx\delta(x)g(x) = g(0) \tag{23}$$

provided $x = 0$ exists within the integration interval. Using the formula for integration by parts, it follows that

$$\int_a^b dx_1 g(x-x_1)\partial_{x_1}\delta(x_1) = \int_a^b dx_1\partial_{x_1}\{\delta(x_1)g(x-x_1)\} \tag{24a}$$

$$-\int_a^b dx_1\delta(x_1)\partial_{x_1}g(x-x_1) = \partial_x g(x) \tag{24b}$$

$$\int_a^b dx_1\partial_{x_1}\{\delta(x-1)g(x-x_1)\} = \{\delta(x_1)g(x-x_1)\}_a^b = 0 \tag{24c}$$

provided $a < x_1 < b$. The integral in $(24a)$ on the right vanishes, due to the vanishing of the δ-function at the limits of the integral, (24c). Hence each differentiation operation on the function is equivalent to a convolution integral involving a corresponding differentiation of the δ-function. In the present context of spatiotemporally dependent functions, we deal with a product of four impulse functions, each for one coordinate

$$\delta(\mathbf{X}) = \delta(x)\delta(y)\delta(z)\delta(ict) \tag{25}$$

In a consistent manner, in $\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}})$ we replace the gradient operator by corresponding differentiations of the components of $\delta(\mathbf{X})$. The Fourier transform $\overset{*}{\boldsymbol{\mathcal{G}}}(\mathbf{X})$ is obtained explicitly as

$$\overset{*}{\boldsymbol{\mathcal{G}}}(\mathbf{X}) = \boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}}\delta(\mathbf{X})) \tag{26}$$

i.e., to derive $\overset{*}{\boldsymbol{\mathcal{G}}}(\mathbf{X})$, one should take $\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}})$ and replace each derivative with the corresponding derivative of the $\delta(\mathbf{X})$ function, *preserving the functional structure of* $\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}})$. Of course, this representation is meaningful only to the extent that $\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}})$ is known and can be applied, i.e., that it is clear how to express it as a polynomial or a rational function. This is consistent with general properties of transformations. If $\boldsymbol{\mathcal{G}}(i\mathbf{K})$ is a slowly varying function (a constant in the limit), then its inverse Fourier transform will be represented by a sharp pulse-like function (a δ-function in the limit), and *vice-versa.* As $\boldsymbol{\mathcal{G}}(i\mathbf{K})$ varies faster, higher derivative terms appear in $\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}})$, and the transform $\overset{*}{\boldsymbol{\mathcal{G}}}(\mathbf{X})$ involves higher derivatives of the δ-function, as shown in (26), resulting in a faster varying integral as in (21). The general expansion of singular functions in terms of series of derivatives of δ-function is discussed in the literature, see Van Bladel [15], also Lindell [16].

The operational nature of

$$\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}}) \cdot \mathbf{A}(\mathbf{X}) = \int (d^4\mathbf{X}_1)\overset{*}{\boldsymbol{\mathcal{G}}}(\mathbf{X}_1) \cdot \mathbf{A}(\mathbf{X} - \mathbf{X}_1) \tag{27a}$$

$$= \int (d^4\mathbf{X}_1)\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}_1}\delta(\mathbf{X}_1)) \cdot \mathbf{A}(\mathbf{X} - \mathbf{X}_1) \tag{27b}$$

coupled with (26) suggests that we refer to the associated constitutive parameters as Volterra operators. Specifically, we refer to operators of the types $\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}})$, $\boldsymbol{\mathcal{G}}(\partial_{\mathbf{X}_1}\delta(\mathbf{X}_1))$, discussed above as Volterra differential operators. Some relevance to operators by the same name discussed in functional analysis texts is obvious, see for example [17].

3. LINEAR INHOMOGENEOUS MEDIA

Heretofore we have exclusively assumed homogeneous linear media. Physical systems are also characterized by spatiotemporally varying media: Presumably, variable constitutive parameters may be associated with different locations and may be time dependent as well, and still possess dispersive properties. From the physical point of view, i.e., according to the physical model imposed on the mathematical formalism, the question arises whether this claim can be consistently adopted? In other words: Do we generally know how to separate the manifestations of dispersion and inhomogeneity? From the mathematical point of view, can we avoid invalid aspects of such a model? Here is an example that clarifies the difficulties of the subject: It is well known, e.g., in waveguide theory, that even in the absence of intrinsically dispersive media, say in vacuum, the geometry causes a different behavior of the system for different frequencies. This is the so called structural dispersion. Similarly in scattering theory when we are dealing with objects whose dimensions are on a par with the wavelength used, we get geometry dependent spectral effects, e.g., modes, resonance effects, cutoff frequencies, etc.. So this begs the question where structural dispersion starts and material dispersion ends, or how to distinguish between the two phenomena. In the case where distinct spatially homogeneous regions can be identified, solutions for unbounded infinite space can often be assumed for each region individually, and the complete solution then be achieved by satisfying the continuity and jump conditions prescribed at the various boundary surfaces. A similar approach might work when initial conditions are assumed. In such boundary surface and initial condition problems, if the dispersion is present in the infinite domain, it is usually referred to as medium dispersion. Any additional effect can then be attributed to the spatiotemporal geometry, i.e., the conditions as stated, and be dubbed structural dispersion. In some cases spatiotemporal variations can be tackled in terms of Doppler effects [18]. However, these approaches are not general. Another approach of limited validity allowing the coexistence of dispersion and inhomogeneity is provided by ray theory, where the problem at hand can be stated in terms of slow variations in space, time, being small relative to the wavelength, period, involved, respectively. In such circumstances, so called short wavelength (or high frequency) approximations can be employed. Usually this appears under the headings "eikonal approximation", "method of characteristics", or the "WKB approximation". In such cases (e.g., see [18]) the constitutive relations are expressed as functions of the two spaces $\mathbf{X}$, and $\mathbf{K}$, simultaneously,

