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COUPLED-MODE ANALYSIS OF A GATING-ASSISTED DIRECTIONAL COUPLER USING SINGULAR PERTURBATION TECHNIQUE

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1. INTRODUCTION

Grating-assisted directional couplers are widely used in the design of optical waveguide filters [1]. A basic configuration of the couplers consists of nonidentical waveguides placed adjacently in which a periodic grating structure is embedded. The phase synchronism in two nonsynchronous waveguides is achieved through the space-harmonic components generated by the grating. When the grating period is specified, a complete power transfer from one guide to the other is obtained at a particular wavelength satisfying the phase-matching condition. This makes the coupler be wavelength-selective.

The power transfer characteristics in grating-assisted directional couplers have been mainly investigated using the coupled-mode theory [2–7], because an exact analytical treatment is rather difficult. Marcuse [2] has pointed out that the grating-assisted couplers should be formulated with the orthogonal coupled-mode theory based on the exact coupler modes to obtain the correct phase-matching condition. However the mode representation by the coupler modes is inconvenient to calculate the power transfer characteristics between two waveguides. Huang and Haus [3] have proposed an orthogonal coupled-mode theory using approximate coupler modes, which are derived by the non-orthogonal coupled-mode theory based on the eigenmodes of each waveguide in isolation. They have used the approximate coupler modes to deduce the phase-matching condition and then transformed the orthogonal coupled-mode equations into the nonorthogonal ones to calculate the power transfer. This approach has been applied to the analysis of a grating-assisted three-waveguide coupler [7].

When each waveguide in isolation is chosen as the basis of a coupled-mode formulation, one notes that two major perturbation effects exist in the grating-assisted couplers. One is the perturbation in the transverse dimension due to the presence of the adjacent waveguide, and the other is the perturbation in the propagation direction due to the presence of periodic grating structure. When these two effects are in the same order of magnitude, the perturbation theory requires the coupler problem be formulated with the accuracy up to the second order of perturbation. This fact suggests that the coupled-mode approach using the approximate coupler modes is not consistent with the perturbation theory. Although the use of the approximate coupler modes greatly simplifies an analytical treatment, its mode representation is correct within the first-order perturbation [8]. More accurate perturbation approach is requested to obtain the nonorthogonal coupled-mode equations which are consistent with the orthogonal ones [2] using the exact coupler modes.

In this paper, we shall develop an accurate nonorthogonal coupled-mode theory for a grating-assisted directional coupler using the singular perturbation technique [9]. The concept of slowly varying amplitude function [10] is introduced to take a proper balance in the two different perturbations mentioned above. The optical fields in the coupler are represented in terms of a linear combination of para-axial wave fields for each isolated waveguide with a reference wavenumber of the surrounding cladding. The para-axial wave fields are expanded using the multiple space-scales [11] and solved so that they satisfy the para-axial wave equation for the coupled structure and the phase matching condition in the respective orders of perturbation. This leads to the asymptotically correct coupled-mode equations based on the

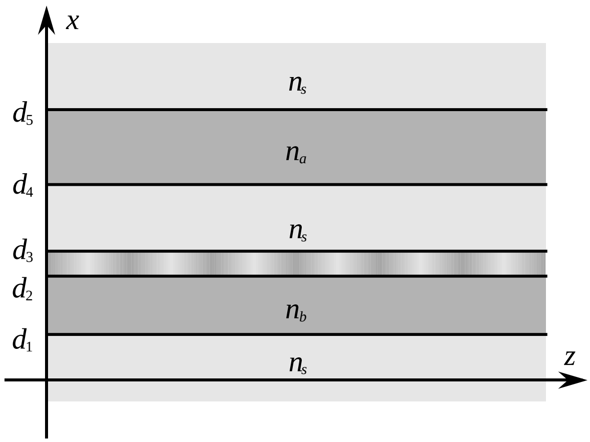

Figure 1. Geometry of the grating-assisted directional coupler.

eigenmodes of each waveguide in isolation. The proposed coupled-mode theory is applied to the analysis of two nonidentical coupled waveguides attached with a sinusoidal grating layer. It is shown that the calculated wavelength response in the power transfer is in close agreement with that obtained by a direct numerical analysis using the Fourier series expansion method [12].

