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ELECTROMAGNETIC INTERACTION OF PARALLEL ARRAYS OF DIPOLE SCATTERERS

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1. INTRODUCTION

For many applications we need to study arrays of scattering and absorbing particles, for example for the design of absorbing coverings, frequency selective surfaces, or photonic crystals. These arrays are often located near conducting surfaces or near interfaces with dielectric media. Also, multilayer arrays are of considerable interest. The key configuration to study here is the system of two parallel arrays of particles. In this paper we address this problem for the case when the arrays can be dense (in the scale of the wavelength), and the separation between the array planes can be arbitrary.

Electromagnetic properties of planar arrays of scatterers can be analytically investigated in several different ways. Most of the difficulties here are related to the calculation of the field which excites every particle in the array, so-called *local field*. The local field acting on an inclusion is the sum of two parts: the external field of the incident wave and the field created by all the other particles of the system at the point where the considered inclusion is located. The problem is to find that second contribution into the local field, that is the field of all the other inclusions. It appears that this part of exciting field (so-called *interaction field*) can be evaluated in regular discrete systems by direct summation of scatterers fields. However, this procedure fails when the full-wave case is considered. In 2D and 3D systems the sums representing the interaction field do not converge because they contain wave-field terms which decrease as $1/r$, where r is the distance from an inclusion to the point where the field is evaluated. The assumption of small losses existing in space saves the situation theoretically, but does not give any practical receipt for the interaction field calculation, because the series converge very slowly.

In this paper we develop an analytical method which allows simple analytical solutions for double arrays of dipole scatterers. The main assumption is that the distance between the elements in an array is smaller than about one half of the wavelength, so that the resonant situation when the interaction field tends to infinity (which happens when the particle separation tends to the wave length) is excluded. The distance between the two arrays can be arbitrary, and also the two arrays can contain different scatterers, for example one array can be made up of inclusions having electric dipole moments and the other one can contain inclusions with magnetic dipole moments or indeed a combination of both electric and magnetic dipole moments. The result is expressed as a sum of dipole fields of a few nearly-located dipoles plus the contribution of all the other dipoles located at distances larger than a certain effective radius expressed in a closed form. With the results of this paper, the reflection and transmission problems can be solved easily provided the polarizabilities of the individual particles are known. This is demonstrated for a double array of magnetic particles, and also for a similar array near a metal screen. In fact, since the field of an array at an arbitrary distance is known, also multilayer structures can be analysed in a similar way. The present theory is restricted to the case of the normal plane-wave incidence. This paper develops the idea originally suggested in [1].

For dense arrays, when the distances between the dipoles are smaller than approximately quarter of the wavelength, the solution can be further simplified. In the analytical formula which expresses the contribution of the dipoles located at distances larger than R, an effective radius $R = R_0$ can be introduced, such that the analytical formula gives the total interaction field (only single central dipole is then considered separately). With this result it becomes easy to solve optimization problems such as choosing the particle polarizability which provides the maximum absorption, etc.

In the known literature, the main attention was on three-dimensional bulk structures, and very extensive literature exists on the effective parameters of composite materials. Single arrays were also investigated numerically (especially in connection to the design of frequency selective surfaces) and analytically in the quasistatic approximation (e.g., [2]). Interactions in layered structures were considered for small distances between layers, in modeling surface properties (e.g., [3, 4]). Usually, these theories were based on the quasistatic approximation or (and) numerical simulations. Also, layered structures with resonant distances between layers were considered in studies ofphotonic band-gap materials, mainly with numerical methods. The exact full-wave solution is available for a single regular array of electric dipoles oriented parallel to the array plane [5].

2. CALCULATION OF LOCAL FIELD

Consider a system of two parallel infinite periodic arrays of scattering particles, see Figure 1. Elements belonging to one array are identical and identically oriented, but they can be different from the other array elements. The shape of the particles can be arbitrary, provided they can be modeled in the dipole approximation. The cell size for both arrays is $a \times b$.

On one hand, the system is investigated in such frequency ranges that the distance between the elements is smaller than approximately $\lambda/2$, where λ is the wavelength of the external field. On the other hand, the characteristic dimension of the particles is smaller than the

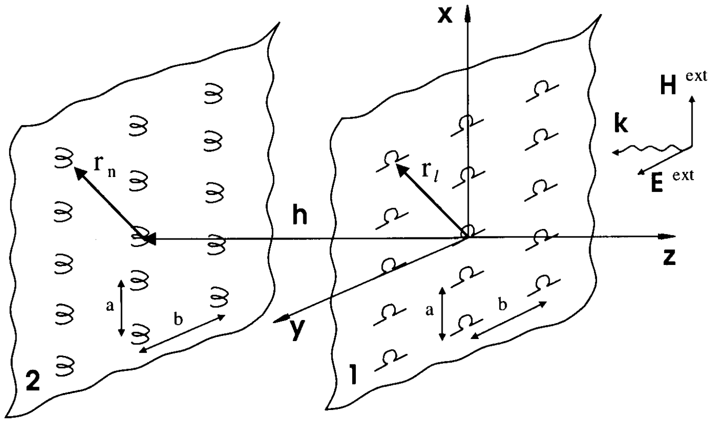

Figure 1. Geometry of the problem.

distance between them. In view of these conditions we assume that the scatterers can be replaced by electric or magnetic dipoles, or a combination of these. Distance between array planes is h, vector $\mathbf{h}$ is parallel to the axis z. The system is under normal plane-wave excitation. We use the standard notations for the free-space wave impedance ($\eta = \sqrt{\mu_0/\epsilon_0}$) and wavenumber ($k = \omega\sqrt{\varepsilon_0\mu_0} = 2\pi/\lambda$). The time dependence is of the form $e^{j\omega t}$.

To find the local field acting on a arbitrarily chosen particle it is necessary to sum up the fields created by all the other particles of the system at the location point of a chosen one. It is obvious that the corresponding sums do not converge in the case when the wave-field terms are taken into account. By assuming that there are small losses in space, the sum can be calculated, but the convergence is very slow because of the oscillating behavior of the sum members. In [1], an alternative method to find the local field avoiding the direct summation of the dipole fields has been proposed. Here we extend that method to double and multilayer arrays, calculating the field created by an array at an arbitrary observation point, not necessary in the array plane. This gives a possibility to study layered structures of dipolar arrays by analytical means.

