Progress In Electromagnetics Research, *PIER 26, 1–41, 2000*

WAVE SCATTERING BY PERIODIC SURFACE AT LOW GRAZING ANGLES: SINGLE GRAZING MODE

M. I. Charnotskii

Science and Technology Corporation
and Environmental Technology Laboratory
NOAA/ERL
325 Broadway, Boulder CO 80303, USA

1. Introduction
2. Integral Equations for the Scattering Problem
2.1 Principal Equations
2.2 Conventional Perturbation Solution
2.3 Low-grazing Scattering Discrepancy
3. One-dimensional Periodic Surface
3.1 Principal Equations
3.2 Conventional Perturbation Theory
3.3 CP Solution in the Presence of a Grazing Mode
4. New Perturbation Expansion in the Presence of a Grazing Mode: GP
4.1 Nonspecular Grazing Order
4.2 New Perturbation Expansion for the Low Grazing Incidence
4.3 Discussion
5. Uniform Perturbation Approximation
5.1 "Natural" Form of GP Approximations
5.2 Uniform Perturbation Approximations
6. Conclusion
Appendix. Details of the Numerical Solution
References

1. INTRODUCTION

We discuss here the specifics of wave scattering by a plane in an average, periodic, perfectly conducting surface when either the incidence or the scattering grazing angle are small. [1] This situation is of significant practical interest for ground-based, active and passive radar studies of the ocean surface, and for long-path propagation along the ocean surface.

For the finite impedance surfaces and the Dirichlet problem, both the numerical results of Barrick [1] for the periodic surface, and the general analytical results of Tatarskii and Charnotskii [2, 3] predict that the continuous scattering amplitude is proportional to the small grazing angle. This result is also supported by the Conventional Perturbation (CP) theory. There is however, a certain controversial aspect when a Neuman problem is considered:

- The numerical results of Barrick [1] show the same type of grazing behavior as for the finite impedance and Dirichlet surfaces. Besides that, Barrick asserts that the reflection coefficient of the slightly rough Neuman surface changes its sign from $+1$ to -1 when the grazing incidence angle tends to zero. This conclusion is supported by theoretical results [5–7] for an average field above a statistically rough surface.

The analytical results of Tatarskii and Charnotskii [2, 3] predict, however, that for a plane in an average, but otherwise arbitrary, Neuman rough surface, the continuous scattering amplitude saturates at a finite limit when the grazing angle approaches zero. This conclusion is supported by the CP theory. Other supporting evidence comes from the model problem of scattering by a single bump at the plane [8], which can be examined using the reflection principle [9].

Scattering by periodic surface with periodic impedance boundary condition was investigated in a series of papers by Urusovskii [10–13]. Principal technique was based on the reduction of the second-kind integral equation for the surface fields to the infinite system of linear algebraic equations. This approach now is almost a standard for this class of problems [14]. Later Urusovskii and Mashashvili [15–16] used this technique to study wave scattering by a periodic interface between two media. Cramer's theorem was used in [11, 12, 15, 16]

[1] A situation where there are two low-grazing modes, including an important backscatter case, is discussed in the accompanying paper [4].

to present a formal solution for the surface fields as a ratio of two infinite determinants. Further approximation presented in [11] and [16], included a special form of perturbation approximation for each of determinants in the Cramer's formula. Unlike our present approach, this type of approximation does not represent a rigorous expansion in terms of any small parameters.

The problem of grazing orders was addressed in [11] and [16]. Urusovskii indicated in [11] that amplitude of a grazing order can exceed the amplitude of the incident wave, and derived a formula for the amplitude of the grazing order that coincides with our Grazing Perturbation (GP) result (4.4). In the limit of large dielectric constant of the second medium formulas (12) and (13) of [16] are also very similar to the GP results (4.4) and (4.7). The value S introduced in [16] is proportional to the effective surface impedance (4.5).

Two recent papers by Urusovskii [17, 18] address wave scattering by periodic and rough surfaces with Dirichlet boundary condition, and use extinction theorem and integral equation of the first kind as a principal approach. The problems associated with the presence of grazing modes neither arises nor discussed there.

Rosich and Wait, [19], developed rigorous perturbation series for wave scattering by periodic surface with impedance boundary condition. Their derivation was based on the Rayleigh hypothesis. Although Rosich and Wait did not discuss it, their results clearly indicate the failure of the perturbation series for the zero impedance, which corresponds to the Neuman boundary condition.

Vinogradov, Zorev et. al, [20] discussed reflection of the x-rays by rough surface at low grazing angles. However, since the media permittivities do not differ significantly for the x-rays, the effects addressed in the present paper do not reveal themselves.

It is reasonable to expect the problem to be simplified by the fact that there is an additional small parameter besides the small angle. For example, one can expect a quick and easy solution when roughness heights are small. However, when the small incidence/scattering angle is combined with the perturbation theory we face a problem with two small parameters: small scattering or incidence angle $\alpha \ll 1$, and small roughness height in the wavelength scale, $kh \ll 1$. If scattering amplitude is an analytical function of these two parameters at zero, then there exists a universal limiting behavior of the scattering amplitude when $\alpha \to 0$ and $kh \to 0$. This limit would not depend on the specific path by which the origin at the (α, kh) plane is approached. It would appear that for the Dirichlet and finite impedance problems this is the case.

When scattering amplitude is not an analytic function of α and kh at zero, it is possible to obtain the different limits when $\alpha \to 0$ and $kh \to 0$, depending on what kind of relation exists between α and kh. We will show that for a Neuman scattering surface with a finite number of propagating diffraction orders this is indeed the case, and for the scattering amplitudes we have:

$$\lim_{\alpha \to 0} \left(\lim_{kh \to 0} \right) \neq \lim_{kh \to 0} \left(\lim_{\alpha \to 0} \right). \tag{1.1}$$

The left-hand part of this formula corresponds to the conventional perturbation (CP) theory, where the small-height assumption is made first, and then the small-angle case is analyzed. The major result of the paper is the development of a new perturbation expansion that corresponds to the right-hand part of (1.1). We call it the GP expansion for brevity.

In Section 2 we introduce the major integral equations for scalar wave scattering by a plane on an average rough surface with Neuman boundary condition. We present the details of the existing discrepancy in the small-grazing-angle behavior predictions by Barrick [1] and Tatarskii/Charnotskii [2, 3].

In Section 3 we introduce the necessary formalism for the one-dimensional periodic surface scattering, and derive the principal equations of our numerical analysis. We discuss the intrinsic limitations of the CP theory in the presence of one low-grazing mode. We show that incorporation of the higher-order term extends the validity domain of the CP formulas, and estimate the validity conditions of CP for one low-grazing mode case.

In Section 4 we develop a new grazing perturbation theory that is valid in the presence of a single grazing mode. We discuss the validity domain of the GP formulas and the role of the higher-order GP terms. We introduce the effective surface impedance and Brewster angle and show that CP and GP together form a complete set of asymptotes with the effective Brewster angle being the border between them.

In Section 5 we propose two sets of formulas that represent the Uniform Perturbation (UP) approximation. UP is valid for the small roughness heights but arbitrary values of the scattering angles.

We present an overview of our major results in the Conclusion. We discuss the limitations associated with surface periodicity and some implications for the nonperiodic, rough-surface, scattering problem.

The Appendix provides some details on our numerical technique.

2. INTEGRAL EQUATIONS FOR THE SCATTERING PROBLEM

2.1 Principal Equations

Consider the 3-D scattering problem for the scattering of a plane wave:

$$E_{IN}(\mathbf{r}, z, \mathbf{q}_0) = \exp(i\mathbf{r}\cdot\mathbf{q}_0 + \nu_0 z), \quad \nu_0^2 = k^2 - q_0^2 \tag{2.1}$$

by the rough surface $\Sigma : \{z = \varsigma(\mathbf{r})\}$.

A scattered wave satisfies the homogeneous Helmholtz's equation

$$[k^2 + \nabla^2]E_{SC}(\mathbf{r}, z, \mathbf{q}_0) = 0 \tag{2.2}$$

in the region $z > \varsigma(\mathbf{r})$, and the Neuman boundary condition

$$\left.\frac{\partial E_{SC}(\mathbf{r}, z, \mathbf{q}_0)}{\partial n(\mathbf{r})} + \frac{\partial E_{IN}(\mathbf{r}, z, \mathbf{q}_0)}{\partial n(\mathbf{r})}\right|_{z=\varsigma(\mathbf{r})} = 0 \tag{2.3}$$

at the surface. For a one-dimensional perfectly conducting surface, this type of boundary condition corresponds to the vertically polarized incident and scattered fields. The scattered field above the highest point of the surface has the form:

$$E_{SC}(\mathbf{r}, z, \mathbf{q}_0) = \iint \frac{d\mathbf{q}}{k\nu(\mathbf{q})} S(\mathbf{q}, \mathbf{q}_0) \exp[i\mathbf{q}\cdot\mathbf{r} + i\nu(\mathbf{q})z], \tag{2.4}$$

where $\nu^2(\mathbf{q}) = k^2 - q^2$. This definition of the scattering amplitude $S(\mathbf{q}, \mathbf{q}_0)$ corresponds to the angular density of the scattered field. A pair of integral equations for the surface field and scattering amplitude can be derived from Green's formula applied to the region $z^* > z > \varsigma(\mathbf{r})$ and a pair of functions, $E_{TOT}(\mathbf{r}, z, \mathbf{q}_0)$ and $\exp[i\mathbf{q}\cdot\mathbf{r} \pm i\nu(\mathbf{q})z]$, and has the following form:

$$\iint d\mathbf{r}E(\mathbf{r},\mathbf{q}_0)[\nu(\mathbf{q})+\mathbf{q}\cdot\nabla\varsigma(\mathbf{r})]\exp[-i\mathbf{q}\cdot\mathbf{r}+i\nu(\mathbf{q})\varsigma(\mathbf{r})] = 8\pi^2\nu_0\delta(\mathbf{q}-\mathbf{q}_0), \tag{2.5}$$

$$\iint d\mathbf{r}E(\mathbf{r},\mathbf{q}_0)[\nu(\mathbf{q})-\mathbf{q}\cdot\nabla\varsigma(\mathbf{r})]\exp[-i\mathbf{q}\cdot\mathbf{r}-i\nu(\mathbf{q})\varsigma(\mathbf{r})] = \frac{8\pi^2}{k}S(\mathbf{q},\mathbf{q}_0). \tag{2.6}$$

Here we introduced the notation $E(\mathbf{r},\mathbf{q}_0)\equiv E_{IN}(\mathbf{r},\varsigma(\mathbf{r}),\mathbf{q}_0)+E_{SC}(\mathbf{r},\varsigma(\mathbf{r}),\mathbf{q}_0)$ for the surface value of the total field.

Integration by part of the terms with $\nabla\varsigma(\mathbf{r})$ in (2.5) and (2.6), allows us to present these equations in the alternative form:

$$\nu(\mathbf{q})\widetilde{E}(\mathbf{q},\mathbf{q}_0)+i\iint d\mathbf{p}\widetilde{E}(\mathbf{q},\mathbf{q}_0)(k^2-\mathbf{p}\cdot\mathbf{q})P[\mathbf{q}-\mathbf{p},\nu(\mathbf{q})] = 2\nu_0\delta(\mathbf{q}-\mathbf{q}_0); \tag{2.7}$$

$$\nu(\mathbf{q})\widetilde{E}(\mathbf{q},\mathbf{q}_0)-i\iint d\mathbf{p}\widetilde{E}(\mathbf{q},\mathbf{q}_0)(k^2-\mathbf{p}\cdot\mathbf{q})P[\mathbf{q}-\mathbf{p},-\nu(\mathbf{q})] = \frac{2}{k}S(\mathbf{q},\mathbf{q}_0). \tag{2.8}$$

Where we use Fourier transform of the surface field (surface source function):

$$\widetilde{E}(\mathbf{p},\mathbf{q}_0)\equiv\frac{1}{4\pi^2}\iint d\mathbf{r}E(\mathbf{r},\mathbf{q}_0)\exp(-i\mathbf{p}\cdot\mathbf{r}), \tag{2.9}$$

and introduce a modified characteristic function of heights:

$$P(\mathbf{P},\pm\nu)\equiv\frac{1}{4\pi^2}\iint d\mathbf{r}\frac{\exp[\pm i\nu\varsigma(\mathbf{r})]-1}{\pm i\nu}\exp(-i\mathbf{p}\cdot\mathbf{r}). \tag{2.10}$$

Note that when $\nu\varsigma\ll 1$, we have:

$$P(\mathbf{p},\pm\nu)\approx\widetilde{\varsigma}(\mathbf{p})\pm i\frac{\nu}{2}\iint d\mathbf{p}\widetilde{\varsigma}(\mathbf{p}-\mathbf{p}')\widetilde{\varsigma}(\mathbf{p}'), \tag{2.11}$$

where, $\widetilde{\varsigma}(\mathbf{q})$ is the Fourier transform of roughness heights $\varsigma(\mathbf{r})=\iint d\mathbf{q}\widetilde{\varsigma}(\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{r})$.