$$\mathbf{B}(\mathbf{X}, \mathbf{K}) = \boldsymbol{\mathcal{G}}(\mathbf{X}, i\mathbf{K}) \cdot \mathbf{A}(\mathbf{X}, \mathbf{K}) \tag{28}$$

Obviously, writing an expression like (28) excludes the use of the Fourier transform in its conventional form (14, 15), and the functions in (28) should not be interpreted as either the spatiotemporal or the spectral functions in (14, 15). This technique leads to the Hamiltonian ray propagation algorithm and is an important tool for those cases where the method is applicable (see for example [14] and in a more tutorial fashion [19–21]). However, a general approach that bridges the above two extreme approaches is not available. Describing inhomogeneous media, even in nondispersive media, say using an expression like

$$\mathbf{B}(\mathbf{X}) = \boldsymbol{\mathcal{G}}(\mathbf{X}) \cdot \mathbf{A}(\mathbf{X}) \tag{29}$$

brings up another perplexing aspect, namely a convolution integral in the $\mathbf{K}$-space spectral domain

$$\overline{\mathbf{B}}(\mathbf{K}) = (2\pi)^{-4} \int (d^4\mathbf{K}_1) \overline{\boldsymbol{\mathcal{G}}}(\mathbf{K}_1) \cdot \overline{\mathbf{A}}(\mathbf{K} - \mathbf{K}_1) \tag{30}$$

analogous to (21). This would suggest a non-local interaction between points in the spectral domain, constituting the creation of new frequencies and wave vectors even in the presence of a linear system. It is well known that the creation of new spectral components, e.g., due to modulation effects, are characteristic of nonlinear systems. In the present case of inhomogeneity, new spectral components, wave vectors and frequencies, must be created to satisfy the system's mathematical equations, involving spatiotemporally dependent coefficients. At a first glance this appears to be enigmatic and unacceptable, but such circumstances have been encountered previously. It is recalled that the only case where new frequency components are created in a linear regime can be found in the context of Doppler effects created by moving surfaces [18] or in general in the presence of time-varying media and/or boundary surfaces. But in such special cases, even though the media themselves, hence the appropriate differential equations, are linear and involving constant coefficients, the system cannot qualify as homogeneous and constant in time! In a figurative manner, we may say that gradients in the structural properties in space act as "lenses", affecting wave vectors, and time variations in media (including boundaries) produce Doppler effects. The problem is therefore narrowed down to the separation of intrinsic nonlinearity and spatiotemporal inhomogeneity in the production of new spectral components. In an attempt to present a consistent approach to the problem, the constitutive equations are now presented in the form

$$\mathbf{B}(\mathbf{X}) = \boldsymbol{\mathcal{G}}(\mathbf{X}, \partial_{\mathbf{X}}) \cdot \mathbf{A}(\mathbf{X}) \tag{31}$$

or, anticipating the nonlinear generalization given subsequently, we use the notation

$$\mathbf{B}(\mathbf{X}) = \boldsymbol{\mathcal{G}}\left(\mathbf{X}, \partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}\right) \cdot \mathbf{A}(\mathbf{X}_1) \tag{32}$$

indicating that the differential operators are applied first, then the change of variable $\mathbf{X}_1 = \mathbf{X}$ is effected. This provides us with a method of combining inhomogeneous media, i.e., differential equations including spatiotemporally variable coefficients, with the differential operators involved with dispersion.

In general both material and structural dispersion is present, as indicated by (32). But unlike the eikonal approximation leading to mixed spaces expressions like in (28), presently (31) or (32) are expressed in terms of spatiotemporal coordinates only. This representation facilitates the corresponding Fourier transform: According to (15), the inverse Fourier transform corresponding to (32) is therefore

$$\overline{\mathbf{B}}(\mathbf{K}) = \int (d^4\mathbf{X}) \boldsymbol{\mathcal{G}}\left(\mathbf{X}, \partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}\right) \cdot \mathbf{A}(\mathbf{X}_1) e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{33a}$$

$$= (2\pi)^{-4} \int (d^4\mathbf{X}) e^{-i\mathbf{K}\cdot\mathbf{X}} \int (d^4\mathbf{K}_1) \boldsymbol{\mathcal{G}}\left(\mathbf{X}, \partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}\right) \cdot \overline{\mathbf{A}}(\mathbf{K}_1) e^{i\mathbf{K}_1\cdot\mathbf{X}_1} \tag{33b}$$

$$= (2\pi)^{-4} \int (d^4\mathbf{X}) \int (d^4\mathbf{K}_1) \boldsymbol{\mathcal{G}}(\mathbf{X}, i\mathbf{K}_1) \cdot \overline{\mathbf{A}}(\mathbf{K}_1) e^{i(\mathbf{K}_1-\mathbf{K})\cdot\mathbf{X}} \tag{33c}$$

