Progress In Electromagnetics Research, *PIER 25, 309–331, 2000*

CONCISE SPECTRAL FORMALISM IN THE ELECTROMAGNETICS OF BIANISOTROPIC MEDIA

E. L. Tan and S. Y. Tan

School of Electrical and Electronic Engineering
Nanyang Technological University
Singapore 639798
Republic of Singapore

1. INTRODUCTION

In recent years, owing to their potential applications, increasing interest has been devoted to the study of electromagnetic wave interactions with complex media. The most general linear complex medium is the bianisotropic medium [1] which embraces many novel artificial materials [2]. Many authors have carried out extensive studies on wave propagation, scattering and radiation in (bi)anisotropic media [3–10]. One of the most powerful method suited for analysis of bianisotropic media utilizes the well-known 4×4 matrix method in spectral domain

[11–28]. Based on the transition matrix obtained from this approach, many problems involving reflection, transmission and even the dyadic Green's functions have been successfully treated. Despite the extensive literature thus far, one still finds that the system matrices (for first-order differential equation) given by most authors are often cumbersome to manipulate. Especially for general complex media, the matrix elements are expressed in very lengthy and exhaustive forms that are prone to making mistakes. Moreover, emphasis on inhomogeneous differential equation has not been as much as that for homogeneous case. The Green's function solutions were usually left specific or implicit, i.e., either not provided explicitly (for general bianisotropic media) or embedded in complicated expressions. This renders lucid interpretation of the results difficult and prohibits identification of key constituents in the solutions. Furthermore, generalization of the matrix approach to extended (bianisotropic) medium was simply presumed upon without much precaution elaborated.

In view of the above complications, various authors have introduced and applied certain compact notations to simplify analysis [29–31]. Specifically, notation using six-vectors and six-dyadics [32] appears to be of much help in allowing general analysis with less symbols and more structured algebraic form. Parallel to their compact formalism, as well as in line with our earlier attempts of unification [33, 34], this paper presents a concise treatment of electromagnetics in bianisotropic media based on source-incorporated 4×4 matrix method in spectral domain. The system matrices are written in very condensed and highly symmetric form protruding their key ingredients along with respective roles. All four types of dyadic Green's functions, i.e., electric-electric, electric-magnetic, magnetic-electric and magnetic-magnetic, for an unbounded bianisotropic medium are derived simultaneously. The singularities and discontinuities associated with spectral expansions are deduced directly based on the theory of distribution [35]. Each Green's dyadic is decomposed into various transverse and longitudinal partitions correspond to transverse and longitudinal fields attributed to transverse and longitudinal sources. The intimate relationships among these partitions are revealed and exploited. Important reciprocity theorems are revisited in spectral domain, stating succinctly the relations between system matrices of original and complementary media. The connections between the Green's functions obtained from 4×4 matrix method and those from reciprocity theorems are clarified. Throughout the following analysis, $e^{-i\omega t}$ time dependence is assumed and suppressed.

2. SOURCE-INCORPORATED 4×4 MATRIX FORMALISM

2.1 Basic Formulation

Consider a homogeneous bianisotropic medium characterized by constitutive relations [1]

$$\begin{bmatrix} \overline{D} \\ \overline{B} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{bmatrix} \cdot \begin{bmatrix} \overline{E} \\ \overline{H} \end{bmatrix} \tag{1}$$

where $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$, $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$ represent respectively the medium permittivity, permeability and magneto-electric dyadics/pseudodyadics. Substituting these relations into source-incorporated Maxwell equations, we have

$$\begin{bmatrix} \nabla \times \overline{E} \\ \nabla \times \overline{H} \end{bmatrix} = \begin{bmatrix} \overline{\overline{0}} & -\overline{\overline{I}} \\ \overline{\overline{I}} & \overline{\overline{0}} \end{bmatrix} \cdot \left(-i\omega \begin{bmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{bmatrix} \cdot \begin{bmatrix} \overline{E} \\ \overline{H} \end{bmatrix} + \begin{bmatrix} \overline{J} \\ \overline{M} \end{bmatrix} \right) \tag{2}$$

where $\overline{\overline{I}}$ is the idemfactor. Let us describe our Cartesian coordinate system by unit vectors $\hat{t}_1$, $\hat{t}_2$ and $\hat{p}$, with $\hat{p}$ chosen as our preferred direction. To facilitate subsequent analysis, it is expedient to partition the dyadics and vectors according to

$$\overline{\overline{A}} = \begin{bmatrix} \overline{\overline{A}}_{\text{tt}}^{(2\times 2)} & \overline{\overline{A}}_{\text{tp}}^{(2\times 1)} \\ \overline{\overline{A}}_{\text{pt}}^{(1\times 2)} & \overline{\overline{A}}_{\text{pp}}^{(1\times 1)} \end{bmatrix} \tag{3}$$

$$\overline{A} = \begin{bmatrix} \overline{A}_{\text{t}}^{(2\times 1)} \\ \overline{A}_{\text{p}}^{(1\times 1)} \end{bmatrix} \tag{4}$$

where subscripts t/p signify quantities transverse/parallel to $\hat{p}$. The size of each submatrix is as indicated in brackets. Henceforth, we will carry out our analysis in the two-dimensional Fourier transform domain which assumes field and source dependence of $e^{i\overline{k}_t \cdot \overline{r}_t}$ with

$$\overline{k}_t = k_{t1}\hat{t}_1 + k_{t2}\hat{t}_2 \tag{5}$$

$$\overline{r}_t = t_1\hat{t}_1 + t_2\hat{t}_2. \tag{6}$$

Applying the decomposition (3)–(4) into Maxwell equations (2) cast in Fourier domain, we obtain two linear algebraic equations and four coupled linear first-order ordinary differential equations as follows.

The algebraic equations relate the longitudinal ($\hat{p}$) components of electric and magnetic fields to their transverse components plus the longitudinal components of current sources:

$$\begin{bmatrix} \overline{E}_\mathrm{p} \\ \overline{H}_\mathrm{p} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\alpha}}_\mathrm{ee} & \overline{\overline{\alpha}}_\mathrm{em} \\ \overline{\overline{\alpha}}_\mathrm{me} & \overline{\overline{\alpha}}_\mathrm{mm} \end{bmatrix} \cdot \begin{bmatrix} \overline{E}_\mathrm{t} \\ \overline{H}_\mathrm{t} \end{bmatrix} + \frac{1}{i\omega} \begin{bmatrix} \overline{\overline{\epsilon}}_\mathrm{pp} & \overline{\overline{\xi}}_\mathrm{pp} \\ \overline{\overline{\zeta}}_\mathrm{pp} & \overline{\overline{\mu}}_\mathrm{pp} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \overline{J}_\mathrm{p} \\ \overline{M}_\mathrm{p} \end{bmatrix}. \tag{7}$$

Here, the four $\overline{\overline{\alpha}}$'s represent 1×2 matrices performing transverse-to-longitudinal transformations of field vectors:

$$\begin{bmatrix} \overline{\overline{\alpha}}_\mathrm{ee} & \overline{\overline{\alpha}}_\mathrm{em} \\ \overline{\overline{\alpha}}_\mathrm{me} & \overline{\overline{\alpha}}_\mathrm{mm} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\epsilon}}_\mathrm{pp} & \overline{\overline{\xi}}_\mathrm{pp} \\ \overline{\overline{\zeta}}_\mathrm{pp} & \overline{\overline{\mu}}_\mathrm{pp} \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} \overline{\overline{0}} & -\overline{\overline{\kappa}}_\mathrm{pt} \\ \overline{\overline{\kappa}}_\mathrm{pt} & \overline{\overline{0}} \end{bmatrix} - \begin{bmatrix} \overline{\overline{\epsilon}}_\mathrm{pt} & \overline{\overline{\xi}}_\mathrm{pt} \\ \overline{\overline{\zeta}}_\mathrm{pt} & \overline{\overline{\mu}}_\mathrm{pt} \end{bmatrix} \right) \tag{8}$$

with

$$\overline{\overline{\kappa}} = \frac{\overline{k}_t}{\omega} \times \overline{\overline{I}} = \frac{1}{\omega} \begin{bmatrix} 0 & 0 & k_{t2} \\ 0 & 0 & -k_{t1} \\ -k_{t2} & k_{t1} & 0 \end{bmatrix} \tag{9}$$

to be partitioned according to (3) (as well). Notice that for convenience sake, the variables in (7) have been written using the same notations as those in (2) although they should be understood as (transverse) Fourier transformed quantities. We shall follow this convention throughout except that in cases where references in space domain are required, the arguments $\overline{r}$ or $\overline{r}_t$ would be included explicitly.