Under the normal plane-wave excitation all the dipoles of each array are equal and in-phase. Let us write the expression for the full-wave fields created by electric and magnetic dipoles. For the electric field of an electric dipole and the magnetic field of a magnetic dipole we have, respectively (see, e.g., [6, p. 411]),

$$\mathbf{E}^p(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\begin{aligned} & k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{-jkr}}{r} \\ & \quad + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\end{aligned}\right\} \tag{1}$$

$$\mathbf{H}^m(\mathbf{r}) = \frac{1}{4\pi\mu_0}\left\{\begin{aligned} & k^2(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\frac{e^{-jkr}}{r} \\ & \quad + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}]\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\end{aligned}\right\} \tag{2}$$

The electric field created by a magnetic dipole and the magnetic field created by an electric dipole are the following:

$$\mathbf{E}^m(\mathbf{r}) = \frac{k^2}{4\pi\sqrt{\epsilon_0\mu_0}}(\mathbf{n}\times\mathbf{m})\frac{e^{-jkr}}{r}\left(1+\frac{1}{jkr}\right) \tag{3}$$

$$\mathbf{H}^p(\mathbf{r}) = -\frac{k^2}{4\pi\sqrt{\epsilon_0\mu_0}}(\mathbf{n}\times\mathbf{p})\frac{e^{-jkr}}{r}\left(1+\frac{1}{jkr}\right) \tag{4}$$

Here $\mathbf{p}$ is the electric dipole moment, $\mathbf{m}$ is the magnetic dipole moment, $\mathbf{n}=\mathbf{r}/r$ is the unit vector in the direction of $\mathbf{r}$, $r=|\mathbf{r}|$, and vector $\mathbf{r}$ points from the observation point to the dipole.

Because the particles of one array can be different from that in the other array, the local field exciting the elements of the system should be found for the two planes separately. This local field at the plane of each array can be written as the sum of the external field $\mathbf{E}^{\text{ext}}=\mathbf{E}_0e^{-j\mathbf{k}\cdot\mathbf{r}}=\mathbf{E}_0e^{jkz}$ ($\mathbf{H}^{\text{ext}}=\mathbf{H}_0e^{jkz}$) and the interaction field $\mathbf{E}^{\text{int}}$, $\mathbf{H}^{\text{int}}$:

$$\begin{aligned} \mathbf{E}_1^{\text{loc}} &= \mathbf{E}^{\text{ext}}|_{z=0}+\mathbf{E}_1^{\text{int}}, \qquad & \mathbf{E}_2^{\text{loc}} &= \mathbf{E}^{\text{ext}}|_{z=-h}+\mathbf{E}_2^{\text{int}} \\ \mathbf{H}_1^{\text{loc}} &= \mathbf{H}^{\text{ext}}|_{z=0}+\mathbf{H}_1^{\text{int}}, \qquad & \mathbf{H}_2^{\text{loc}} &= \mathbf{H}^{\text{ext}}|_{z=-h}+\mathbf{H}_2^{\text{int}} \end{aligned} \tag{5}$$

Note that the geometry of the problem is such that the local field acting on the particles in one plane is the same for all of them. This allows to place the origin of the Cartesian coordinate system (x,y,z) at the center of an arbitrary particle and restrict the analysis to the local field for the particle at the origin and for the particle at $(0,0,-h)$.

To calculate the interaction field we remove one dipole and sum up the fields generated at its location by all the other dipoles of both arrays. Electric interaction fields are given by

$$\begin{aligned}
\mathbf{E}_1^{\text{int}} &= {\sum_l}' \mathbf{E}^p(\mathbf{r}_l) + \sum_n \{\mathbf{E}^p(\mathbf{r}_n + \mathbf{h}) + \mathbf{E}^m(\mathbf{r}_n + \mathbf{h})\} \\
\mathbf{E}_2^{\text{int}} &= {\sum_n}' \mathbf{E}^p(\mathbf{r}_n) + \sum_l \{\mathbf{E}^p(\mathbf{r}_l - \mathbf{h}) + \mathbf{E}^m(\mathbf{r}_l - \mathbf{h})\}
\end{aligned} \tag{6}$$

Magnetic interaction fields are given by similar relations if one replaces $\mathbf{E}$ by $\mathbf{H}$, index p by index m and conversely (here and in the following we do not explicitly present this case to save space). The first terms in (6) are the contributions of all the scatterers in the same array where the local field is determined, except of one particle (this exception is denoted by the prime sign). Note that we do not consider the electric field created by magnetic dipoles here because for our case its contribution into the local field vanishes due to the symmetry of the problem. The second sum in (6) is the field created by the scatterers of the other array.

According to the present method, which is in a way similar to the Lorenz–Lorentz–Clausius–Mossotti theory, to find the interaction field avoiding direct summation we choose a circle of radius R on the planes of both arrays, and for each plane replace the discrete dipole moment distribution outside the circle by a continuous distribution of dipole moments. The continuous distributions (surface densities of dipole moments) are connected to the discrete dipole moments as

$$\mathbf{p}_s = \frac{\mathbf{p}}{S_0}, \qquad \mathbf{m}_s = \frac{\mathbf{m}}{S_0} \tag{7}$$

where $\mathbf{p}_s$ and $\mathbf{m}_s$ denote the electric and magnetic dipole surface densities, respectively. S_0 is the cell area ($S_0 = ab$ for rectangular cells). Next, from every sum in the expression for the interaction field (6) we extract the contributions of distant dipoles ($r_i > R$). These extracted sums we replace by integrals which are the fields of the homogenized dipole moment distributions:

$$\sum_i{}' \mathbf{E}^p(\mathbf{r}_i) \approx \sum_{\substack{i \\ r_i < R}}{}' \mathbf{E}^p(\mathbf{r}_i) + \frac{1}{S_0} \int\limits_{r_i > R} \mathbf{E}^p(\mathbf{r}_i)\, dS \tag{8}$$

$$\sum_i \{\mathbf{E}^p(\mathbf{r}_i \pm \mathbf{h}) + \mathbf{E}^m(\mathbf{r}_i \pm \mathbf{h})\} \approx \sum_{\substack{i\\ r_i<R}} \{\mathbf{E}^p(\mathbf{r}_i \pm \mathbf{h}) + \mathbf{E}^m(\mathbf{r}_i \pm \mathbf{h})\} + \frac{1}{S_0}\int\limits_{r_i>R} \{\mathbf{E}^p(\mathbf{r}_i \pm \mathbf{h}) + \mathbf{E}^m(\mathbf{r}_i \pm \mathbf{h})\}\, dS \tag{9}$$

Now it is convenient to introduce the following notations:

$$\begin{aligned} \mathbf{E}^{\text{sheet}}(\mathbf{p}, \pm\mathbf{h}) &= \frac{1}{S_0}\int\limits_{r_i>R} \mathbf{E}^p(\mathbf{r}_i \pm \mathbf{h})\, dS \\ \mathbf{E}^{\text{sheet}}(\mathbf{m}, \pm\mathbf{h}) &= \frac{1}{S_0}\int\limits_{r_i>R} \mathbf{E}^m(\mathbf{r}_i \pm \mathbf{h})\, dS \end{aligned} \tag{10}$$

$$\begin{aligned} \mathbf{E}^{\text{hole}}(\mathbf{p}, \pm\mathbf{h}) &= \sum_{\substack{i\\ r_i<R}} \mathbf{E}^p(\mathbf{r}_i \pm \mathbf{h}), \qquad h \neq 0 \\ \mathbf{E}^{\text{hole}}(\mathbf{m}, \pm\mathbf{h}) &= \sum_{\substack{i\\ r_i<R}} \mathbf{E}^m(\mathbf{r}_i \pm \mathbf{h}), \qquad h \neq 0 \end{aligned} \tag{11}$$

$\mathbf{E}^{\text{sheet}}$ is the field created by the continuous distribution sheet with a hole of radius R, at the axis of the hole and at a distance h from its center. $\mathbf{E}^{\text{hole}}$ is the field created by the discrete dipoles inside the circle. In the case $h = 0$ we must replace $\sum$ by $\sum'$. It means that the dipole for which the local field is determined does not take part in the generation of this field (which is applied to this very dipole).