$$= (2\pi)^{-8} \int (d^4\mathbf{X}) \int (d^4\mathbf{K}_1) \int (d^4\mathbf{K}_2) \overline{\boldsymbol{\mathcal{G}}}(\mathbf{K}_2, i\mathbf{K}_1) \cdot \overline{\mathbf{A}}(\mathbf{K}_1) e^{i(\mathbf{K}_1+\mathbf{K}_2-\mathbf{K})\cdot\mathbf{X}} \tag{33d}$$

the bars indicating the transform of $\mathbf{A}(\mathbf{X}_1)$ with respect to $\mathbf{X}_1$, and $\boldsymbol{\mathcal{G}}(\mathbf{X}, i\mathbf{K}_1)$ with respect to $\mathbf{X}$. Exploiting the definition (25) of the δ-function we now have

$$\delta(\mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}) = (2\pi)^{-4} \int (d^4\mathbf{X}) e^{i(\mathbf{K}_1+\mathbf{K}_2-\mathbf{K})\cdot\mathbf{X}} \tag{34}$$

and finally this yields a new extended convolution integral

$$\overline{\mathbf{B}}(\mathbf{K}) = (2\pi)^{-4} \int (d^4\mathbf{K}_1) \int (d^4\mathbf{K}_2) \overline{\boldsymbol{\mathcal{G}}}(\mathbf{K}_2, i\mathbf{K}_1) \cdot \overline{\mathbf{A}}(\mathbf{K}_1) \delta(\mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}) \tag{35a}$$

$$= (2\pi)^{-4} \int (d^4\mathbf{K}_1)\overline{\boldsymbol{\mathcal{G}}}(\mathbf{K}-\mathbf{K}_1, i\mathbf{K}_1)\cdot\overline{\mathbf{A}}(\mathbf{K}_1) \tag{35b}$$

$$= (2\pi)^{-4} \int (d^4\mathbf{K}_1)\overline{\boldsymbol{\mathcal{G}}}(\mathbf{K}_1, i\mathbf{K}_1)\cdot\overline{\mathbf{A}}(\mathbf{K}-\mathbf{K}_1) \tag{35c}$$

displaying both material and structural dispersion. A word of caution must be added to the above formulas. The substitution $\partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}} \Leftrightarrow i\mathbf{K}_1$ in (33b) does not alter the functional structure. It is simply a substitution. On the other hand, in (33d) the step leading from $\boldsymbol{\mathcal{G}}(\mathbf{X}, i\mathbf{K}_1)$ to $\overline{\boldsymbol{\mathcal{G}}}(\mathbf{K}_2, i\mathbf{K}_1)$ actually involves a Fourier transformation. Consequently $\overline{\boldsymbol{\mathcal{G}}}(\mathbf{K}_1, i\mathbf{K}_1)$ in $(35c)$ is to be understood as a Fourier transformed function with respect to the first argument only. If the need to preserve the identity of various arguments arises, one has to trace the functions and their arguments back through (33–35).

Inasmuch as (35) constitutes an integral representation, and in view of our previous success in replacing the integral representation by a Volterra differential operator, as in (27), it is wished to recast (35) in terms of a differential operator representation. Consider the expression

$$\overline{\mathbf{B}}(\mathbf{K}) = \boldsymbol{\mathcal{G}}\left(i\partial_{\mathbf{K}_1}\Big|_{\mathbf{K}_1=\mathbf{K}}, i\mathbf{K}\right)\cdot\overline{\mathbf{A}}(\mathbf{K}_1) \tag{36}$$

where the derivative operator is, similarly to (3), the four-gradient in $\mathbf{K}$-space (see [10, 21]),

$$\partial_{\mathbf{K}} = (\partial_{\mathbf{k}}, -ic\partial_\omega) \tag{37}$$

Applying (15) to $\overline{\mathbf{A}}(\mathbf{K}_1)$ in (36), and inserting $\boldsymbol{\mathcal{G}}\left(i\partial_{\mathbf{K}_1}\Big|_{\mathbf{K}_1=\mathbf{K}}, i\mathbf{K}\right)$ within the integration, and then applying the differential operator, yields

$$\overline{\mathbf{B}}(\mathbf{K}) = \int (d^4\mathbf{X})\boldsymbol{\mathcal{G}}\left(i\partial_{\mathbf{K}_1}\Big|_{\mathbf{K}_1=\mathbf{K}}, i\mathbf{K}\right)\cdot\mathbf{A}(\mathbf{X})e^{-i\mathbf{K}_1\cdot\mathbf{X}} \tag{38a}$$

$$= \int (d^4\mathbf{X})\boldsymbol{\mathcal{G}}(\mathbf{X}, i\mathbf{K})\cdot\mathbf{A}(\mathbf{X})e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{38b}$$

$$= \int (d^4\mathbf{X})\boldsymbol{\mathcal{G}}\left(\mathbf{X}, -\partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}\right)\cdot\mathbf{A}(\mathbf{X})e^{-i\mathbf{K}\cdot\mathbf{X}_1} \tag{38c}$$

carefully written in a way that shows that the differential operator $\boldsymbol{\mathcal{G}}\left(\mathbf{X}, -\partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}\right)$ acts on the exponential only, and it is understood that the integration with respect to $\mathbf{X}$ can only be applied after the operator acts. Compare now the last result $(38c)$ to $(33a)$. Somewhat similarly to (24), we apply integration by parts to $(38c)$, obtaining