Introducing an antisymmetric 2×2 (partitioned) matrix

$$\overline{\overline{\Gamma}}_\mathrm{a} = (\hat{p} \times \overline{\overline{I}})_\mathrm{tt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{10}$$

we have four differential equations given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}p} \begin{bmatrix} \overline{E}_\mathrm{t} \\ \overline{H}_\mathrm{t} \end{bmatrix} &= i\omega \begin{bmatrix} \overline{\overline{\Lambda}}_\mathrm{ee} & \overline{\overline{\Lambda}}_\mathrm{em} \\ \overline{\overline{\Lambda}}_\mathrm{me} & \overline{\overline{\Lambda}}_\mathrm{mm} \end{bmatrix} \cdot \begin{bmatrix} \overline{E}_\mathrm{t} \\ \overline{H}_\mathrm{t} \end{bmatrix} \\ &\quad + \begin{bmatrix} \overline{\overline{0}} & \overline{\overline{\Gamma}}_\mathrm{a} \\ -\overline{\overline{\Gamma}}_\mathrm{a} & \overline{\overline{0}} \end{bmatrix} \cdot \left(\begin{bmatrix} \overline{J}_\mathrm{t} \\ \overline{M}_\mathrm{t} \end{bmatrix} + \begin{bmatrix} \overline{\overline{\beta}}_\mathrm{ee} & \overline{\overline{\beta}}_\mathrm{em} \\ \overline{\overline{\beta}}_\mathrm{me} & \overline{\overline{\beta}}_\mathrm{mm} \end{bmatrix} \cdot \begin{bmatrix} \overline{J}_\mathrm{p} \\ \overline{M}_\mathrm{p} \end{bmatrix} \right). \end{aligned} \tag{11}$$

Here, the four $\overline{\overline{\beta}}$'s complement those $\overline{\overline{\alpha}}$'s above and represent 2×1 matrices performing longitudinal-to-transverse transformations of source vectors:

$$\begin{bmatrix} \overline{\overline{\beta}}_{\mathrm{ee}} & \overline{\overline{\beta}}_{\mathrm{em}} \\ \overline{\overline{\beta}}_{\mathrm{me}} & \overline{\overline{\beta}}_{\mathrm{mm}} \end{bmatrix} = \left(\begin{bmatrix} \overline{\overline{0}} & -\overline{\overline{\kappa}}_{\mathrm{tp}} \\ \overline{\overline{\kappa}}_{\mathrm{tp}} & \overline{\overline{0}} \end{bmatrix} - \begin{bmatrix} \overline{\overline{\epsilon}}_{\mathrm{tp}} & \overline{\overline{\xi}}_{\mathrm{tp}} \\ \overline{\overline{\zeta}}_{\mathrm{tp}} & \overline{\overline{\mu}}_{\mathrm{tp}} \end{bmatrix} \right) \cdot \begin{bmatrix} \overline{\overline{\epsilon}}_{\mathrm{pp}} & \overline{\overline{\xi}}_{\mathrm{pp}} \\ \overline{\overline{\zeta}}_{\mathrm{pp}} & \overline{\overline{\mu}}_{\mathrm{pp}} \end{bmatrix}^{-1}. \tag{12}$$

Observe the highly symmetric form of expression in (12) compare to (8). Using these $\overline{\overline{\beta}}$'s and $\overline{\overline{\alpha}}$'s, the four $\overline{\overline{\Lambda}}$'s can be expressed succinctly as

$$\begin{aligned} \begin{bmatrix} \overline{\overline{\Lambda}}_{\mathrm{ee}} & \overline{\overline{\Lambda}}_{\mathrm{em}} \\ \overline{\overline{\Lambda}}_{\mathrm{me}} & \overline{\overline{\Lambda}}_{\mathrm{mm}} \end{bmatrix} = \begin{bmatrix} \overline{\overline{0}} & \overline{\overline{\Gamma}}_{\mathrm{a}} \\ -\overline{\overline{\Gamma}}_{\mathrm{a}} & \overline{\overline{0}} \end{bmatrix} \cdot \Bigg(& \begin{bmatrix} \overline{\overline{\beta}}_{\mathrm{ee}} & \overline{\overline{\beta}}_{\mathrm{em}} \\ \overline{\overline{\beta}}_{\mathrm{me}} & \overline{\overline{\beta}}_{\mathrm{mm}} \end{bmatrix} \cdot \begin{bmatrix} \overline{\overline{\epsilon}}_{\mathrm{pp}} & \overline{\overline{\xi}}_{\mathrm{pp}} \\ \overline{\overline{\zeta}}_{\mathrm{pp}} & \overline{\overline{\mu}}_{\mathrm{pp}} \end{bmatrix} \\ & \cdot \begin{bmatrix} \overline{\overline{\alpha}}_{\mathrm{ee}} & \overline{\overline{\alpha}}_{\mathrm{em}} \\ \overline{\overline{\alpha}}_{\mathrm{me}} & \overline{\overline{\alpha}}_{\mathrm{mm}} \end{bmatrix} - \begin{bmatrix} \overline{\overline{\epsilon}}_{\mathrm{tt}} & \overline{\overline{\xi}}_{\mathrm{tt}} \\ \overline{\overline{\zeta}}_{\mathrm{tt}} & \overline{\overline{\mu}}_{\mathrm{tt}} \end{bmatrix} \Bigg). \end{aligned} \tag{13}$$

Equation (11) constitutes the basis of source-incorporated 4×4 matrix formalism [19, 29, 30]. As is evident from this equation, the key ingredients of various system matrices $\overline{\overline{\alpha}}$, $\overline{\overline{\beta}}$, $\overline{\overline{\Lambda}}$ have been clearly identified. Their concise and 'pretty' appearance are mostly appreciated in view of the previous expressions of matrix elements via quite a number of auxiliary notations in rather elaborated forms [24, 28]. Note that the usual duality relationships which can be handy in the course of their derivation are immediately apparent from our presentation above [20, 25].

At this point, it seems appropriate to introduce some shorthand notations for (11), (7), (8), (12), and (13) as

$$\frac{\mathrm{d}}{\mathrm{d}p}\overline{\mathrm{f}}_{\mathrm{t}} = i\omega\overline{\overline{\Lambda}} \cdot \overline{\mathrm{f}}_{\mathrm{t}} + \overline{\overline{\Gamma}}_{\mathrm{v}} \cdot (\overline{\mathrm{s}}_{\mathrm{t}} + \overline{\overline{\beta}} \cdot \overline{\mathrm{s}}_{\mathrm{p}}) \tag{14}$$

$$\overline{\mathrm{f}}_{\mathrm{p}} = \overline{\overline{\alpha}} \cdot \overline{\mathrm{f}}_{\mathrm{t}} + \frac{1}{i\omega}\overline{\overline{M}}_{\mathrm{pp}}^{-1} \cdot \overline{\mathrm{s}}_{\mathrm{p}} \tag{15}$$

$$\overline{\overline{\alpha}} = \overline{\overline{M}}_{\mathrm{pp}}^{-1} \cdot (\overline{\overline{K}}_{\mathrm{pt}} - \overline{\overline{M}}_{\mathrm{pt}}) \tag{16}$$

$$\overline{\overline{\beta}} = (\overline{\overline{K}}_{\mathrm{tp}} - \overline{\overline{M}}_{\mathrm{tp}}) \cdot \overline{\overline{M}}_{\mathrm{pp}}^{-1} \tag{17}$$

$$\overline{\overline{\Lambda}} = \overline{\overline{\Gamma}}_{\mathrm{v}} \cdot (\overline{\overline{\beta}} \cdot \overline{\overline{M}}_{\mathrm{pp}} \cdot \overline{\overline{\alpha}} - \overline{\overline{M}}_{\mathrm{tt}}). \tag{18}$$

The definitions of various symbols follow readily from above, e.g.,

$$\overline{\overline{\Gamma}}_{\mathrm{v}} = \begin{bmatrix} \overline{\overline{0}} & \overline{\overline{\Gamma}}_{\mathrm{a}} \\ -\overline{\overline{\Gamma}}_{\mathrm{a}} & \overline{\overline{0}} \end{bmatrix} \tag{19}$$

$$\overline{\overline{K}}_{\rm pt} = \begin{bmatrix} \overline{\overline{0}} & -\overline{\overline{\kappa}}_{\rm pt} \\ \overline{\overline{\kappa}}_{\rm pt} & \overline{\overline{0}} \end{bmatrix} \tag{20}$$

$$\overline{\overline{K}}_{\rm tp} = \begin{bmatrix} \overline{\overline{0}} & -\overline{\overline{\kappa}}_{\rm tp} \\ \overline{\overline{\kappa}}_{\rm tp} & \overline{\overline{0}} \end{bmatrix} \tag{21}$$

and the subscripted $\overline{\overline{M}}$'s group together the respective partitions of medium constitutive dyadics. (Note that the size of zero matrices $\overline{\overline{0}}$ should vary accordingly.) $\overline{\rm f}_{\rm t}$ and $\overline{\rm s}_{\rm t}$ will be termed below as transverse field and source vectors respectively. Prior to looking into the inhomogeneous equation (14), let us tentatively omit the source terms and revisit briefly its homogeneous solutions (treated extensively in the literature since [11]).

2.2 Homogeneous Solutions

Within a source-free homogeneous medium, $\overline{\overline{\Lambda}}$ is a constant 4×4 matrix in the homogeneous differential equation

$$\frac{\rm d}{{\rm d}p}\overline{\rm f}_{\rm t} = i\omega\overline{\overline{\Lambda}} \cdot \overline{\rm f}_{\rm t}. \tag{22}$$

This equation admits nontrivial solutions of $e^{ik_p p}$ dependence subjected to the (Booker) dispersion relation

$$\det(\omega\overline{\overline{\Lambda}} - k_p\overline{\overline{\rm I}}_4) = 0. \tag{23}$$

Since (23) yields a quartic equation for k_p, there are four eigenvalues and four eigenvectors associated with it. As a result, the general solution for the transverse field vector can be written as

$$\overline{\rm f}_{\rm t}(p) = \overline{\overline{\Psi}} \cdot \overline{\overline{\rm P}}(p) \cdot \overline{\rm c} = \overline{\overline{\Psi}} \cdot \overline{\rm w}(p) \tag{24}$$

where the eigenvalues k_{pj} ($j = 1, 2, 3, 4$) are included in phase matrix $\overline{\overline{\rm P}}$ while their corresponding eigenvectors form the columns of eigenvector matrix $\overline{\overline{\Psi}}$:

$$\overline{\overline{\rm P}}(p) = {\rm diag}[e^{ik_{pj}p}] \tag{25}$$

$$\overline{\overline{\Psi}} = \begin{bmatrix} \overline{e}_1 & \overline{e}_2 & \overline{e}_3 & \overline{e}_4 \\ \overline{h}_1 & \overline{h}_2 & \overline{h}_3 & \overline{h}_4 \end{bmatrix}. \tag{26}$$

In accordance with the composition of $\overline{\mathrm{f}}_{\mathrm{t}}$, each eigenvector is seen to be made up of partial tangential electric ($\overline{e}_j$) and magnetic ($\overline{h}_j$) eigenfields which are individually two-component column vectors. Furthermore, $\overline{\mathrm{c}}$ is a four-component coefficient vector containing the unknown constants (independent of p) to be specified by primary excitations or determined from boundary and/or radiation conditions. These unknowns can be lumped together with the exponential phase factors into another wave vector $\overline{\mathrm{w}}$ whose four components are now dependent on p. For either unknown representation, equation (24) states that the total tangential field vector can be viewed as a weighted sum of four partial eigenfields.