Equation (2.7) allows to calculate the source function $\hat{E}(\mathbf{q},\mathbf{q}_0)$, and then equation (2.8) is used to find the scattering amplitude $S(\mathbf{q},\mathbf{q}_0)$. Note also that, as was discussed in [21], the roughness height enters our equations only in the exponential form.

For the Dirichlet problem using a similar procedure one can derive the following set of equations:

$$\widetilde{F}(\mathbf{q},\mathbf{q}_0) + i\nu(\mathbf{q})\iint d\mathbf{p}\widetilde{F}(\mathbf{p},\mathbf{q}_0)P[\mathbf{q}-\mathbf{p},\nu(\mathbf{q})] = 2\nu_0\delta(\mathbf{q}-\mathbf{q}_0) \quad (2.12)$$

$$-\widetilde{F}(\mathbf{q},\mathbf{q}_0) + i\nu(\mathbf{q})\iint d\mathbf{p}\widetilde{F}(\mathbf{p},\mathbf{q}_0)P[\mathbf{q}-\mathbf{p},-\nu(\mathbf{q})] = \frac{2}{k}S(\mathbf{q},\mathbf{q}_0) \quad (2.13)$$

where $\widetilde{F}(\mathbf{q},\mathbf{q}_0)$ is the Fourier transform of the surface source function:

$$F(\mathbf{r},\mathbf{q}_0) \equiv i\frac{\partial E_{TOT}[\mathbf{r},\varsigma(\mathbf{r}),\mathbf{q}_0]}{\partial n(\mathbf{r})}\sqrt{1+|\nabla\varsigma(\mathbf{r})|^2}.$$

2.2 Conventional Perturbation Solution

A conventional perturbation solution seeks the surface field and scattering amplitude in the form of the Taylor functional series in $\varsigma(\mathbf{r})$

$$\begin{aligned}\widetilde{E}(\mathbf{p},\mathbf{q}_0) &= \widetilde{E}^{(0)}(\mathbf{p},\mathbf{q}_0) + \widetilde{E}^{(1)}(\mathbf{q},\mathbf{q}_0) + \widetilde{E}^{(2)}(\mathbf{p},\mathbf{q}_0) + \dots \\ S(\mathbf{p},\mathbf{q}_0) &= S^{(0)}(\mathbf{p},\mathbf{q}_0) + S^{(1)}(\mathbf{q},\mathbf{q}_0) + S^{(2)}(\mathbf{p},\mathbf{q}_0) + \dots\end{aligned} \quad (2.14)$$

Substitution of (2.14) into (2.10) and (2.11) and equating of the similar orders in ς provide the following results for up to the second order in heights:

$$\begin{aligned}\widetilde{E}(\mathbf{q},\mathbf{q}_0) =& 2\delta(\mathbf{q}-\mathbf{q}_0) - 2i\widetilde{\varsigma}(\mathbf{q}-\mathbf{q}_0)\frac{(k^2-\mathbf{q}\cdot\mathbf{q}_0)}{\nu(\mathbf{q})}\frac{1}{\nu(\mathbf{q})} \\ &\iint d\mathbf{p}\widetilde{\varsigma}(\mathbf{p}-\mathbf{q}_0)\widetilde{\varsigma}(\mathbf{q}-\mathbf{p})\Big\{\nu(\mathbf{q})(k^2-\mathbf{q}\cdot\mathbf{q}_0) \\ &- 2\frac{(k^2-\mathbf{p}\cdot\mathbf{q}_0)(k^2-\mathbf{q}\cdot\mathbf{p})}{\nu(\mathbf{p})}\Big\},\end{aligned} \quad (2.15)$$

and

$$\begin{aligned}S(\mathbf{q},\mathbf{q}_0) =& k\nu_0\delta(\mathbf{q}-\mathbf{q}_0) - 2ik\widetilde{\varsigma}(\mathbf{q}-\mathbf{q}_0)(k^2-\mathbf{q}\cdot\mathbf{q}_0) \\ &- 2k\iint d\mathbf{p}\widetilde{\varsigma}(\mathbf{p}-\mathbf{q}_0)\widetilde{\varsigma}(\mathbf{q}-\mathbf{p})\frac{(k^2-\mathbf{p}\cdot\mathbf{q}_0)(k^2-\mathbf{q}\cdot\mathbf{p})}{\nu(\mathbf{p})}.\end{aligned} \quad (2.16)$$

For the Dirichlet scattering problem (horizontal polarization), we have the standard result:

$$S(\mathbf{q},\mathbf{q}_0) = -k\nu_0\delta(\mathbf{q}-\mathbf{q}_0) + 2ik\nu_0\nu(\mathbf{q})\widetilde{\varsigma}(\mathbf{q}-\mathbf{q}_0) + 2k\nu_0\nu(\mathbf{q})\iint d\mathbf{p}\widetilde{\varsigma}(\mathbf{p}-\mathbf{q}_0)\widetilde{\varsigma}(\mathbf{q}-\mathbf{p})\nu(\mathbf{p}). \quad (2.17)$$

The notable feature of CP formulas for the Neuman problem (2.15), (2.16) is a singularity of at $|\mathbf{q}| = k$. It appears that it will be present in all the higher-order terms of the series as well. However, this is an integrable singularity, and it causes no major problems.

2.3 Low-grazing Scattering Discrepancy

Our analytical capabilities are quite limited for the nonperturbation case due, to the lack of an closed form solution to the problem. However, we are still able to draw some general conclusions regarding the low-grazing-angle behavior of scattered field by examining the principal integral equations (2.7) and (2.8). If we subtract (2.7) from (2.8), the scattering amplitude can be presented in the following form:

$$S(\mathbf{q},\mathbf{q}_0) = k\nu_0\delta(\mathbf{q}-\mathbf{q}_0) - \frac{ik}{2}\iint d\mathbf{p}\widetilde{E}(\mathbf{p},\mathbf{q}_0)(k^2-\mathbf{p}\cdot\mathbf{q}) \{P[\mathbf{q}-\mathbf{p},-\nu(\mathbf{q})]P[\mathbf{q}-\mathbf{p},-\nu(\mathbf{q})]\}. \quad (2.18)$$

However, if we add (2.7) to (2.8), then the following presentation of the scattering amplitude can be derived:

$$S(\mathbf{q},\mathbf{q}_0) = -k\nu_0\delta(\mathbf{q}-\mathbf{q}_0) + k\nu(\mathbf{q})\widetilde{E}(\mathbf{q},\mathbf{q}_0) + \frac{ik}{2}\iint d\mathbf{p}\widetilde{E}(\mathbf{p},\mathbf{q}_0) (k^2-\mathbf{p}\cdot\mathbf{q})\{P[\mathbf{q}-\mathbf{p},-\nu(\mathbf{q})]P[\mathbf{q}-\mathbf{p},-\nu(\mathbf{q})]\} \quad (2.19)$$

Equations (2.18) and (2.19) are both exact and give the same values of the scattering amplitude for any conditions. If we formally let the scattering angle $\alpha \to 0$ in (2.18), which implies that $\mathbf{q} \to k\hat{\mathbf{q}}$, and $\nu(\mathbf{q}) \approx k\alpha \ll k$, we have:

$$S(k\hat{\mathbf{q}},\mathbf{q}_0) = k\nu_0\delta(k\hat{\mathbf{q}}-\mathbf{q}_0) - ik^2\iint d\mathbf{p}\widetilde{E}(\mathbf{p},\mathbf{q}_0)(k-\mathbf{p}\cdot\hat{\mathbf{q}})\widetilde{\varsigma}(k\hat{\mathbf{q}}-\mathbf{p}). \quad (2.20)$$

Since the first term in the right-hand part of (2.20) represents the reflected field for the plane Neuman surface, the integral in the right-hand part of (2.20) represents the diffusive component of the scattered field. This means that $S(\mathbf{q},\mathbf{q}_0) \propto \alpha^0$ at small grazing angles. Note also that this result is not limited by the small heights assumption, but is supported by the perturbation theory results (2.16). This conclusion was presented by Tatarskii and Charnotskii [2, 3].

Another point of view was presented by Barrick in [1]. Actually his analysis was done for a periodic surface, but we present the general case interpretation here. If we formally let $\alpha \to 0$ in (2.19) then:

$$\begin{aligned} S(k\hat{\mathbf{q}},\mathbf{q}_0) = &- k\nu_0\delta(k\hat{\mathbf{q}}-\mathbf{q}_0) + k^2\sin\alpha\Big\{\widetilde{E}(k\hat{\mathbf{q}},\mathbf{q}_0) \\ &-\frac{k}{2}\iint d\mathbf{p}\widetilde{E}(\mathbf{p},\mathbf{q}_0)(k-\mathbf{p}\cdot\hat{\mathbf{q}}) \\ &\iint d\mathbf{p}'\widetilde{\varsigma}(k\hat{\mathbf{q}}-\mathbf{p}')\widetilde{\varsigma}(\mathbf{p}'-\mathbf{p})\Big\}. \end{aligned} \tag{2.21}$$

This means that at small observation angles the Neuman rough surface behaves as a plane with the Dirichlet boundary condition, and the diffusive scattered field at small scattering angles is proportional to the small grazing angle α. These conclusions are based on the notion that the bracketed term at the right-hand part of (2.21) is bounded. It is not possible to check this assumption for the general case, but if one uses the CP result (2.15) for the surface one can see that it is not true. This is not surprising, of course, since the -1 reflection coefficient and $S(\mathbf{q},\mathbf{q}_0) \propto \alpha^0$ are in clear contradiction to the CP result (2.16).

This solution implies also that the surface field satisfies the equation:

$$\iint d\mathbf{p}'\widetilde{E}(k\hat{\mathbf{q}}-\mathbf{p}',\mathbf{q}_0)(\mathbf{p}'\cdot\hat{\mathbf{q}})\hat{\varsigma}(\mathbf{p}') = -2i\frac{\nu_0}{k}\delta(k\hat{\mathbf{q}}-\mathbf{q}_0), \tag{2.22}$$

which follows from equation (2.7) when $\alpha \to 0$. It is difficult to reach any conclusions regarding the existence of a solution for this equation; however, it is quite obvious that this equation does not have a solution in the form of a conventional perturbation series (2.14).

It is instructive to note that although both derivations start from the exact integral equations (2.7) and (2.8), Tatarskii/Charnotskii's arguments rely to some extent on the CP results, while Barrick's analysis uses equation (2.22), which does not have a CP solution. This implies that the CP solution needs to be examined more accurately for a small grazing angle case. The rest of the paper is devoted to this examination. We limit our discussion to the case of a one-dimensional periodic surface from this point on, but we will address the possible consequences of this restricted approach at the end of the paper.

The Dirichlet and finite-impedance scattering problems do not carry this type of controversy with them since both Tatarskii/Charnotskii's and Barrick's analyses result in the same $O(\alpha)$ dependence for scattering amplitude at the small grazing angles. Therefore, the following discussion is concentrated on the Neuman problem.

3. ONE-DIMENSIONAL PERIODIC SURFACE

In this section we introduce the necessary formalism for scattering from a periodic one- dimensional surface, and examine the validity domain of the CP approximation in the presence of a grazing scattered mode.

3.1 Principal Equations

Consider a surface with 1-D periodic heights function:

$$\varsigma(\mathbf{r}) = \varsigma(x, y) = \eta(x), \quad \eta(x + L) = \eta(x). \tag{3.1}$$

In this case the Fourier transform of the height function is replaced by Fourier series:

$$\hat{\eta}(p) = \sum_{n=-\infty}^{\infty} \eta_n \delta(p - n\kappa), \quad \kappa = \frac{2\pi}{L}. \tag{3.2}$$

The surface and scattered fields are pseudoperiodic, meaning that Fourier transform of the total surface field has the form

$$\widetilde{E}(\mathbf{q}, \mathbf{q}_0) \equiv \widetilde{E}(q_x, q_y, q_{0_x}, q_{0_y}) = \delta(q_y - q_{0_y}) \sum_n e_n(q_{0_x}) \delta(q_x - q_n),$$

and the scattering amplitude can be presented as follows:

$$S(\mathbf{q}, \mathbf{q}_0) \equiv S(q_x, q_y, q_{0_x}, q_{0_y}) = k\delta(q_y - q_{0_y}) \sum_{m=-\infty}^{\infty} T_m(q_{0_x}) \nu_m \delta(q - q_m), \tag{3.3}$$

where $q_n = q_{0_x} + n\kappa$, and $\nu_m = \nu(q_{m_x}, q_{0_y})$ are the horizontal and vertical components of the wave vectors of diffraction orders. This definition of the discrete scattering amplitudes allows us to present the scattered field above the highest point of the surface as a sum of plane waves:

$$E_{SC}(x, z, q_0) = \sum_{m=-\infty}^{\infty} T_m(q_0) \exp(iq_m x + i\nu_m z), \tag{3.4}$$

where from this point on we assume that $q_{0_y} = 0$, and drop the subscript x for brevity.