2. FORMULATION

A grating-assisted directional coupler under consideration is schematically shown in Fig. 1. It consists of two nonidentical dielectric slab waveguides a and b, which are situated parallel to each other with a separation distance $d_4 - d_3$ in a surrounding dielectric of refractive index n_s. The waveguide a is a regular one with thickness $d_5 - d_4$ and refractive index n_a, whereas the upper surface of the waveguide b with thickness $d_2 - d_1$ and refractive index n_b is attached with a grating layer of thickness $d_3 - d_2$ and period p. The individual waveguides a and b are assumed to support only one guided mode when in isolation. The relative permittivity in the grating layer is supposed to be independent of x and given as ${n_c}^2 + {n_d}^2\, g(z)$ where $g(z)$ is the grating profile function:

$$g(z) = \cos \frac{2\pi}{p} z. \tag{1}$$

The geometry is uniform in the y-direction. Then the relative permittivity distribution of the waveguide system is defined as

$$\varepsilon_r(x,z) = \begin{cases} {n_a}^2 & \text{for } d_4 < x < d_5 \\ {n_c}^2 + {n_d}^2\, g(z) & \text{for } d_2 < x < d_3 \\ {n_b}^2 & \text{for } d_1 < x < d_2 \\ {n_s}^2 & \text{otherwise} \end{cases}. \tag{2}$$

The permeability of free space is assumed over the whole layer. We examine the two-dimensional ($\partial/\partial y = 0$) TE wave propagating in the z-direction.

Let the wave function $\psi(x,z)$ be the y-components of the electric field. Then $\psi(x,z)$ satisfies the Helmholtz equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + {k_s}^2\Big(1 + \Delta\varepsilon_a(x) + \Delta\varepsilon_b(x) + \Delta\varepsilon_g(x)\, g(z)\Big)\right]\psi(x,z) = 0 \tag{3}$$

with

$$\Delta\varepsilon_a(x) = \begin{cases} \dfrac{{n_a}^2 - {n_s}^2}{{n_s}^2} & \text{for } d_4 < x < d_5 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$\Delta\varepsilon_b(x) = \begin{cases} \dfrac{{n_c}^2 - {n_s}^2}{{n_s}^2} & \text{for } d_2 < x < d_3 \\ \dfrac{{n_b}^2 - {n_s}^2}{{n_s}^2} & \text{for } d_1 < x < d_2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$\Delta\varepsilon_g(x) = \begin{cases} \dfrac{{n_d}^2}{{n_s}^2} & \text{for } d_2 < x < d_3 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and the boundary conditions on the composite waveguide structure with $\varepsilon_r(x,z)$, where k_s is the wavenumber in the cladding layer with refractive index n_s

The influence of geometrical perturbations on the wave function $\psi(x,z)$ is much more sensitive [10] in the transverse direction than in the direction of wave propagation. Taking into account this fact, the slowly varying amplitude representation for $\psi(x,z)$ is introduced as follows:

$$\psi(x,z) = \phi(x,z)\, e^{-j\, k_s\, z}. \tag{7}$$

It follows that $|\partial^2\phi/\partial z^2| \ll k_s|\partial\phi/\partial z|$, since the index difference between core and cladding is relatively small for optical waveguides of single mode.

To formulate the coupled-mode analysis, we decompose $\phi(x,z)$ as

$$\phi(x,z) = \phi_a(x,z) + \phi_b(x,z). \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (3), the original wave equation is transformed [9] into the coupled wave equations as follows:

$$\begin{aligned}&\left(\frac{\partial^2}{\partial x^2} - j2k_s\frac{\partial}{\partial z} + {k_s}^2\Delta\varepsilon_a(x)\right)\phi_a(x,z)\\ &\qquad = -\left(\frac{\partial^2}{\partial z^2}\phi_a(x,z) + {k_s}^2\Delta\varepsilon_a(x)\,\phi_b(x,z)\right)\end{aligned} \tag{9}$$

$$\begin{aligned}&\left(\frac{\partial^2}{\partial x^2} - j2k_s\frac{\partial}{\partial z} + {k_s}^2\Delta\varepsilon_b(x)\right)\phi_b(x,z)\\ &\qquad = -\left[\left(\frac{\partial^2}{\partial z^2} + {k_s}^2\Delta\varepsilon_g(x)\,g(z)\right)\phi_b(x,z) + {k_s}^2\Delta\varepsilon_b(x)\,\phi_a(x,z)\right]\\ &\qquad\quad - {k_s}^2\Delta\varepsilon_g(x)\,g(z)\,\phi_a(x,z).\end{aligned} \tag{10}$$

It is verified from the uniqueness theorem that the total wave function $\psi(x,z)$ satisfies Eq. (3) when $\phi_a(x,z)$ and $\phi_b(x,z)$ satisfy Eqs. (9) and (10). The terms in the right hand side of Eqs. (9) and (10) may be regarded as the perturbations to the para-axial wave equations for the slowly varying amplitude functions. Those perturbations include the correction terms $\partial^2\phi_a/\partial z^2$ and $\partial^2\phi_b/\partial z^2$ to the para-axial approximation, the correction term ${k_s}^2\Delta\varepsilon_g(x)\,g(z)\,\phi_b$ in the presence of the grating layer, the dominant coupling terms ${k_s}^2\Delta\varepsilon_a(x)\,\phi_b$ and ${k_s}^2\Delta\varepsilon_b(x)\,\phi_a$ without the grating layer, and the higher order coupling term ${k_s}^2\Delta\varepsilon_g(x)\,g(z)\,\phi_a$ through the grating layer. We introduce a nondimensional small parameter δ to identify the order of magnitude of these perturbations and rewrite Eqs. (9) and (10) as follows:

$$\begin{aligned}&\left(\frac{\partial^2}{\partial x^2} - j2k_s\frac{\partial}{\partial z} + {k_s}^2\Delta\varepsilon_a(x)\right)\phi_a(x,z)\\ &\qquad = -\delta\left(\frac{\partial^2}{\partial z^2}\phi_a(x,z) + {k_s}^2\Delta\varepsilon_a(x)\,\phi_b(x,z)\right)\end{aligned} \tag{11}$$

$$\begin{aligned}&\left(\frac{\partial^2}{\partial x^2} - j2k_s\frac{\partial}{\partial z} + {k_s}^2\Delta\varepsilon_b(x)\right)\phi_b(x,z)\\ &\qquad = -\delta\left[\left(\frac{\partial^2}{\partial z^2} + {k_s}^2\Delta\varepsilon_g(x)\,g(z)\right)\phi_b(x,z) + {k_s}^2\Delta\varepsilon_b(x)\,\phi_a(x,z)\right]\\ &\qquad\quad - \delta^2\,{k_s}^2\Delta\varepsilon_g(x)\,g(z)\,\phi_a(x,z).\end{aligned} \tag{12}$$

In order to apply a perturbation approach to Eqs. (11) and (12), we introduce the multiple space scales [11]; $z_0 = z, z_1 = \delta\, z, z_2 = \delta^2\, z, \cdots, z_n = \delta^n\, z, \cdots$ and expand the wave function $\phi_\nu(x,z)$ $(\nu = a, b)$ as follows:

$$\phi_\nu(x,z) = \sum_{n=0}^{\infty} \delta^n\, \phi_{\nu,n}(x, z_{s,0}) \tag{13}$$

where the notation $z_{s,n}$ abbreviates the sequence of the space scales: $z_n, z_{n+1}, z_{n+2}, \cdots$. Substituting Eq. (13) into Eqs. (11) and (12) and making use of the relation of the derivative expansion: $\partial/\partial z = \partial/\partial z_0 + \delta\, \partial/\partial z_1 + \delta^2\, \partial/\partial z_2 + \cdots$, we obtain a set of equations to be solved in the respective orders of perturbation as

$$L_a\, \phi_{a,n}(x, z_{s,0}) = \begin{cases} 0 & \text{for } n = 0 \\ -\displaystyle\sum_{m=1}^{n} T_m\, \phi_{a,n-m}(x, z_{s,0}) - {k_s}^2 \Delta\varepsilon_a(x)\, \phi_{b,n-1}(x, z_{s,0}) & \\ & \text{for } n \geq 1 \end{cases} \tag{14}$$

$$L_b\, \phi_{b,n}(x, z_{s,0}) = \begin{cases} 0 & \text{for } n = 0 \\ -T_1\, \phi_{b,0}(x, z_{s,0}) - {k_s}^2 \Delta\varepsilon_b(x)\, \phi_{a,0}(x, z_{s,0}) & \\ \quad - {k_s}^2 \Delta\varepsilon_g(x)\, g(z_{s,0})\, \phi_{b,0}(x, z_{s,0}) & \text{for } n = 1 \\ -\displaystyle\sum_{m=1}^{n} T_m\, \phi_{b,n-m}(x, z_{s,0}) - {k_s}^2 \Delta\varepsilon_b(x)\, \phi_{a,n-1}(x, z_{s,0}) & \\ -{k_s}^2 \Delta\varepsilon_g(x)\, g(z_{s,0}) \Big(\phi_{a,n-2}(x, z_{s,0}) + \phi_{b,n-1}(x, z_{s,0})\Big) & \\ & \text{for } n \geq 2 \end{cases} \tag{15}$$

with

$$L_\nu \equiv \frac{\partial^2}{\partial x^2} - j2k_s \frac{\partial}{\partial z_0} + {k_s}^2 \Delta\varepsilon_\nu(x) \qquad (\nu = a, b) \tag{16}$$

$$T_n \equiv \sum_{m=1}^{n} \frac{\partial^2}{\partial z_{m-1} \partial z_{n-m}} - j2k_s \frac{\partial}{\partial z_n}. \tag{17}$$

Equations (14) and (15) are solved successively from the zero-order equations, because the right hand sides are given by the solutions of the lower-order equations.