Note that in arriving at (24), it has been assumed that $\overline{\overline{\Lambda}}$ is similar to a diagonal matrix and the inverse of $\overline{\overline{\Psi}}$ exists. In addition, let us assume in the sequel that k_{p1} and k_{p2} have positive imaginary parts while k_{p3} and k_{p4} have negative imaginary parts. For lossless medium, we may need to apply the concept of slight loss limit to identify the sign of those imaginary parts [36]. Since these k_{pj} appear in the exponents as $e^{ik_{pj}p}$, one can regard the eigenfields $\overline{e}_1$, $\overline{h}_1$ (for k_{p1}) and $\overline{e}_2$, $\overline{h}_2$ (for k_{p2}) as outward-bounded waves which remain bounded as $p \to +\infty$. Likewise, $\overline{e}_3$, $\overline{h}_3$ (for k_{p3}) and $\overline{e}_4$, $\overline{h}_4$ (for k_{p4}) correspond to inward-bounded waves that are still bounded as $p \to -\infty$. Note that it is actually not very appropriate to term the waves as 'outgoing' and 'incoming' or 'outward-propagating' and 'inward-propagating'. This is because for general (bi)anisotropic media, phase propagation direction (according to real parts of k_{pj}) may not coincide with energy flow direction [29]. In fact, outward-bounded waves may be incoming at infinity! Therefore, the radiation condition should in general based on bounded solutions which require all waves to be sufficiently bounded. Under the above association, we can partition the phase matrix and eigenvector matrix into 2×2 submatrices as

$$\overline{\overline{\mathrm{P}}} = \begin{bmatrix} \overline{\overline{\mathrm{P}}}^{>} & \overline{\overline{0}} \\ \overline{\overline{0}} & \overline{\overline{\mathrm{P}}}^{<} \end{bmatrix} \tag{27}$$

$$\overline{\overline{\Psi}} = \begin{bmatrix} \overline{\overline{\mathrm{e}}}^{>} & \overline{\overline{\mathrm{e}}}^{<} \\ \overline{\overline{\mathrm{h}}}^{>} & \overline{\overline{\mathrm{h}}}^{<} \end{bmatrix} \tag{28}$$

where $>$ and $<$ in the superscripts refer to outward-bounded and inward-bounded waves respectively. Likewise, the coefficient vector and wave vector can also be decomposed into 2×1 partitions

$$\bar{\mathrm{c}} = \begin{bmatrix} \bar{\mathrm{c}}^{>} \\ \bar{\mathrm{c}}^{<} \end{bmatrix} \tag{29}$$

$$\overline{\mathrm{w}} = \begin{bmatrix} \overline{\mathrm{w}}^{>} \\ \overline{\mathrm{w}}^{<} \end{bmatrix}. \tag{30}$$

These unknown coefficients are to be solved implicitly via the dyadic Green's functions in the next section.

3. DYADIC GREEN'S FUNCTIONS

We now return to the inhomogeneous differential equation (14) for sources embedded in a bianisotropic medium. Taking into account both electric and magnetic source types located at $\overline{r} = \overline{r}'$ (prime for source coordinates), these sources can be represented in the distribution sense in spectral domain by

$$\begin{bmatrix} \overline{J}(p) \\ \overline{M}(p) \end{bmatrix} = \int \mathrm{d}p' \begin{bmatrix} \overline{\overline{I}}\,\delta(p-p') & \overline{\overline{0}} \\ \overline{\overline{0}} & \overline{\overline{I}}\,\delta(p-p') \end{bmatrix} \cdot \begin{bmatrix} \overline{J}(p') \\ \overline{M}(p') \end{bmatrix} \tag{31}$$

where $\delta(p-p')$ denotes the one-dimensional Dirac delta function. Due to linearity of Maxwell equations, there are four types of dyadic Green's functions which relate the electric and magnetic fields directly to the electric and magnetic sources according to (all quantities are still in Fourier domain)

$$\begin{bmatrix} \overline{E}(p) \\ \overline{H}(p) \end{bmatrix} = \int \mathrm{d}p' \begin{bmatrix} \overline{\overline{G}}_{\mathrm{ee}}(p,p') & \overline{\overline{G}}_{\mathrm{em}}(p,p') \\ \overline{\overline{G}}_{\mathrm{me}}(p,p') & \overline{\overline{G}}_{\mathrm{mm}}(p,p') \end{bmatrix} \cdot \begin{bmatrix} \overline{J}(p') \\ \overline{M}(p') \end{bmatrix}. \tag{32}$$

Henceforth, each Green's dyadic is to be partitioned in the manner of (3) and gathered as

$$\overline{\overline{\mathrm{G}}}_{\mathrm{tt}} = \begin{bmatrix} \overline{\overline{G}}_{\mathrm{eett}} & \overline{\overline{G}}_{\mathrm{emtt}} \\ \overline{\overline{G}}_{\mathrm{mett}} & \overline{\overline{G}}_{\mathrm{mmtt}} \end{bmatrix}.$$

Similar partitioning and grouping apply to $\overline{\overline{\mathrm{G}}}_{\mathrm{tp}}$, $\overline{\overline{\mathrm{G}}}_{\mathrm{pt}}$, $\overline{\overline{\mathrm{G}}}_{\mathrm{pp}}$, as well as the idemfactor of (31):

$$\overline{\overline{\mathrm{I}}}_{\mathrm{u}} = \begin{bmatrix} \overline{\overline{\mathrm{I}}}_{\mathrm{t}} & \overline{\overline{0}} \\ \overline{\overline{0}} & \overline{\overline{\mathrm{I}}}_{\mathrm{t}} \end{bmatrix} = \overline{\overline{\mathrm{I}}}_{4} \tag{34}$$

$$\overline{\overline{\mathrm{I}}}_{\mathrm{t}} = \overline{\overline{I}}_{\mathrm{tt}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{35}$$

As indicated by the subscripts, each $\overline{\overline{\mathrm{G}}}$ subscripted (t/p)(p/t) specifies the (transverse / longitudinal) fields attributed to the (longitudinal / transverse) sources, e.g.,

$$\overline{\mathrm{f}}_{\mathrm{t}}(p) = \int \mathrm{d}p' \, \overline{\overline{\mathrm{G}}}_{\mathrm{tt}}(p, p') \cdot \overline{\mathrm{s}}_{\mathrm{t}}(p'). \tag{36}$$

We will determine all these Green's functions in the following.

3.1 Singularities and Discontinuities

In general, the dyadic Green's functions can be expanded into three regions as

$$\overline{\overline{\mathrm{G}}} = \overline{\overline{\mathrm{G}}}^0 \, \delta(p - p') + \overline{\overline{\mathrm{G}}}^> \, U(p - p') + \overline{\overline{\mathrm{G}}}^< \, U(p' - p). \tag{37}$$

Here, the $\overline{\overline{\mathrm{G}}}^0$ part accounts for the singularities that may be required for complete expansion in source regions. $U(\pm p \mp p')$ are the Heaviside unit step functions pertaining to $p \gtrless p'$. Consider first the transverse-transverse Green's functions $\overline{\overline{\mathrm{G}}}_{\mathrm{tt}}$ which take the same (superscripted $0\,, >\,, <$) form of expansion as (37). Applying such expansion into (14) and upon taking the derivative in the sense of distributions, e.g., [35]

$$\frac{\mathrm{d}}{\mathrm{d}p} U(\pm p \mp p') = \pm \delta(p - p'), \tag{38}$$

we arrive at the deductions [34]

$$\overline{\overline{\mathrm{G}}}^0_{\mathrm{tt}} = \overline{\overline{0}} \tag{39}$$

$$[\, \overline{\overline{\mathrm{G}}}^>_{\mathrm{tt}} - \overline{\overline{\mathrm{G}}}^<_{\mathrm{tt}} \,]_{p=p'} = \overline{\overline{\Gamma}}_{\mathrm{v}}. \tag{40}$$

Equation (39) asserts that the transverse-transverse Green's functions can be specified solely in terms of $\overline{\overline{\mathrm{G}}}^{\gtrless}_{\mathrm{tt}}$. From (40), the discontinuities in $\overline{\overline{\mathrm{G}}}^{\gtrless}_{\mathrm{tt}}$ will lead to discontinuities in the transverse field vectors across the source plane $p = p'$, i.e., the elementary boundary conditions. For the transverse-longitudinal Green's functions $\overline{\overline{\mathrm{G}}}_{\mathrm{tp}}$, their singularity factors and discontinuities are deduced as

$$\overline{\overline{\mathrm{G}}}^0_{\mathrm{tp}} = \overline{\overline{0}} \tag{41}$$

$$[\, \overline{\overline{\mathrm{G}}}^>_{\mathrm{tp}} - \overline{\overline{\mathrm{G}}}^<_{\mathrm{tp}} \,]_{p=p'} = \overline{\overline{\Gamma}}_{\mathrm{v}} \cdot \overline{\overline{\beta}}. \tag{42}$$

Again, on the basis of (41), we can infer that $\overline{\overline{\mathrm{G}}}_{\mathrm{tp}}^{\gtrless}$ alone are sufficient for complete expansion. Furthermore, from (42), one is able to appreciate the (extra) roles of various $\overline{\overline{\beta}}$'s in characterizing the discontinuities in $\overline{\overline{\mathrm{G}}}_{\mathrm{tp}}^{\gtrless}$, apart from their previous part in the transformation (14). Combining (42) and (40), we find that they indeed yield the complete transverse discontinuity relations obtained earlier in space domain [34]. Here, we arrive at these relations in a more condensed manner based on direct algebra in spectral domain.