$$\overline{\mathbf{B}}(\mathbf{K}) = \int (d^4\mathbf{X})\boldsymbol{\mathcal{G}}\left(\mathbf{X}, -\partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}\right) \cdot \mathbf{A}(\mathbf{X})e^{-i\mathbf{K}\cdot\mathbf{X}_1} \tag{39a}$$

$$= \int (d^4\mathbf{X})\boldsymbol{\mathcal{G}}\left(\mathbf{X}, -\partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}\right) \cdot \mathbf{A}(\mathbf{X}_1)e^{-i\mathbf{K}\cdot\mathbf{X}_1} \tag{39b}$$

$$+ \int (d^4\mathbf{X})\boldsymbol{\mathcal{G}}\left(\mathbf{X}, \partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}\right) \cdot \mathbf{A}(\mathbf{X}_1)e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{39c}$$

indicating that (38c), (33a) will be identical, provided (39b) vanishes. For this to happen, $\mathbf{A}(\mathbf{X})e^{-i\mathbf{K}\cdot\mathbf{X}}$ must be a constant, i.e., $\mathbf{A}(\mathbf{X}) = e^{i\mathbf{K}\cdot\mathbf{X}}$ Obviously this case is too specialized and trivial to be considered a general result, moreover, it probably is not a solution of a wave equation involving spatiotemporal inhomogeneity. However, in the context of the eikonal approximation it might be an approximate solution

$$\mathbf{A}(\mathbf{X}) = \mathbf{A}_0(\mathbf{X})e^{i\theta(\mathbf{X})} \tag{40}$$

etc., where $\mathbf{A}_0(\mathbf{X})$ is a slowly varying amplitude, whose derivatives are negligible in comparison to the derivative of the exponential. The phase $\theta(\mathbf{X})$ is a function whose derivatives satisfy

$$\partial_{\mathbf{X}}\theta(\mathbf{X}) = \mathbf{K}(\mathbf{X}) \tag{41}$$

where $\mathbf{K}(\mathbf{X})$ is slowly varying. Accordingly, the solutions to the differential equations are stipulated as locally plane waves with slowly varying parameters (e.g., see [14, 19]). In the present context the representation (36) might have some merit. It is important to note that in (32) and (36) we deal with one and the same function $\boldsymbol{\mathcal{G}}$ with the appropriate arguments indicated. Consequently (32, 36) can be written as (28), which reintroduces the changes in **K**-, and **X**-, spaces simultaneously, leading to a mixed spaces dispersion relation

$$F(\mathbf{K}, \mathbf{X}) = 0 \tag{42}$$

from which the Hamiltonian ray equations are derived (e.g., see [14, 19]).

4. NONLINEAR HOMOGENEOUS MEDIA

Once again, we are dealing with an indeterminate mathematical set of equations (12), and have to provide supplementary constitutive relations. It is conceivable that the physical model itself be nonlinear, in that case (12) is inadequate in its present form. Nevertheless, without detracting from the generality of the present discussion, it will be assumed here that the nonlinearity enters by way of the supplemental relations, as is the case for Maxwell's equations. Therefore (13) must now be replaced by constitutive relations which will account for material nonlinearity. As in the linear case, the constitutive relations can be provided by additional equations, or by some arbitrary models based on empirical results, etc..

The introduction of Volterra's functional series [22] provides a general framework for including nonlinear properties. However, as we go into some detail, it must be noted that higher order terms describe nonlinear interactions of increasing complexity. Hence for any concrete application of the model, some truncation of the infinite series must be effected. It is amply shown in the literature that Volterra's series are the functional analog of the expansion of a function in terms of a Taylor's series. Both cases constitute an approximation when the series is truncated, keeping a finite number of leading terms. Therefor the Volterra series model is sometimes characterized as describing weak nonlinearity (see for example [10] for some earlier references). Volterra's series consist of functionals,

$$\mathbf{B}(\mathbf{X}) = \sum_{n=1}^{\infty} \mathbf{B}^{(n)}(\mathbf{X}) = \sum_{n=1}^{\infty} \mathbf{P}^{(n)}\{\mathbf{X}, \mathbf{A}\} \tag{43}$$

such that each term of the series is given by a generalization of the convolution integral in the form

$$\mathbf{B}^{(n)}(\mathbf{X}) = \int (d^4\mathbf{X}_1) \dots \int (d^4\mathbf{X}_n) \\ \overset{*}{\boldsymbol{\mathcal{G}}}{}^{(n)}(\mathbf{X}_1, \dots, \mathbf{X}_n) \because \mathbf{A}(\mathbf{X} - \mathbf{X}_1) \dots \mathbf{A}(\mathbf{X} - \mathbf{X}_n) \tag{44}$$

where the cluster $\because$ indicates a tensor multiplication of the order prescribed by the arguments. The Fourier transform of (44) is given by

$$\overline{\mathbf{B}}^{(n)}(\mathbf{K}) = (2\pi)^{4(n-1)} \int (d^4\mathbf{K}_1) \ldots \int (d^4\mathbf{K}_{n-1})$$
$$\overset{*}{\overline{\boldsymbol{\mathcal{G}}}}{}^{(n)}(\mathbf{K}_1, \ldots, \mathbf{K}_n) \because \overline{\mathbf{A}}(\mathbf{K}_1) \ldots \overline{\mathbf{A}}(\mathbf{K}_n) \tag{45}$$