Then, the local field can be represented in terms of $\mathbf{E}_0$, $\mathbf{E}^{\text{sheet}}$ and $\mathbf{E}^{\text{hole}}$:

$$\begin{aligned} \mathbf{E}_1^{\text{loc}} &= \mathbf{E}_0 + \mathbf{E}^{\text{sheet}}(\mathbf{p}_1, 0) + \mathbf{E}^{\text{sheet}}(\mathbf{p}_2, \mathbf{h}) + \mathbf{E}^{\text{sheet}}(\mathbf{m}_2, \mathbf{h}) \\ &\quad + \mathbf{E}^{\text{hole}}(\mathbf{p}_1, 0) + \mathbf{E}^{\text{hole}}(\mathbf{p}_2, \mathbf{h}) + \mathbf{E}^{\text{hole}}(\mathbf{m}_2, \mathbf{h}) \\ \mathbf{E}_2^{\text{loc}} &= \mathbf{E}_0 e^{-jkh} + \mathbf{E}^{\text{sheet}}(\mathbf{p}_2, 0) + \mathbf{E}^{\text{sheet}}(\mathbf{p}_1, -\mathbf{h}) + \mathbf{E}^{\text{sheet}}(\mathbf{m}_1, -\mathbf{h}) \\ &\quad + \mathbf{E}^{\text{hole}}(\mathbf{p}_2, 0) + \mathbf{E}^{\text{hole}}(\mathbf{p}_1, -\mathbf{h}) + \mathbf{E}^{\text{hole}}(\mathbf{m}_1, -\mathbf{h}) \end{aligned} \tag{12}$$

Magnetic local field can be expressed analogously. Next we calculate $\mathbf{E}^{\text{sheet}}$ and $\mathbf{H}^{\text{sheet}}$. To evaluate the integrals we split the excited in particles electric and magnetic dipole moments into components which are parallel and orthogonal to the array planes and consider them separately. After some algebra, the integrals reduce to simple combinations of elementary functions. For co-directed dipoles parallel to the array plane we find:

$$\mathbf{E}^{\text{sheet}}(\mathbf{p},\mathbf{h}) = -j\omega\mathbf{p}_s\frac{\eta}{4}\left\{1-\frac{1}{jk\sqrt{R^2+h^2}} + \frac{h^2}{R^2+h^2}\left(1+\frac{1}{jk\sqrt{R^2+h^2}}\right)\right\}e^{-jk\sqrt{R^2+h^2}} \quad (13)$$

$$\mathbf{H}^{\text{sheet}}(\mathbf{m},\mathbf{h}) = -\frac{j\omega\mathbf{m}_s}{4\eta}\left\{1-\frac{1}{jk\sqrt{R^2+h^2}} + \frac{h^2}{R^2+h^2}\left(1+\frac{1}{jk\sqrt{R^2+h^2}}\right)\right\}e^{-jk\sqrt{R^2+h^2}} \quad (14)$$

$$\mathbf{E}^{\text{sheet}}(\mathbf{m},\mathbf{h}) = \frac{j\omega}{2}\frac{\mathbf{h}\times\mathbf{m}_s}{\sqrt{R^2+h^2}}e^{-jk\sqrt{R^2+h^2}} \quad (15)$$

$$\mathbf{H}^{\text{sheet}}(\mathbf{p},\mathbf{h}) = -\frac{j\omega}{2}\frac{\mathbf{h}\times\mathbf{p}_s}{\sqrt{R^2+h^2}}e^{-jk\sqrt{R^2+h^2}} \quad (16)$$

For dipoles orthogonal to the plane we have:

$$\mathbf{E}^{\text{sheet}}(\mathbf{p},\mathbf{h}) = -\frac{j\omega\mathbf{p}_s\eta}{2}\left\{1+\frac{1}{jk\sqrt{R^2+h^2}}\right\}\frac{R^2}{R^2+h^2}e^{-jk\sqrt{R^2+h^2}} \quad (17)$$

$$\mathbf{H}^{\text{sheet}}(\mathbf{m},\mathbf{h}) = -\frac{j\omega\mathbf{m}_s}{2\eta}\left\{1+\frac{1}{jk\sqrt{R^2+h^2}}\right\}\frac{R^2}{R^2+h^2}e^{-jk\sqrt{R^2+h^2}} \quad (18)$$

$$\mathbf{E}^{\text{sheet}}(\mathbf{m},\mathbf{h}) = \mathbf{H}^{\text{sheet}}(\mathbf{p},\mathbf{h}) = 0 \quad (19)$$

As a check, we can observe that for $h = 0$ these results reduce to that obtained earlier in [1]. Also, all the limiting cases give the expected correct results. In particular, for $h \to \infty$ we have simple plane-wave fields created by the averaged polarizations. Also for $R = 0$ we have a uniform polarized sheet (electric or magnetic current sheet), and the results correctly give the plane-wave field excited by a current plane. In Appendix we show the details of the calculations of the field created by the continuous electric dipole distribution sheet with a hole of radius R for co-directed dipoles parallel to the array plane ($\mathbf{E}^{\text{sheet}}(\mathbf{p},\mathbf{h})$, (13)).

3. APPROXIMATE ANALYTICAL FORMULA

For dense arrays only a few nearly located inclusions must be considered as separate scatterers in the calculations of the interaction field, and the main contribution can be evaluated as that from the averaged dipole moment density, see [1]. We have a freedom to choose the radius R inside which we consider the inclusions individually. If we now replace all the particles of a square-cell array but the reference one by a "sheet with a hole" (which means we choose the hole radius R to be smaller than the distance between the particles a), we can find such a radius of the hole that the results given for $h = 0$ by formulas (13), (17) (or (14), (18)) coincide at zero frequency with the known static results. This takes place for $R = R_0 = a/(4C_{\rm par})$ for dipoles parallel to the plane and $R_0 = a/(2C_{\rm ort})$ for dipoles orthogonal to the plane, where $C_{\rm par}$ and $C_{\rm ort}$ are the known interaction constants at zero frequency, see the next section. It is interesting to note that this value is the same for dipoles both parallel and orthogonal to the plane (in fact, $R_0 \approx a/1.438$). Next, the contributions into the interaction field from both arrays at low frequencies can be approximately found from (13)–(19) substituting this special value of R. Curves calculated by this approximate method are compared with the exact solution in the next Section (surprizingly good agreement with the exact solution is observed). In principle, one could establish an even more accurate model defining the effective hole radius as a function of the distance h to the array plane. However, the calculated results show that there would be only a small improvement but far more complications.