By the same token, but resorting to (15) along with the above deductions, we obtain the singularity factors and the '$\gtrless$' parts of longitudinal-transverse and longitudinal-longitudinal Green's functions as

$$\overline{\overline{\mathrm{G}}}_{\mathrm{pt}}^{0} = \overline{\overline{0}} \tag{43}$$

$$\overline{\overline{\mathrm{G}}}_{\mathrm{pt}}^{\gtrless} = \overline{\overline{\alpha}} \cdot \overline{\overline{\mathrm{G}}}_{\mathrm{tt}}^{\gtrless} \tag{44}$$

$$\overline{\overline{\mathrm{G}}}_{\mathrm{pp}}^{0} = \frac{1}{i\omega} \overline{\overline{M}}_{\mathrm{pp}}{}^{-1} \tag{45}$$

$$\overline{\overline{\mathrm{G}}}_{\mathrm{pp}}^{\gtrless} = \overline{\overline{\alpha}} \cdot \overline{\overline{\mathrm{G}}}_{\mathrm{tp}}^{\gtrless}. \tag{46}$$

Unlike the case of $\overline{\overline{\mathrm{G}}}_{\mathrm{pt}}$ and those before, one finds from (45) that $\overline{\overline{\mathrm{G}}}_{\mathrm{pp}}$ contain explicit extra source point singularities $\overline{\overline{\mathrm{G}}}_{\mathrm{pp}}^{0}$ which must be augmented to $\overline{\overline{\mathrm{G}}}_{\mathrm{pp}}^{\gtrless}$ for complete expansion. These results are in accordance with those expressed in space domain earlier [9, 10, 34]. Here, the singularities have been derived based on algebra (involving $\overline{\overline{\kappa}}$) directly from the two-dimensional spectral domain (rather than extracted from the three-dimensional spectral domain), followed by (simple) inverse Fourier transform to space domain (rather than directly in space domain involving ∇). Furthermore, applying the discontinuity relations (40) and (42) into (44) and (46), we get

$$[\,\overline{\overline{\mathrm{G}}}_{\mathrm{pt}}^{>} - \overline{\overline{\mathrm{G}}}_{\mathrm{pt}}^{<}\,]_{p=p'} = \overline{\overline{\alpha}} \cdot \overline{\overline{\Gamma}}_{\mathrm{v}} \tag{47}$$

$$[\,\overline{\overline{\mathrm{G}}}_{\mathrm{pp}}^{>} - \overline{\overline{\mathrm{G}}}_{\mathrm{pp}}^{<}\,]_{p=p'} = \overline{\overline{\alpha}} \cdot \overline{\overline{\Gamma}}_{\mathrm{v}} \cdot \overline{\overline{\beta}}. \tag{48}$$

These complementary relations describe the discontinuities undergone in the normal components of dyadics (hence fields) attributed to transverse and longitudinal current components at plane $p = p'$. Note their compact form of expressions as presented here in spectral domain.

From (47), one would recognize the roles of various $\overline{\overline{\alpha}}$'s in characterizing the discontinuities in $\overline{\overline{\mathrm{G}}}_{\mathrm{pt}}^{\gtrless}$, besides the transformation (15).

3.2 Eigenfield Expansions

In addition to the foregoing singularities and discontinuities, the distributional analysis of (14) also states that [34]

$$\frac{\mathrm{d}}{\mathrm{d}p}\overline{\overline{\mathrm{G}}}_{\mathrm{tt}}^{\gtrless} = i\omega\overline{\overline{\Lambda}} \cdot \overline{\overline{\mathrm{G}}}_{\mathrm{tt}}^{\gtrless} \tag{49}$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\overline{\overline{\mathrm{G}}}_{\mathrm{tp}}^{\gtrless} = i\omega\overline{\overline{\Lambda}} \cdot \overline{\overline{\mathrm{G}}}_{\mathrm{tp}}^{\gtrless}. \tag{50}$$

Evidently, these differential equations are in the form of (22) thus suggesting the representations of field antecedents using $\overline{\overline{\Psi}}$ and $\overline{\overline{\mathrm{P}}}$ of (25)–(26), e.g.,

$$\overline{\overline{\mathrm{G}}}_{\mathrm{tt}}^{>} U(p-p') + \overline{\overline{\mathrm{G}}}_{\mathrm{tt}}^{<} U(p'-p) = \overline{\overline{\Psi}} \cdot \overline{\overline{\mathrm{P}}}(p) \cdot \overline{\overline{\mathrm{U}}}_{pp'} \cdot \overline{\overline{\sigma}}(p'). \tag{51}$$

Here, $\overline{\overline{\sigma}}$ denotes the source consequents to be determined as functions of source coordinates. $\overline{\overline{\mathrm{U}}}_{pp'}$ specifies the appropriate region for the respective bounded eigenwaves (originated from the source consequents) as

$$\overline{\overline{\mathrm{U}}}_{pp'} = \begin{bmatrix} \overline{\overline{\mathrm{I}}}_{\mathrm{t}}\, U(p-p') & \overline{\overline{0}} \\ \overline{\overline{0}} & \overline{\overline{\mathrm{I}}}_{\mathrm{t}}\, U(p'-p) \end{bmatrix}. \tag{52}$$

Substituting (51) into the discontinuity relation (40), we immediately obtain

$$\overline{\overline{\sigma}}(p') = \overline{\overline{\mathrm{I}}}_{\mathrm{v}} \cdot \overline{\overline{\mathrm{P}}}^{-1}(p') \cdot \overline{\overline{\Psi}}^{-1} \cdot \overline{\overline{\Gamma}}_{\mathrm{v}} \tag{53}$$

where

$$\overline{\overline{\mathrm{I}}}_{\mathrm{v}} = \begin{bmatrix} \overline{\overline{\mathrm{I}}}_{\mathrm{t}} & \overline{\overline{0}} \\ \overline{\overline{0}} & -\overline{\overline{\mathrm{I}}}_{\mathrm{t}} \end{bmatrix}. \tag{54}$$

To determine $\overline{\overline{\mathrm{G}}}_{\mathrm{tp}}^{\gtrless}$ in (50), we follow the same steps as $\overline{\overline{\mathrm{G}}}_{\mathrm{tt}}^{\gtrless}$ by introducing another unknown like $\overline{\overline{\sigma}}$ to be utilized in the discontinuity relation (42). This leads us directly to

$$\overline{\overline{\mathrm{G}}}_{\mathrm{tp}}^{\gtrless} = \overline{\overline{\mathrm{G}}}_{\mathrm{tt}}^{\gtrless} \cdot \overline{\overline{\beta}}. \tag{55}$$

From above, we notice that once $\overline{\overline{\sigma}}$ is available from (53), one can immediately obtain $\overline{\overline{\mathrm{G}}}_{\mathrm{tt}}^{\gtrless}$, $\overline{\overline{\mathrm{G}}}_{\mathrm{tp}}^{\gtrless}$, $\overline{\overline{\mathrm{G}}}_{\mathrm{pt}}^{\gtrless}$ and $\overline{\overline{\mathrm{G}}}_{\mathrm{pp}}^{\gtrless}$ from equations (51), (55), (44) and (46) respectively. Hence all partitions of all types of dyadic Green's functions have been determined simultaneously! Moreover, these equations also reveal the intimate relationships among various partitions of three-dimensional Green's dyadics and assert that all other partitions (' $\gtrless$ ' parts) can be deduced readily from the transverse-transverse Green's functions. In particular, combining the 0, $>$ and $<$ parts, we can write the relations among various partitions of total Green's dyadics as

$$\overline{\overline{\mathrm{G}}}_{\mathrm{tp}} = \overline{\overline{\mathrm{G}}}_{\mathrm{tt}} \cdot \overline{\overline{\beta}} \tag{56}$$

$$\overline{\overline{\mathrm{G}}}_{\mathrm{pt}} = \overline{\overline{\alpha}} \cdot \overline{\overline{\mathrm{G}}}_{\mathrm{tt}} \tag{57}$$

$$\overline{\overline{\mathrm{G}}}_{\mathrm{pp}} = \frac{1}{i\omega}\overline{\overline{M}}_{\mathrm{pp}}^{-1}\delta(p-p') + \overline{\overline{\alpha}} \cdot \overline{\overline{\mathrm{G}}}_{\mathrm{tt}} \cdot \overline{\overline{\beta}}. \tag{58}$$

Furthermore, (56) and (57) yield directly

$$\overline{\overline{\mathrm{G}}}_{\mathrm{pt}} \cdot \overline{\overline{\beta}} = \overline{\overline{\alpha}} \cdot \overline{\overline{\mathrm{G}}}_{\mathrm{tp}} \tag{59}$$

In many problems involving planar structures supporting planar elements with planar excitations, the transverse-transverse Green's functions alone are sufficient for most analysis [37]. For cases when longitudinal elements or excitations are present, one would require to consider other partitions as well, but these can be easily derived using various $\overline{\overline{\alpha}}$'s and $\overline{\overline{\beta}}$'s, taking into account the singularities as (58).