The energy conservation principle can be presented as follows:

$$\sum_{\text{Prop}} |T_m(q_0)|^2 \nu_m = \nu_0. \tag{3.5}$$

Here the sum is over the propagating modes only: $\{m : |q_m| < k\}$. This equation merely means that the power inflow across the plane $z = const$ above the surface from the incident wave is equal to the power outflow through the same plane carried by the scattered waves.

Integral equations (2.7) and (2.8) can now be transformed to linear algebraic form

$$2\nu_0 \delta_{m,0} = e_m(q_0)\nu_m + i \sum_{n=-\infty}^{\infty} e_n(q_0)(k^2 - q_m q_n) p_{m-n}(\nu_m), \tag{3.6}$$

$$2\nu_m T_m(q_0) = e_m(q_0)\nu_m - i \sum_{n=-\infty}^{\infty} e_n(q_0)(k^2 - q_m q_n) p_{m-n}(-\nu_m). \tag{3.7}$$

Equations (3.6) and (3.7) hold for $m = 0, \pm 1, \pm 2, \ldots$. We use in (3.6) and (3.7) the following Fourier series comparable to (2.10):

$$\exp(\pm i\nu\eta(x)) = 1 \pm i\nu \sum_{n=-\infty}^{\infty} p_n(\pm\nu) \exp(in\kappa x). \tag{3.8}$$

Note that when $\nu\eta(x) \ll 1$, similar to (2.11) we have

$$p_m(\pm\nu) \approx \eta_m \pm \frac{i\nu}{2} \sum_{p=-\infty}^{\infty} \eta_{m-p}\eta_p. \tag{3.9}$$

As with the rough surface case equations (2.18) and (2.19), in order to calculate the scattering amplitude when the surface field is known, instead of (3.7) one can use the difference of (3.6) and (3.7):

$$T_m(q_0) = \delta_{m,0} - \frac{i}{2\nu} \sum_{n=-\infty}^{\infty} e_n(q_0)(k^2 - q_m q_n)[p_{m-n}(\nu_m) + p_{m-n}(-\nu_m)], \tag{3.10}$$

or the sum of (3.6) and (3.7):

$$\begin{aligned} T_m(q_0) = &-\delta_{m,0} + e_m(q_0) + \frac{i}{2\nu_m} \sum_{n=-\infty}^{\infty} e_n(q_0)(k^2 - q_m q_n) \\ &[p_{m-n}(\nu_m) - p_{m-n}(-\nu_m)]. \end{aligned} \tag{3.11}$$

For the further development of perturbation solutions the last formula is more convenient since the sum at the right-hand part contains the terms $[p_{m-n}(\nu_m) - p_{m-n}(-\nu_m)]$, which are at least of the second order in heights as it follows from (3.9). Therefore, in most cases the first two terms at the right-hand part of (3.11) will be sufficient.

For the **Dirichlet case**, from equations (2.12) and (2.13) we have results similar to those of (3.6) and (3.7):

$$2\nu_0 \delta_{m,0} = \nu_m f_m(q_0) + i\nu_m \sum_{n=-\infty}^{\infty} p_{m-n}(\nu_m) f_n(q_0), \tag{3.12}$$

$$2\nu_m T_m(q_0) = -\nu_m f_m(q_0) + i\nu_m \sum_{n=-\infty}^{\infty} p_{m-n}(-\nu_m) f_n(q_0). \tag{3.13}$$

3.2 Conventional Perturbation Theory

The conventional perturbation series for the surface fields and scattering amplitudes have the following form:

$$\begin{aligned} e_m(q_0) &= e_m^{(0)}(q_0) + e_m^{(1)}(q_0) + \dots, \\ T_m(q_0) &= T_m^{(0)}(q_0) + T_m^{(1)}(q_0) + \dots, \end{aligned} \tag{3.14}$$

where $^{(n)}$-th term is $O(k^n \eta^n)$. The first three orders of conventional perturbation solutions of (3.6) and (3.11) for the surface fields and scattering amplitudes are presented as follows:

$$e_m(q_0) = 2\delta_{m,0} - 2i\frac{(k^2 - q_m q_0)}{\nu_m}\eta_m + \sum_{p=-\infty}^{\infty}\eta_{m-p}\eta_p\left[(k^2 - q_m q_0) - 2\frac{(k^2 - q_m q_p)(k^2 - q_p q_0)}{\nu_m \nu_p}\right], \quad (3.15)$$

$$T_m(q_0) = \delta_{m,0} - 2i\frac{(k^2 - q_m q_0)}{\nu_m}\eta_m - 2\sum_{p=-\infty}^{\infty}\eta_{m-p}\eta_p\frac{(k^2 - q_m q_p)(k^2 - q_p q_0)}{\nu_p \nu_m}. \quad (3.16)$$

These formulas can be derived directly from the rough surface perturbation result (2.15) and (2.16) when the Fourier transform of the heights is given by (3.2). The leading terms of (3.15) can also be obtained from the exact set of equations (3.6) if we assume that $k\eta \ll 1$, and $|e_0| \gg |e_m|$. In this case, (3.6) can be simplified to:

$$\begin{cases} e_0(q_0)\nu_0 = 2\nu_0 \\ e_m(q_0)\nu_m + ie_0(q_0)(k^2 - q_m q_0)\eta_m = 0 \end{cases}. \quad (3.17)$$

The solution of (3.17) readily provides the first two terms at the right-hand part of (3.15). The higher-order terms can be obtained under the same assumptions if one keeps higher- order terms when approximating equation (3.6).

It is well known that changes in the d.c. component of the height function η_0 affect the scattering amplitudes in a very simple way, namely:

$$T_m(q_0)|_{\eta_0} = T_m(q_0)|_{\eta_0=0}\exp[-i(\nu_m + \nu_0)\eta_0].$$

Unfortunately, the perturbation theory fails to take advantage of this fundamental property of the scattering amplitude, and η_0 enters formula (3.16) the same way as do all the other components η_m. In particular, it creates the linear-in-heights term for the specular reflection coefficient $T_0(q_0)$. We will overcome this drawback of the perturbation theory in the following discussion by assuming that $\eta_0 = 0$. Of course this can always be accomplished by the proper choice of coordinate origin.

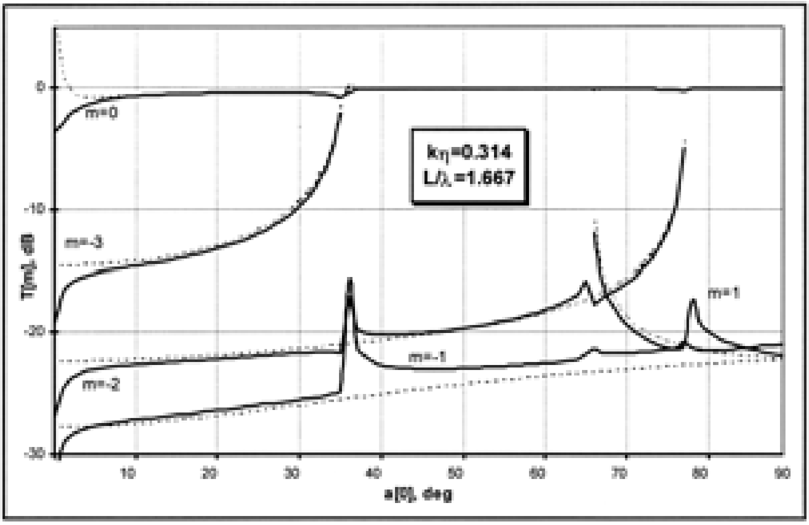

Figure 1. Comparison of the numerical and the first-order CP results for the Neuman problem.

The graph of Fig. 1 presents the numerical solution of equations (3.6) and (3.11) (solid curves) for a wide range of incidence angles. The details of the numerical solution can be found in the Appendix. The low-grazing-angle modes appear at grazing incidence angles of $\alpha_0 \approx 78.5°$, $\alpha_0 \approx 66.4°$, and $\alpha_0 \approx 36.9°$, when the $m = -2$, $m = 1$, and $m = -3$ diffraction orders are at grazing, and at $\alpha_0 = 0°$ when the incident and specular waves are at grazing. One can see that mode magnitudes increase significantly when a certain mode approaches grazing. In the theory of diffraction gratings this effect is called a Wood anomaly [22]. At first glance it appears that the first-order CP provides a good approximation for the grazing order; however, a more detailed investigation finds that this is not true for very small grazing angles. It is also evident that when one mode reaches the grazing position it substantially affects the rest of the scattering spectrum. Quite naturally, the first-order perturbation theory fails to reflect this coupling. The most distinct feature of the chart, however, is a complete failure of the perturbation theory at the low-grazing incident angles.

Figure 2 presents the same data as Fig. 1 but for the Dirichlet problem. Unlike in the Neuman case, these curves are almost featureless. The accuracy of the perturbation theory is uniform across the range

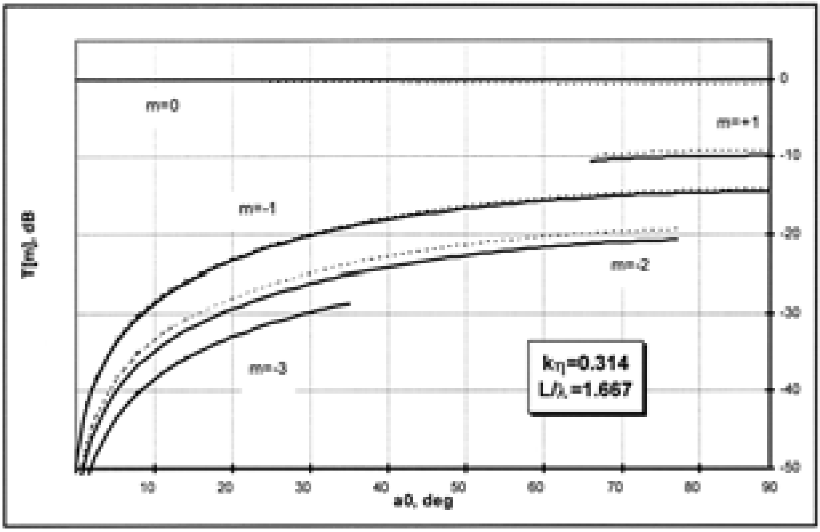

Figure 2. Same as Fig. 1 but for the Dirichlet problem. Charts for $m = +1$, -1, and -2 are displaced vertically by $+15$, $+10$, and $+5\,\mathrm{dB}$.

of incidence angles. If the vicinities of the low-grazing-angle events are excluded, the accuracy of the perturbation theory is about the same for the Neuman and Dirichlet cases.

3.3 CP Solution in the Presence of a Grazing Mode

It is clear from (3.16) and Fig. 1 that when no grazing modes are present the CP solution is valid for $k\eta \ll 1$. However, the presence of an additional small parameter, small grazing angle of the diffraction order, affects the validity domain of the CP solution.

Since we are concerned with the case when the number of propagating modes is not large, we typically can have only one grazing propagating mode. This case is discussed in detail in the present paper. Under certain conditions on the radiation wavelength, grating period, and incidence angle (Littrow mounting) it is also possible that two grazing modes present. Such a case is discussed in the accompanying paper [4].

Suppose now that for certain $m = m^*$ the m^*-th diffraction order is close to grazing, i.e., we have $\nu_{m^*} \ll k$. The corresponding scattering grazing angle $\alpha_{m^*} = \sin^{-1}(\frac{\nu_{m^*}}{k}) \ll 1$.

When the **incident wave is not grazing** it is quite clear that for the sum in (3.15) contains the term:

$$-2\eta_{m-m^*}\eta_{m^*}\frac{(k^2-q_m q_{m^*})(k^2-q_{m^*}q_0)}{\nu_{m^*}\nu_m}=O\left(\frac{k^2\eta^2}{\alpha_{m^*}}\right). \qquad (3.18)$$

It is also apparent that the multiple sums corresponding to the higher orders of CP contain the terms proportional to $\alpha_{m^*}^{-2}$, $\alpha_{m^*}^{-3}$, and so on.