This approximate method gives reasonable results for grids with square cells only. The reason is that in other cases the round shape of the hole does not correspond to the cell shape.

4. NUMERICAL EXAMPLES

In the numerical examples we consider arrays with square cells ($a = b$). The contributions into electric and magnetic interaction fields from one array of electric dipoles are evaluated for different values of the parameters ka and h/a. Two cases are considered, that of the parallel and perpendicular orientations of the dipole moments with respect to the array plane.

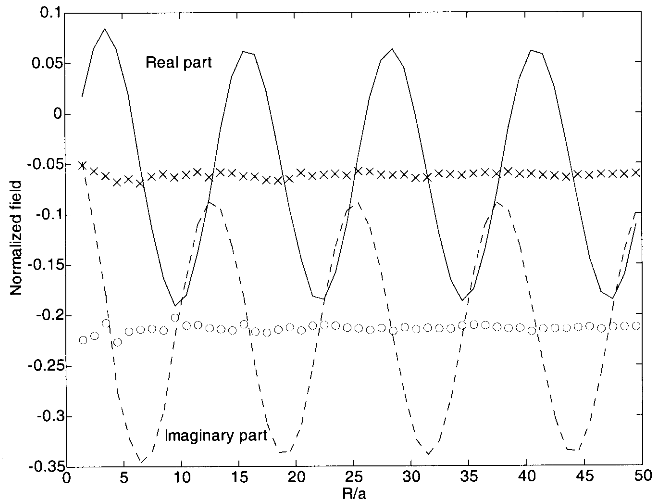

Figure 2. Behavior of the sum of dipole electric fields for $ka = 0.5$ and $h = a$. Parallel orientation of electric dipoles.

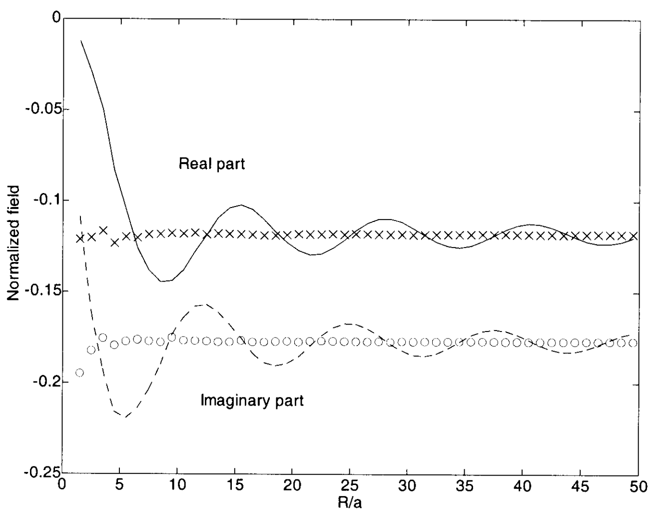

Figure 3. Behavior of the sum of dipole magnetic fields for $ka = 0.5$ and $h = a$. Parallel orientation of electric dipoles.

Let us study the behavior of the dipole field sums $\mathbf{E}^{\rm hole}$, $\mathbf{H}^{\rm hole}$. The calculated results for the parallel orientation of dipoles are presented on Figures 2 and 3. On Figure 2 the normalized electric field is given in the form of the interaction constant $C = E\epsilon_0 a^3/p$ (as defined by Collin in [5]). The solid and dashed lines represent, respectively, the real and imaginary parts of the normalized field created by all the discrete dipoles located in the "hole" of one plane. The frequency is chosen so that $ka = 0.5$. The distance from the array plane to the observation point $h = a$. One can see that when the hole radius R increases, this part of the interaction field does not converge to any value, but it is an oscillating function of R. However, when the field of the external to the "hole" area $\mathbf{E}^{\rm sheet}$ is added, the behavior of the total one-plane interaction field $\mathbf{E}^{\rm hole} + \mathbf{E}^{\rm sheet}$ is completely different: good convergence is observed. The corresponding results are presented on Figure 2 as $\times$ and $\circ$ points marking the real and imaginary parts of C, respectively. Figure 3 shows the same results for the magnetic field of the parallel-oriented electric dipoles. The normalizing factor for the magnetic field is defined as $\eta\epsilon_0 a^3/p$.

Figures 4–9 (solid lines) show the normalized electric and magnetic fields created by one array of the electric dipoles (both parallel- and orthogonal-oriented) as functions of the parameters ka and h/a. The results given by an approximate analytical formula introduced in the previous Section are shown by crosses $\times$. Here the contribution of the central dipole has been subtracted from the calculated values of the fields. Because at large distances from the array plane the interaction field behaves as a plane wave $\exp(-jkh)$, the normalization factors for the electric and magnetic fields of the array are chosen as $\epsilon_0 a^3 \exp(jkh)/p$ and $\eta\epsilon_0 a^3 \exp(jkh)/p$, respectively. In this form the results show the "amplitude" of the wave, which of course depends on the distance from the plane. The fields have been calculated in the following manner. At first, the hole radius R was chosen so large that the oscillations of the total interaction field $\mathbf{E}^{\rm hole} + \mathbf{E}^{\rm sheet}$ (or $\mathbf{H}^{\rm hole} + \mathbf{H}^{\rm sheet}$) become negligible, see Figures 2, 3. Then for that R the fields have been calculated and normalized.

On Figure 4 the values of the normalized electric field of parallel-oriented electric dipoles are depicted. One can see that for $ka = 0.1$ and $h = 0$ the magnitude of the normalized field approaches its exact static value (approximately 0.36). When the distance from the array plane h increases, the magnitude of the real part of the field becomes smaller and smaller. This is the reactive field which corresponds to the local, non-propagating part of the array field. The imaginary part corresponds to the magnitude of the plane wave excited by the averaged plane current. For $ka = 1.0$ this part of the field is larger because the current $\mathbf{J}_s = j\omega\mathbf{p}_s$ which radiates that plane wave is proportional to the frequency. From these results we can estimate how close to the array plane the far-zone plane wave if formed: the field is nearly a plane wave at distances of about several grid periods. On Figure 5 the same results for the normalized magnetic field of the same arrays of electric dipoles are shown. Just at the array plane (at $h = 0$) the magnetic field is zero, at large distances it becomes the plane-wave field (the imaginary parts tend to the same limits as that of the normalized electric field, compare with Figure 4). The reactive field has an extremum at about one half of the grid period.