subject to the constraint

$$\mathbf{K} = \mathbf{K}_1 + \ldots + \mathbf{K}_n \tag{46}$$

The proof of (45) is deferred to the next section, where inhomogeneity is included. In analogy to the linear case, $\overset{*}{\overline{\boldsymbol{\mathcal{G}}}}{}^{(n)}(\mathbf{K}_1, \ldots, \mathbf{K}_n)$ in (45) is the $4 \times n$ order Fourier transform of $\overset{*}{\boldsymbol{\mathcal{G}}}{}^{(n)}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ in (44). By the same arguments presented above for the linear case (22), we can rearrange or, if necessary, approximate the transformed function in the form

$$\overset{*}{\overline{\boldsymbol{\mathcal{G}}}}{}^{(n)}(\mathbf{K}_1, \ldots, \mathbf{K}_n) = \boldsymbol{\mathcal{G}}^{(n)}(i\mathbf{K}_1, \ldots, i\mathbf{K}_n) \tag{47}$$

In view of (44–47), and the above discussion (22, 26, 27), relating the integral and the differential operators, it follows that a differential operator can be defined such that

$$\mathbf{B}^{(n)}(\mathbf{X}) = \boldsymbol{\mathcal{G}}^{(n)} \left(\partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}, \ldots, \partial_{\mathbf{X}_n}\Big|_{\mathbf{X}_n=\mathbf{X}} \right) \because \mathbf{A}(\mathbf{X}_1) \ldots \mathbf{A}(\mathbf{X}_n) \tag{48}$$

displaying the nonlinear Volterra differential operator. The representation (48) allows for a wide range of nonlinear terms, e.g., if an operator involves some constant multiplying the function, times the square of the four-gradient operation, then we obtain

$$\left(a\Big|_{\mathbf{X}_1=\mathbf{X}}, \partial_{\mathbf{X}_2}\Big|_{\mathbf{X}_2=\mathbf{X}}, \partial_{\mathbf{X}_3}\Big|_{\mathbf{X}_3=\mathbf{X}} \right) \mathbf{A}(\mathbf{X}_1)\mathbf{A}(\mathbf{X}_2)\mathbf{A}(\mathbf{X}_3)$$
$$= a\mathbf{A}(\mathbf{X})(\partial_{\mathbf{X}}\mathbf{A}(\mathbf{X}))^2 \tag{49}$$

where presently A denotes an arbitrary scalar function. Obviously (48) is a convenient and flexible notation for incorporating nonlinear constitutive relations.

Periodic waves are of particular interest (see [13], also [23] for earlier work), because in nonlinear media interactions of periodic waves create periodic waves, i.e., change the balance of amplitudes of the already existing periodic waves. Thus a periodic wave

$$A(\mathbf{X}) = \sum_p A_{0p} e^{ip\mathbf{K}\cdot\mathbf{X}} \tag{50}$$

subjected to (48) yields

$$\begin{aligned} B(\mathbf{X}) &= \sum_m B_{0m} e^{im\mathbf{K}\cdot\mathbf{X}} \\ &= \boldsymbol{\mathcal{G}}\left(\partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}, \ldots, \partial_{\mathbf{X}_n}\Big|_{\mathbf{X}_n=\mathbf{X}}\right) \mathbf{A}(\mathbf{X}_1)\cdot \ldots \cdot \mathbf{A}(\mathbf{X}_n) \qquad (51a) \\ &= \boldsymbol{\mathcal{G}}(ip_1\mathbf{K}, \ldots, ip_n\mathbf{K}) \sum_{p_1} A_{0p_1} e^{ip_1\mathbf{K}\cdot\mathbf{X}} \cdot \ldots \cdot \sum_{p_n} A_{0p_n} e^{ip_n\mathbf{K}\cdot\mathbf{X}} \qquad (51b) \end{aligned}$$

and the orthogonality of the exponentials prescribes

$$m = p_1 + \ldots + p_n \tag{52}$$

thus prescribing relations between the amplitudes of the various harmonics.

5. NONLINEAR INHOMOGENEOUS MEDIA

Following (32), we now introduce inhomogeneity into the medium, appropriately modifying (48)

$$\mathbf{B}^{(n)}(\mathbf{X}) = \boldsymbol{\mathcal{G}}^{(n)}\left(\mathbf{X}, \partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}, \ldots, \partial_{\mathbf{X}_n}\Big|_{\mathbf{X}_n=\mathbf{X}}\right) \because \mathbf{A}(\mathbf{X}_1)\ldots\mathbf{A}(\mathbf{X}_n) \tag{53}$$