In the zero-order problem, the wave equations for $\phi_{a,0}(x, z_{s,0})$ and $\phi_{b,0}(x, z_{s,0})$ are decoupled and the analytical steps to derive the zero-order solutions are the same as those for an isolated waveguide under the para-axial approximation. Noting that each waveguide in isolation supports only one guided mode, the solutions to the zero-order equation are obtained as follows:

$$\phi_{\nu,0}(x, z_{s,0}) = a_{\nu,0}(z_{s,1})\, u_{\nu,0}(x)\, e^{-j\,\beta_{\nu,0}\,z_0} \qquad (\nu = a, b) \tag{18}$$

with

$$u_{a,0}(x) = \eta_a \begin{cases} e^{-\alpha_a(x-d_5)} & \text{for } x > d_5 \\ \cos\kappa_a\,(x-d_4) + \dfrac{\alpha_a}{\kappa_a}\sin\kappa_a\,(x-d_4) & \text{for } d_4 < x < d_5 \\ e^{\alpha_a(x-d_4)} & \text{for } x < d_4 \end{cases} \tag{19}$$

$$u_{b,0}(x) = \eta_b \begin{cases} \dfrac{\kappa_g \sec\kappa_g\,(d_3-d_2)}{\kappa_g + \alpha_b \tan\kappa_g\,(d_3-d_2)} e^{-\alpha_b(x-d_3)} & \text{for } x > d_3 \\ \cos\kappa_g\,(x-d_2) - \dfrac{\alpha_b - \kappa_g \tan\kappa_g\,(d_3-d_2)}{\kappa_g + \alpha_b \tan\kappa_g\,(d_3-d_2)} \sin\kappa_g\,(x-d_2) & \text{for } d_2 < x < d_3 \\ \cos\kappa_b\,(x-d_2) + \dfrac{\alpha_b - \kappa_b \tan\kappa_b\,(d_2-d_1)}{\kappa_b + \alpha_b \tan\kappa_b\,(d_2-d_1)} \sin\kappa_b\,(x-d_2) & \text{for } d_1 < x < d_2 \\ \dfrac{\kappa_b \sec\kappa_b\,(d_2-d_1)}{\kappa_b + \alpha_b \tan\kappa_b\,(d_2-d_1)} e^{\alpha_b(x-d_1)} & \text{for } x < d_1 \end{cases} \tag{20}$$

$$\alpha_\nu = \sqrt{2k_s\beta_{\nu,0}} \tag{21}$$

$$\kappa_\nu = \sqrt{{k_0}^2\left({n_\nu}^2 - {n_s}^2\right) - 2k_s\beta_{\nu,0}} \tag{22}$$

$$\kappa_g = \sqrt{{k_0}^2\left({n_c}^2 - {n_s}^2\right) - 2k_s\beta_{b,0}} \tag{23}$$

where $a_{\nu,0}(z_{s,1})$ is the modal amplitude, $u_{\nu,0}(x)$ is the modal profile function, η_ν is the normalization constant defined so that $\int_{-\infty}^{\infty} |u_{\nu,0}(x)|^2\, dx = 1$, and $\beta_{\nu,0}$ is the correction of propagation constant to the para-axial approximation that satisfies

$$\tan\left(\kappa_a \frac{d_5 - d_4}{2}\right) = \frac{\alpha_a}{\kappa_a} \qquad \text{for } \beta_{a,0} \tag{24}$$

$$\tan\left[\kappa_g\,(d_3-d_2)\right] = -\frac{\kappa_g\left[2\alpha_b\kappa_b + \left({\alpha_b}^2 - {\kappa_b}^2\right)\tan\kappa_b\,(d_2-d_1)\right]}{\kappa_b\left({\alpha_b}^2 - {\kappa_g}^2\right) - \alpha_b\left({\kappa_g}^2 + {\kappa_b}^2\right)\tan\kappa_b\,(d_2-d_1)} \quad \text{for } \beta_{b,0}. \tag{25}$$

The dependences of the modal amplitude $a_{\nu,0}(z_{s,1})$ on $z_{s,1}$ are determined from the analysis of the higher-order equations.

Since the coupling between waveguides a and b occurs under a phase synchronism through the attached grating layer, we assume that $\beta_{a,0} > \beta_{b,0}$ and $[2\pi/p - (\beta_{a,0} - \beta_{b,0})]/\beta_{a,0} = O(\delta^n)$ $(n \geq 1)$. Then the profile function $g(z)$ is rewritten using the multiple space-scales as follows:

$$g(z_{s,0}) = \cos\Big[(\beta_{a,0} - \beta_{b,0})\, z_0 + \Delta\beta\, z_n\Big] \tag{26}$$

where $\Delta\beta$ is the deviation from the zero-order phase-matching condition

$$\Delta\beta = \frac{2\pi}{p} - (\beta_{a,0} - \beta_{b,0}). \tag{27}$$

For the analysis of higher-order wave equations, we introduce the Fourier transform with respect to the zero-order space-scale z_0 as

$$\widetilde{f}(\beta) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z_0)\, e^{j\,\beta\, z_0}\, dz_0 \tag{28}$$

where the tilde indicates the variable in the Fourier transformed domain. Then the first-order wave equations (14) and (15) are transformed into