Figure 6 represents the normalized electric field of the orthogonal-oriented dipoles. Because no plane wave is excited in such systems, the interaction field tends to zero when $h \to \infty$. For the case of $h = 0$ the exact static solution can be found in a similar way as in [5] for the dipoles parallel to the plane (see also [2]). For the perpendicular orientation, the result is of the opposite sign and twice as large as in the previous case (approximately -0.72). We observe that for small ka the limiting value agrees with the known static result. Figures 7–9 show the dependency of the normalized fields on ka. These results allow to conclude that the approximate analytical formula which uses the static solution to determine the equavalent hole radius R_0 gives in fact very good results up to $ka \approx 1.5$, where dynamic effects are already quite essential. From the shapes of the curves one can see that the real and imaginary parts of the interaction constants are subject to the Kramers–Kronig relations, as should be.

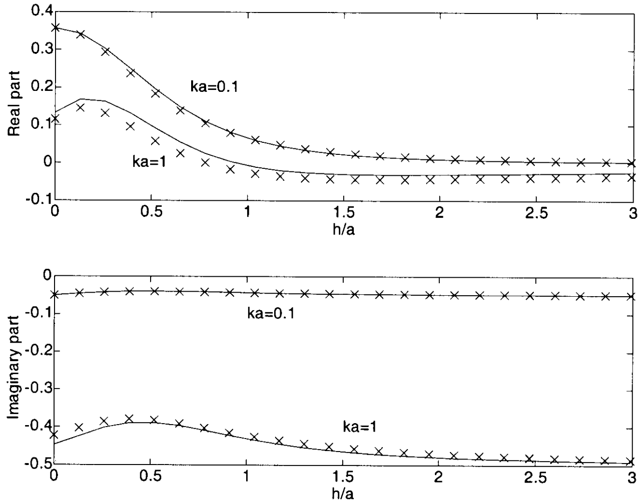

Figure 4. Real and imaginary parts of the normalized electric field $E\varepsilon_0 a^3 \exp(jkh)/p$ as functions of h/a. Electric dipoles are parallel to the array plane. Exact (solid lines) and approximate $(\times)$ results.

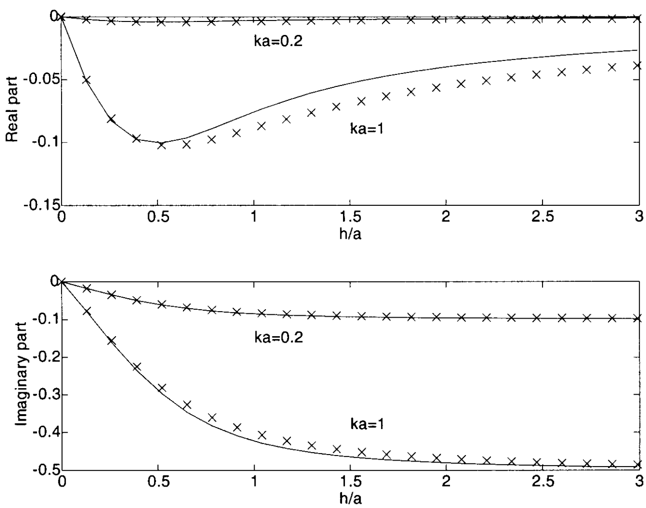

Figure 5. Real and imaginary parts of the normalized magnetic field $H\eta\varepsilon_0 a^3 \exp(jkh)/p$ as functions of h/a. Electric dipoles are parallel to the array plane. Exact (solid lines) and approximate $(\times)$ results.

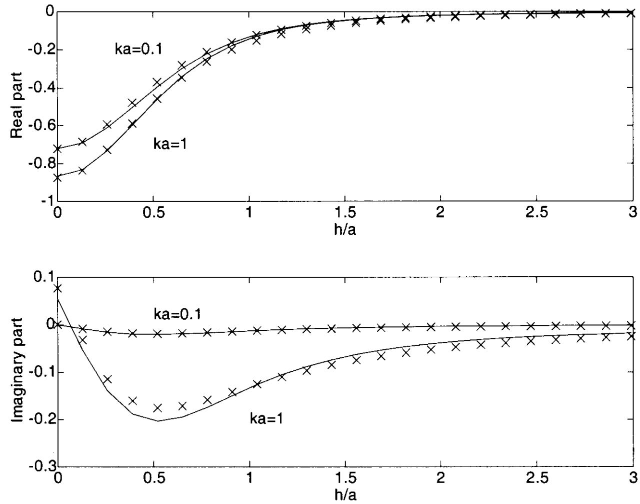

Figure 6. Real and imaginary parts of the normalized electric field $E\varepsilon_0 a^3 \exp(jkh)/p$ as functions of h/a. Electric dipoles are orthogonal to the array plane. Exact (solid lines) and approximate $(\times)$ results.

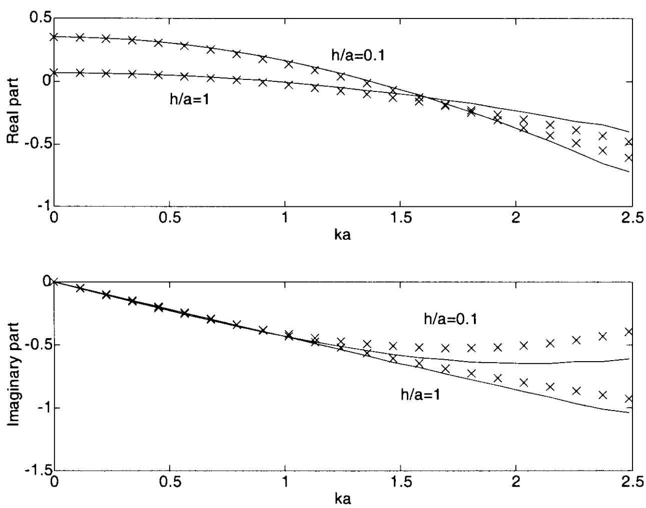

Figure 7. Real and imaginary parts of the normalized electric field $E\varepsilon_0 a^3 \exp(jkh)/p$ as functions of ka. Electric dipoles are parallel to the array plane. Exact (solid lines) and approximate $(\times)$ results.

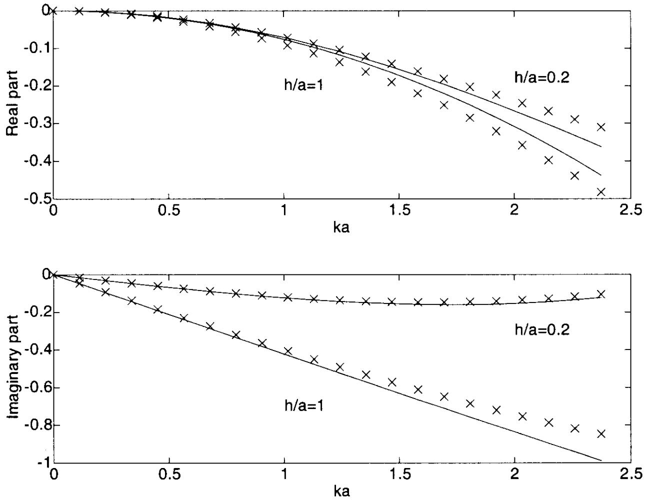

Figure 8. Real and imaginary parts of the normalized magnetic field $H\eta\varepsilon_0 a^3 \exp(jkh)/p$ as functions of ka. Electric dipoles are parallel to the array plane. Exact (solid lines) and approximate $(\times)$ results.

