

HEAVISIDE OPERATIONAL RULES APPLICABLE TO ELECTROMAGNETIC PROBLEMS

I. V. Lindell

Electromagnetics Laboratory
Helsinki University of Technology
P. O. Box 3000, FIN-02015 HUT, Espoo, Finland

1. Introduction

- 1.1 Heaviside and the Operator Calculus
- 1.2 Operator Formulation
- 1.3 Series Expansion

2. Operational Rules

- 2.1 Rules from Laplace Transforms
- 2.2 Rules from Integral Identities
- 2.3 Rules from Fourier Transforms
- 2.4 Multidimensional Rules
 - 2.4.1 *Two-dimensional Rules*
 - 2.4.2 *Three-dimensional Rules*
 - 2.4.3 *Extension of Operators*

3. Other Methods

- 3.1 Heaviside Shifting
- 3.2 Generalized Heaviside Shifting
- 3.3 Quadratic Shifting Rule
- 3.4 Convolutions

4. Multipole Fields and Sources

- 4.1 Operator Representation of Multipoles
- 4.2 Synthesis of Radiation Patterns

5. Conclusion

Appendixes

References

1. INTRODUCTION

1.1 Heaviside and the Operator Calculus

In analyzing different electromagnetic problems, Oliver Heaviside (1850–1925) was led to a simple way of solving differential equations through what he called algebration. He treated basic differential operators like d/dz as algebraic quantities and simplified his expressions by applying known algebraic methods. To interpret the results he applied ingenious techniques which involved such unorthodox concepts as divergent series. Being simple and straightforward without the ballast of full mathematical rigor, the solutions thus obtained were, however, checked through other analytical means. To make his method known, Heaviside submitted a three-part manuscript to the *Proceedings of the Royal Society*. After the publication of two parts with no refereeing¹ [3, 4], the third part [5] was given to a referee who rejected it because of lack of rigor in the analysis. The essence of the third part can now only be found in condensed form in his book [6].

Heaviside considered his operator method effective when applied properly. He checked numerically his steps of analysis by summing up series (in making repetitious multi-digit additions and multiplications Heaviside would probably have appreciated even the simplest modern hand-held calculator). Because of checking the results through other analytic means, his papers were full of alternative methods. Heaviside was aware of the incompleteness of his arguments and he often invited mathematicians to study the basics in a more rigorous manner. Let us quote [3]

It proved itself to be a powerful (if somewhat uncertain) kind of mathematical machinery. We may, for example, do in a line or two, work whose verification by ordinary methods may be very lengthy. On the other hand, the very reverse may be the case. I have, however, convinced myself that the subject is one that deserves to be thoroughly examined and elaborated by mathematicians, so that the method brought into general use in mathematical physics, not to supplant ordinary methods, but to supplement them; in short to be used when it is found to be useful.

¹ At that time no paper of a Fellow of Royal Society was refereed for the *Proceedings* though all *Transactions* papers were refereed [1, 2].

After some hesitation, the elaboration by mathematicians eventually led to the branch of Laplace transforms² [7] and, later, to other operator methods [8–11]. Also, the Heaviside operator method in its original form has found supporters after adding more rigorous arguments [12–18]. However, even without the complete rigor, the Heaviside operational calculus has proven to be a helpful tool for finding the first results to new problems, difficult to tackle otherwise [19, 20].

The idea behind the operator calculus was not originated by Heaviside [21–24]. Actually, already G. A. Leibnitz noticed the similarity between Taylor series and algebraic power expansions³. The concept of the differential operator was born when the n th derivative of a function, $d^n f/dz^n$, was written in the form $(d/dz)^n f$ (in the sequel we write ∂_z in short for d/dz). In the early 1800's when rewriting expressions like

$$\alpha \frac{d^m f}{dz^m} + \beta \frac{d^n f}{dz^n} = (\alpha \partial_z^m + \beta \partial_z^n) f, \quad (1)$$

it was realized that linear differential operators with constant coefficients obeyed the commutative and distributive laws of algebra and that explained their close relation. Negative powers of the differential operator ∂_z were introduced already by Leibnitz. Fractional powers came into picture through the work of Liouville in the 1830's. From 1830's to the 1860's English mathematicians, most of all R. Murphy, D. F. Gregory and G. Boole, finally developed the symbolic operator method. After this there was little progress before the topic was taken up by Heaviside in the late 1880's and operators of the form $F(\partial_z)$, where $F(x)$ can be any function, were considered by him. Heaviside, a self-taught scientist, got his background in differential operators most probably from Boole's book *Treatise on differential equations* of 1865.

1.2 Operator Formulation

To demonstrate Heaviside's operator approach associated with transmission lines in the time domain, let us use modern notation.

² Heaviside knew about the basic form of the Laplace transform [6], Vol. 3, p.236, but he did not make use of it.

³ "Symbolismus memorabilis calculi algebraici et infinitesimalis in comparatione potentiarum et differentiarum," published in 1710.

The transmission-line equation for the voltage $U(z, t)$ on a conventional transmission line with line parameters R, L, G, C reads

$$\partial_z^2 U(z, t) - (R + L\partial_t)(G + C\partial_t)U(z, t) = \partial_z u(z, t). \quad (2)$$

Assuming a concentrated series voltage source $u(z, t) = U_o(t)\delta(z)$ with time-harmonic excitation $U_o(t) = U_o e^{j\omega t}$, the solution in the region $z > 0$ is of the form $U(z, t) = U(z)e^{j\omega t}$ and the equation is simplified to

$$\partial_z^2 U(z) - \gamma^2 U(z) = U_o \delta'(z), \quad \gamma = \sqrt{(R + j\omega L)(G + j\omega C)}. \quad (3)$$

The solution for $z > 0$ in this case can be simply found by ordinary techniques as

$$U(z, t) = -\frac{1}{2}e^{-\gamma z}U_o e^{j\omega t} = -\frac{1}{2}e^{-\gamma z}U_o(t). \quad (4)$$

Now Heaviside went forward to write the solution of (2) in a similar form but replacing $j\omega$ by ∂_t which makes the coefficient γ an operator $\gamma(\partial_t)$

$$U(z, t) = -\frac{1}{2}e^{-z\gamma(\partial_t)}U_o(t), \quad \gamma(\partial_t) = \sqrt{(R + L\partial_t)(G + C\partial_t)}. \quad (5)$$

The solution is now in the typical Heavisidean form $F(\partial_t)U_o(t)$. There is an operator $F(\partial_t)$, a function of ∂_t , operating on $U_o(t)$, a function of time. For the basic solution, one can take the impulse function, whence the problem lies in finding an interpretation to the strange mathematical quantity $F(\partial_t)\delta(t)$. The operator here appears quite complicated: an exponential function of a square-root of a quotient of linear functions of ∂_t , nowadays labeled as a pseudo-differential operator. To interpret this in a computational form, Heaviside used various methods, most often series expansions.