For a sufficiently small grazing angle the term presented by the formula (3.18) can become larger than the second term in (3.15) which is $O(k\eta)$. This essentially ruins the original assumption of the perturbation theory that the higher-order in $k\eta$ terms are smaller than the lower-order ones. Obviously this means that the validity of the CP theory is restricted somehow by the presence of the grazing diffraction mode. We now need to estimate the condition under which the CP theory is valid. It seems natural to impose the condition that the second term in (3.15) is larger than the one given by (3.18). The results would be the inequality $\alpha_{m^*}>k\eta$. However, if we keep the (3.18) term in our CP formulas we should be concerned about the value of the neglected higher-order terms only. More detailed analysis of the higher terms of the CP series shows that among the terms proportional to the $\alpha_{m^*}^{-2}$ the leading one has the order $O(\frac{k^4\eta^4}{\alpha_{m^*}^2})$, and the leading term proportional to $\alpha_{m^*}^{-3}$ is $O(\frac{k^6\eta^6}{\alpha_{m^*}^3})$, and so on. This gives the following validity condition for the CP:

$$\alpha_{m^*}>k^2\eta^2 \qquad (3.19)$$

which is a less restrictive inequality than $\alpha_{m^*}>k\eta$. Hence, it makes sense to add a simple term in the form of (3.18) to the first-order CP formulas to expand the validity domain of the result. The sketch of the validity domain of the CP approximation at the $\log(\alpha_{m^*},k\eta)$ plane is presented in Fig. 3.

For the grazing order itself, $m=m^*$, the first order in heights term is $O(\frac{k\eta}{\alpha_{m^*}})$. It is easy to show that the sum in (3.15) cannot produce the term proportional to the $\alpha_{m^*}^{-2}$, but the third order in heights term contains the $O(\frac{k^3\eta^3}{\alpha_{m^*}^2})$ term. Therefore the CP formula for the grazing order is valid under the same condition (3.19) as was derived for the non-grazing modes.

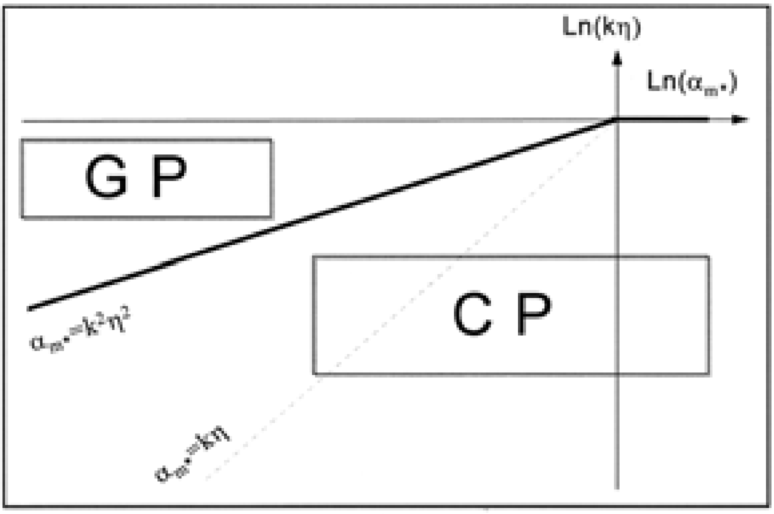

Figure 3. Validity domains of the perturbation approximations in the presence of a single grazing mode.

Finally, scattering amplitudes in the CP approximation can be presented in the following form:

$$T_0(q_0) = 1 - 2\sum_{n\neq 0} \frac{(k^2 - q_0 q_n)^2}{\nu_0 \nu_n} |\eta_n|^2 + O\left(\frac{k^4\eta^4}{\alpha_{m^*}^2}\right), \tag{3.20}$$

$$T_{m^*}(q_0) = -2i\eta_{m^*} \frac{(k^2 - q_{m^*} q_0)}{\nu_{m^*}} + O\left(\frac{k^3\eta^3}{\alpha_{m^*}^2}\right), \tag{3.21}$$

$$T_m(q_0) = -i\eta_m \frac{(k^2 - q_m q_0)}{\nu_m} - 2\eta_{m-m^*}\eta_{m^*} \frac{(k^2 - q_m q_{m^*})(k^2 - q_{m^*} q_0)}{\nu_{m^*}\nu_m} + O\left(\frac{k^4\eta^4}{\alpha_{m^*}^2}\right), \qquad m \neq 0, m^*. \tag{3.22}$$

We allowed some extra terms in the formula (3.20) for the specular mode in order to comply with the energy conservation principle (3.5). It is also possible, of course, to simply use the equation (3.16) under condition (3.19), but this formula would include many extra terms that are smaller than the accuracy of the result. Addition of the higher-order terms of the perturbation series would not expand the validity domain of these formulas given by (3.19), but probably can improve the accuracy of the result inside this domain.

It is interesting that near the boundary of the validity domain, when $k\eta = O(\sqrt{\alpha_{m^*}})$, we have $T_{m^*}(q_0) = O(\frac{1}{\sqrt{\alpha_{m^*}}}) > 1$, which means that the grazing mode can have a larger amplitude than the incident plane wave. This situation was brought up in [11]. It does not violate the energy conservation principle (3.5), since the energy flux across the plane $z = const$ associated with this mode, according to (3.5), is $|T_{m^*}(q_0)|^2 \nu_{m^*}$, and is bounded when $\alpha_{m^*} \to 0$.

Figure 4 provides a more detailed illustration of our discussion for the occurrence of grazing modes at incidence angles $\alpha_0 \approx 66.4^\circ$ when $m^* = 1$ mode is at grazing. Numerical data show that all the modes, including the grazing mode, saturate to the finite limits when $\alpha_1 \to 0$. One can see that despite the visual impression of Fig. 1, the first-order perturbation fails to predict the saturation level of the grazing mode, but provides adequate results for the moderately small grazing angles. Formulas (3.20–3.22) are substantially more accurate in the transition region between the two saturation levels for the nongrazing modes, but predict the erroneous $\alpha_{m^*}^{-1}$ growth for the $\alpha_{m^*} \to 0$.

Regarding the specular mode, we note that the zero-order CP result $T_0(q_0) \approx 1$ would be quite adequate for the entire range of grazing angles due to the small heights considered. However, the inclusion of the second-order terms, which is usually justified based on energy conservation reasons, blows up the result at the same angles where the CP approximation for the nonspecular modes becomes invalid. It is important to note that when $\alpha_{m^*} \to 0$, but under condition (3.19), the scattering amplitude for the grazing mode $T_{m^*}(q_0)$ indeed becomes larger than the scattering amplitude for all the other modes, including the specular. This situation will be used in the following section to construct a new grazing perturbation expansion.

For the **grazing incidence case** we have $\alpha_0 = \sin^{-1}(\frac{\nu_0}{k}) \ll 1$. Due to the reciprocity there is no need to analyze the nongrazing orders: the validity condition for the CP series is the same as for the nongrazing incidence and grazing scattering and is given by (3.19).

For the specular reflection coefficient $T_0(q_0)$ the sum in (3.16) has the order $O(\frac{k^2\eta^2}{\alpha_0})$, and it is possible to show that the term proportional to the α_0^{-2} is $O(\frac{k^4\eta^4}{\alpha_0^2})$. This indicates that the validity condition (3.19) for the CP series stands for the specular reflection as well. Estimations for the higher orders support this conclusion. For the generic scattered mode the leading term is $O(k\eta)$, and the sum in (3.15) generates no terms proportional to α_0^{-1}. The lowest order in heights term

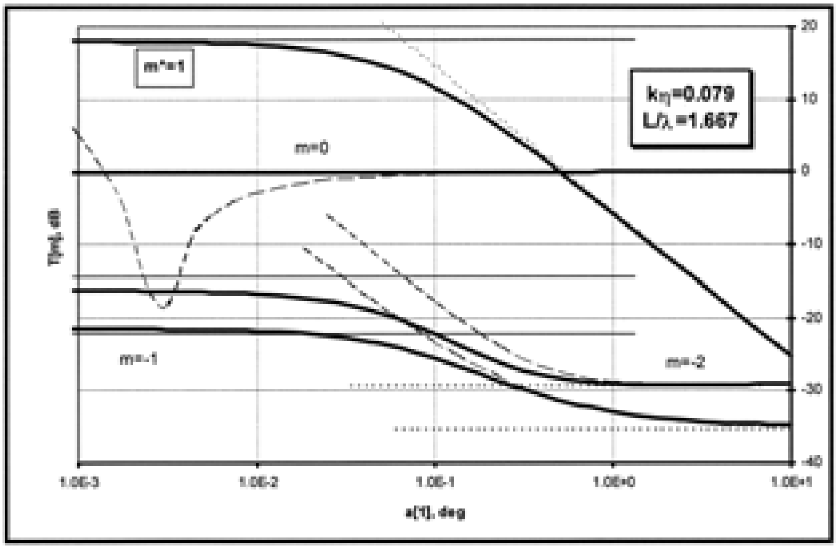

Figure 4. Scattering amplitudes for the $m^* = 1$ grazing mode. Heavy solid curves – numerical solution. Dotted lines – first-order perturbation. Dashed curves – CP formulas (3.19–3.21). Light solid lines – GP formulas (4.9) and (4.10).

that is proportional to α_0^{-1} has the order $O(\frac{k^3\eta^3}{\alpha_0})$. This proves that the validity condition (3.19) for the CP series stands for the grazing incidence as a whole.

Finally, the set of CP grazing incidence formulas that are valid under condition (3.19) with $m^* = 0$ is as follows:

$$T_0(q_0) = 1 - 2\sum_{n\neq 0} \frac{(k^2 - q_0 q_n)^2}{\nu_0 \nu_n} |\eta_n|^2 + O\left(\frac{k^4\eta^4}{\alpha_{m^*}^2}\right), \tag{3.23}$$

$$T_m(q_0) = -2i\eta_m \frac{(k^2 - q_m q_0)}{\nu_m} + O\left(\frac{k^3\eta^3}{\alpha_0}\right), \quad m \neq 0. \tag{3.24}$$

Note that formula (3.24) is reciprocal to (3.21), as expected.

Figure 5 presents a comparison of the numerical results to the CP formulas (3.23), (3.24). The nonspecular modes decrease linearly with α_0 when $\alpha_0 \to 0$. Formulas (3.23) and (3.24) provide an adequate approximation at the moderately small incidence angles while in the

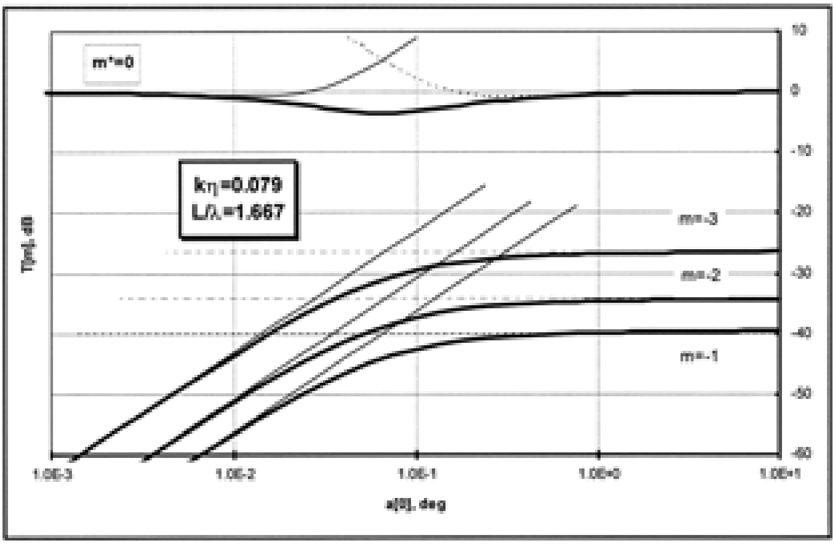

Figure 5. Scattering amplitudes for grazing incidence, $m^* = 0$. Heavy solid curves – numerical solution. Dotted lines – CP formulas (3.23) and (3.24). Light solid lines – GP formulas (4.19) and (4.20).

validity domain (3.19) but fail to match this decrease for the extremely small angles.

The most interesting feature of the grazing forward scattering is revealed on the following two charts, Figs. 6 and 7, where we present the numerical solution for quadratic components and the phase of the reflection coefficient $T_0(q_0)$ near grazing for different roughness heights. The complex reflection coefficient rotates on $180°$ at the complex plane and changes its value from $+1$ to -1. The magnitude of the reflection coefficient does not change significantly in this process, which is also evident from the charts in Fig. 4. It is interesting that the shapes of the curves are nearly the same for the wide range of roughness heights. One can also notice that the critical angle where the phase is $90°$ is proportional to $(k\eta)^2$, which is consistent with our estimation (3.19).