For convenience, let us write the partitions of $\overline{\overline{\Psi}}^{-1}$ as

$$\overline{\overline{\Psi}}^{-1} = \begin{bmatrix} \overline{\overline{\mathrm{e}}}^{>} & \overline{\overline{\mathrm{e}}}^{<} \\ \overline{\overline{\mathrm{h}}}^{>} & \overline{\overline{\mathrm{h}}}^{<} \end{bmatrix}^{-1} = \begin{bmatrix} \overline{\overline{\Phi}}_{\mathrm{e}}^{>} & \overline{\overline{\Phi}}_{\mathrm{h}}^{>} \\ \overline{\overline{\Phi}}_{\mathrm{e}}^{<} & \overline{\overline{\Phi}}_{\mathrm{h}}^{<} \end{bmatrix} \tag{60}$$

where the 2×2 submatrices $\overline{\overline{\Phi}}_{\mathrm{e}}^{\gtrless}$ and $\overline{\overline{\Phi}}_{\mathrm{h}}^{\gtrless}$ can be expressed explicitly in terms of $\overline{\overline{\mathrm{e}}}^{\gtrless}$ and $\overline{\overline{\mathrm{h}}}^{\gtrless}$ if desired, i.e.,

$$\overline{\overline{\Phi}}_{\mathrm{e}}^{\gtrless} = \left[\overline{\overline{\mathrm{e}}}^{\gtrless} - \overline{\overline{\mathrm{e}}}^{\lessgtr} \cdot \overline{\overline{\mathrm{h}}}^{\lessgtr -1} \cdot \overline{\overline{\mathrm{h}}}^{\gtrless}\right]^{-1} \tag{61}$$

$$\overline{\overline{\Phi}}_{\mathrm{e}}^{\gtrless} = \left[\overline{\overline{\mathrm{h}}}^{\gtrless} - \overline{\overline{\mathrm{h}}}^{\lessgtr} \cdot \overline{\overline{\mathrm{e}}}^{\lessgtr -1} \cdot \overline{\overline{\mathrm{e}}}^{\gtrless}\right]^{-1}. \tag{62}$$

To exemplify, we have the transverse-transverse parts of electric-electric Green's dyadic given by

$$\overline{\overline{\mathrm{G}}}_{\mathrm{eett}}^{>} = -\overline{\overline{\mathrm{e}}}^{>} \cdot \overline{\overline{\mathrm{P}}}^{>}(p - p') \cdot \overline{\overline{\Phi}}_{\mathrm{h}}^{>} \cdot \overline{\overline{\Gamma}}_{\mathrm{a}} \tag{63}$$

$$\overline{\overline{\mathrm{G}}}_{\mathrm{eett}}^{<} = \overline{\overline{\mathrm{e}}}^{<} \cdot \overline{\overline{\mathrm{P}}}^{<}(p - p') \cdot \overline{\overline{\Phi}}_{\mathrm{h}}^{<} \cdot \overline{\overline{\Gamma}}_{\mathrm{a}} \tag{64}$$

Combining with other partitions and upon taking the inverse Fourier transform, their complete expansion in space domain reads

$$\begin{aligned}\overline{\overline{\mathrm{G}}}_{\mathrm{ee}}(\overline{r}, \overline{r}') = & \frac{\overline{\overline{\mu}}_{\mathrm{pp}}}{i\omega(\overline{\overline{\epsilon}}_{\mathrm{pp}} \cdot \overline{\overline{\mu}}_{\mathrm{pp}} - \overline{\overline{\xi}}_{\mathrm{pp}} \cdot \overline{\overline{\zeta}}_{\mathrm{pp}})} \delta(\overline{r} - \overline{r}')\hat{p}\hat{p}' \\ & + \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk_{t1} \int_{-\infty}^{\infty} dk_{t2}\, e^{i\overline{k}_t \cdot (\overline{r}_t - \overline{r}_t')} \\ & \left[\overline{\overline{\mathrm{G}}}_{\mathrm{ee}}^{>} U(p - p') + \overline{\overline{\mathrm{G}}}_{\mathrm{ee}}^{<} U(p' - p)\right]\end{aligned} \tag{65}$$

where $\delta(\overline{r} - \overline{r}')$ is the three-dimensional Dirac delta function. From (63)–(64), it is evident that once the phase and eigenvector matrices for source-free medium have been worked out, the dyadic Green's functions can be determined readily. These matrices have been studied extensively in the past and they can be computed easily today in view of the availability of many robust eigenpackages. For some simpler medium, they even can be solved analytically at hand since the matrices are of order 4×4 only. Note that our solutions have been expressed in terms of the inverse of eigenvector matrix directly, not indirectly in terms of the transition matrix as in usual practice [19–22]. Such representations manifest much natural (complementary) sense, i.e., eigenvector matrix is associated with field antecedents, while its inverse is associated with source consequents. Still, other representations are available, for example those employing adjoint operators or reciprocity theorems [1, 4, 29, 30] Their connections are to be revealed in the next section.

4. RECIPROCITY THEOREMS

As demonstrated in [32], Lorentz and modified reciprocity theorems can be expressed simply through six-vector formalism. In the spectral domain, they can also be written down simply, perhaps simpler since one deals mostly with algebras and less with differentials. Let us discuss those theorems based on the source-incorporated 4×4 matrix of Section 2.

4.1 Complementary Medium and Wavenumbers

We first introduce a set of differential equations, fields, sources and other notations (superscripted with C) complementary to those in the previous sections, e.g.,

$$\frac{\mathrm{d}}{\mathrm{d}p}\bar{\mathrm{f}}_{\mathrm{t}}^{C} = i\omega\overline{\overline{\Lambda}}^{C}\cdot\bar{\mathrm{f}}_{\mathrm{t}}^{C} + \overline{\overline{\Gamma}}_{\mathrm{v}}\cdot(\bar{\mathrm{s}}_{\mathrm{t}}^{C} + \overline{\overline{\beta}}^{C}\bar{\mathrm{s}}_{\mathrm{p}}^{C}) \tag{66}$$

$$\overline{\overline{\Lambda}}^{C} = \overline{\overline{\Gamma}}_{\mathrm{v}}\cdot(\overline{\overline{\beta}}^{C}\cdot\overline{\overline{M}}_{\mathrm{pp}}^{C}\cdot\overline{\overline{\alpha}}^{C} - \overline{\overline{M}}_{\mathrm{tt}}^{C}). \tag{67}$$

Consider next the expansion of $\frac{\mathrm{d}}{\mathrm{d}p}(\bar{\mathrm{f}}_{\mathrm{t}}^{CT}\cdot\overline{\overline{\Gamma}}_{\mathrm{u}}\cdot\bar{\mathrm{f}}_{\mathrm{t}})$ where

$$\overline{\overline{\Gamma}}_{\mathrm{u}} = \begin{bmatrix}\overline{\overline{0}} & \overline{\overline{\Gamma}}_{\mathrm{a}}\\ \overline{\overline{\Gamma}}_{\mathrm{a}} & \overline{\overline{0}}\end{bmatrix}. \tag{68}$$

Making use of the inhomogeneous differential equations (14) and (66), we have

$$\frac{\mathrm{d}}{\mathrm{d}p}(\bar{\mathrm{f}}_{\mathrm{t}}^{CT}\cdot\overline{\overline{\Gamma}}_{\mathrm{u}}\cdot\bar{\mathrm{f}}_{\mathrm{t}}) = i\omega\,\bar{\mathrm{f}}_{\mathrm{t}}^{CT}\cdot(\overline{\overline{\Gamma}}_{\mathrm{u}}\cdot\overline{\overline{\Lambda}} + \overline{\overline{\Lambda}}^{CT}\cdot\overline{\overline{\Gamma}}_{\mathrm{u}})\cdot\bar{\mathrm{f}}_{\mathrm{t}} + \bar{\mathrm{f}}_{\mathrm{t}}^{CT}\cdot\overline{\overline{\mathrm{I}}}_{\mathrm{v}}\cdot(\bar{\mathrm{s}}_{\mathrm{t}} + \overline{\overline{\beta}}\cdot\bar{\mathrm{s}}_{\mathrm{p}}) - \bar{\mathrm{f}}_{\mathrm{t}}^{T}\cdot\overline{\overline{\mathrm{I}}}_{\mathrm{v}}\cdot(\bar{\mathrm{s}}_{\mathrm{t}}^{C} + \overline{\overline{\beta}}^{C}\cdot\bar{\mathrm{s}}_{\mathrm{p}}^{C}). \tag{69}$$

Evidently, the first source-free term in the right hand side of (69) will vanish if

$$\overline{\overline{\Lambda}}^{C} = \overline{\overline{\Gamma}}_{\mathrm{u}}\cdot\overline{\overline{\Lambda}}^{T}\cdot\overline{\overline{\Gamma}}_{\mathrm{u}}. \tag{70}$$