By inspection of (19) and (30), or the more general form (35), it is seen that the inhomogeneity introduces an additional fourfold integral in the spectral domain. We therefore expect the Fourier transform of (53) to show the same augmentation compared to the homogeneous case (45, 46). Similarly to (33), we now have

$$\begin{aligned} \overline{\mathbf{B}}^{(n)}(\mathbf{K}) = \int (d^4\mathbf{X}) \boldsymbol{\mathcal{G}}^{(n)}\left(\mathbf{X}, \partial_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}, \ldots, \partial_{\mathbf{X}_n}\Big|_{\mathbf{X}_n=\mathbf{X}}\right) \\ \because \mathbf{A}(\mathbf{X}_1)\ldots\mathbf{A}(\mathbf{X}_n) e^{-i\mathbf{K}\cdot\mathbf{X}} \qquad \text{(54a)} \end{aligned}$$

$$= (2\pi)^{-4n} \int (d^4\mathbf{X}) \int (d^4\mathbf{K}_1) \ldots$$
$$\int (d^4\mathbf{K}_n) \boldsymbol{\mathcal{G}}^{(n)} \left(\mathbf{X}, \partial_{\mathbf{X}_1} \Big|_{\mathbf{X}_1 = \mathbf{X}}, \ldots, \partial_{\mathbf{X}_n} \Big|_{\mathbf{X}_n = \mathbf{X}} \right)$$
$$\because \overline{\mathbf{A}}(\mathbf{K}_1) \ldots \overline{\mathbf{A}}(\mathbf{K}_n) e^{i(\mathbf{K}_1 \cdot \mathbf{X}_1 + \ldots + \mathbf{K}_n \cdot \mathbf{X}_n - \mathbf{K} \cdot \mathbf{X})} \tag{54b}$$
$$= (2\pi)^{-4n} \int (d^4\mathbf{X}) \int (d^4\mathbf{K}_1) \ldots$$
$$\int (d^4\mathbf{K}_n) \boldsymbol{\mathcal{G}}^{(n)} (\mathbf{X}, i\mathbf{K}_1, \ldots, i\mathbf{K}_n)$$
$$\because \overline{\mathbf{A}}(\mathbf{K}_1) \ldots \overline{\mathbf{A}}(\mathbf{K}_n) e^{i\mathbf{X} \cdot (\mathbf{K}_1 + \ldots + \mathbf{K}_n - \mathbf{K})} \tag{54c}$$
$$= (2\pi)^{-4n} \int (d^4\mathbf{X}) \int (d^4\mathbf{K}_1) \ldots$$
$$\int (d^4\mathbf{K}_n) \boldsymbol{\mathcal{G}}^{(n)} (\mathbf{X}, i\mathbf{K}_1, \ldots, i\mathbf{K}_n)$$
$$\because \overline{\mathbf{A}}(\mathbf{K}_1) \ldots \overline{\mathbf{A}}(\mathbf{K}_n) e^{i\mathbf{X} \cdot (\mathbf{K}_1 + \ldots + \mathbf{K}_n - \mathbf{K})} \tag{54d}$$
$$= (2\pi)^{-4(n+1)} \int (d^4\mathbf{Q}) \int (d^4\mathbf{K}_1) \ldots$$
$$\int (d^4\mathbf{K}_n) \boldsymbol{\mathcal{G}}^{(n)} (\mathbf{Q}, i\mathbf{K}_1, \ldots, i\mathbf{K}_n)$$
$$\because \overline{\mathbf{A}}(\mathbf{K}_1) \ldots \overline{\mathbf{A}}(\mathbf{K}_n) \delta(\mathbf{Q} + \mathbf{K}_1 + \ldots + \mathbf{K}_n - \mathbf{K}) \tag{54e}$$

and finally we obtain

$$\overline{\mathbf{B}}^{(n)}(\mathbf{K}) = (2\pi)^{-4n} \int (d^4\mathbf{K}_1) \ldots \int (d^4\mathbf{K}_n) \boldsymbol{\mathcal{G}}^{(n)} (\mathbf{Q}, i\mathbf{K}_1, \ldots, i\mathbf{K}_n)$$
$$\because \overline{\mathbf{A}}(\mathbf{K}_1) \ldots \overline{\mathbf{A}}(\mathbf{K}_n) \tag{55}$$

with the associated constraint

$$\mathbf{Q} = \mathbf{K} - (\mathbf{K}_1 + \ldots + \mathbf{K}_n) \tag{56}$$

Compare now (55, 56) to (45, 46), respectively, noting (47). Also compare (55, 56) to the linear case (35).

In this section we have discussed the medium's nonlinearity and inhomogeneity, both phenomena contributing to the appearance of new spectral components.

6. SPECIAL RELATIVITY AND MINKOWSKI'S CONSTITUTIVE THEORY

The role of constitutive relations in Special Relativity is closely intertwined with the concept of covariance (form-invariance) of the basic physical model in all inertial systems, i.e., the idea that there is no preferred inertial system, hence "laws of nature" have the same form (i.e., formulas possess the same functional structure) in all such systems. The obvious candidate for such a discussion is Maxwell's Electromagnetism, but the following ideas should be applicable to other branches of physics as well, provided such a covariance principle can be identified.

The Lorentz covariance of Maxwell's equations applies to the indeterminate fundamental set (1), e.g., see [1, 2]. Accordingly, if (1) is declared in one inertial system, then in another we have, corresponding to (1), a set with the same functional structure, i.e.,

$$\partial_{\mathbf{X}'} \times \mathbf{E}' = -\partial_{t'}\mathbf{B}' - \mathbf{j}'_m \tag{57}$$

etc., where presently $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$ etc., and the native coordinate system is $\mathbf{X}' = \{\mathbf{x}', ict'\}$. The Lorentz transformation $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$, and the ensuing transformation $\partial_{\mathbf{x}'} = \partial_{\mathbf{x}'}[\partial_{\mathbf{X}}]$ for the components of the four-gradient operator $\partial_{\mathbf{X}} = \partial_{\mathbf{X}}(\partial_{\mathbf{x}}, -\frac{i}{c}\partial_t)$ are explicitly given by