A situation where there are two low-grazing modes, including an important backscatter case, is discussed in the accompanying paper [4].

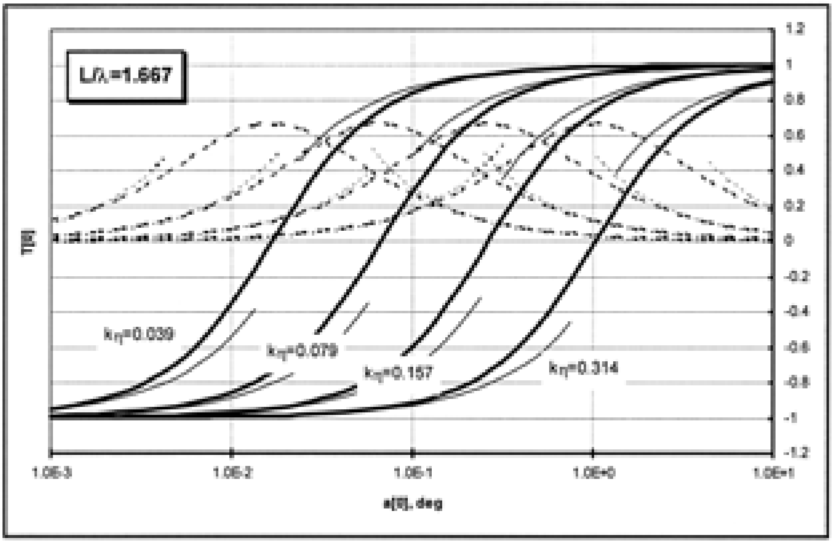

Figure 6. Reflection coefficient for grazing incidence, $m^* = 0$. Solid curves – real part. Dotted curves – imaginary part. Light solid and dotted curves – CP and GP results.

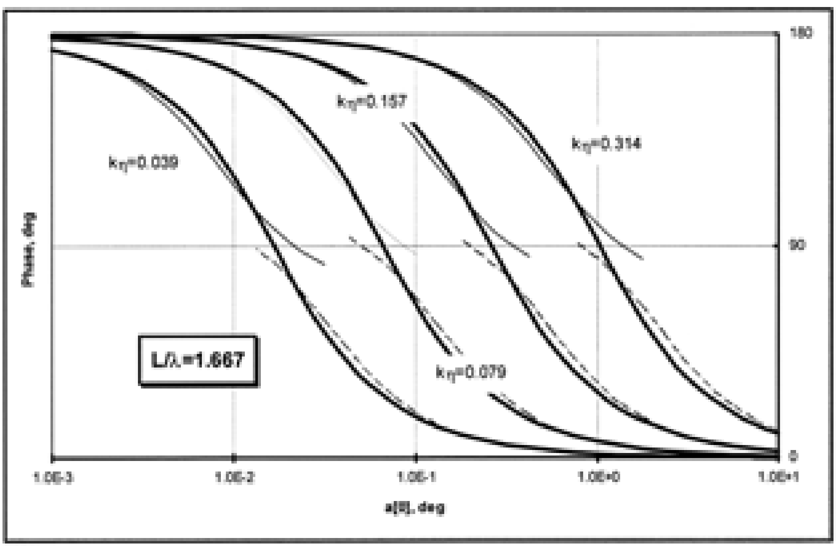

Figure 7. Phase of the reflection coefficient for the grazing incidence. Heavy solid curves – numerical data. Dashed curves – CP. Light solid lines – GP.

4. NEW PERTURBATION EXPANSION IN THE PRESENCE OF A GRAZING MODE: GP

In this section we introduce a new perturbation solution of the principal equations (3.6), (3.7) for the case when the m^*-th diffraction order propagates at a small grazing angle. Unlike in the CP theory, our development here corresponds to the right-hand part of (1.1). In other words, we allow $\alpha_{m^*} \to 0$ first, and after that take advantage of the fact that $k\eta < 1$.

We have to consider two separate cases: Nongrazing incidence when $m^* \neq 0$, discussed in Section 4.1; and the grazing incidence case when the specular order is grazing and $m^* = 0$, considered in Section 4.2.

4.1 Nonspecular Grazing Order

Numerical results presented in Fig. 4 clearly indicate that the CP theory is invalid in the limit of a small grazing angle. The rationale for this deficiency is quite clear from (3.6). For the cases when the m^* order is not specular, equations (3.6) for the Fourier components of the surface field take the following form:

$$\begin{cases} \nu_m e_m(q_0) - i\kappa \sum_{n=-\infty}^{\infty} e_n(q_0)[\kappa(m-m^*)(n-m^*) \\ \qquad \pm k(m+n-2m^*)]p_{m-n}(\nu_m) = 2\nu_0\delta_{m,0}, \quad m \neq m^* \\ \pm ik\kappa \sum_{n\neq m^*} e_n(q_0)(m^*-n)\eta_{m^*-n} = 0, \qquad m = m^*. \end{cases} \tag{4.1}$$

Here, we formally set $\nu_{m^*} = 0$ in (3.6), and use $q_n \approx \pm k + (n-m^*)\kappa$. We also used expansion (3.9) in the m^*-th equation of (4.1). It is important to note that the e_{m^*} does not enter the m^*-th equation of (4.1). This set of linear equations does not explicitly contain any small parameters related to the small grazing angle α_{m^*}. Hence, if the matrix of the system (4.1) is not singular, we should be able to obtain the leading order term of the expansion of the surface field in terms of α_{m^*} just by solving (4.1), and the solution would have a zero-order in terms of $\alpha_{m^*} \to 0$.

Consider now perturbation expansion for the solution of (4.1) in $k\eta$. If we assume that the CP expansion of the surface fields in the form (3.14) is valid, and use (3.9) everywhere in (4.1), then from the first set of equations in (4.1) we see that $e_n^{(0)} = 2\delta_{n,0}$, and $e_n = O(k\eta)$. This is quite similar to the CP result (3.15), of course. Obviously however, this solution cannot satisfy the m^*-th equation of (4.1) since there appear to be no terms to compensate for the relatively large contribution of the $e_0^{(0)}$ to the sum at the left-hand part.

Since the CP approach is not valid for (4.1) we are forced to look for a somewhat different perturbation expansion that still makes use of the fact that $k\eta \ll 1$. The equation corresponding to $m = 0$ is the only equation in the system (4.1) that has the nonzero right-hand part. The rest of the system is homogeneous, meaning that it merely propagates and redistributes the perturbation introduced in the $m = 0$ equation. For the CP, theory the right-hand part of the $m = 0$ equation is equal to the single term $\nu_0 e_0^{(0)}(q_0)$ at the left-hand part since the rest of the terms are at least $O(k\eta)$. This type of balance is mandated by the basic form (3.14) of the CP series, and necessarily results in the inconsistency with the m^*-th equation of (4.1) as we have already seen. Resolution of this discrepancy lies in the acceptance that there are some other terms in the $m = 0$ equation that are responsible for the balance of the right-hand part. It is clear that these additional crucial terms should have the order $O[(k\eta)^{-1}]$. The obvious candidate is the $m = m^*$, of course. Hence, finally we seek a GP expansion in the form:

$$e_m(q_0) = \delta_{m,m^*} e_m^{(-1)}(q_0) + e_m^{(0)}(q_0) + e_m^{(1)}(q_0) + \ldots, \qquad (4.2)$$

where $e_m^{(p)}(q_0) = O[(k\eta)^p]$. This form of the new perturbation series will get a more rigorous justification later in this section, when we discuss the validity domain of the GP expansion.

We substitute (4.2) into (4.1), use the expansion (3.9) for $p_{m-n}(\nu_m)$, and keep the leading terms in $k\eta$ to obtain the following set of homogeneous in $k\eta$ equations:

$$\begin{cases} \nu_m e_m^{(0)}(q_0) \pm ik\kappa(m^* - m)\eta_{m-m^*} e_{m^*}^{(-1)}(q_0) = 2\nu_0 \delta_{m,0}, & m \neq m^* \\ \displaystyle\sum_{n \neq m^*} (m^* - n)\eta_{m^*-n} e_n^{(0)}(q_0) = 0, & m = m^*. \end{cases} \qquad (4.3)$$

This is a closed set of equations for $e_{m^*}^{(-1)}(q_0)$ and $e_m^{(0)}(q_0)$, $m \neq m^*$. All terms in each equation of the set are of the same order in $k\eta$. Hence, there are no small parameters in either $k\eta$ nor α_{m^*} left in the problem, and provided that the system (4.3) is not singular, it furnishes the bounded leading-order solution for small α_{m^*} and $k\eta$ in the sense of the right-hand part of (1.1). Urusovskii presented the proof of the boundness of the surface sources in the presence of a grazing mode based on a different technique in [10] and [11].

In order to solve (4.3) we use the first series of equations in (4.3) to represent $e_m^{(0)}(q_0)$ for $m \neq m^*$ in terms of $e_{m^*}^{(-1)}(q_0)$, and substitute the result into the m^*-th equation. The solution for $e_{m^*}^{(-1)}(q_0)$ is then readily obtained from the m^*-th equation in the form:

$$e_{m^*}^{(-1)}(q_0) = \mp \frac{2i\kappa m^* \eta_{m^*}}{Z}, \tag{4.4}$$

where

$$Z_{EFF} = k\kappa^2 \sum_{p \neq 0} \frac{p^2}{\sqrt{2pk\kappa - p^2\kappa^2}} |\eta_p|^2 \tag{4.5}$$

is the effective impedance of the Neuman surface. The reason for this notation will become clear in the following sections. We note now only that $Z_{EFF} = O(k^2\eta^2)$, and does not depend on m^*, direction of the incident wave, and α_{m^*}. Still, Z_{EFF} is not a pure geometrical characteristic of the surface because it depends on the radiation frequency. Note also that our single-grazing-mode assumption warrants that the denominators in the sum at the right-hand part of (4.5) are nonzero. In our calculation examples for $k\eta = 0.079$ we have $Z_{EFF} = 0.00045 - 0.011i$. Pending further explanation we also introduce an effective Brewster angle related to the effective impedance in a conventional way:

$$\beta_{EFF} \equiv \sin^{-1}(|Z_{EFF}|) \approx |Z_{EFF}|. \tag{4.6}$$

Our solution (4.4) for $e_{m^*}^{(-1)}(q_0)$ has the $O[(k\eta)^{-1}]$ order, as was expected. It also is Bragg-like proportional to the η_{m^*}. This can be a problem for a grating with a sparse spectrum, e.g., for sinusoidal grating. If $\eta_{m^*} = 0$, then the scattering to the m^*-th mode is governed by multiple-scattering effects, and our approximation needs to be modified. Note also that (4.4) vanishes when $m^* = 0$, i.e., for the grazing incidence. This case requires a separate analysis and will be presented in the next section. Formula (4.4) was obtained in [11] using approach outlined in Introduction.

The leading terms for the nongrazing orders, $m \neq m^*$, can be obtained by substitution of (4.4) into the first subset of (4.3) and have the following form:

$$e_m^{(0)}(q_0) = 2\delta_{m,0} + 2k\frac{\kappa^2 m^*(m - m^*)\eta_{m^*}\eta_{m-m^*}}{\nu_m Z_{EFF}}, \quad m \neq m^*. \tag{4.7}$$

This result has the expected $O[(k\eta)^0]$ order, and does not have the Bragg-type form. Instead it looks more like the double-scattering type [compare to the last term in (3.15)] with the first Bragg scattering into the grazing mode, and the second Bragg scattering of the grazing mode into the m-th mode. Formulas (4.4) and (4.7) can be obtained as the appropriate limiting case of the formulas (12) and (13) of [16].

Similar to the small-grazing-angle expansion of (3.6), that was used to derive (4.1) we present equation (3.11) for the scattering amplitudes in the following form:

$$\begin{aligned} T_m(q_0) = & -\delta_{m,0} + e_m(q_0) - \frac{i\kappa}{2\nu_m}\sum_{n=-\infty}^{\infty} e_n(q_0)[\kappa(m - m^*)(n - m^*) \\ & \pm k(m + n - 2m^*)][p_{m-n}(\nu_m) - p_{m-n}(-\nu_m)] \end{aligned} \tag{4.8}$$

This formula contains no explicit small parameters associated with the small grazing angle.