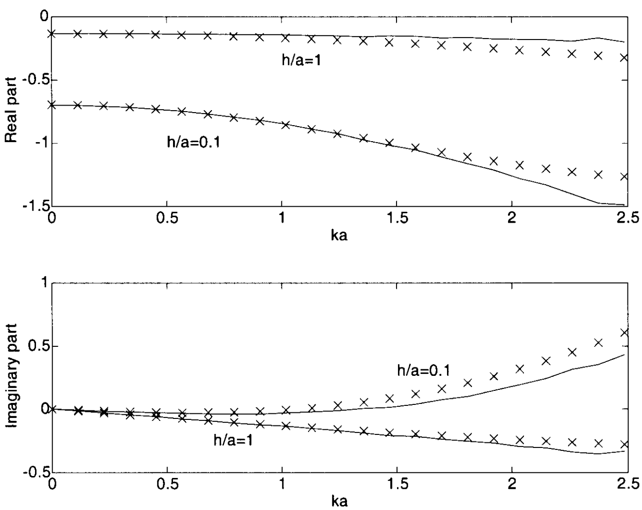

Figure 9. Real and imaginary parts of the normalized electric field $E\varepsilon_0 a^3 \exp(jkh)/p$ as functions of ka. Electric dipoles are orthogonal to the array plane. Exact (solid lines) and approximate $(\times)$ results.

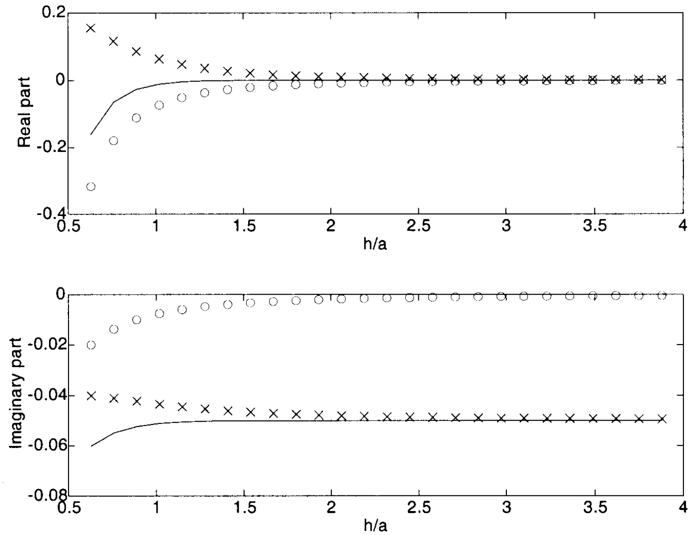

Figure 10. Real and imaginary parts of the normalized electric field $E\varepsilon_0 a^3 \exp(jkh)/p$ as functions of h/a for $ka = 0.1$. Dipoles are parallel to the array plane. The field of the sheet with a hole $(\times)$, the field of the central dipole $(\circ)$ and their sum (solid lines).

In the previous examples, the contribution of the central dipole has been removed, mainly because in calculations of the local field in a single array that reference particle is absent. In that way we could see how that local field behaves at small distances from the plane. However, when one calculates the contribution of the inclusions in one plane to the local field for an inclusion in the other plane, the first plane must be a complete plane with all the particles present. The next two Figures (10, 11) give a comparison of the contributions from the central dipole and that from all the other dipoles to the field of an array measured at distance h from the array plane. Figure 10 is for dense arrays ($ka = 0.1$). We observe that the central dipole contribution into the reactive field (the real part of the normalized field) dominates at small distances from the plane. The rest of the plane gives a contribution of the opposite sign, and at distances larger than about $h = a$ the two parts cancel, and only the plane-wave field (the imaginary part of the field) remains. Clearly, the contribution to the plane wave from one individual dipole is negligible.

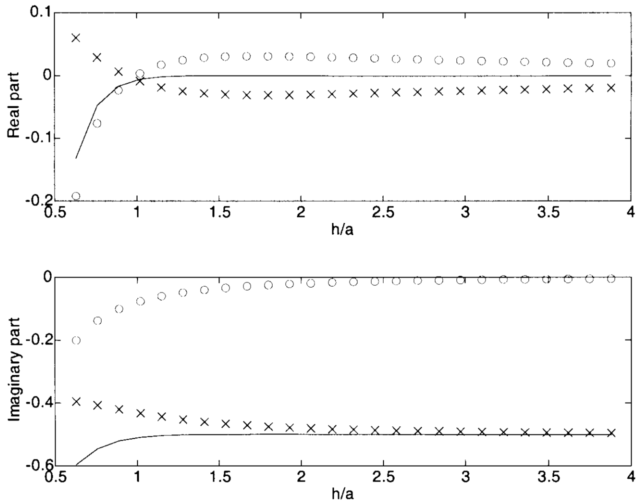

Figure 11. Same as in Figure 10 for $ka = 1$.

Figure 11 corresponds to sparse arrays ($ka = 1$). Also in this case we see that the reactive field of the plane quickly vanishes (see solid line, the real part of the field) as the contributions from the central dipole and that from all the other dipoles cancel each other. In this case the discrete character of the array is seen at much larger (in terms of the ratio h/a) distances from the plane. However, it is very important that also in this case the total field of the plane (solid lines) reduces to a plane wave already quite close to the plane: that happens at $h > 2a$, just like for dense arrays.