1.3 Series Expansion

To demonstrate Heaviside's series approach to handling operator expressions let us consider a simpler operator from his own introductory example in [3], which in modern notation (adding the unit step function $\theta(z)$ always omitted by Heaviside) reads

$$F(z) = \sqrt{\frac{\partial_z}{\partial_z + \beta}} \theta(z). \quad (6)$$

Expanding the operator in inverse powers of ∂_z as

$$F(z) = \left(1 - \frac{1}{2} \frac{\beta}{\partial_z} + \frac{1 \cdot 3}{2^2 \cdot 2!} \frac{\beta^2}{\partial_z^2} - \frac{1 \cdot 3 \cdot 5}{2^3 3!} \frac{\beta^3}{\partial_z^3} + \dots \right) \theta(z), \quad (7)$$

and applying an operational rule (n -fold integration, derived elsewhere) for each term,

$$\frac{1}{\partial_z^n} \theta(z) = \frac{z^n}{n!} \theta(z), \quad (8)$$

Heaviside arrives at the series

$$F(z) = \left(1 - \frac{1}{2} \frac{\beta z}{1!} + \frac{1 \cdot 3}{2^2 \cdot 2!} \frac{\beta^2 z^2}{2!} - \frac{1 \cdot 3 \cdot 5}{2^3 3!} \frac{\beta^3 z^3}{3!} + \dots \right) \theta(z). \quad (9)$$

After some ingenious reasoning Heaviside identifies the series with the Taylor expansion of a known function and finally writes the solution in analytic form as

$$F(z) = e^{-\beta z/2} I_0(\beta z/2) \theta(z), \quad (10)$$

where $I_0(x)$ is the modified Bessel function. After finding the solution for an operator expression like (6) he finally proceeds to prove that it satisfies the original equation where the operator expression came from. Thus, the operator formalism can be at best defined as a method helping one to find solutions to equations.

The step from the operator formulation to its interpretation is a crucial one and requires some operational rules which are the topic of the present paper. In this form, the Heaviside operational calculus has been used to finding solutions for some electromagnetic problems for which other methods have failed [20].

2. OPERATIONAL RULES

Rules like (8) needed in the operator analysis can be found by different means. First of all, connection to Laplace transforms gives the possibility to make use of existing tables of Laplace transforms, e.g., [25–28]. Other rules can be found from tables of integrals, e.g., [29–32]. The Fourier transform pair gives a method to transform a given operational rule to another. Most of the present rules are considered

through the mathematically rigorous two-sided Laplace transform in [8], a widely ignored source of information. Here we consider mainly one-dimensional rules with z as the variable, but also some multi-dimensional rules with $\boldsymbol{\rho}$ (2D) and \mathbf{r} (3D) as the vector variable are briefly discussed in a special section.

2.1 Rules from Laplace Transforms

Although the present form of Laplace transform technique was developed after the Heaviside operator method, it serves as a convenient means to introduce the operational formulas. The definition of the Laplace transform $f(p) \rightarrow F(q)$ reads

$$F(q) = \int_0^{\infty} f(p)e^{-pq} dp. \quad (11)$$

Writing $q = \partial_z$ and operating on a function $g(z)$ defines an operator $F(\partial_z)$ as

$$F(\partial_z)g(z) = \int_0^{\infty} f(p)e^{-p\partial_z} g(z) dp. \quad (12)$$

Applying the Taylor expansion of an exponential function,

$$e^{-pq} = \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} q^n, \quad (13)$$

the Taylor expansion for the function $g(z)$ can be expressed in the operational form

$$g(z-p) = \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} \partial_z^n g(z) = e^{-p\partial_z} g(z). \quad (14)$$

Applying this, the function $f(p)$ has the connection to the operator $F(\partial_z)$ as

$$F(\partial_z)g(z) = \int_0^{\infty} f(p)g(z-p) dp = \int_{-\infty}^{\infty} f(p)\theta(p)g(z-p) dp = [f(z)\theta(z)] * g(z), \quad (15)$$

i.e., through the convolution of the functions $f(z)\theta(z)$ and $g(z)$. Here, $\theta(z)$ denotes the Heaviside unit step function

$$\theta(z) = 0, \quad z < 0, \quad \theta(0) = 1/2, \quad \theta(z) = 1, \quad z > 0 \tag{16}$$

and differentiation of the step function gives the Dirac delta function,

$$\partial_z \theta(z) = \delta(z). \tag{17}$$

Because the operators considered here only depend on ∂_z and not on the variable z , two operators $F_1(\partial_z)$, $F_2(\partial_z)$ always commute,

$$F_1(\partial_z)F_2(\partial_z) = F_2(\partial_z)F_1(\partial_z). \tag{18}$$

This can be well understood when the operators are expressed through their Taylor expansions

$$F(\partial_z) = \sum_{n=0}^{\infty} F_n \partial_z^n, \tag{19}$$

because any two terms of such expansions commute.

Taking the function $g(z)$ as a sequence approaching the delta function $\delta(z - z_o)$, the relation (15) becomes

$$F(\partial_z)\delta(z - z_o) = f(z - z_o)\theta(z - z_o). \tag{20}$$

This and its special case

$$F(\partial_z)\delta(z) = f(z)\theta(z) \tag{21}$$

form the basis of the Heaviside operator method. Typically, a field from a point source can be expressed in the form of a pseudo-differential operator $F(\partial_z)$ operating on the delta function. The task is to find the function $f(z)$, which defines the response for $z > 0$. Applying tables of Laplace transforms, e.g., in [27], we can write rules for different pseudo-differential operators operating on the delta function. A collection of such rules is included in the Appendix.