We substitute (4.4) and (4.7) into (4.8) and retain the leading orders in $k\eta$ to attain:

$$T_m(q_o) \approx \delta_{m,0} + 2k\frac{\kappa^2 m^*(m - m^*)\eta_{m^*}\eta_{m-m^*}}{\nu_m Z_{EFF}}, \quad m \neq m^*; \tag{4.9}$$

$$T_{m^*}(q_0) \approx \mp\frac{2i\kappa m^*\eta_{m^*}}{Z_{EFF}}. \tag{4.10}$$

Formulas (4.9) and (4.10) are the major results for the GP expansion. Both formulas are nonlinear in terms of the roughness Fourier coefficients, and predict the finite limit for the scattering amplitude at grazing. Note that the scattering amplitude for the grazing order increases when the roughness decreases. Obviously, this cannot be allowed in the limit of vanishing roughness. Hence, the validity domain of our GP expansion should be somehow limited at the small roughness side.

We estimate the validity domain of (4.9) and (4.10) by matching these formulas to the CP results (3.20–3.22). For $m^* \neq 0$, the CP specular reflection coefficient is given by (3.20), and $T_0(q_0) = 1 + O(\frac{k^2\eta^2}{\alpha_{m^*}})$. Formula (4.9) for $m = 0$ gives $T_0(q_0) = 1 + O(1)$, and there is a match at $\alpha_{m^*} = O(k^2\eta^2)$. For grazing mode, equation (3.21) gives $T_{m^*}(q_0) = O(\frac{k^2\eta^2}{\alpha_{m^*}})$, and this matches the $T_{m^*}(q_0) = O(\frac{1}{k\eta})$ [formula (4.10)] at the same boundary $\alpha_{m^*} = O(k^2\eta^2)$. Finally, for the generic mode we have from (3.22): $T_m(q_0) = O(k\eta) + O(\frac{k^2\eta^2}{\alpha_{m^*}})$, which again matches the $T_m(q_0) = O(1)$ [formula (4.9)] at $\alpha_{m^*} = O(k^2\eta^2)$.

We recall the small height condition $k\eta < 1$ that was used through the development in this section, and conclude that our GP formulas (4.9) and (4.10) are valid under condition

$$\alpha_{m^*} < k^2\eta^2 < 1. \tag{4.11}$$

This domain is complementary to the validity domain (3.19) of the CP theory, as indicated in Fig. 3. Recalling our definition of the effective Brewster angle (4.6), we can also assert that the border between the CP and GP domains is:

$$\alpha_{m^*} \approx \beta_{EFF}. \tag{4.12}$$

Note that the estimation of the critical grazing angle $\alpha_{m^*} = O(k\eta)$ presented in [16] is not correct.

Light, solid, horizontal lines at the left-hand part of the chart in Fig. 4 represent the GP asymptotes calculated according to (4.9) and (4.10). The effective Brewster angle for all charts in Figs. 4–5 is $\beta_{EFF} \approx 0.067°$, which is close to the position where the CP and GP asymptotes intersect.

Note that the accuracy of the GP asymptotes varies for the different diffracted modes. This issue will be addressed at the end of this section.

4.2 New Perturbation Expansion for the Low Grazing Incidence

For the low-grazing-angle incidence we have $m^* = 0$, and the specular component is at the low grazing angle. Since we assume that the number of propagating modes is not large, there are no other modes propagating at a low grazing angle.

Similar to (4.1), we set $\nu_0 = 0$ and $q_n \approx \pm k + \kappa n$ at the left-hand part of the exact equation (3.6) to obtain a new set of equations for the surface field:

$$\begin{cases} \nu_m e_m(q_0) - i\kappa \sum_{n=-\infty}^{\infty} e_n(q_0)[\kappa mn \pm k(m+n)] p_{m-n}(\nu_m) = 0, \quad m \neq 0 \\ \qquad \mp ik\kappa \sum_{n\neq 0} e_n(q_0) n \eta_{-n} = 2\nu_0, \; m = 0. \end{cases} \tag{4.13}$$

If we assume that all $e_m(q_0) \propto \nu_0$, then we can eliminate all small parameters associated with the small incidence angle from equations (4.13). Using reasoning quite similar to that used in preceding equation (4.2), we seek the GP in $k\eta$ solution of (4.13) in the form:

$$e_m(q_0) = \delta_{m,0} e_m^{(-2)}(q_0) + e_m^{(-1)}(q_0) + e_m^{(0)}(q_0) + \ldots \tag{4.14}$$

We substitute (4.14) into (4.13) and keep only the leading terms in $k\eta$ to attain the system of equation that contains no small parameters when $k\eta \to 0$:

$$\begin{cases} \nu_m e_m^{(-1)}(q_0) \mp i\kappa k \eta_m m e_0^{(-2)}(q_0) = 0, \qquad m \neq 0 \\ \qquad \mp i\kappa k \sum_{n\neq 0} n\eta_{-n} e_n^{(-1)}(q_0) = 2\nu_0, \quad m = 0. \end{cases} \tag{4.15}$$

In order to solve (4.15), we represent $e_m^{(-1)}(q_0)$ for $m \neq 0$ in terms of $e_0^{(-2)}(q_0)$ from the first series of equations in (4.15), and substitute the result into the $m = 0$-th equation. The solution for $e_0^{(-2)}(q_0)$ is then readily obtained from the $m = 0$ equation of (4.15) as follows:

$$e_0^{(-2)}(q_0) = \frac{2\nu_0}{kZ_{EFF}}. \tag{4.16}$$

The leading terms for the nongrazing orders $e_m^{(-1)}(q_0)$ can be obtained by substitution of (4.16) into the first subset of (4.15), and have the following form:

$$e_m^{(-1)}(q_0) = \pm \frac{2i\nu_0 m\kappa\eta_m}{\nu_m Z_{EFF}}, \quad m \neq 0. \tag{4.17}$$

It is worth noting that the leading terms of the surface field have lower orders in roughness heights than for the finite incidence case, and all the Fourier components of the surface field vanish as $O(\alpha_0)$, when the incidence angle approaches zero.

When the incidence angle is small, equations (3.11) for the scattering amplitude can be presented as follows:

$$T_m(q_0) = -\delta_{m,0} + e_m(q_0) - \frac{i\kappa}{2\nu_m} \sum_{n=-\infty}^{\infty} e_n(q_0) \\ [\kappa mn \pm k(m+n)][p_{m-n}(\nu_m) - p_{m-n}(-\nu_m)]. \tag{4.18}$$

We substitute (4.16) and (4.17) in (4.18) and keep the leading terms in $k\eta$ to obtain the formulas for the scattering amplitudes:

$$T_m(q_0) = \pm \frac{2i\nu_0 \kappa m \eta_m}{\nu_m Z_{EFF}}, \quad m \neq 0 \tag{4.19}$$

$$T_0(q_0) \approx -1 + \frac{2\nu_0}{k Z_{EFF}}. \tag{4.20}$$

GP asymptotes calculated using (4.19) and (4.20) are presented as thin solid light lines at the left-hand part of Fig. 5, and appear to be a good approximation to the direct numerical solution for small incidence angles. It is clear from (4.19) and (4.20) that when $\alpha_0 \to 0$ all diffraction orders besides the specular vanish, and the amplitude for the specular order approaches -1, which is the typical value for the Dirichlet problem rather than for the Neuman problem considered here. This matches our numerical results presented in Fig. 6. The value of the effective Brewster angle, $\beta_{EFF} \approx 0.067°$, matches the position of the minimum for the reflection coefficient $T_0(q_0)$, and the intersections of the CP and GP asymptotes in Fig. 5. It is also easy to see from Figs. 6 and 7 that the Brewster angle changes as a square of heights, as expected.

In order to determine the validity domain of new formulas we match them to the CP formulas (3.23) and (3.24). For the generic scattered order from (3.23) we have $T_m(q_0) = O(k\eta)$, and from (4.19) $T_m(q_0) = O(\frac{\alpha_0}{k\eta})$. These values match at $\alpha_0 \propto k^2\eta^2$. For the reflection coefficient formula (3.22) gives $T_0(q_0) = 1 + O(\frac{k^2\eta^2}{\alpha_0})$, and from (4.20) we have $T_0(q_0) = -1 + O(\frac{\alpha_0}{k^2\eta^2})$. These values match at $\alpha_0 \propto k^2\eta^2$ also.

4.3 Discussion

This analysis, together with our discussion at the end of the previous section, shows that for the case when a single grazing mode is allowed, the CP formulas (3.20–3.24) and the GP expansion (4.9), (4.10) and (4.19), (4.20) form a complete set of asymptotes for the small roughness heights. It is also important to note that the extra, second order in heights term should be included into the CP formula (3.22) in order to have only two asymptotic domains: $\alpha_{m^*} > k^2\eta^2$ and $\alpha_{m^*} < k^2\eta^2 < 1$. If only the first order in heights terms are used in (3.22), then the intermediate asymptotic domain $k^2\eta^2 < \alpha_{m^*} < k\eta$ has to be considered (see Fig. 3).

Formulas (4.9), (4.10) and (4.19), (4.20) present the leading terms of the GP expansion. The following terms have the higher power in $k\eta$ and/or α_{m^*}. Figure 4 shows that for certain modes the leading GP terms are not very accurate, even at extremely small grazing angles. The reason for this is that the heights are not small enough. One can achieve better accuracy for the small grazing angles but for moderately small heights if the higher-order terms of the series (4.2) are used to solve (4.1). This was done in [8] and showed the expected accuracy improvement for moderate heights up to $k\eta = 1.57$. We were unable to continue our numerical calculations due to the increasing errors in the energy balance, but GP results remained stable and showed the correct trend.

One of the important features of the scattering problem is reciprocity, which is valid for arbitrary heights and incidence and/or scattering angles. It is easy to check that the conventional perturbation expansion (3.16) is reciprocal. Since our GP formulas for the scattering amplitude (4.9), (4.10), (4.19), and (4.20) are essentially rigorous expansions of the exact solution in terms of small angle and small height, we expect that reciprocity is preserved. Exhaustive examination in [8] proved that it is indeed the case.

5. UNIFORM PERTURBATION APPROXIMATION

In this section we propose a more natural form of the GP approximation that uses the real physical parameters of the problem rather than the rigorous but formal presentations obtained in Section 4. Using CP and GP expansions as starting points, we develop a uniform approximation that is valid for small heights and arbitrary grazing angles.

This uniform approximation tends toward the CP and GP results in the appropriate limits, and provides a smooth solution in the transition region between the CP and GP domains.

5.1 "Natural" Form of GP Approximations

Final results of our GP approximation for scattering amplitudes are presented by the formulas (4.9), (4.10), (4.19), and (4.20). These formulas are essentially the exact leading terms of the series expansions of scattering amplitudes for $\alpha_{m^*} \to 0$ and $k\eta \to 0$ when $\alpha_{m^*} \ll k^2\eta^2$. Therefore, the dependence on the small grazing angle and heights is presented in the polynomial form in ν_{m^*} and in the rational form in η_n. We now propose the formulas that use the more "natural" parameters such as horizontal components of the wave vectors. These formulas do not represent the exact leading terms of the appropriate expansions. However, the differences are of the same order as the higher-order terms of the rigorous series, which we neglect anyway. We believe that this natural form of the solution makes it easier to link the discrete scattering amplitudes of the grating problem to the continuous scattering amplitudes of the rough surface problem.

What we actually intend to do here is replace the exact (in the leading order in ν_{m^*} and $k\eta$) set of equations (4.3) for the **non-grazing incidence** by the set:

$$\begin{cases} \nu_m e_m(q_0) + i(k^2 - q_m q_{m^*})\eta_{m-m^*} e_{m^*}(q_0) = 2\nu_0 \delta_{m,0}, & m \neq m^*, \\ i \sum\limits_{n \neq m^*} (k^2 - q_{m^*} q_n)\eta_{m^*-n} e_m(q_0) = 0, & m = m^*. \end{cases} \tag{5.1}$$

The solution for the scattering amplitudes can be obtained from (5.1) and (3.11) :

$$T_m(q_0) \approx \delta_{m,0} - \frac{2(k^2 - q_{m^*} q_m)(k^2 - q_{m^*} q_0)\eta_{m-m*}\eta_{m^*}}{k\nu_m Z_{EFF}(q_{m^*})}. \tag{5.2}$$

$$T_{m^*}(q_0) = \frac{-2i(k^2 - q_{m^*} q_0)\eta_{m^*}}{k Z_{EFF}(q_{m^*})}. \tag{5.3}$$

Here, we introduce the "natural" form of the effective impedance:

$$Z_{EFF}(q_0) \approx \sum_{n \neq 0} \frac{(k^2 - q_{m^*} q_{m^*+n})^2}{k\nu_{m^*+n}} |\eta_n|^2. \tag{5.4}$$

This form of the effective impedance depends on the incidence angle, and reduces to (4.5) when $\alpha_{m^*} \to 0$.