5. ARRAYS OF MAGNETIC DIPOLE SCATTERERS

5.1 Two Arrays of Magnetic Dipoles with Scalar Polarizabilities

In this section we consider a system of two parallel square-cells arrays of magnetically polarizable dipole particles. Geometry of the problem is depicted on Figure 1. The system is illuminated by a plane wave $\mathbf{H}^{\text{ext}} = \mathbf{H}_0 e^{jkz}$. The particles of the two arrays have scalar magnetic polarizabilities which are equal to χ_1 and χ_2, where the index refers to the first and the second array, respectively. Induced magnetic dipole moments of the particles are determined by the values of the local fields:

$$\mathbf{m}_1 = \chi_1 \mathbf{H}_1^{\rm loc}, \qquad \mathbf{m}_2 = \chi_2 \mathbf{H}_2^{\rm loc} \tag{20}$$

The expressions for the local fields are given analogously to

$$\begin{aligned}\mathbf{H}_1^{\rm loc} &= \mathbf{H}_0 + \mathbf{H}^{\rm sheet}(\mathbf{m}_1, 0) + \mathbf{H}^{\rm hole}(\mathbf{m}_1, 0)\\ &\quad + \mathbf{H}^{\rm sheet}(\mathbf{m}_2, \mathbf{h}) + \mathbf{H}^{\rm hole}(\mathbf{m}_2, \mathbf{h})\\ \mathbf{H}_2^{\rm loc} &= \mathbf{H}_0 e^{-jkh} + \mathbf{H}^{\rm sheet}(\mathbf{m}_2, 0) + \mathbf{H}^{\rm hole}(\mathbf{m}_2, 0)\\ &\quad + \mathbf{H}^{\rm sheet}(\mathbf{m}_1, \mathbf{h}) + \mathbf{H}^{\rm hole}(\mathbf{m}_1, \mathbf{h})\end{aligned} \tag{21}$$

Using formula (14) with $R = R_0 = a/1.438$ (see Section 3) we can rewrite relations (21) in the following form:

$$\begin{aligned}\mathbf{H}_1^{\rm loc} &= \mathbf{H}_0 + \alpha\mathbf{m}_1 + \beta\mathbf{m}_2\\ \mathbf{H}_2^{\rm loc} &= \mathbf{H}_0 e^{-jkh} + \alpha\mathbf{m}_2 + \beta\mathbf{m}_1\end{aligned} \tag{22}$$

where the interaction coefficients α and β read:

$$\alpha = -\frac{j\omega}{4\eta S_0}\left\{1 - \frac{1}{jkR_0}\right\} e^{-jkR_0} \tag{23}$$

$$\begin{aligned}\beta = &-\frac{j\omega}{4\eta S_0}\left\{1 - \frac{1}{jk\sqrt{R_0^2+h^2}}\right.\\ &\left. + \frac{h^2}{R_0^2+h^2}\left(1 + \frac{1}{jk\sqrt{R_0^2+h^2}}\right)\right\} e^{-jk\sqrt{R_0^2+h^2}}\\ &- \frac{1}{4\pi\mu_0}\left\{\frac{1}{h^3} + \frac{jk}{h^2} - \frac{k^2}{h}\right\} e^{-jkh}\end{aligned} \tag{24}$$

Parameter α determines the interaction between the particles of one array, and parameter β — interaction between the two arrays. The second term in (24) takes into account the contribution of the central dipole from the other array. Using (20), (22)–(24) we find

$$\mathbf{m}_1 = \frac{\mathbf{H}_0}{\Delta}\left\{\frac{1}{\chi_2} - \alpha + \beta e^{-jkh}\right\}, \quad \mathbf{m}_2 = \frac{\mathbf{H}_0}{\Delta}\left\{\left(\frac{1}{\chi_1} - \alpha\right) e^{-jkh} + \beta\right\} \tag{25}$$

where $\Delta = (\chi_1^{-1} - \alpha)(\chi_2^{-1} - \alpha) - \beta^2$. To calculate the far-zone reflected or transmitted field the parallel-oriented magnetic dipoles belonging to one plane can be replaced by a magnetic current sheet carrying the surface current $\mathbf{J}_m = j\omega\mathbf{m}/S_0$. Magnetic field created by that current sheet is $\mathbf{H} = -\mathbf{J}_m e^{-jk|z|}/2\eta$, for the case when the sheet is located at $z = 0$. Now we can write the reflected field as a sum of two parts created by the first and the second arrays:

$$\mathbf{H}^{\text{ref}} = \mathbf{H}_1^{\text{ref}} + \mathbf{H}_2^{\text{ref}} = -\frac{j\omega}{2\eta S_0}\left(\mathbf{m}_1 + \mathbf{m}_2 e^{-jkh}\right\} \tag{26}$$

Hence, the reflection coefficient is

$$\mathcal{R} = \left.\frac{H^{\text{ref}}}{H^{\text{ext}}}\right|_{z=0} = -\frac{j\omega}{2\eta S_0 \Delta}\left\{\frac{e^{-j2kh}}{\chi_1} + \frac{1}{\chi_2} - \alpha(1 + e^{-j2kh}) + 2\beta e^{-jkh}\right\} \tag{27}$$

5.2 Reflection From an Array Near a Metal Screen

The above solution can be used to find the reflection coefficient from an array of magnetic particles near a metal screen. This problem is relevant to the design of radar absorbers based on artifical magnetics. Let us consider the case of one array of magnetic dipole particles with scalar polarizabilities χ. The array plane is located at distance $h/2$ from a metal screen. The incident field is the same as assumed in the previous section. In this case we can use Figure 1 assuming that plane one is the plane of the dipole scatterers and plane two is the mirror image of that scatterers in the ideally conducting plane. The value of the magnetic dipole moment of the particles is

$$\mathbf{m} = \chi\mathbf{H}^{\text{loc}} \tag{28}$$

where the local field can be written in terms of $\mathbf{H}_0$ and $\mathbf{H}^{\text{int}}$:

$$\mathbf{H}^{\text{loc}} = \mathbf{H}_0(1 + e^{-jkh}) + \mathbf{H}^{\text{int}} \tag{29}$$

Here the factor $1 + e^{-jkh}$ is needed to account for the reflection of the incident wave in the screen. $\mathbf{H}^{\text{int}}$ is the interaction field, which is proportional to the magnetic moment of the particles:

$$\mathbf{H}^{\text{int}} = \alpha\mathbf{m} \tag{30}$$

The interaction coefficient α has been studied above, and it reads

$$\alpha = -\frac{j\omega}{4\eta S_0}\left\{1-\frac{1}{jkR_0}\right\}e^{-jkR_0} - \frac{1}{4\pi\mu_0}\left\{\frac{1}{h^3}+\frac{jk}{h^2}-\frac{k^2}{h}\right\}e^{-jkh}$$
$$-\frac{j\omega}{4\eta S_0}\left\{1-\frac{1}{jk\sqrt{R_0^2+h^2}}\right.$$
$$\left.+\frac{h^2}{R_0^2+h^2}\left(1+\frac{1}{jk\sqrt{R_0^2+h^2}}\right)\right\}e^{-jk\sqrt{R_0^2+h^2}} \tag{31}$$

where $R_0 = a/1.438$. After simple algebra we find the induced dipole moment

$$\mathbf{m} = \frac{\chi\left(1+e^{-jkh}\right)}{1-\alpha\chi}\mathbf{H}_0 \tag{32}$$

Using (28)–(31) we find the reflected magnetic field

$$\mathbf{H}^{\text{ref}} = \mathbf{H}_0 e^{-jkh} - \frac{j\omega\mathbf{m}}{2\eta S_0}\left(1+e^{-jkh}\right) \tag{33}$$

and the reflection coefficient

$$\mathcal{R} = \left.\frac{H^{\text{ref}}}{H^{\text{ext}}}\right|_{z=0} = e^{-jkh} - \frac{j\omega\chi\left(1+e^{-jkh}\right)^2}{2\eta S_0(1-\alpha\chi)} \tag{34}$$

The analytical result for the reflection coefficient allows to define requirements for the optimum values of particle polarizabilities needed to achieve specific goals. Obviously, the reflective properties strongly depend on the interaction constant which defines parameter α in (34).