$$\begin{aligned}\widetilde{L}_a(\beta)\, \widetilde{\phi}_{a,1}(x,\beta,z_{s,1}) =& u_{a,0}(x)\, \delta(\beta - \beta_{a,0}) \left({\beta_{a,0}}^2 + j2k_s \frac{\partial}{\partial z_1} \right) a_{a,0}(z_{s,1}) \\ &- {k_s}^2 \Delta\varepsilon_a(x)\, u_{b,0}(x)\, \delta(\beta - \beta_{b,0})\, a_{b,0}(z_{s,1}) \qquad (29) \\ \widetilde{L}_b(\beta)\, \widetilde{\phi}_{b,1}(x,\beta,z_{s,1}) =& u_{b,0}(x)\, \delta(\beta - \beta_{b,0}) \left({\beta_{b,0}}^2 + j2k_s \frac{\partial}{\partial z_1} \right) a_{b,0}(z_{s,1}) \\ &- {k_s}^2 \Delta\varepsilon_b(x)\, u_{a,0}(x)\, \delta(\beta - \beta_{a,0})\, a_{a,0}(z_{s,1}) \\ &- k_s^2 \Delta\varepsilon_g(x) u_{b,0}(x) \widetilde{g}(\beta - \beta_{b,0}, z_{s,1}) a_{b,0}(z_{s,1}) \qquad (30)\end{aligned}$$

where $\delta(\beta - \beta_{\nu,0})$ denotes the delta function and $\widetilde{L}_\nu(\beta)$ is the linear operator defined as

$$\widetilde{L}_\nu(\beta) \equiv \frac{d^2}{dx^2} - 2k_s\beta + {k_s}^2 \Delta\varepsilon_\nu(x). \tag{31}$$

The linear differential equations (29) and (30) are singular at $\beta = \beta_{a,0}$ and $\beta = \beta_{b,0}$, and the solutions are allowed only when the solvability conditions are satisfied. Fredholm's alternative theorem [11] is used to derive the solvability conditions. Let $h_\nu(x;\beta)$ be the solution to the homogeneous part of Eqs. (29) and (30) which is given by

$$h_\nu(x;\beta) = \begin{cases} u_{\nu,0}(x) & \text{for } \beta = \beta_{\nu,0} \\ 0 & \text{for } \beta \neq \beta_{\nu,0} \end{cases}. \tag{32}$$

Then the theorem claims that the right hand sides of Eqs. (29) and (30) should be orthogonal to $h_a(x;\beta)$ and $h_b(x;\beta)$, respectively, since the operator $\widetilde{L}_\nu(\beta)$ is self-adjoint. These are the solvability conditions which are written as follows:

$$\int_{-\infty}^{\infty} h_a^*(x;\beta)\left[u_{a,0}(x)\,\delta(\beta-\beta_{a,0})\left({\beta_{a,0}}^2 + j2k_s\frac{\partial}{\partial z_1}\right)a_{a,0}(z_{s,1})\right.$$
$$\left. -{k_s}^2\Delta\varepsilon_a(x)\,u_{b,0}(x)\,\delta(\beta-\beta_{b,0})\,a_{b,0}(z_{s,1})\right]dx = 0 \tag{33}$$

$$\int_{-\infty}^{\infty} h_b^*(x;\beta)\left[u_{b,0}(x)\,\delta(\beta-\beta_{b,0})\left({\beta_{b,0}}^2 + j2k_s\frac{\partial}{\partial z_1}\right)a_{b,0}(z_{s,1})\right.$$
$$-\,{k_s}^2\Delta\varepsilon_b(x)\,u_{a,0}(x)\,\delta(\beta-\beta_{a,0})\,a_{a,0}(z_{s,1})$$
$$\left. -{k_s}^2\Delta\varepsilon_g(x)\,u_{b,0}(x)\,\widetilde{g}(\beta-\beta_{b,0},z_{s,1})\,a_{b,0}(z_{s,1})\right]dx = 0 \tag{34}$$

where the asterisk indicates the complex conjugate. Since $\beta_{a,0} \neq \beta_{b,0}$, the solvability conditions (33) and (34) lead to

$$\frac{\partial a_{\nu,0}(z_{s,1})}{\partial z_1} = -j\,\beta_{\nu,1}\,a_{\nu,0}(z_{s,1}) \quad (\nu = a,b) \tag{35}$$

with

$$\beta_{\nu,1} = -\frac{{\beta_{\nu,0}}^2}{2k_s} \tag{36}$$

where $\beta_{\nu,1}$ gives the first-order correction to the propagation constant. When the solvability conditions (35) are satisfied, the solutions to the first-order equations (29) and (30) are obtained in the original space domain as follows:

$$\phi_{a,1}(x,z_{s,0}) = a_{a,1}(z_{s,1})u_{a,0}(x)e^{-j\,\beta_{a,0}\,z_0} + a_{b,0}(z_{s,1})u_{ab,1}(x)e^{-j\,\beta_{b,0}z_0} \tag{37}$$