From (15) we can also obtain the convolution rule

$$F_1(\partial_z)F_2(\partial_z)\delta(z) = [F_1(\partial_z)\delta(z)] * [F_2(\partial_z)\delta(z)] = [f_1(z)\theta(z)] * [f_2(z)\theta(z)] \tag{22}$$

for two operators satisfying

$$F_1(\partial_z)\delta(z) = f_1(z)\theta(z), \quad F_2(\partial_z)\delta(z) = f_2(z)\theta(z). \quad (23)$$

The rules derived through Laplace transform give functions which vanish for $z < 0$. They can be modified by changing the sign of z to give a function vanishing in $z > 0$ (note that $\delta(-z) = \delta(z)$):

$$F(-\partial_z)\delta(z) = f(-z)\theta(-z). \quad (24)$$

For example, taking the simple case

$$\frac{1}{\partial_z + B}\delta(z) = e^{-Bz}\theta(z), \quad (25)$$

and, by changing the signs of both z and B , we have a second choice

$$\frac{1}{\partial_z + B}\delta(z) = -e^{-Bz}\theta(-z). \quad (26)$$

Summing these gives

$$\frac{1}{\partial_z + B}\delta(z) = \frac{1}{2}e^{-Bz}\text{sgn}(z), \quad (27)$$

while the difference leads to vanishing of the left side, or, more exactly, to

$$\frac{0}{\partial_z + B} = e^{-Bz}[\theta(z) + \theta(-z)] = e^{-Bz}, \quad \Rightarrow \quad (\partial_z + B)e^{-Bz} = 0, \quad (28)$$

which is also true but does not help much. The other three results give different representations of the Heaviside operator expression corresponding to three different solutions for the differential equation,

$$(\partial_z + B)f(z) = \delta(z). \quad (29)$$

They satisfy different conditions in infinity but have the same unit step discontinuity at $z = 0$. This example shows us that the operator formulation may have different interpretations due to different additional conditions.

If the function $f(z)$ in (21) is even, i.e., if $f(-z) = f(z)$, we can combine (21) and (24) to give the rule

$$\frac{1}{2} [F(\partial_z) + F(-\partial_z)] \delta(z) = f(z), \quad (30)$$

where the step function has disappeared from the right-hand side. Similarly, if the function $f(z)$ is odd, we can combine

$$\frac{1}{2} [F(\partial_z) - F(-\partial_z)] \delta(z) = f(z). \quad (31)$$

In both cases it is assumed that the left-hand side is not zero.

2.2 Rules from Integral Identities

Other operational rules not based on the Laplace transform can be found from integral tables. Basically, any integral identity of the form

$$\int_{-\infty}^{\infty} f(z) e^{-qz} dz = F(q) \quad (32)$$

can be transformed to an operational rule of the form

$$F(\partial_z) \delta(z) = f(z). \quad (33)$$

The main difference to rules obtained from the Laplace transforms is that functions on the right side are not limited by the unit step function $\theta(z)$ but may extend from $-\infty$ to $+\infty$. This is one point which makes the Heaviside operator approach more general than the Laplace transform approach.

As an example let us consider the following integral identity involving the Gaussian function, [31] 2. 3. 15. 11,

$$\int_{-\infty}^{\infty} e^{-pz^2} e^{-qz} dz = \sqrt{\frac{\pi}{p}} e^{q^2/4p}. \quad (34)$$

As before, replacing q by ∂_z and operating on $\delta(z)$ we have the rule

$$e^{\partial_z^2/4p} \delta(z) = \sqrt{\frac{p}{\pi}} e^{-pz^2}. \quad (35)$$

Integrating the right-hand side from $-\infty$ to ∞ gives unity for all values of p . The function is a delta sequence which approaches the delta function for $p \rightarrow \infty$. The operator on the left can be called a broadening operator since it broadens the delta function to a Gaussian function [33]. For $p \rightarrow \infty$ the operator approaches unity and, thus, the right-hand side has the delta-function limit.

Another operational rule is obtained from the identity [8]

$$\int_{-\infty}^{\infty} \frac{e^{-qz}}{1 + e^{-\alpha z}} dz = \frac{\pi/\alpha}{\sin(\pi q/\alpha)}, \quad (36)$$

which gives us

$$\frac{\pi/\alpha}{\sin(\pi \partial_z/\alpha)} \delta(z) = \frac{1}{1 + e^{-\alpha z}}. \quad (37)$$

The function on the right is a rounded step function because it has the values 0, 1 for $z \rightarrow -\infty, +\infty$, respectively. Its derivative is a delta sequence for $\alpha \rightarrow \infty$, whence the derivative of the operator is another broadening operator.

Further, from the integral identity [31], (2. 3. 11. 9)

$$\int_{-\infty}^{\infty} \frac{(z + \sqrt{z^2 - a^2})^\nu + (z - \sqrt{z^2 - a^2})^\nu}{\sqrt{z^2 - a^2}} e^{-qz} dz = 2a^\nu K_\nu(qa), \quad (38)$$

by defining

$$\tau = \cos^{-1}(z/a), \quad \Rightarrow \quad z \pm \sqrt{z^2 - a^2} = e^{\pm j\tau}, \quad (39)$$

we have

$$\int_{-\infty}^{\infty} \frac{2a^\nu \cos(\nu\tau)}{\sqrt{z^2 - a^2}} e^{-qz} dz = \int_{-\infty}^{\infty} \frac{2a^\nu T_\nu(z/a)}{\sqrt{z^2 - a^2}} e^{-qz} dz = 2a^\nu K_\nu(qa), \quad (40)$$

where $T_\nu(\xi)$ is the generalization of the Chebyshev polynomial $T_n(\xi)$. The arising operational rule can be expressed as

$$K_\nu(a\partial_z)\delta(z) = \frac{T_\nu(z/a)}{\sqrt{z^2 - a^2}}. \quad (41)$$

Making $a = -jb$ imaginary gives us

$$H_\nu^{(1)}(b\partial_z)\delta(z) = \frac{2}{\pi j^{\nu+1}} \frac{T_\nu(jz/b)}{\sqrt{z^2 + b^2}}. \tag{42}$$

Taking the real part of this leads to

$$J_\nu(b\partial_z)\delta(z) = \frac{j^{-\nu-1}T_\nu(jz/b) + j^{\nu+1}T_\nu(-jz/b)}{\pi\sqrt{z^2 + b^2}}. \tag{43}$$

As a last example we take an identity from [32], p.453 (2. 18. 1. 10)

$$\int_{-a}^a \frac{e^{-qx}}{\sqrt{a^2 - x^2}} T_n(x/a) dx = (-1)^n \pi I_n(aq), \tag{44}$$

where $I_n(\xi)$ denotes the modified Bessel function. This gives rise to the rule

$$I_n(a\partial_z)\delta(z) = \frac{(-1)^n}{\pi} \frac{T_n(z/a)}{\sqrt{a^2 - z^2}} \theta(a^2 - z^2). \tag{45}$$

Since the right side is a function vanishing for $|z| > a$, this rule can also be obtained through Laplace transform tables.