6. CONCLUSION

In this paper, the field created by a planar regular array of densely packed particles has been analytically evaluated. The array elements are assumed to be arbitrary-directed dipoles. No quasi-static approximation has been made, all field components taken into account. Direct summation of the dipole fields cannot be used here since the series do not converge. A physical model based on the assumption that the far-located dipoles can be replaced by a homogenized sheet of surface dipole densities has been used to solve the problem. This approach

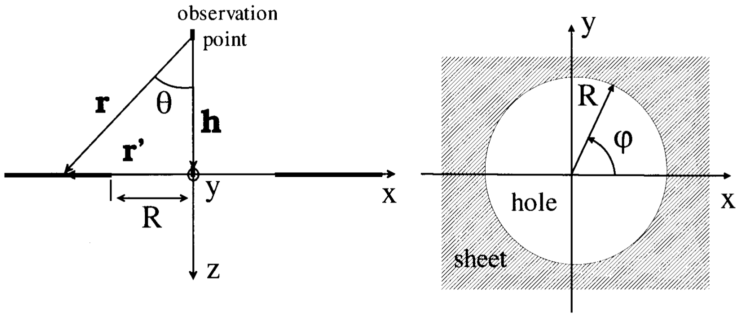

Figure 12. To the calculation of $\mathbf{E}^{\text{sheet}}(\mathbf{p},\mathbf{h})$.

leads to simple explicit formulas for the array field which have a clear physical interpretation. For dense arrays with square cells, an approximate formula has been suggested which expresses the array field in terms of very simple elementary functions and allows analytical investigations of complicated reflection and transmission problems for layered grid structures.

The results are illustrated by numerical examples which demonstrate how the array field depends on the problem parameters and the frequency. The distance from the grid plane at which the field becomes a plane wave (as if the grid would bear the averaged homogeneous current) can be easily estimated from these data. The theory has been applied to the reflection problem for two arrays of magnetic dipoles and for an array of magnetically polarizable particles near a metal screen. Explicit analytical results for the reflection coefficient are given.

APPENDIX: CALCULATION OF $\mathbf{E}^{\text{sheet}}(\mathbf{p},\mathbf{h})$ FOR CO-DIRECTED DIPOLES PARALLEL TO THE ARRAY PLANE

For the geometry of the problem we refer to Figure 12. The part of the interaction field due to the distributed dipole moments can be expressed for this case as

$$\mathbf{E}^{\text{sheet}}(\mathbf{p},\mathbf{h}) = \frac{1}{S_0}\int\limits_{r'>R} \mathbf{E}^p(\mathbf{r}'+\mathbf{h})\,dS = \frac{1}{S_0}\int\limits_{\sqrt{R^2+h^2}}^{\infty}\int\limits_{0}^{2\pi} \mathbf{E}^p(\mathbf{r})r\,d\varphi\,dr \tag{35}$$

After substitution of (1) we can write:

$$\mathbf{E}^{\text{sheet}}(\mathbf{p},\mathbf{h}) = \frac{1}{4\pi\epsilon_0 S_0} \int\limits_{\sqrt{R^2+h^2}}^{\infty} \int\limits_{0}^{2\pi} \left\{ k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n} + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^2}+\frac{jk}{r}\right)\right\} e^{-jkr}\, d\varphi\, dr \tag{36}$$

Vector $\mathbf{n}=\mathbf{r}/r$ is the unit vector with the components:

$$\begin{cases} n_x = \sin\theta\cos\varphi \\ n_y = \sin\theta\sin\varphi \\ n_z = \pm\cos\theta = \pm h/r \end{cases} \tag{37}$$

In relations (37) the upper sign coresponds to the location of the observation point in region $z<0$, and the lower sign for that in region $z>0$. Let us consider the case when $\mathbf{p}=p\mathbf{x}_0$, where $\mathbf{x}_0$ is the unit vector in the direction of the x-axis, then

$$\begin{aligned} (\mathbf{n}\times\mathbf{p})\times\mathbf{n} &= p(n_y^2+n_z^2)\mathbf{x}_0 - pn_xn_y\mathbf{y}_0 - pn_xn_z\mathbf{z}_0 \\ 3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p} &= p(3n_x^2-1)\mathbf{x}_0 + 3pn_xn_y\mathbf{y}_0 + 3pn_xn_z\mathbf{z}_0 \end{aligned} \tag{38}$$

Note that y- and z-components in $\mathbf{E}^{\text{sheet}}(\mathbf{p},\mathbf{h})$ vanish after integration, thus the electric field is also x-directed. In addition to this, one can see that because the term n_z appears in (38) only as n_z^2, the values of the field are the same at both sides of the sheet:

$$E^{\text{sheet}}(p,h) = \frac{p}{4\pi\epsilon_0 S_0} \int\limits_{\sqrt{R^2+h^2}}^{\infty} \int\limits_{0}^{2\pi} \left\{ k^2(\sin^2\theta\sin^2\varphi+\cos^2\theta) + (3\sin^2\theta\cos^2\varphi-1)\left(\frac{1}{r^2}+\frac{jk}{r}\right)\right\} e^{-jkr}\, d\varphi\, dr \tag{39}$$

Integrating (39) over φ, we have after some transformations:

$$E^{\text{sheet}}(p,h) = \frac{p}{4\epsilon_0 S_0} \int\limits_{\sqrt{R^2+h^2}}^{\infty} \left\{ k^2 + \frac{jkr+1}{r^2} - jkh^2\,\frac{jkr+2}{r^3} - h^2\,\frac{jkr+3}{r^4} \right\} e^{-jkr} dr \tag{40}$$

Using the identity

$$\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{r^{\gamma}}\right) = -\frac{jkr+\gamma}{r^{\gamma+1}}e^{-jkr} \tag{41}$$

we can integrate by parts to find

$$\begin{aligned} E^{\text{sheet}}(p,h) = -\frac{jkp}{4\epsilon_0 S_0}\Bigg\{1 - \frac{1}{jk\sqrt{R^2+h^2}} + \frac{h^2}{R^2+h^2} \\ + \frac{h^2}{jk(R^2+h^2)^{3/2}}\Bigg\} e^{-jk\sqrt{R^2+h^2}} \end{aligned} \tag{42}$$

This result can be easily generalized for arbitrary $\mathbf{p} = p_x\mathbf{x}_0 + p_y\mathbf{y}_0$, and the final result is (13).

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