$$\phi_{b,1}(x,z_{s,0}) = a_{b,1}(z_{s,1})u_{b,0}(x)\,e^{-j\beta_{b,0}z_0} + a_{a,0}(z_{s,1})u_{ba,1}(x)\,e^{-j\beta_{a,0}z_0}$$
$$+ a_{b,0}(z_{s,1})u_{g,1}^{(+1)}(x)e^{-j(\beta_{a,0}z_0+\Delta\beta z_n)}$$
$$+ a_{b,0}(z_{s,1})u_{g,1}^{(-1)}(x)e^{j[(\beta_{a,0}-2\beta_{b,0})z_0+\Delta\beta z_n]} \tag{38}$$

with

$$u_{\nu\mu,1}(x) = -{k_s}^2 \widetilde{L}_{\nu}^{-1}(\beta_{\mu,0})\, \Delta\varepsilon_{\nu}(x)\, u_{\mu,0}(x) \quad (\nu, \mu = a, b \text{ and } \nu \neq \mu) \tag{39}$$

$$u_{g,1}^{(+1)}(x) = -\frac{1}{2}{k_s}^2 \widetilde{L}_{b}^{-1}(\beta_{a,0})\, \Delta\varepsilon_{g}(x)\, u_{b,0}(x) \tag{40}$$

$$u_{g,1}^{(-1)}(x) = -\frac{1}{2}{k_s}^2 \widetilde{L}_{b}^{-1}(-\beta_{a,0} + 2\beta_{b,0})\, \Delta\varepsilon_{g}(x)\, u_{b,0}(x) \tag{41}$$

where $a_{\nu,1}(z_{s,1})$ denotes the modal amplitude in the first-order and $\widetilde{L}_{\nu}^{-1}(\beta)$ is the inverse operator of $\widetilde{L}_{\nu}(\beta)$ with $\beta \neq \beta_{\nu,0}$. In the right hand sides of Eqs. (37) and (38), the first terms are the solutions to the homogeneous parts of Eqs. (29) and (30) and other terms represent the particular solutions. Since the particular solutions of Eqs. (29) and (30) are easily obtained by a standard procedure for ordinary differential equations, they are expressed in terms of the inverse operator $\widetilde{L}_{\nu}^{-1}(\beta)$ to simplify the notations.

Performing the same analytical steps as described above, we obtain the solvability conditions for the second-order wave equations as follows:

$$\begin{aligned} \frac{\partial a_{a,1}(z_{s,1})}{\partial z_1} + \frac{\partial a_{a,0}(z_{s,1})}{\partial z_2} =& - j\, \beta_{a,1}\, a_{a,1}(z_{s,1}) - j\, \beta_{a,2}\, a_{a,0}(z_{s,1}) \\ & - j\, \xi_{ab,2}\, a_{b,0}(z_{s,1})\, e^{-j\,\Delta\beta\, z_n} \end{aligned} \tag{42}$$

$$\begin{aligned} \frac{\partial a_{b,1}(z_{s,1})}{\partial z_1} + \frac{\partial a_{b,0}(z_{s,1})}{\partial z_2} =& - j\, \beta_{b,1}\, a_{b,1}(z_{s,1}) - j\, \beta_{b,2}\, a_{b,0}(z_{s,1}) \\ & - j\, \xi_{ba,2}\, a_{a,0}(z_{s,1})\, e^{j\,\Delta\beta\, z_n} \end{aligned} \tag{43}$$

with

$$\beta_{a,2} = -\,\frac{\beta_{a,0}\beta_{a,1}}{k_s} + \frac{k_s}{2}\int_{-\infty}^{\infty} \Delta\varepsilon_a(x)\, u_{a,0}^*(x)\, u_{ba,1}(x)\, dx \tag{44}$$

$$\begin{aligned} \beta_{b,2} =& -\,\frac{\beta_{b,0}\beta_{b,1}}{k_s} + \frac{k_s}{2}\int_{-\infty}^{\infty} \Delta\varepsilon_b(x)\, u_{b,0}^*(x)\, u_{ab,1}(x)\, dx \\ & + \frac{k_s}{4}\int_{-\infty}^{\infty} \Delta\varepsilon_g(x)\, u_{b,0}^*(x) \left(u_{g,1}^{(+1)}(x) + u_{g,1}^{(-1)}(x) \right) dx \end{aligned} \tag{45}$$

$$\xi_{ab,2} = \frac{k_s}{2}\int_{-\infty}^{\infty} \Delta\varepsilon_a(x)\, u_{a,0}^*(x)\, u_{g,1}^{(+1)}(x)\, dx \tag{46}$$

$$\xi_{ba,2} = \frac{k_s}{4}\int_{-\infty}^{\infty} \Delta\varepsilon_g(x)\, u_{b,0}^*(x)\, \left(u_{a,0}(x) + u_{ba,1}(x)\right)\, dx. \tag{47}$$

The higher-order ($n \geq 3$) wave equations in Eqs. (14) and (15) are solved by following the same analytical procedure described above.