2.3 Rules from Fourier Transforms

Fourier transforms can also be applied to find operational rules. The transform pair has the symmetric form

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{jkz} dz, \tag{46}$$

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{-jkz} dk. \tag{47}$$

Replacing k by $j\partial_z$ in (46) and z by $-j\partial_k$ in (47) and operating on the respective delta functions $\delta(z)$, $\delta(k)$, we have

$$F(j\partial_z)\delta(z) = \frac{1}{\sqrt{2\pi}} f(z), \quad f(-j\partial_k)\delta(k) = \frac{1}{\sqrt{2\pi}} F(k). \tag{48}$$

These expressions show the reciprocity between the two functions F and f and that they can be used for transforming one operational rule to another one.

The relations (48) give a possibility to invert given operator relations. For example, if the former of (48) has the form (37), we can identify the two functions as

$$F(z) = \frac{\pi}{\alpha \sin(-j\pi z/\alpha)}, \quad f(z) = \frac{\sqrt{2\pi}}{1 + e^{-\alpha z}}. \quad (49)$$

Thus, the inverse operational rule can be written as

$$f(-j\partial_z)\delta(z) = \frac{1}{\sqrt{2\pi}}F(z), \quad \Rightarrow \quad \frac{\sqrt{2\pi}}{1 + e^{j\alpha\partial_z}}\delta(z) = \frac{\pi}{\alpha\sqrt{2\pi} \sin(-j\pi z/\alpha)}. \quad (50)$$

Another form for this is

$$\frac{1}{\cos(\alpha\partial_z/2)}\delta(z) = e^{j\alpha\partial_z/2} \frac{1}{\alpha \sin(-j\pi z/\alpha)} = \frac{1}{\alpha \cosh(\pi z/\alpha)}. \quad (51)$$

Thus, a new operational rule was quite easily obtained. Because the integral from $-\infty$ to ∞ of the right-hand side equals 1, this again is an example of a broadened delta function. The delta function is obtained as the limit $\alpha \rightarrow 0$.

An operator $O(\partial_z)$ broadening the delta function can be formed from many functions $O(z)$ which are even in z , grow as $|z| \rightarrow \infty$, and whose power expansion starts with 1 [33]. For example, in terms of the modified Bessel function we write

$$O_a(\partial_z) = \frac{2}{a\partial_z} I_1(a\partial_z). \quad (52)$$

Applying (45), the broadened delta function in this case can be obtained by integrating the Chebyshev function $T_1(x) = x$

$$\begin{aligned} \delta_a(z) &= O_a(\partial_z)\delta(z) = \frac{1}{a\partial_z} I_1(a\partial_z)\delta(z) = - \int_{-\infty}^z \frac{2T_1(z'/a)}{\pi a \sqrt{a^2 - z'^2}} \theta(a^2 - z'^2) dz' \\ &= \frac{2}{\pi a^2} \sqrt{a^2 - z^2} \theta(a^2 - z^2). \end{aligned} \quad (53)$$

The broadened delta has the form one half of an ellipse between $-a$ and $+a$.

Because all operator expressions containing constants and the operator ∂_z commute, all formulas in Appendix, written for discontinuous delta and step functions, can be transformed to involve the broadened delta function in convolutionary form. For example, operating with any broadening operator $O_\alpha(\partial_z)$ giving the broadened delta function $\delta_\alpha(z) = O_\alpha(\partial_z)\delta(z)$, because of (22) we can transform the rule

$$\frac{1}{\sqrt{\partial_z^2 + B^2}}\delta(z) = J_0(Bz)\theta(z) \tag{54}$$

to the rule

$$\begin{aligned} \frac{1}{\sqrt{\partial_z^2 + B^2}}\delta_\alpha(z) &= \frac{1}{\sqrt{\partial_z^2 + B^2}}O_\alpha(\partial_z)\delta(z) = [J_0(Bz)\theta(z)] * \delta_\alpha(z) \\ &= \int_0^z J_0(Bz')\delta_\alpha(z - z')dz'. \end{aligned} \tag{55}$$

Some more exotic rules can be found as follows. Identifying the functions from

$$\partial_z^{-1}\delta(z) = \theta(z), \quad \Rightarrow \quad F(z) = \frac{j}{z}, \quad f(z) = \sqrt{2\pi}\theta(z), \tag{56}$$

leads to the inverse operational rule

$$\theta(-j\partial_z)\delta(z) = \frac{j}{2\pi z}. \tag{57}$$

This can be understood as a definition of the step-function operator. To check the result, we can start from the rule (37), whose Fourier complement is

$$\frac{1}{1 + e^{j\alpha\partial_z}}\delta(z) = \frac{j}{2\alpha \sinh(\pi z/\alpha)}. \tag{58}$$

For $\alpha \rightarrow \infty$ the operator on the left-hand side is seen to approach $\theta(-j\partial_z)$ and the right-hand side becomes $j/2\pi z$.