A sufficient condition for this to be met is provided by the modified reciprocity theorem which conceives a complementary medium having constitutive parameters given by [38, 1]

$$\begin{bmatrix}\overline{\overline{\epsilon}}^{C} & \overline{\overline{\xi}}^{C}\\ \overline{\overline{\zeta}}^{C} & \overline{\overline{\mu}}^{C}\end{bmatrix} = \begin{bmatrix}\overline{\overline{I}} & \overline{\overline{0}}\\ \overline{\overline{0}} & -\overline{\overline{I}}\end{bmatrix}\cdot\begin{bmatrix}\overline{\overline{\epsilon}} & \overline{\overline{\xi}}\\ \overline{\overline{\zeta}} & \overline{\overline{\mu}}\end{bmatrix}^{T}\cdot\begin{bmatrix}\overline{\overline{I}} & \overline{\overline{0}}\\ \overline{\overline{0}} & -\overline{\overline{I}}\end{bmatrix}. \tag{71}$$

Writing out in terms of their partitions, they read

$$\overline{\overline{M}}_{\mathrm{tt}}^{C} = \overline{\overline{\mathrm{I}}}_{\mathrm{v}}\cdot\overline{\overline{M}}_{\mathrm{tt}}^{T}\cdot\overline{\overline{\mathrm{I}}}_{\mathrm{v}} \tag{72}$$

$$\overline{\overline{M}}_{\mathrm{tp}}^{C} = \overline{\overline{\mathrm{I}}}_{\mathrm{v}}\cdot\overline{\overline{M}}_{\mathrm{pt}}^{T}\cdot\overline{\overline{\mathrm{I}}}_{\mathrm{a}}$$

$$\overline{\overline{M}}_{\mathrm{pt}}^{C} = \overline{\overline{\mathrm{I}}}_{\mathrm{a}}\cdot\overline{\overline{M}}_{\mathrm{tp}}^{T}\cdot\overline{\overline{\mathrm{I}}}_{\mathrm{v}}$$

$$\overline{\overline{M}}_{\mathrm{pp}}^{C} = \overline{\overline{\mathrm{I}}}_{\mathrm{a}}\cdot\overline{\overline{M}}_{\mathrm{pp}}^{T}\cdot\overline{\overline{\mathrm{I}}}_{\mathrm{a}} \tag{75}$$

where

$$\overline{\overline{\mathrm{I}}}_{\mathrm{a}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{76}$$

Moreover, in view of the definitions in (67), one also requires the complementary transverse wave numbers besides the constitutive parameters,

$$\overline{\overline{K}}{}^{C}_{\mathrm{tp}} = \overline{\overline{\mathrm{I}}}_{\mathrm{v}} \cdot \overline{\overline{K}}{}^{T}_{\mathrm{pt}} \cdot \overline{\overline{\mathrm{I}}}_{\mathrm{a}} \tag{77}$$

$$\overline{\overline{K}}{}^{C}_{\mathrm{pt}} = \overline{\overline{\mathrm{I}}}_{\mathrm{a}} \cdot \overline{\overline{K}}{}^{T}_{\mathrm{tp}} \cdot \overline{\overline{\mathrm{I}}}_{\mathrm{v}}. \tag{78}$$

This would allow us to relate $\overline{\overline{\alpha}}^{C}$ and $\overline{\overline{\beta}}^{C}$ to those of original medium as

$$\overline{\overline{\alpha}}^{C} = \overline{\overline{\mathrm{I}}}_{\mathrm{a}} \cdot \overline{\overline{\beta}}^{T} \cdot \overline{\overline{\mathrm{I}}}_{\mathrm{v}} \tag{79}$$

$$\overline{\overline{\beta}}^{C} = \overline{\overline{\mathrm{I}}}_{\mathrm{v}} \cdot \overline{\overline{\alpha}}^{T} \cdot \overline{\overline{\mathrm{I}}}_{\mathrm{a}}. \tag{80}$$

Notice that all expressions from (72) to (80) involve operations in the form

$$\overline{\overline{\mathrm{I}}}_{(\mathrm{v,a})} \cdot \overline{\overline{A}}^{T} \cdot \overline{\overline{\mathrm{I}}}_{(\mathrm{v,a})}.$$

Such an operation is termed adjugate transpose or complementary operation defined for six-dyadic in [32], c.f. (71). Here, we adapt this operation so that it can apply to various nonsquare $\overline{\overline{A}}$'s utilizing the adjugation matrix of appropriate size, i.e. $\overline{\overline{\mathrm{I}}}_{\mathrm{v}}$ or $\overline{\overline{\mathrm{I}}}_{\mathrm{a}}$.

Equations (70), (79) and (80) describe succinctly the relationships between the system matrices of complementary and original media. Writing out their partitions, we have for instance

$$\overline{\overline{\Lambda}}{}^{C}_{\mathrm{ee}} = \overline{\overline{\Gamma}}_{\mathrm{a}} \cdot \overline{\overline{\Lambda}}{}^{T}_{\mathrm{mm}} \cdot \overline{\overline{\Gamma}}_{\mathrm{a}} \tag{81}$$

$$\overline{\overline{\Lambda}}{}^{C}_{\mathrm{em}} = \overline{\overline{\Gamma}}_{\mathrm{a}} \cdot \overline{\overline{\Lambda}}{}^{T}_{\mathrm{em}} \cdot \overline{\overline{\Gamma}}_{\mathrm{a}} \tag{82}$$

$$\overline{\overline{\Lambda}}{}^{C}_{\mathrm{me}} = \overline{\overline{\Gamma}}_{\mathrm{a}} \cdot \overline{\overline{\Lambda}}{}^{T}_{\mathrm{me}} \cdot \overline{\overline{\Gamma}}_{\mathrm{a}} \tag{83}$$

$$\overline{\overline{\Lambda}}{}^{C}_{\mathrm{mm}} = \overline{\overline{\Gamma}}_{\mathrm{a}} \cdot \overline{\overline{\Lambda}}{}^{T}_{\mathrm{ee}} \cdot \overline{\overline{\Gamma}}_{\mathrm{a}}. \tag{84}$$

Incorporating such relations for $\overline{\overline{\Lambda}}^{C}$ into its dispersion relation

$$\det(\omega\overline{\overline{\Lambda}}^{C} - k_p^{C}\overline{\overline{\mathrm{I}}}_4) = 0, \tag{85}$$

we find that (85) is identical to (23) when

$$k_p^C = -k_p. \tag{86}$$

Together with (77)–(78) which implies

$$\overline{k}_t^C = -\overline{k}_t, \tag{87}$$

they state that waves propagating along a certain direction in an original medium correspond to that propagating along opposite direction in the complementary medium. This reversal of wave numbers must be invoked together with the above complementary of medium parameters in the spectral domain, although other operations on wave numbers and medium parameters can be considered as well, e.g., formally adjoint or conjugate system [30, 39, 40] When condition (70) is satisfied, equation (69) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}p}(\overline{\mathrm{f}}_t^{CT} \cdot \overline{\overline{\Gamma}}_\mathrm{u} \cdot \overline{\mathrm{f}}_\mathrm{t}) = \overline{\mathrm{f}}_\mathrm{t}^{CT} \cdot \overline{\overline{\mathrm{I}}}_\mathrm{v} \cdot (\overline{\mathrm{s}}_\mathrm{t} + \overline{\overline{\beta}} \cdot \overline{\mathrm{s}}_\mathrm{p}) - \overline{\mathrm{f}}_\mathrm{t}^T \cdot \overline{\overline{\mathrm{I}}}_\mathrm{v} \cdot (\overline{\mathrm{s}}_\mathrm{t}^C + \overline{\overline{\beta}}^C \cdot \overline{\mathrm{s}}_\mathrm{p}^C). \tag{88}$$

We shall investigate the properties of the partitions of complementary Green's dyadics based on this equation.

4.2 Complementary Dyadic Green's Functions

Let us first carry out the integration on (88) with respect to all p. Due to the radiation condition of bounded solutions (with at least slight loss) at infinity, the left hand side vanishes leaving

$$\int \mathrm{d}p\, \overline{\mathrm{f}}_\mathrm{t}^{CT} \cdot \overline{\overline{\mathrm{I}}}_\mathrm{v} \cdot (\overline{\mathrm{s}}_\mathrm{t} + \overline{\overline{\beta}} \cdot \overline{\mathrm{s}}_\mathrm{p}) - \int \mathrm{d}p\, \overline{\mathrm{f}}_\mathrm{t}^T \cdot \overline{\overline{\mathrm{I}}}_\mathrm{v} \cdot (\overline{\mathrm{s}}_\mathrm{t}^C + \overline{\overline{\beta}}^C \cdot \overline{\mathrm{s}}_\mathrm{p}^C) = 0. \tag{89}$$

From here, the well-known modified reciprocity principle becomes evident, although they merely specify the reactions between transverse fields and transverse sources, as well as between transverse fields and longitudinal sources. The reactions involving longitudinal fields can be derived via $\overline{\overline{\alpha}}$, $\overline{\overline{\alpha}}^C$, etc.