$$\mathbf{x}' = \widetilde{\mathbf{U}} \cdot (\mathbf{x} - \mathbf{v}t), \quad t' = \gamma(t - \mathbf{x} \cdot \mathbf{v}/c^2) \tag{58a}$$

$$\partial_{\mathbf{x}'} = \widetilde{\mathbf{U}} \cdot (\partial_{\mathbf{x}} + \mathbf{v}\partial_t/c^2), \quad \partial_{t'} = \gamma(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}) \tag{58b}$$

$$\gamma = (1 - \beta^2)^{-1/2}, \quad \beta = v/c, \quad v = |\mathbf{v}| \tag{58c}$$

$$\widetilde{\mathbf{U}} = \widetilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}}, \quad \hat{\mathbf{v}} = \mathbf{v}/v \tag{58d}$$

see also [10].

The four-vectors current/charge densities $\mathbf{J}_{e,m} = \{\mathbf{j}_{e,m}, \rho_{e,m}\}$, for either electric or magnetic entities, are related by the Lorentz transformation $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$, i.e., corresponding to (58) we now have

$$\mathbf{J}'_{e,m}(\mathbf{X}') = \mathbf{J}'_{e,m}[\mathbf{J}_{e,m}(\mathbf{X})] \tag{59}$$

and for brevity we introduce the notation

$$\mathbf{J}_{\mathbf{v}}(\mathbf{X}) = \mathbf{J}'_{e,m}(\mathbf{X}') \tag{60}$$

noting that $\mathbf{J_v}(\mathbf{X})$ is defined by (59), but its functional structure is different from both $\mathbf{J}'_{e,m}(\mathbf{X}')$ and $\mathbf{J}_{e,m}(\mathbf{X})$.

The fields appearing in (1) and the corresponding (57) are related by formulas which can be summarized as follows:

$$\mathbf{M}'(\mathbf{X}') = \widetilde{\mathbf{V}} \cdot (\mathbf{M}(\mathbf{X}) + \mathbf{v} \times \mathbf{N}(\mathbf{X})) \tag{61a}$$

$$\mathbf{M} = \{\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}\}, \quad \mathbf{N} = \{\mathbf{B}, -\mathbf{E}/c^2, \mathbf{H}/c^2, \mathbf{D}\} \tag{61b}$$

$$\widetilde{\mathbf{V}} = \gamma\widetilde{\mathbf{I}} + (1-\gamma)\hat{\mathbf{v}}\hat{\mathbf{v}} \tag{61c}$$

where the field elements are respectively chosen from the vectorial quadruplets, with the proper spatiotemporal coordinates native to the inertial frame of reference in question. For brevity (61) is summarized as

$$\mathbf{M}'(\mathbf{X}') = \mathbf{M_v}(\mathbf{X}) \tag{62}$$

Similarly the Lorentz transformation $\partial_{\mathbf{X}'} = \partial_{\mathbf{X}'}[\partial_{\mathbf{X}}]$ for the spatiotemporal derivatives, explicitly given in (58b), is abbreviated as

$$\partial_{\mathbf{X}'} = \partial_{\mathbf{X}}^{\mathbf{v}} \tag{63}$$

By substitution of the set of equations (60, 62, 63) into the set of primed Maxwell's equations (57), one derives the set (1), and vice-versa. Therefore, as previously discussed [10], any two of these sets will yield the third, and any two sets are therefore equivalent to the Special Relativity postulate of the covariance (form-invariance) of Maxwell's equations.

Following these ideas is the essence of the Minkowski methodology. Accordingly, the idea of the covariance of Maxwell's equations is preserved if we are to start from the comoving frame of reference equations (57) (i.e., (1) with properly primed quantities), then substitute the comoving frame of reference constitutive relations, and finally use the Lorentz transformations for the operators, as in (63), and substitute (60, 62) for the densities and fields. This provides a complete algorithm. Nothing more is necessary for the process of considering constitutive relations in a non-comoving (often dubbed as the "laboratory") inertial frame of reference.

The reader should be aware of the fact that the present approach slightly differs from the original one described by Sommerfeld [12], although still within the same framework of the Minkowski method. Thus Sommerfeld starts with the Maxwell equations in the laboratory frame of reference, i.e., (1), adds the constitutive relations in the co-moving frame of reference, and substitutes (62) into them. Inasmuch as the constitutive parameters are treated as constants, the question of dispersion and inhomogeneity, central to the present analysis, does not arise. This approach is not adequate for more general cases where products of fields appear, as in the present case when nonlinearity is involved and dealt with by invoking the Volterra series differential operators.