Another exotic operational rule is obtained as the Fourier complement of the triviality

$$\delta(z) = \delta(z), \quad \Rightarrow \quad F(z) = 1, \quad f(z) = \sqrt{2\pi}\delta(z), \tag{59}$$

in the form

$$\delta(-j\partial_z)\delta(z) = \delta(j\partial_z)\delta(z) = \frac{1}{2\pi}, \quad (60)$$

which defines the delta-function operator. This rule can also be obtained as a limit from (35), which defines the Gaussian broadened version and reduces to (60) for $p \rightarrow \infty$. In fact, denoting $1/4p = P$, (35) can be expressed as

$$\sqrt{\frac{P}{\pi}} e^{-P(j\partial_z)^2} \delta(z) = \delta_p(j\partial_z)\delta(z) = \frac{1}{2\pi} e^{-z^2/4P}. \quad (61)$$

Here we denote the gaussian delta function by $\delta_p(j\partial_z)$. For $P \rightarrow \infty$ the broadening operator becomes unity, $\delta_p(j\partial_z) \rightarrow \delta(j\partial_z)$ and the rule (60) results as the limit. In the other limit $P \rightarrow 0$ (61) becomes the triviality $\delta(z) = \delta(z)$.

2.4 Multidimensional Rules

In some cases the previous one-dimensional operational rules can be combined to make multidimensional rules. For example, the Gaussian operator rule (35) directly generates the following three-dimensional rule:

$$\begin{aligned} e^{\nabla^2/4p} \delta(\mathbf{r}) &= e^{\partial_x^2/4p} \delta(x) e^{\partial_y^2/4p} \delta(y) e^{\partial_z^2/4p} \delta(z) \\ &= \sqrt{\frac{p}{\pi}} e^{-px^2} \sqrt{\frac{p}{\pi}} e^{-py^2} \sqrt{\frac{p}{\pi}} e^{-pz^2} = \left(\frac{p}{\pi}\right)^{3/2} e^{-pr^2}. \end{aligned} \quad (62)$$

However, in general the rules in spaces of larger dimensions must be generated independently.

Laplace transform is rarely considered in a multidimensional form. In contrast, the Fourier transform pair is frequently applied in three dimensions as

$$F(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{j\mathbf{k}\cdot\mathbf{r}} dV_r, \quad (63)$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int F(\mathbf{k}) e^{-j\mathbf{k}\cdot\mathbf{r}} dV_k. \quad (64)$$

Again, this gives us a method to transform one operational rule to another in three dimensions, whence (48) becomes

$$F(j\nabla)\delta(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} f(\mathbf{r}), \quad f(-j\nabla)\delta(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} F(\mathbf{r}). \quad (65)$$

In two dimensions the corresponding formulas are

$$F(j\nabla_t)\delta(\boldsymbol{\rho}) = \frac{1}{2\pi}f(\boldsymbol{\rho}), \quad f(-j\nabla_t)\delta(\boldsymbol{\rho}) = \frac{1}{2\pi}F(\boldsymbol{\rho}). \quad (66)$$

The two-dimensional differential operator is defined by

$$\nabla_t = \mathbf{u}_x\partial_x + \mathbf{u}_y\partial_y = \nabla - \mathbf{u}_z\partial_z. \quad (67)$$

Integral identities written in two or three dimensions can be applied to find operational rules.

2.4.1 Two-dimensional Rules

As an example of an operational rule in two dimensions let us consider the following two-dimensional integral identity in polar coordinates [32] (2. 12. 8. 3)

$$\int_0^\infty \int_0^{2\pi} \frac{1}{2\pi\rho} e^{-B\rho} e^{-jK\rho\cos\varphi} \rho d\rho d\varphi = \int_0^\infty e^{-B\rho} J_0(K\rho) d\rho = \frac{1}{\sqrt{K^2 + B^2}}. \quad (68)$$

Denoting $B = jk$ and $K^2 = \mathbf{K} \cdot \mathbf{K}$, with $\mathbf{K} \cdot \boldsymbol{\rho} = K\rho\cos\varphi$, this can be written in the form of a two-dimensional Fourier integral over the whole $\boldsymbol{\rho}$ plane S_ρ :

$$\frac{1}{\sqrt{\mathbf{K} \cdot \mathbf{K} - k^2}} = \int \frac{e^{-jk\rho}}{2\pi\rho} e^{-j\mathbf{K} \cdot \boldsymbol{\rho}} dS_\rho. \quad (69)$$

Replacing \mathbf{K} by $-j\nabla_t$, we can write the operational rule from the integral over the xy plane as

$$\begin{aligned} \frac{1}{\sqrt{-(\nabla_t^2 + k^2)}}\delta(\boldsymbol{\rho}) &= \int \frac{e^{-jk\rho'}}{2\pi\rho'} e^{-\boldsymbol{\rho}' \cdot \nabla_t} \delta(\boldsymbol{\rho}) dS_{\rho'} \\ &= \int \frac{e^{-jk\rho'}}{2\pi\rho'} \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') dS_{\rho'} = \frac{e^{-jk\rho}}{2\pi\rho}. \end{aligned} \quad (70)$$

The sign in the exponent $e^{-jk\rho}$ corresponds to the outgoing-wave condition.

The special case $k = 0$ gives the rule

$$\frac{1}{\sqrt{-\nabla_t^2}}\delta(\boldsymbol{\rho}) = \frac{1}{2\pi\rho}. \quad (71)$$

Making the affine transformation [34]

$$\boldsymbol{\rho} \rightarrow \overline{\overline{A}}^{-1/2} \cdot \boldsymbol{\rho}, \quad \nabla_t \rightarrow \overline{\overline{A}}^{1/2} \cdot \nabla_t, \quad \delta(\boldsymbol{\rho}) \rightarrow \frac{\delta(\boldsymbol{\rho})}{\sqrt{\det \overline{\overline{A}}}}, \quad (72)$$

where $\overline{\overline{A}}$ is a symmetric and positive-definite two-dimensional dyadic and the inverse and determinant⁴ must be understood in two-dimensional sense. Because the inverse in this case has the form [34]

$$\overline{\overline{A}}^{-1} = \frac{1}{\det \overline{\overline{A}}} \overline{\overline{A}}_x^\times \mathbf{u}_z \mathbf{u}_z, \quad (73)$$

we obtain the generalized rule

$$\frac{1}{\sqrt{-\overline{\overline{A}} : \nabla_t \nabla_t}} \delta(\boldsymbol{\rho}) = \frac{1}{2\pi \sqrt{\overline{\overline{A}} : (\mathbf{u}_z \times \boldsymbol{\rho})(\mathbf{u}_z \times \boldsymbol{\rho})}}. \quad (74)$$