We next define the complementary Green's dyadics $\overline{\overline{\mathrm{G}}}_\mathrm{tt}^C$, etc., in the manner analogous to (36) and substitute them into (89) for

$$\begin{aligned}\int \mathrm{d}p \int \mathrm{d}p' [\overline{\overline{\mathrm{G}}}_\mathrm{tt}^C(p,p') \cdot \overline{\mathrm{s}}_\mathrm{t}^C(p') + \overline{\overline{\mathrm{G}}}_\mathrm{tp}^C(p,p') \cdot \overline{\mathrm{s}}_\mathrm{p}^C(p')]^T \\ \cdot \overline{\overline{\mathrm{I}}}_\mathrm{v} \cdot [\overline{\mathrm{s}}_\mathrm{t}(p) + \overline{\overline{\beta}} \cdot \overline{\mathrm{s}}_\mathrm{p}(p)] - [\overline{\overline{\mathrm{G}}}_\mathrm{tt}(p,p') \cdot \overline{\mathrm{s}}_\mathrm{t}(p') \\ + \overline{\overline{\mathrm{G}}}_\mathrm{tp}(p,p') \cdot \overline{\mathrm{s}}_\mathrm{p}(p')]^T \cdot \overline{\overline{\mathrm{I}}}_\mathrm{v} \cdot [\overline{\mathrm{s}}_\mathrm{t}^C(p) + \overline{\overline{\beta}}^C \cdot \overline{\mathrm{s}}_\mathrm{p}^C(p)] = 0.\end{aligned} \tag{90}$$

Using the inter-partition relations (56)–(59) along with those for complementary-versus-original discussed in the previous subsection, we can deduce from (90)

$$\overline{\overline{\mathrm{G}}}^{C}_{\mathrm{tt}}(p', p) = \overline{\overline{\mathrm{I}}}_{\mathrm{v}} \cdot \overline{\overline{\mathrm{G}}}^{T}_{\mathrm{tt}}(p, p') \cdot \overline{\overline{\mathrm{I}}}_{\mathrm{v}} \tag{91}$$

$$\overline{\overline{\mathrm{G}}}^{C}_{\mathrm{tp}}(p', p) = \overline{\overline{\mathrm{I}}}_{\mathrm{v}} \cdot \overline{\overline{\mathrm{G}}}^{T}_{\mathrm{pt}}(p, p') \cdot \overline{\overline{\mathrm{I}}}_{\mathrm{a}} \tag{92}$$

$$\overline{\overline{\mathrm{G}}}^{C}_{\mathrm{pt}}(p', p) = \overline{\overline{\mathrm{I}}}_{\mathrm{a}} \cdot \overline{\overline{\mathrm{G}}}^{T}_{\mathrm{tp}}(p, p') \cdot \overline{\overline{\mathrm{I}}}_{\mathrm{v}} \tag{93}$$

$$\overline{\overline{\mathrm{G}}}^{C}_{\mathrm{pp}}(p', p) = \overline{\overline{\mathrm{I}}}_{\mathrm{a}} \cdot \overline{\overline{\mathrm{G}}}^{T}_{\mathrm{pp}}(p, p') \cdot \overline{\overline{\mathrm{I}}}_{\mathrm{a}}. \tag{94}$$

Again, note the usage of $\overline{\overline{\mathrm{I}}}_{\mathrm{v}}$ and $\overline{\overline{\mathrm{I}}}_{\mathrm{a}}$ as before in describing succinctly the relationships among complementary and original Green's dyadic partitions. These relations are useful particularly for reciprocal medium which is invariant under complementary operation, since one can derive certain part (e.g., tp) simply from the other (e.g., pt) without having to compute everything from scratch [41, 42]. Writing out explicitly, say for $\overline{\overline{\mathrm{G}}}^{C}_{\mathrm{tp}}$, we have

$$\overline{\overline{G}}^{C}_{\mathrm{eetp}}(p', p) = \overline{\overline{G}}^{T}_{\mathrm{eept}}(p, p') \tag{95}$$

$$\overline{\overline{G}}^{C}_{\mathrm{emtp}}(p', p) = -\overline{\overline{G}}^{T}_{\mathrm{mept}}(p, p') \tag{96}$$

$$\overline{\overline{G}}^{C}_{\mathrm{metp}}(p', p) = -\overline{\overline{G}}^{T}_{\mathrm{empt}}(p, p') \tag{97}$$

$$\overline{\overline{G}}^{C}_{\mathrm{mmtp}}(p', p) = \overline{\overline{G}}^{T}_{\mathrm{mmpt}}(p, p'). \tag{98}$$

Hence, apart from the subscripts for partitioning, the relations for partitioned Green's dyadics are seen to coincide with those for the total Green's dyadics given in space domain [8, 32]. On the other hand, since our dyadic Green's function solutions obtained previously in Section 3 are expressed in terms of the inverse of eigenvector matrix, c.f. (53), the satisfaction of (91), etc., does not seem to be immediately apparent. This point will be clarified below.

4.3 Biorthogonality Relations

Consider the source-free case for (88), that is

$$\frac{\mathrm{d}}{\mathrm{d}p}(\overline{\mathrm{f}}^{CT}_{\mathrm{t}} \cdot \overline{\overline{\Gamma}}_{\mathrm{u}} \cdot \overline{\mathrm{f}}_{\mathrm{t}}) = 0. \tag{99}$$

Recall from Section 2 that the eigenfields $\bar{\mathrm{f}}_{\mathrm{t}}$ and similarly $\bar{\mathrm{f}}_{\mathrm{t}}^{C}$ have the longitudinal dependence of $e^{ik_p p}$ and $e^{ik_p^C p}$. Corresponding to each eigenvalue for k_{pj} and $k_{pj'}^{C}$, we substitute $\bar{\mathrm{f}}_{\mathrm{t}j}$ and $\bar{\mathrm{f}}_{\mathrm{t}j'}^{C}$ into (99) obtaining

$$(k_{pj} + k_{pj'}^{C})\bar{\mathrm{f}}_{\mathrm{t}j'}^{CT} \cdot \bar{\bar{\Gamma}}_{\mathrm{u}} \cdot \bar{\mathrm{f}}_{\mathrm{t}j} = 0. \tag{100}$$

This indeed gives the well-known biorthogonality relation [29, 30]

$$\bar{\mathrm{f}}_{\mathrm{t}j'}^{CT} \cdot \bar{\bar{\Gamma}}_{\mathrm{u}} \cdot \bar{\mathrm{f}}_{\mathrm{t}j} = 0 \qquad \text{if} \quad k_{pj} \neq -k_{pj'}^{C}. \tag{101}$$

Now since $\bar{\mathrm{f}}_{\mathrm{t}j}$ can be written as one column of $\bar{\bar{\Psi}}$ in (24) and similarly for $\bar{\mathrm{f}}_{\mathrm{t}j'}^{C}$ in terms of $\bar{\bar{\Psi}}^{C}$, we can write (101) as

$$\bar{\bar{\Psi}}^{CT} \cdot \bar{\bar{\Gamma}}_{\mathrm{u}} \cdot \bar{\bar{\Psi}} = \bar{\bar{\mathrm{N}}}. \tag{102}$$

In conformity with (101), $\bar{\bar{\mathrm{N}}}$ will be a diagonal matrix if the eigenvalues have been arranged properly such that $k_{pj} = -k_{pj'}^{C}$ for $j = j'$, and all eigenvalues are distinct. When there are degenerate eigenvalues, one can adopt some convenient orthogonalization schemes to construct a set of orthogonal eigenfields. Then, the diagonal elements in $\bar{\bar{\mathrm{N}}}$ just represent the corresponding normalization factors for subsequent Green's function solutions. In any case, whether fully diagonal or not, we have

$$\bar{\bar{\Psi}}^{-1} = \bar{\bar{\mathrm{N}}}^{-1} \cdot \bar{\bar{\Psi}}^{CT} \cdot \bar{\bar{\Gamma}}_{\mathrm{u}}. \tag{103}$$

Using this specification of $\bar{\bar{\Psi}}^{-1}$, the source consequents in (53) can be rewritten as (when $\bar{\bar{\mathrm{N}}}$ is diagonal)

$$\bar{\bar{\sigma}}(p') = \bar{\bar{\mathrm{I}}}_{\mathrm{v}} \cdot \bar{\bar{\mathrm{N}}}^{-1} \cdot \bar{\bar{\mathrm{P}}}^{CT}(p') \cdot \bar{\bar{\Psi}}^{CT} \cdot \bar{\bar{\mathrm{I}}}_{\mathrm{v}} \tag{104}$$

Solution (104) takes the form of those extensively studied in the literature utilizing either Lorentz, modified or other (adjoint) reciprocity theorems. From the analytical point of view, this form is really preferable since one does not have to compute the tiresome inverse, but merely needs to perform the complementary of medium parameters plus the reversal of wave numbers to get $\bar{\bar{\Psi}}^{C}$. However, from the numerical (generic coding) point of view, (104) would be less in favor compare to (53) especially in dealing with degenerate cases since one demands somewhat careful preprocessings and orthogonalizations.

Furthermore, there is no explicit analytical expressions for us to simply complementary-plus-reversal if we just take the numerical eigenpackages for granted, so that one might have to recompute the eigensolutions for complementary medium. On the other hand, the inverse of eigenvector matrix is in direct conformity with our earlier assumption of their existence and diagonability, which follows immediately once a set of linearly independent (but not necessary biorthogonal yet) eigenvectors are available. During actual solving of source responses, one may not even have to find the explicit inverse, but only requires its product with a (current) vector. From here, we see that the key ingredients of our solutions for electromagnetic problems are those eigenvector matrices, specifically their submatrices $\overline{\overline{\mathrm{e}}}^{\gtrless}$ and $\overline{\overline{\mathrm{h}}}^{\gtrless}$ which constitute the field antecedents as well as source consequents. Having determined these matrices from homogeneous solutions, one can directly obtain the dyadic Green's functions pertaining to point source embedded in general bianisotropic media.