The solvability conditions in each order of perturbation are combined to deduce the required coupled-mode equations. Since the modal amplitudes $a_{\nu,n}(z_{s,1})$ ($\nu = a, b; n = 0, 1$) are associated with the common profile function $u_{\nu,0}(x)$, new modal amplitudes are defined by

$$a_\nu(z_{s,1}) = a_{\nu,0}(z_{s,1}) + \delta\, a_{\nu,1}(z_{s,1}) + O(\delta^2) \quad (\nu = a, b). \tag{48}$$

Using Eqs. (35), (42), and (48), the evolution of $a_a(z_{s,1})$ in the original space scale z can be expressed as

$$\begin{aligned}\frac{d\, a_a(z_{s,1})}{dz} =& \delta \frac{\partial\, a_{a,0}(z_{s,1})}{\partial z_1} + \delta^2 \left(\frac{\partial\, a_{a,1}(z_{s,1})}{\partial z_1} + \frac{\partial\, a_{a,0}(z_{s,1})}{\partial z_2} \right) + O(\delta^3) \\ =& - j\, \delta\, \beta_{a,1} \Big(a_{a,0}(z_{s,1}) + \delta\, a_{a,1}(z_{s,1}) \Big) - j\, \delta^2\, \beta_{a,2}\, a_{a,0}(z_{s,1}) \\ & - j\, \delta^2\, \xi_{ab,2}\, a_{b,0}\, e^{-j\,\Delta\beta\, z_n} + O(\delta^3) \\ =& - j\, \left(\delta\, \beta_{a,1} + \delta^2\, \beta_{a,2} \right) a_a(z_{s,1}) - j \delta^2 \xi_{ab,2} a_b\, e^{-j\,\Delta\beta\, z_n} + O(\delta^3).\end{aligned} \tag{49}$$

Similarly, from Eqs. (35), (43), and (48) we have

$$\frac{d\, a_b(z_{s,1})}{dz} = -j\, \left(\delta\, \beta_{b,1} + \delta^2\, \beta_{b,2} \right) a_b(z_{s,1}) - j\, \delta^2\, \xi_{ba,2}\, a_a\, e^{j\,\Delta\beta\, z_n} + O(\delta^3). \tag{50}$$

Omitting the terms of higher order denoted $O(\delta^3)$, Eqs. (49) and (50) yield the coupled-mode equations in the second-order of perturbations as follows:

$$\frac{d\, a_a(z)}{dz} = -j\, (\beta_{a,1} + \beta_{a,2})\, a_a(z) - j\, \xi_{ab,2}\, a_b(z)\, e^{-j\,\Delta\beta\, z} \tag{51}$$

$$\frac{d\, a_b(z)}{dz} = -j\, (\beta_{b,1} + \beta_{b,2})\, a_b(z) - j\, \xi_{ba,2}\, a_a(z)\, e^{j\,\Delta\beta\, z} \tag{52}$$

where the multiple space-scale $z_{s,1}$ is transformed back into the original space-scale z by letting $\delta = 1$. From the self-phase modulation coefficients in Eqs. (51) and (52), we obtain the phase-matching condition with the second-order accuracy

$$(\beta_{a,0} + \beta_{a,1} + \beta_{a,2}) - (\beta_{b,0} + \beta_{b,1} + \beta_{b,2}) = \frac{2\pi}{p}. \tag{53}$$

When the condition is satisfied at a certain wavelength of optical wave, a complete power transfer from one waveguide to the other is attained after a propagation length

$$l = \frac{\pi}{2\sqrt{\xi_{ab,2}\,\xi_{ba,2}}} \tag{54}$$

where l is referred to the coupling length of the coupler.

3. NUMERICAL EXAMPLES

To validate the proposed coupled-mode theory, the power transfer characteristics in the grating-assisted directional coupler as shown in Fig. 1 are analyzed and the results are compared with those obtained by more rigorous numerical solution method. The values of geometrical parameters are chosen as $d_5 - d_4 = 0.35\ \mu\mathrm{m}$, $d_4 - d_3 = 0.6\ \mu\mathrm{m}$, $d_3 - d_2 = 0.05\ \mu\mathrm{m}$, $d_2 - d_1 = 0.2\ \mu\mathrm{m}$, $n_a = n_b = 3.4$, $n_s = 3.2$, ${n_c}^2 = ({n_b}^2 + {n_s}^2)/2$, ${n_d}^2 = ({n_b}^2 - {n_s}^2)/2$. When the wavelength of optical field is specified, the propagation constants $\beta_{\nu,0}$ to $\beta_{\nu,2}$ are calculated from Eqs. (24), (25), (36), (44), and (45). The center wavelength for the optical filtering is chosen to be $\lambda_0 = 0.83\ \mu\mathrm{m}$. Equation (53) determines the grating period $p = 21.42\ \mu\mathrm{m}$ which satisfies the phase-matching condition at $\lambda = \lambda_0$. Figure 2 shows the wavelength response of the output power in waveguide b at $z = 1.66\,\mathrm{mm}$ for the excitation of waveguide a at $z = 0$. The propagation length $z = 1.66\,\mathrm{mm}$ corresponds to the coupling length determined by Eq. (54) for the center wavelength λ_0. The solid line shows the result of the coupled-mode analysis. For the same configurations of waveguides, we have performed a numerical analysis using the Fourier series expansion method [12] for optical waveguides. The original waveguide system was assumed to be in a periodic cell with period $12\,\lambda_0$ in the x direction, the optical fields were expanded by a Fourier series of 100 terms, and the periodic waveguide transitions in the z direction were approximated by a series of 2325 step transitions with each step length $p/30$. The results of the numerical analysis are plotted in Fig. 2 by crosses. It is seen that the wavelength response predicted by the present theory are in good agreement with that of the Fourier series expansion method.