More rules can be formed from [8], p.407:

$$\frac{1}{\nabla_t^2 - (\partial_t/v)^2} \delta(\boldsymbol{\rho}) \delta(vt) = -\frac{1}{2\pi \sqrt{(vt)^2 - \rho^2}} \theta(vt - \rho), \quad (75)$$

$$\frac{1}{\nabla_t^2 - (\partial_t/v)^2 + \gamma^2} \delta(\boldsymbol{\rho}) \delta(vt) = -\frac{\cosh(\gamma \sqrt{(vt)^2 - \rho^2})}{2\pi \sqrt{(vt)^2 - \rho^2}} \theta(vt - \rho), \quad (76)$$

$$\frac{1}{\nabla_t^2 + k^2} \left(\frac{\partial_x}{\sqrt{\partial_x^2 + k^2}} + \frac{\partial_y}{\sqrt{\partial_y^2 + k^2}} \right) \delta(\boldsymbol{\rho}) = J_0(k\rho) \theta(x) \theta(y). \quad (77)$$

The Green-function rules are

$$\frac{1}{\nabla_t^2} \delta(\boldsymbol{\rho}) = \frac{1}{2\pi} \ln \rho, \quad (78)$$

$$\frac{1}{\nabla_t^2 + k^2} \delta(\boldsymbol{\rho}) = \frac{-1}{4j} H_0^{(2)}(k\rho). \quad (79)$$

The Fourier complement rules of these are obtained easily as

$$\ln \sqrt{-\nabla_t^2} \delta(\boldsymbol{\rho}) = -\frac{1}{2\pi \rho^2}, \quad (80)$$

$$H_0^{(2)} \left(a \sqrt{\nabla_t^2} \right) \delta(\boldsymbol{\rho}) = \frac{j}{\pi^2} \frac{1}{\rho^2 + a^2}. \quad (81)$$

⁴ Two-dimensional determinant was denoted by $\text{spm} \overline{\overline{A}}$ in [34].

2.4.2 Three-dimensional Rules

Let us start from the integral formula ([31], p.387, (2. 5. 3. 12))

$$\int_0^\infty \frac{\sin Kr}{Kr} dr = \frac{\pi}{2K}. \tag{82}$$

Substituting

$$\frac{\sin Kr}{Kr} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi e^{-jKr \cos \theta} \sin \theta d\theta d\varphi = \frac{1}{4\pi} \int_{4\pi} e^{-j\mathbf{K}\cdot\mathbf{r}} d\Omega, \tag{83}$$

where θ is the angle between \mathbf{K} and \mathbf{r} vectors, we can expand

$$\frac{1}{\sqrt{\mathbf{K}\cdot\mathbf{K}}} = \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{1}{2\pi^2 r'^2} e^{-j\mathbf{K}\cdot\mathbf{r}'} r'^2 \sin \theta dr' d\theta d\varphi = \int \frac{e^{-j\mathbf{K}\cdot\mathbf{r}'}}{2\pi^2 r'^2} dV'. \tag{84}$$

This defines the following three-dimensional operational rule:

$$\frac{1}{\sqrt{-\nabla^2}} \delta(\mathbf{r}) = \frac{1}{2\pi^2 r^2}. \tag{85}$$

The same can also be derived by starting from the well-known Green-function equation

$$(\nabla^2 + k^2) \frac{e^{-jkr}}{4\pi r} = -\delta(\mathbf{r}), \quad \frac{1}{-(\nabla^2 + k^2)} \delta(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r}. \tag{86}$$

and finding the reciprocal rule by the Fourier transform method. Changing $k \rightarrow ja$ and identifying the functions as

$$F(j\nabla) = \frac{1}{(j\nabla)^2 + a^2}, \quad \Rightarrow \quad F(\mathbf{r}) = \frac{1}{r^2 + a^2}, \tag{87}$$

$$f(\mathbf{r}) = (2\pi)^{3/2} \frac{e^{ar}}{4\pi r} \quad \Rightarrow \quad f(-j\nabla) = (2\pi)^{3/2} \frac{e^{a\sqrt{-\nabla^2}}}{4\pi\sqrt{-\nabla^2}}, \tag{88}$$

we arrive at the rule

$$\frac{e^{a\sqrt{-\nabla^2}}}{\sqrt{-\nabla^2}} \delta(\mathbf{r}) = \frac{1}{2\pi^2(r^2 + a^2)}, \tag{89}$$

which is a generalization of (85).

As another example, the Fourier complement of the trivial rule $\delta(\mathbf{r}) = \delta(\mathbf{r})$ can be written as

$$\delta(j\nabla)\delta(\mathbf{r}) = \frac{1}{(2\pi)^3}, \quad (90)$$

which is also obtained as the product of three rules of the form (60), if we define

$$\delta(j\nabla) = \delta(j\partial_x)\delta(j\partial_y)\delta(j\partial_z). \quad (91)$$

2.4.3 Extension of Operators

Multidimensional rules can also be derived by extending one-dimensional rules so that constants are replaced by operators. For example, let us take the following operational rule from the Appendix,

$$\frac{e^{-\sqrt{\partial_z^2 - B^2}a}}{\sqrt{\partial_z^2 - B^2}} \left(\frac{\partial_z - \sqrt{\partial_z^2 - B^2}}{\partial_z + \sqrt{\partial_z^2 - B^2}} \right)^\nu \delta(z) = \left(\frac{z-a}{z+a} \right)^\nu I_{2\nu} \left(B\sqrt{z^2 - a^2} \right) \theta(z-a). \quad (92)$$

Now we replace the parameter B by the operator ∂_t/v and operate on $\delta(t)$:

$$\begin{aligned} & \frac{e^{-\sqrt{\partial_z^2 - (\partial_t/v)^2}a}}{\sqrt{\partial_z^2 - (\partial_t/v)^2}} \left(\frac{\partial_z - \sqrt{\partial_z^2 - (\partial_t/v)^2}}{\partial_z + \sqrt{\partial_z^2 - (\partial_t/v)^2}} \right)^n \delta(z)\delta(t) \\ &= \left(\frac{z-a}{z+a} \right)^n I_{2n} \left(\sqrt{z^2 - a^2} \partial_t/v \right) \delta(t)\theta(z-a). \end{aligned} \quad (93)$$

The last expression can be written in terms of another rule and we obtain

$$\begin{aligned} & \frac{e^{-\sqrt{\partial_z^2 - (\partial_t/v)^2}a}}{\sqrt{\partial_z^2 - (\partial_t/v)^2}} \left(\frac{\partial_z - \sqrt{\partial_z^2 - (\partial_t/v)^2}}{\partial_z + \sqrt{\partial_z^2 - (\partial_t/v)^2}} \right)^n \delta(z)\delta(t) \\ &= \left(\frac{z-a}{z+a} \right)^n \frac{v}{\pi\sqrt{z^2 - a^2}} T_{2n} \left(\frac{vt}{\sqrt{z^2 - a^2}} \right) \theta(z^2 - a^2 - (vt)^2)\theta(z-a). \end{aligned} \quad (94)$$

In spite of their importance, we do not continue with the multidimensional rules but leave them to the topic of another paper.