For the **grazing incidence** when no other grazing modes are allowed, we can modify equations (4.13) similar to (5.1), and instead of (4.19) and (4.20) we will have:

$$T_m(q_0) = \frac{-2i\nu_0(k^2 - q_0 q_m)\eta_m}{\nu_m k Z_{EFF}(q_0)}, \tag{5.5}$$

$$T_0(q_0) = -1 + \frac{2\nu_0}{kZ_{EFF}(q_0)}. \tag{5.6}$$

Note that despite the modifications we made, formula (5.5) is reciprocal to (5.3).

5.2. Uniform Perturbation Approximations

In this section we will combine the CP solutions with the GP solutions to create a uniform perturbation solution, which is valid for small heights with or without the grazing order present and reduces to the CP or GP solution when applicable.

We consider the **nongrazing incidence** first, and assume that only one, m^*-th, grazing mode is present, hence $\nu_{m^*} \ll k$. There are two small parameters, α_{m^*} and $k\eta$, now in the basic equation set (3.6), and we intend to develop an approximation that uses no assumptions regarding the relative value of these two parameters. In order to do this we retain in the exact equation set (3.6) the terms that have been kept in (3.17) or (5.1). The result is the following set of equations:

$$\begin{cases} \nu_0 e_0(q_0) + i(k^2 - q_0 q_{m^*})\eta_{-m^*} e_{m^*}(q_0) = 2\nu_0, \quad m = 0, \\ \nu_{m^*} e_{m^*}(q_0) + i(k^2 - q_{m^*} q_0)\eta_{m^*} e_0(q_0) \\ \quad + i \displaystyle\sum_{n \neq 0, m^*} (k^2 - q_{m^*} q_0)\eta_{m^*-n} e_n(q_0) = 0, \quad m = m^*, \\ \nu_m e_m(q_0) + i(k^2 - q_m q_0)\eta_m e_0(q_0) \\ \quad + i(k^2 - q_m q_{m^*})\eta_{m-m^*} e_{m^*}(q_0) = 0, \quad m \neq 0, m^*. \end{cases} \tag{5.7}$$

The exact solution of this set is:

$$e_0(q_0) = \frac{2\nu_0}{\Delta}\left[\nu_{m^*} + \sum_{n \neq 0, m^*} \frac{(k^2 - q_{m^*} q_n)^2}{\nu_n} |\eta_{m^*-n}|^2\right], \tag{5.8}$$

$$e_{m^*}(q_0) = -\frac{2\nu_0}{\Delta}\Bigg[i(k^2 - q_{m^*}q_0) + \sum_{n\neq 0,m^*} \frac{(k^2 - q_{m^*}q_n)(k^2 - q_nq_0)}{\nu_n}\eta_n\eta_{m^*-n}\Bigg], \quad (5.9)$$

$$e_m(q_0) = -\frac{i}{\nu_m}[(k^2 - q_mq_0)\eta_m e_0(q_0) + (k^2 - q_mq_{m^*})\eta_{m-m^*}e_{m^*}(q_0)], \quad (5.10)$$

where

$$\Delta = \nu_0\nu_{m^*} + \nu_0 \sum_{n\neq m^*} \frac{(k^2 - q_{m^*}q_n)^2}{\nu_n}|\eta_{m^*-n}|^2 - i(k^2 - q_{m^*}q_0)\eta_{-m^*} \sum_{n\neq 0,m^*} \frac{(k^2 - q_{m^*}q_n)(k^2 - q_nq_0)}{\nu_n}\eta_n\eta_{m^*-n}. \quad (5.11)$$

We use these results in (3.11) to obtain formulas for scattering amplitudes. These formulas tend to correct CP or GP limits when $k\eta \to 0$ or $\nu_{m^*} \to 0$ as expected. However, after close examination it becomes clear that these formulas contain some terms that can be omitted without sacrificing this asymptotic correctness. These simplified UP formulas have the following form:

$$T_0(q_0) = 1 - \frac{2(k^2 - q_{M^*}q_0)^2|\eta_{m^*}|^2}{\nu_0[\nu_{m^*} + kZ_{EFF}(q_{m^*})]}, \quad (5.12)$$

$$T_{m^*}(q_0) = \frac{-2i(k^2 - q_{m^*}q_0)\eta_{m^*}}{\nu_{m^*} + kZ_{EFF}(q_{m^*})}, \quad (5.13)$$

$$T_m(q_0) = \frac{-2i\nu_{m^*}(k^2 - q_mq_0)\eta_m - 2(k^2 - q_{m^*}q_0)(k^2 - q_{m^*}q_0)\eta_{m-m^*}\eta_{m^*}}{\nu_m[\nu_{m^*} + kZ_{EFF}(q_{m^*})]}, \quad m \neq 0, m^*. \quad (5.14)$$

When $k^2\eta^2 < \alpha_{m^*}$ these formulas tend to the CP limits (3.20–3.22). When $k^2\eta^2 > \alpha_{m^*}$ formulas (5.12–5.14) approach the GP formulas (4.9), (4.10) or their natural form (5.5) and (5.6). This further supports our estimation $\alpha_{m^*} = O(k^2\eta^2)$ of the border between the CP and GP domains. Note also that formula (5.14) is self-reciprocal.

Figure 8 presents a comparison of the UP approximation (5.12–5.14) to numerical results, CP formulas (3.20–3.22), and GP formulas

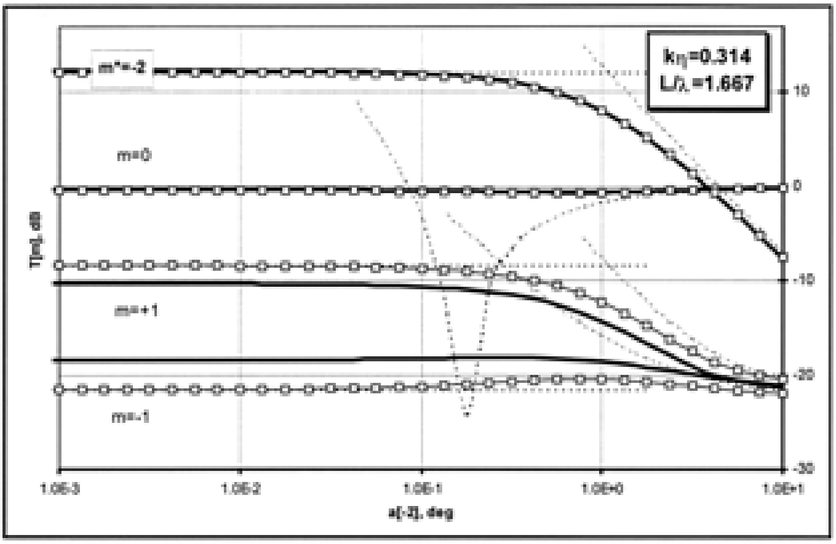

Figure 8. Comparison of the UP-I approximation to numerical solution for $k\eta = 0.314$ and $m^* = -2$. Heavy solid curves – numerical data. Dashed curves – CP and GP results. Light solid curves marked with squares – UP-I approximation.

(5.5) and (5.6). As expected, the UP approximation joins the CP and GP curves smoothly. The uniform approximation appears to be extremely accurate for the four-decades range of grazing angles when $k\eta = 0.079$, and $\beta_{EFF} \approx 0.067°$. This includes the transition region $\alpha_{m^*} = O(k^2\eta^2)$ where both CP and GP are not valid. In order to reveal some differences between the UP and numerical solution on the chart we had to increase the heights four times to $k\eta = 0.314$.

It is interesting that the increase in errors is due mostly to the inaccuracy of the GP asymptotes. As we discussed in a previous section, this can be fixed by including the second- order terms in the GP formulas. It is possible, of course, to build the uniform approximation based on these more accurate asymptotes. The accuracy for the grazing order remains extremely high even for the $k\eta = 0.314$. The effective Brewster angle for this case is $\beta_{EFF} \approx 1.1°$, which is very close to the intersection point of the CP and CP asymptotes for the grazing and $m^* = 1$ modes.

For the **grazing incidence** we combine the major terms of the CP equations (3.17) and the GP equations (4.15) to form the following equation set for the surface field:

$$\begin{cases} \nu_0 e_0(q_0) + i\sum_{n\neq 0}(k^2 - q_0 q_n)\eta_{-n} e_n(q_0) = 2\nu_0, & m = 0 \\ \nu_m e_m(q_0) + i(k^2 - q_m q_0)\eta_m e_0(q_0) = 0, & m \neq 0. \end{cases} \tag{5.15}$$

The solution of (5.15) for the surface fields is straightforward, and after substitution in (3.11) for the specular reflection coefficient we have:

$$T_0(q_0) = \frac{\nu_0 - kZ_{EFF}(q_0)}{\nu_0 + kZ_{EFF}(q_0)}. \tag{5.16}$$

Comparison of this formula to the reflection coefficient of the plane interface with the impedance boundary justifies our definitions of the effective impedance (4.5) and (5.4). This formula provides a smooth transition from the CP reflection coefficient $+1$ to the GP reflection coefficient -1. For $k^2\eta^2 < \alpha_0$, formula (5.16) reduces to the GP result (3.23) with correct orders up to $O[(k\eta)^2]$. When $k^2\eta^2 > \alpha_0$, formula (5.16) reduces to (4.20) or (5.6) with two correct terms.

For the generic scattered mode we have:

$$T_m(q_0) = \frac{-2i\nu_0(k^2 - q_0 q_m)\eta_m}{\nu_m[\nu_0 + kZ_{EFF}(q_0)]}, \quad m \neq 0. \tag{5.17}$$

Formula (5.17) is reciprocally coupled to the earlier UP result (5.13). For $k^2\eta^2 < \alpha_0$ (5.17) gives the major term of (3.24). When $k^2\eta^2 > \alpha_0$ formula (5.17) reduces to (4.19) or (5.5).

Figure 9 presents a comparison of the UP approximation to the numerical results for the grazing incidence. We had to use the moderate height, $k\eta = 0.314$, $\beta_{EFF} \approx 1.1^\circ$ to be able to show any difference between the UP and exact solution.

The UP appears to be even more accurate for the reflection coefficient, $T_0(q_0)$. In order to show the differences between the exact and UP solutions for $T_0(q_0)$ we had to further increase the heights to $k\eta = 0.628$. Figure 10 presents a comparison of the exact and the UP results for the complex reflection coefficient. The UP approximation describes the smooth change of the reflection coefficient from $+1$ to -1 through the rotation at the complex plane without significant

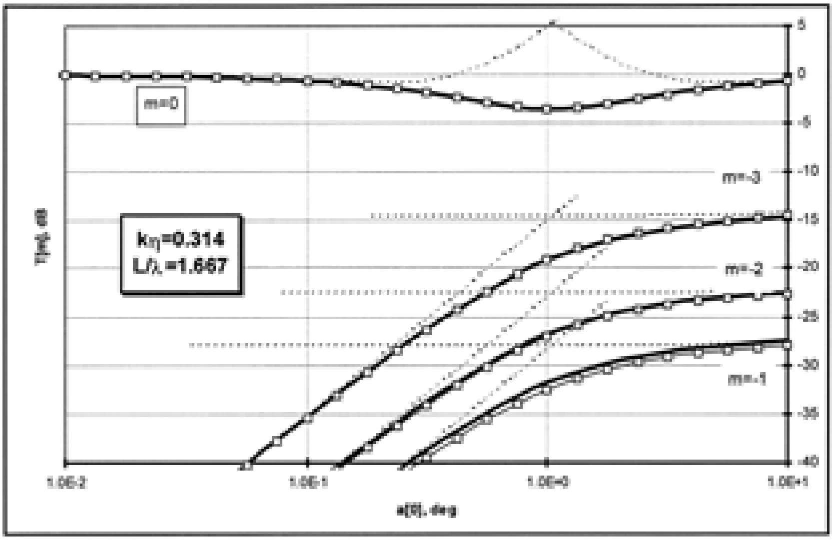

Figure 9. Same as Fig. 8, but for the grazing incidence, $m^* = 0$.

changes in the magnitude of the $T_0(q_0)$ when the incidence angle approaches zero. The phase of the reflection coefficient is presented in Fig. 11, and also shows the remarkable accuracy of the UP approximation for moderate heights. Note that the value of the critical angle where phase is close to $90°$ is very close to the effective Brewster angle, and increases as a square of height consistently with (4.6).

6. CONCLUSION

We found that for the Neuman periodic surface with a finite number of propagating scattered modes, the conventional perturbation theory is invalid when one of the modes propagates at a low grazing angle.

A new grazing perturbation theory was developed that is valid when the grazing angle tends to zero and the heights are small in the wavelength scale. The heights dependence in these new formulas is highly nonlinear. The GP solution represents the exact asymptotes of the scattered field in the limit of small grazing angle and for small heights. Some formulas of the grazing perturbation theory agree with the results of [11] and [16].