In more general terms, we start from the system's equations (cf. (10)), here primes are used to indicate the comoving frame of reference

$$\boldsymbol{\mathcal{A}}(\partial_{\mathbf{X}'}) \cdot \mathbf{A}'(\mathbf{X}') = -\boldsymbol{\mathcal{B}}(\partial_{\mathbf{X}'}) \cdot \mathbf{B}'(\mathbf{X}') + \mathbf{w}'(\mathbf{X}') \tag{64}$$

and perform the necessary transformations, obtaining

$$\boldsymbol{\mathcal{A}}(\partial^{\mathbf{v}}_{\mathbf{X}}) \cdot \mathbf{A}_{\mathbf{v}}(\mathbf{X}') = -\boldsymbol{\mathcal{B}}(\partial^{\mathbf{v}}_{\mathbf{X}}) \cdot \mathbf{B}'(\mathbf{X}') + \mathbf{w}_{\mathbf{v}}(\mathbf{X}') \tag{65}$$

Note that $\mathbf{B}'(\mathbf{X}')$ is not changed, because it will be substituted by the constitutive relations. We use

$$\mathbf{B}'(\mathbf{X}') = \sum_n \mathbf{B}'^{(n)}(\mathbf{X}') \tag{66a}$$

$$\mathbf{B}'^{(n)} = \boldsymbol{\mathcal{G}}^{(n)} \left(\mathbf{X}', \partial_{\mathbf{X}'_1}\Big|_{\mathbf{X}'_1=\mathbf{X}'}, \ldots, \partial_{\mathbf{X}'_n}\Big|_{\mathbf{X}'_n=\mathbf{X}'} \right) \because \mathbf{A}'(\mathbf{X}'_1) \ldots \mathbf{A}'(\mathbf{X}'_n) \tag{66b}$$

The series (66a) must be properly understood as a sum of terms as given in (66b), each term involving the tensor product of n fields, determined by the ordinal number n relevant to the term under consideration. This includes the linear case as $n = 1$ where only one term and one field are involved. Substituting $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$ for the inhomogeneity coefficients in $\boldsymbol{\mathcal{G}}^{(n)}$ is denoted by an index $\mathbf{v}$ as shown below. Following the outline above, we recast (66b) as

$$\mathbf{B}'^{(n)} = \boldsymbol{\mathcal{G}}^{(n)}_{\mathbf{v}} \left(\mathbf{X}, \partial^{\mathbf{v}}_{\mathbf{X}_1}\Big|_{\mathbf{X}_1=\mathbf{X}}, \ldots, \partial^{\mathbf{v}}_{\mathbf{X}_n}\Big|_{\mathbf{X}_n=\mathbf{X}} \right) \because \mathbf{A}_{\mathbf{v}}(\mathbf{X}_1) \ldots \mathbf{A}_{\mathbf{v}}(\mathbf{X}_n) \tag{67}$$

and substitute (66a) with (67) into (65). This provides a complete and determinate representation of the physical model and the associated constitutive relations in arbitrary inertial systems (the "laboratory" frame of reference) in which the medium is in motion.

The new Volterra differential operators, presently given for media that are both dispersive and inhomogeneous, provide a rational and practical framework for dealing with constitutive relations within the framework of the Minkowski methodology. In the past, spatiotemporal integral expressions like (21) for the linear case, and even more so (44) for the nonlinear case, raised questions regarding the appropriate formulation, and statement of the limits, in view of the light-cone relativistic causality considerations: Inasmuch as (21), (44) describe non-local interactions, an event $\mathbf{X}$ is affected by all events $\mathbf{X}_1, \ldots, \mathbf{X}_n$ referred to in the integration. The distance between two events must satisfy the postulate that the upper bound for signal speed is c, the speed of light in free space. To be causally related, events must be within the so-called light cone, defined by $|\mathbf{X} - \mathbf{X}_n| \leq 0$. The implementation of this condition in (21, 44) is complicated, and affects the corresponding transforms (20, 45). In the homogeneous cases (11, 48), for linear, nonlinear, systems, respectively, and the corresponding (32) and (53), all the interactions are local. As long as such models are valid, light cone relativistic causality considerations do not arise.

The derivation of explicit expressions for first order velocity effects is nowadays easily facilitated with the aid symbolic mathematics applications. Treating all expressions as functions of $\mathbf{v}$, first order effects are obtained by effecting a Taylor series approximation and keeping the leading order velocity terms, e.g.,

$$\mathbf{A}_\mathbf{v}(\mathbf{X}) = \mathbf{A}(\mathbf{X}) + \left((\mathbf{v} \cdot \partial_\mathbf{v}) \Big|_{\mathbf{v}=0} \right) \mathbf{A}_\mathbf{v}(\mathbf{X}) + \mathcal{O}(\beta^2) \tag{68}$$

where the first term is simply $\mathbf{A}'(\mathbf{X}')$ with the primes dropped. Expressions become considerably simpler when expanded and higher order velocity terms canceled.

7. SUMMARY AND CONCLUDING REMARKS

The thrust of the present investigation is the consistent definition of constitutive relations in the framework of a physical field theory, be it mechanical or otherwise. The Maxwellian theory of Electromagnetism provides a good working model for highlighting the problems and indicating the solutions. Essentially we are interested in macroscopic field models, relating various measurable fields, which are spatiotemporally dependent on the location four-vector **X**. The physical model is usually indeterminate, and requires additional relations between the unknown fields in order to achieve a solution.

Media properties are, generally speaking, dispersion, inhomogeneity, and anisotropicity, and all these might involve nonlinearity as well. While many applications treat the problems at hand heuristically, it is desired to provide a general consistent approach to constitutive relations. This is achieved here by using the Volterra differential operators defined above. These differential operators indicate local interactions, thus obviating the need to use non-local integral expressions.

In the context of Special Relativity, the last point is even more important in view of relativistic causality considerations, usually referred to in terms of the "light cone" concept.

The present discussion is not strictly rigorous in the mathematical sense, and is rather intended for more application-oriented investigators. No doubt there is much to add to the present discussion in order to chart the domains of validity of the various representations.

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