3. OTHER METHODS

There are other methods to find new operational rules from old ones. This section considers some of them.

3.1 Heaviside Shifting

The method called Heaviside shifting ([6] Vol. 2, p.294) is a complement to the position shifting operator $e^{a\partial_z}$ introduced in Section 2.1 through the Taylor expansion. Making partial differentiations we can take the exponential function $e^{\alpha z}$ through the operator as

$$\partial_z [e^{\alpha z} f(z)] = e^{\alpha z} [\alpha f(z) + \partial_z f(z)] = e^{\alpha z} (\partial_z + \alpha) f(z), \quad (95)$$

which can be interpreted as shifting the operator ∂_z by α . This can be easily generalized for $n = 0, 1, 2, \dots$ as

$$\partial_z^n [e^{\alpha z} f(z)] = e^{\alpha z} (\partial_z + \alpha)^n f(z). \quad (96)$$

This formula works also for negative integers, because substituting $f(z) = (\partial_z + \alpha)^{-n} g(z)$ in the above and operating by ∂_z^{-n} we have

$$\partial_z^{-n} [e^{\alpha z} g(z)] = e^{\alpha z} (\partial_z + \alpha)^{-n} g(z). \quad (97)$$

Applying the Taylor expansion, this can be generalized to any operator $F(\partial_z)$ in the form

$$F(\partial_z) e^{\alpha z} f(z) = e^{\alpha z} F(\partial_z + \alpha) f(z). \quad (98)$$

Operating on a delta function this can also be written as

$$F(\partial_z + \alpha) \delta(z) = e^{-\alpha z} F(\partial_z) \delta(z), \quad (99)$$

which is a convenient tool for working with operator expressions.

For example, the rule for $n > 0$

$$\frac{1}{(\partial_z + \alpha)^n} \delta(z) = e^{-\alpha z} \frac{1}{\partial_z^n} \delta(z) = e^{-\alpha z} \frac{z^{n-1}}{(n-1)!} \theta(z) \quad (100)$$

is straightforwardly obtained from that with $\alpha = 0$. As another example, let us first derive the following rule through convolution:

$$\frac{1}{\sqrt{\partial_z + \beta_1} \sqrt{\partial_z + \beta_2}} \delta(z) = \left(\frac{1}{\sqrt{\partial_z + \beta_1}} \delta(z) \right) * \left(\frac{1}{\sqrt{\partial_z + \beta_2}} \delta(z) \right)$$

$$\begin{aligned}
&= \left(\frac{e^{-\beta_1 z}}{\sqrt{\pi z}} \theta(z) \right) * \left(\frac{e^{-\beta_2 z}}{\sqrt{\pi z}} \theta(z) \right) \\
&= \frac{e^{-\beta_2 z}}{\pi} \int_0^z \frac{e^{-(\beta_1 - \beta_2)z'}}{\sqrt{z'(z - z')}} dz' \\
&= \frac{e^{-\beta_2 z}}{\pi} \int_0^2 \frac{e^{-(\beta_1 - \beta_2)zx/2}}{\sqrt{x(2 - x)}} dx, \tag{101}
\end{aligned}$$

and substitute the last integral from ([30], (3.364)) to arrive at the rule

$$\frac{1}{\sqrt{(\partial_z + \beta_1)(\partial_z + \beta_2)}} \delta(z) = e^{-(\beta_1 + \beta_2)z/2} I_0 \left(\frac{(\beta_1 - \beta_2)z}{2} \right) \theta(z). \tag{102}$$

The same formula can be obtained by denoting $\beta_1 = \alpha + \beta$, $\beta_2 = \alpha - \beta$ and applying Heaviside shifting and an operational rule valid for $\alpha = 0$:

$$\begin{aligned}
\frac{1}{\sqrt{(\partial_z + \alpha)^2 - \beta^2}} \delta(z) &= e^{-\alpha z} \frac{1}{\sqrt{\partial_z^2 - \beta^2}} \delta(z) = e^{-\alpha z} J_0(j\beta z) \theta(z) \\
&= e^{-\alpha z} I_0(\beta z) \theta(z). \tag{103}
\end{aligned}$$

Shifting by an imaginary parameter $\alpha = j\beta$ can also be applied with the variants

$$F(\partial_z + j\beta) \delta(z) = e^{-j\beta z} F(\partial_z) \delta(z), \tag{104}$$

$$[F(\partial_z + j\beta) + F(\partial_z - j\beta)] \delta(z) = 2 \cos(\beta z) F(\partial_z) \delta(z), \tag{105}$$

$$[F(\partial_z + j\beta) - F(\partial_z - j\beta)] \delta(z) = -2j \sin(\beta z) F(\partial_z) \delta(z). \tag{106}$$

The shifting rule can be compared to that of the Taylor expansion,

$$f(z + z') = e^{z' \partial_z} f(z), \quad \leftrightarrow \quad F(\partial_z + \alpha) \delta(z) = e^{-\alpha z} F(\partial_z) \delta(z). \tag{107}$$

Actually, denoting $\partial_z = q$ and $z = -\partial_q$, the shifting rule equals the Taylor expansion:

$$e^{-\alpha z} F(\partial_z) \delta(z) = e^{\alpha \partial_q} F(q) \delta(z) = F(q + \alpha) \delta(z). \tag{108}$$