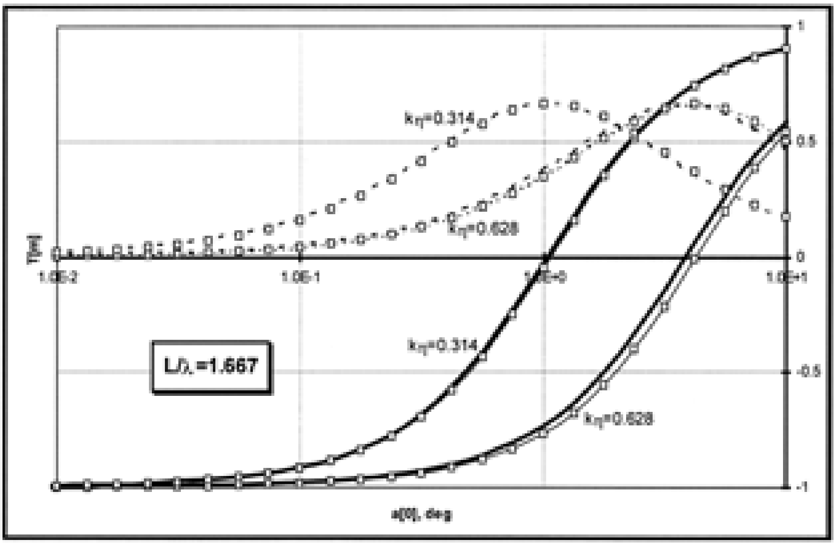

Figure 10. Quadratic components of the reflection coefficient $T_0(q_0)$ at small grazing angles. Heavy curves – numerical solution. Curves marked with squares – UP. Solid curves – real part, dashed curves – imaginary part.

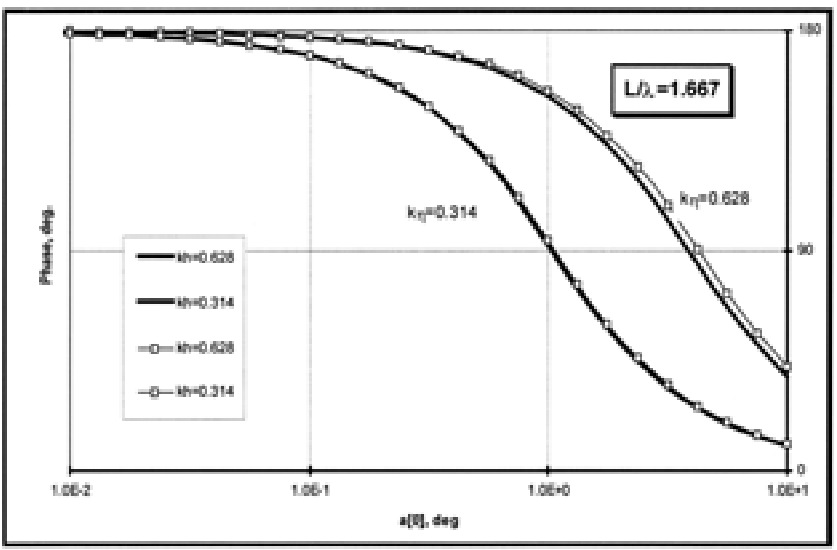

Figure 11. Phase of the reflection coefficient $T_0(q_0)$ at small grazing angles. Heavy curves – numerical solution. Curves marked with squares – UP.

We found that it is necessary to add the second-order term to the conventional first-order perturbation formula to extend the validity domain of the conventional perturbation approximation in the presence of the single grazing mode.

Grazing perturbation theory, and conventional perturbation theory, together form a complete set of asymptotes for the scattering amplitudes when heights are small in the wavelength scale. This explains the contradiction between the results of Barrick [1] and Tatarskii/Charnotskii [2, 3] regarding the small grazing angle behavior of the scattering amplitudes: for a single grazing mode, Barrick's result is valid under condition (4.11), and Tatarskii/Charnotskii's result is valid under condition (3.19).

We developed a uniform perturbation theory that combines the conventional and grazing perturbation formulas and is valid for small heights with or without the grazing modes present. We found that the peculiarities of scattering by the Neuman surface in the presence of a single grazing mode can be accurately explained by introduction of the effective surface impedance, which depends on the surface shape and the radiation wavelength but is independent of the direction of the incident wave. The Brewster angle associated with the effective impedance is a critical angle that separates the validity domains of Barrick's and Tatarskii/Charnotskii's results. This angle is proportional to the square of the heights when a single grazing mode is present.

Uniform perturbation theory describes a smooth transition of the reflection coefficient for the slightly rough Neuman surface from the value of $+1$ – for the incidence angles larger than the effective Brewster angle – to the value of -1 – for incidence angles smaller than the effective Brewster angle. In the complex plane, this change corresponds to a rotation on 180° without significant changes in the magnitude of the reflection coefficient.

A finite number of propagating modes is crucial for the present development. Transition to the rough surface case is achieved through the increase of the surface period. This is associated with the increase in the number of the propagating modes. Formally, this adds a new small parameter to the problem: the angular separation between adjacent modes, or the ratio of the wavelength to the period. This third small parameter requires that the asymptotic analysis be performed in the 3-D parameters space.

Earlier investigations [5–7] led to the introduction of the effective surface impedance for statistically homogeneous random surfaces. In the case of a statistically homogeneous random periodic surface, the average value of the effective impedance (5.4) coincides with the effective impedance of [5–7]. However, substantial differences exist between [5-7] and the present approach:

- The investigation in [5–7] considered the statistically homogeneous surface, but present analysis deals with the periodic surface.
- The effective impedance in [5–7] is a result of certain approximations employed to perform the averaging for the reflection coefficient and/or scattering crossection. Effective impedance in GP theory appears naturally for the exact asymptote in the clearly defined domain and without any kind of averaging.
- Averaging of the GP or UP formulas for a random periodic surface does not lead directly to the results of [5–7].
- Effective impedance is not universal: it does not exist for the two-mode geometry [4], which is more suitable for the long-period limit leading to the true rough surface case.

APPENDIX. DETAILS OF THE NUMERICAL SOLUTION

In order to obtain a numerical solution to equation (3.6), the number of equations and the number of Fourier coefficients of the surface field were limited to

$$m_{\min} \le m.n \le m_{\max}, \tag{A.1}$$

where the lower boundary of the mode range, $m_{\min}$, was equal to the minimal number of propagating modes minus three to four times the number of propagating modes. The upper boundary was determined in a similar way. For example in the case of Fig. 5 the mode range was $(-16;+13)$.

All dimensional parameters were normalized by the wavenumber k, and the final formula for the matrix coefficients was as follows:

$$s_{m,n} = \widetilde{\nu}_m \delta_{m,n} + i(1 - \widetilde{q}_m \widetilde{q}_n) p_{m-n}(k\widetilde{\nu}_m), \tag{A.2}$$

where $\widetilde{\ }$ denotes the normalized variables. Fourier coefficients $p_n(k\widetilde{\nu}_m)$ were calculated using the 128-point FFT, and the same number of points was used to calculate the Fourier coefficients of heights η_n entering the CP and GP formulas.

The model height profile used for all computation has the following form:

$$\eta(x) = \frac{\eta}{5}[\cos(\kappa x) + \sin(2\kappa x) - \sin(3\kappa x) + \cos(4\kappa x) - \sin(5\kappa x)]. \quad \text{(A.3)}$$

The $k\eta$ values used for computation are marked at the charts. The grating period $L = 2\pi\kappa^{-1}$ was fixed at $L = 1.667\lambda$ for all single grazing mode cases.

The number of Fourier components for the surface model (A.3) was sufficient to escape the sparse-spectrum problem discussed in section 4.1 for all the CP and GP formulas. We also avoided using too many high frequency terms, to keep the surface slopes moderate. Since in many cases the scattering amplitudes depend strongly on a single Fourier component of the surface, we chose to keep the amplitudes for all the components equal, thus avoiding interference between the scattering processes and the surface features. In one case, (Fig. 2), this caused all the scattering amplitudes to have practically the same magnitude, and we had to artificially shift the curves to demonstrate their fine features.

The square complex matrix $\{s_{m,n}\}$ was inverted using the regular LU decomposition technique, and the solution for e_n was substituted into the normalized formula (3.11), which was also truncated by (A.1) to calculate the scattering amplitudes.

The energy conservation was controlled during the calculations, and typically the error was less than 10^{-5} for $k\eta \leq 0.628$.

ACKNOWLEDGMENTS

This document has been generated as part of a joint NOAA/DOD – Advanced Sensor Application Program. The author thanks V. I. Tatarskii, I. M. Fuks, A. G. Voronovich, and V. U. Zavorotny for their helpful comments.

REFERENCES

1. Barrick, D. E., "Grazing angle behavior of scatter and propagation above any rough surface," *IEEE Trans. Antennas Propag.,* Vol. 46, 73–83, 1998.
2. Tatarskii, V. I. and M. I. Charnotskii, "On the universal behavior of scattering from a rough surface for small grazing angles," *IEEE Trans. Antennas Propag.,* Vol. 46, 67–72, 1998.

3. Tatarskii, V. I. and M. I. Charnotskii, "Universal behavior of scattering amplitudes for scattering from a plane in average rough surface for small grazing angles," *Waves in Random Media,* Vol. 8, 29–40, 1998.
4. Charnotskii, M. I., "Wave scattering by periodic surface at low grazing angles: two grazing modes," *J. Electromagn. Waves Appl.,* 1999, (accompanying paper).
5. Bass, F. G. and I. M. Fuks, *Wave Scattering from Statistically Rough Surfaces,* Pergamon Press, New York, Ch. 2, 1979.
6. Fuks, I. M. and V. I. Tatarskii, "Scattering crossection from a rough surface for small grazing angles," *Proc. IGARSS,* IV, 2279–2283, Seattle, 1998.
7. Fuks, I. M., V. I. Tatarskii, and D. E. Barrick, "Behavior of scattering from a rough surface at small grazing angles," *Waves in Random Media,* 1999 (in press).
8. Charnotskii, M. I., "Wave scattering at low grazing angles: New perturbation theory," *NOAA Tech. Memo,* ERL ETL-291, 1998.
9. Twersky, V., "On scattering of electromagnetic waves by rough surface," *IRE Trans. Antennas Propag.,* Vol. 5, 81–90, 1957.
10. Urusovskii, I. A., "Sound scattering by a sinusoidally uneven surface characterized by normal acoustic conductivity," *Soviet Physics-Acoustics,* Vol. 5, 362–369, 1959.
11. Urusovskii, I. A., "Diffraction of sound on a periodically uneven and inhomogeneous surface," *Soviet Physics-Doklady,* Vol. 5, 345–348, 1960.
12. Urusovskii, I. A., "Diffraction by periodic surface," *Soviet Physics-Acoustics,* Vol. 10, 287–293, 1965.
13. Urusovskii, I. A., "Wave diffraction by a sinusoidal surface," *Soviet Physics-Acoustics,* Vol. 11, 75–80, 1965.
14. Maystre, D., "Rigorous vector theories of diffraction gratings," *Progress in Optics XXI,* E. Wolf, ed., North-Holland, Amsterdam, 1984.
15. Mashashvili, E. S. and I. A. Urusovskii, "Wave diffraction at a periodically uneven surface," *Soviet Physics-Acoustics,* Vol. 15, 79–85, 1969.
16. Mashashvili, E. S. and I. A. Urusovskii, "Diffraction of a plane wave by a periodic uneven interface between media," *Soviet Physics-Acoustics,* Vol. 15, 217–222, 1969.
17. Urusovskii, I. A., "Theory of wave diffraction by a periodic rough surface," *Soviet Physics-Acoustics,* Vol. 37, 522–525, 1991.
18. Urusovskii, I. A., "Diffraction of a plane wave by rough surface," *Soviet Physics-Acoustics,* Vol. 38, 507–512, 1992.

19. Rosich, R. K., and J. R. Wait, "A general perturbation solution for reflection from two-dimensional periodic surfaces," *Radio Science,* Vol. 12, 719–729, 1977.
20. Vinogradov, A. V., N. N. Zorev, I. V. Kozhevnikov, and I. G. Yakushkin, "Phenomenon of total external reflection of x rays," *Sov. Phys. JETP,* Vol. 82, 1225–1229, 1985.
21. Tatarskii, V. I., "Formulation of rough-surface scattering theory in terms of phase factors and approximate solutions based on this formulation," *Waves in Random Media,* Vol. 7, 557–576, 1997.
22. Hutley, M. C., *Diffraction Gratings,* Academic Press, London, New York, Ch. 6, 1982.