For the sake of comparison, we have calculated the power transfer characteristics using the well-known improved coupled-mode theory

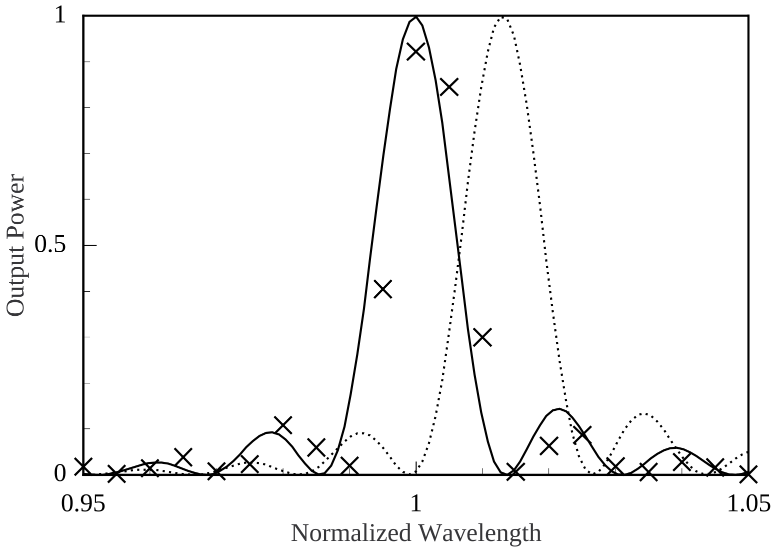

Figure 2. Comparison of the wavelength response of the output power in waveguide b at the coupling length $l = 1.66\,\text{mm}$ as the function of normalized wavelength λ/λ_0, where $d_5 - d_4 = 0.35$ μm, $d_4 - d_3 = 0.6$ μm, $d_3 - d_2 = 0.05$ μm, $d_2 - d_1 = 0.2$ μm, $n_a = n_b = 3.4$, $n_s = 3.2$, ${n_c}^2 = ({n_b}^2 + {n_s}^2)/2$, ${n_d}^2 = ({n_b}^2 - {n_s}^2)/2$, $p = 21.42$ μm, and the center wavelength $\lambda_0 = 0.83$ μm. The solid and dotted curves are obtained by the present formulation and the improved coupled-mode theory, respectively, and the crosses are the numerical results of the Fourier series expansion method [12].

[13] which is widely used in the design of optical waveguide components. To evaluate accurately the coupling coefficients, the two waveguides with relative permittivity distributions ${n_s}^2(1 + \Delta\varepsilon_a(x))$ and ${n_s}^2(1 + \Delta\varepsilon_b(x))$ and the associated eigenmode fields were adopted as the basis [13] of the coupled-mode formulation. The result calculated for the same grating period $p = 21.42$ μm and propagation length $z = 1.66\,\text{mm}$ as described above is shown in Fig. 2 by the dotted line. It is seen that the wavelength response in the power transfer is different from those of the rigorous numerical analysis and the present coupled-mode analysis. This is because the self-phase modulation terms in the improved coupled-mode theory are short of accuracy for obtaining the phase-matching condition. The precise coupled-mode analysis presented here suggests that for an accurate treatment of grating-assisted

directional couplers, the effects of grating and adjacent waveguide on the coupling should be incorporated in proper manner by taking into account the order of perturbation.

4. CONCLUSION

We have presented an accurate nonorthogonal coupled-mode theory for a grating-assisted directional coupler using the singular perturbation technique. In this approach, the optical fields in the coupler are represented by a linear combination of para-axial wave fields for each isolated waveguide with a reference wavenumber of the surrounding cladding. The para-axial wave fields are expanded using the multiple space-scales and solved so that they satisfy the wave equation for the original coupled structure in the respective order of perturbation. The manipulation of the perturbation analysis is straightforward. From the analysis of up to second-order perturbation, accurate coupled-mode equations based on the eigenmodes of each waveguide in isolation have been derived. The proposed coupled-mode theory has been used to analyze two nonidentical coupled waveguides attached with a sinusoidal grating layer. The calculated wavelength response in the power transfer shows a close agreement with that obtained by a direct numerical analysis using the Fourier series expansion method.

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