5. CONCLUSION

Based on the source-incorporated 4×4 matrix method in spectral domain, this paper has presented a unified and concise formalism of electromagnetics in bianisotropic media. The system matrices have been written in very condensed and highly symmetric form protruding their key ingredients. All four types of dyadic Green's functions for an unbounded bianisotropic medium have been obtained simultaneously in pronounced explicit expressions. The singularities, discontinuities and the spectral expansions of all Green's dyadic partitions have been derived most simply and compactly. Along the derivation, much appreciation has been given to the roles of system matrices in performing transformations and characterizing discontinuities. The intimate relationships among various Green's dyadic partitions have been revealed and exploited, asserting that all other partitions can be deduced readily from the transverse-transverse Green's functions. Important reciprocity theorems have been revisited in spectral domain, stating succinctly the relations between system matrices of original and complementary media. The connections between the Green's functions obtained from 4×4 matrix method and those from reciprocity theorems have been clarified, comparing the source consequents expressed as the inverse of eigenvector matrix and those in terms of complementary eigenvector matrix. Much recognition has been given to the roles of eigenvector submatrices in field antecedents and source consequents. With the concise formalism presented here, it is hoped that subsequent works related to (homogeneous or multilayered) bianisotropic media can be dealt with in a less laborious manner.

REFERENCES

1. Kong, J. A., *Electromagnetic Wave Theory*, Wiley, New York, 1990.
2. Lindell, I. V., A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-Isotropic Media*, Artech House, London, 1994.
3. Tai, C. T., *Dyadic Green's Functions in Electromagnetic Theory*, IEEE Press, New York, 1993.
4. Collin, R. E., *Field Theory of Guided Waves*, IEEE Press, New York, 1991.
5. Chew, W. C., *Waves and Fields in Inhomogeneous Media*, Van Nostrand, New York, 1990.
6. Ali, S. M., T. M. Habashy, and J. A. Kong, "Spectral-domain dyadic Green's function in layered chiral media," *J. Opt. Soc. Am. A.,* Vol. 9, No. 3, 413–423, 1992.
7. Graglia, R. D., P. L. E. Uslenghi, and R. E. Zich, "Dispersion relation for bianisotropic materials and its symmetry properties," *IEEE Trans. Antennas Propagat.,* Vol. 39, No. 1, 83–90, 1991.
8. Monzon, J. C., "On surface integral representations: validity of Huygens' principle and the equivalence principle in inhomogeneous bianisotropic media," *IEEE Trans. Microwave Theory Tech.,* Vol. 41, No. 11, 1995–2001, 1993.
9. Hanson, G. W., "A numerical formulation of dyadic Green's functions for planar bianisotropic media with application to printed transmission lines," *IEEE Trans. Microwave Theory Tech.,* Vol. 44, No. 1, 144–151, 1996.
10. Zhuck, N. P., and A. S. Omar, "Scalarization approach in the theory of em waves in a plane-layered bianisotropic medium: general formulations," *Progress In Electromagnetics Research,* PIER 12, 219–249, 1996.
11. Berreman, D. W., "Optics in stratified and anisotropic media: 4×4-Matrix Formulation," *J. Opt. Soc. Am.,* Vol. 62, No. 4, 502–510, 1972.
12. Yeh, P., "Electromagnetic propagation in birefringent layered media," *J. Opt. Soc. Am.,* Vol. 69, No. 5, 742–756, 1979.

13. Vassell, M. O., "Structure of optical guided modes in planar multilayers of optically anisotropic materials," *J. Opt. Soc. Am.,* Vol. 64, No. 2, 166–173, 1974.
14. Lin-Chung, P. J., and S. Teitler, "4×4 Matrix formalisms for optics in stratified anisotropic media," *J. Opt. Soc. Am. A,* Vol. 1, No. 7, 703–705, 1984.
15. Krowne, C. M., "Fourier transformed matrix method of finding propagation characteristics of complex anisotropic layered media," *IEEE Trans. Microwave Theory Tech.,* Vol. 32, No. 12, 1617–1625, 1984.
16. Morgan, M. A., D. L. Fisher, and E. A. Milne, "Electromagnetic scattering by stratified inhomogeneous anisotropic media," *IEEE Trans. Antennas Propagat,* Vol. 35, No. 2, 191–197, 1987.
17. Titchener, J. B., and J. R. Willis, "The reflection of electromagnetic waves from stratified anisotropic media," *IEEE Trans. Antennas Propagat,* Vol. 39, No. 1, 35-39, 1991.
18. Graglia, R. D., P. L. E. Uslenghi, and R. E. Zich, "Reflection and transmission for planar structures of bianisotropic media," *Electromagnetics,* Vol. 11, 193–208, 1991.
19. Tsalamengas, J. L., "Electromagnetic fields of elementary dipole antennas embedded in stratified general gyrotropic media," *IEEE Trans. Antennas Propagat,* Vol. 37, No. 3, 399–403, 1989.
20. Tsalamengas, J. L., "Interaction of Electromagnetic Waves with General Bianisotropic Slabs," *IEEE Trans. Microwave Theory Tech.,* Vol. 40, No. 10, 1870–1878, 1992.
21. Hsia, I. Y., and N. G. Alexopoulos, "Radiation characteristics of hertzian dipole antennas in a nonreciprocal superstrate-substrate structure," *IEEE Trans. Antennas Propagat,* Vol. 40, No. 7, 782–790, 1992.
22. Mesa, F. L., R. Marques, and M. Horno, "A general algorithm for computing the bidimensional spectral Green's dyad in multilayered complex bianisotropic media: the equivalent boundary method," *IEEE Trans. Microwave Theory Tech.,* Vol. 39, No. 9, 1640–1649, 1991.
23. Bastiaansen, H. J. M., N. H. G. Baken, and H. Blok, "Domain-integral analysis of channel waveguides in anisotropic multilayered media," *IEEE Trans. Microwave Theory Tech.,* Vol. 40, No. 10, 1918–1926, 1992.
24. Olyslager, F., and D. D. Zutter, "Rigorous full-wave analysis of electric and dielectric waveguides embedded in a multilayered bianisotropic medium," *Radio Sci.,* Vol. 28, No. 5, 937–946, 1993.

25. Plaza, G., F. Mesa, and M. Horno, "Computation of propagation characteristics of chiral layered waveguides," *IEEE Trans. Microwave Theory Tech.,* Vol. 45, No. 4, 519–526, 1997.
26. Yang, H. D., "A spectral recursive transformation method for electromagnetic waves in generalized anisotropic layered media," *IEEE Trans. Antennas Propagat,* Vol. 45, No. 3, 520–526, 1997.
27. Yin, W., W. Wan, and W. B. Wang, "Radiation from a dipole antenna on two-layer grounded Faraday chiral substrates," *J. Electrom. Waves Applicat.,* Vol. 9, No. 7/8, 1027–1044, 1995.
28. Yin, W., G. H. Nan, and I. Wolff, "The combined effects of chiral operation in multilayered bianisotropic substrates," *Progress In Electromagnetics Research,* PIER 20, 153–178, 1998.
29. Felsen, L. B., and N. Marcuvitz, *Radiation and Scattering of Waves*, Prentice-Hall, New Jersey, 1973.
30. Altman, C., and K. Suchy, *Reciprocity, Spatial Mapping and Time Reversal in Electromagnetics*, Kluwer, DorDrecht, 1991.
31. Lindell, I. V., *Methods for Electromagnetic Field Analysis*, Clarendon, Oxford, 1992.
32. Lindell, I. V., and A. H. Sihvola, "Six-vector formalism in electromagnetics of bi-anisotropic media," *J. Electrom. Waves Applicat.,* Vol. 9, No. 7/8, 887–903, 1995.
33. Tan, E. L., and S. Y. Tan, "A unified representation of the dyadic Green's functions for planar, cylindrical and spherical multilayered biisotropic media," *Progress In Electromagnetics Research,* PIER 20, 75–100, 1998.
34. Tan, E. L., and S. Y. Tan, "On the eigenfunction expansions of the dyadic Green's functions for bianisotropic media," *Progress In Electromagnetics Research,* PIER 20, 227–247, 1998.
35. Tan, E. L., and S. Y. Tan, "Singularities and discontinuities in the eigenfunction expansions of the dyadic Green's functions for biisotropic media," *Progress In Electromagnetics Research,* PIER 19, 301–318, 1998.
36. Cottis, P. G., and G. D. Kondylis, "Properties of the dyadic Green's function for an unbounded anisotropic medium," *IEEE Trans. Antennas Propagat,* Vol. 43, No. 2, 154–161, 1995.
37. Sorrentino, R. (Ed.) *Numerical Methods for Passive Microwave and Millimeter Wave Structures*, IEEE Press, New York, 1989.
38. Kong, J. A., and D. K. Cheng, "Modified reciprocity theorem for bianisotropic media," *Proc. IEE*, Vol. 117, No. 2, 349–350, 1970.
39. Altman, C., and K. Suchy, "Generalization of an eigenmode scattering theorem," *J. Electrom. Waves Applicat.,* Vol. 5, No. 7, 685–700, 1991.

40. Altman, C., and K. Suchy, "Mode scattering reciprocity relations in waveguides with (bi)anisotropic media," *J. Electrom. Waves Applicat.,* Vol. 10, No. 9, 1311–1323, 1996.
41. Vegni, L., R. Cicchetti, and P. Capece, "Spectral dyadic Green's function formulation for planar integrated structures," *IEEE Trans. Antennas Propagat,* Vol. 36, No. 8, 1057–1065, 1988.
42. Toscano, A., and L. Vegni, "Electromagnetic waves in planar integrated pseudochiral Ω structures," *Progress In Electromagnetics Research,* PIER 9, 181–218, 1994.