3.2 Generalized Heaviside Shifting

An important consequence of the Heaviside shifting rule (99) can be obtained by differentiating or integrating the operator with respect to the parameter α :

$$(\partial_\alpha)^{\pm 1} F(\partial_z + \alpha)\delta(z) = (-z)^{\pm 1} e^{-\alpha z} F(\partial_z)\delta(z). \quad (109)$$

Letting now $\alpha \rightarrow 0$, we have the rule for the derivative or the integral of the operator $F(\partial_z)$:

$$\partial_{\partial_z}^{\pm 1} F(\partial_z)\delta(z) = (-z)^{\pm 1} F(\partial_z)\delta(z) = (-z)^{\pm 1} f(z). \quad (110)$$

As an exotic example, starting from the rule

$$\theta(j\partial_z)\delta(z) = \frac{1}{2j\pi z}, \quad (111)$$

by differentiating we obtain

$$\delta(j\partial_z)\delta(z) = \partial_{j\partial_z}\theta(j\partial_z)\delta(z) = -j(-z)\frac{1}{2j\pi z} = \frac{1}{2\pi}. \quad (112)$$

(110) with the plus sign can be further generalized to higher-order derivatives and their linear combinations:

$$\sum_{n=0}^{\infty} A_n \partial_{\partial_z}^n F(\partial_z)\delta(z) = \sum_{n=0}^{\infty} (-z)^n A_n F(\partial_z)\delta(z) = \sum_{n=0}^{\infty} (-z)^n A_n f(z). \quad (113)$$

If the infinite sum represents the Taylor expansion of a function $g(\partial_{\partial_z})$, we can compactly write

$$g(\partial_{\partial_z})F(\partial_z)\delta(z) = g(-z)F(\partial_z)\delta(z) = g(-z)f(z), \quad (114)$$

which gives us an additional method to generate new rules out of a given operational rule $F(\partial_z)\delta(z) = f(z)$. Because the exponential function $g(x) = e^{\alpha x}$ brings us back to the Heaviside shifting method (99):

$$e^{\alpha \partial_{\partial_z}} F(\partial_z)\delta(z) = F(\partial_z + \alpha)\delta(z) = e^{-\alpha z} F(\partial_z)\delta(z), \quad (115)$$

(114) appears to be a generalization of (99).

3.3 Quadratic Shifting Rule

A quadratic shifting rule can be obtained from [25], p.228, eq. (12) and it can be used to transform the rule $F(\partial_z)\delta(z) = f(z)\theta(z)$ to

$$F\left(\sqrt{\partial_z^2 + \beta^2}\right)\delta(z) = f(z)\theta(z) - \beta \int_0^z f\left(\sqrt{z^2 - z'^2}\right) J_1(\beta z') dz' \theta(z), \quad (116)$$

This can also be written as

$$\begin{aligned} [F\left(\sqrt{\partial_z^2 + \beta^2}\right) - F(\partial_z)]\delta(z) &= \int_0^z f\left(\sqrt{z^2 - z'^2}\right) \partial_{z'} J_0(\beta z') dz' \theta(z) \\ &= - \int_0^z f(z') \partial_{z'} J_0\left(\beta \sqrt{z^2 - z'^2}\right) dz' \theta(z). \end{aligned} \quad (117)$$

To test the quadratic rule, we start from $(\partial_z)^{-1}\delta(z) = \theta(z)$ and transform it to

$$\frac{1}{\sqrt{\partial_z^2 + \beta^2}}\delta(z) = \theta(z) - \beta \int_0^z J_1(\beta z') dz' \theta(z) = J_0(\beta z)\theta(z), \quad (118)$$

a well-known result.

3.4 Convolutions

Some integral identities involving convolutions are obtained easily through operational analysis. For example, take the rule

$$\left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^m \delta(z) = \beta^m \frac{m}{z} J_m(\beta z)\theta(z), \quad (119)$$

and apply it on both sides of the identity

$$\begin{aligned} &\left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^m \left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^n \delta(z) \\ &= \left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^{m+n} \delta(z). \end{aligned} \quad (120)$$

Writing the left-hand side as a convolution, we obtain the integral identity

$$\int_0^z \frac{mn}{z'(z-z')} J_m(\beta z') J_n(\beta(z-z')) dz' = \frac{m+n}{z} J_{m+n}(\beta z), \quad (121)$$

given in [32], p.214 (2. 1. 33. 8).

Another similar-looking identity is obtained applying the operational rule

$$\frac{\left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^m}{\sqrt{\partial_z^2 + \beta^2}} \delta(z) = \beta^m J_m(\beta z) \theta(z) \quad (122)$$

and (119) to both sides of

$$\begin{aligned} & \frac{\left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^m \left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^n}{\sqrt{\partial_z^2 + \beta^2}} \delta(z) \\ &= \frac{\left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^{m+n}}{\sqrt{\partial_z^2 + \beta^2}} \delta(z) \end{aligned} \quad (123)$$

whence we have

$$\int_0^z J_m(\beta(z-z')) J_n(\beta z') \frac{n}{z'} dz' = J_{m+n}(\beta z), \quad (124)$$

another well-known integral identity.

4. MULTIPOLE FIELDS AND SOURCES

As a simple application of the operator technique in electromagnetics, let us consider two-dimensional multipole sources (sources located at a point on the xy plane) and their representation in terms of operators operating on the monopole, a delta-function source. The same can be extended to three dimensions at the expense of a somewhat more complicated notation.

4.1 Operator Representation of Multipoles

The two-dimensional electromagnetic TM field has an axial electric field $\mathbf{E}(\boldsymbol{\rho}) = \mathbf{u}_z E(\boldsymbol{\rho})$ which is generated by an axial current distribution $\mathbf{J}(\boldsymbol{\rho}) = \mathbf{u}_z J(\boldsymbol{\rho})$. It satisfies the Helmholtz equation

$$\left(\nabla_t^2 + k^2\right) E(\boldsymbol{\rho}) = jk\eta J(\boldsymbol{\rho}), \quad k = \omega\sqrt{\mu\epsilon}, \quad \eta = \sqrt{\mu/\epsilon}. \quad (125)$$

The field can be expressed in terms of the two-dimensional basis functions ⁵ $w_n(\boldsymbol{\rho})$ as [35]

$$E(\boldsymbol{\rho}) = E \sum_{n=-\infty}^{\infty} \alpha_n w_n(\boldsymbol{\rho}), \quad w_n(\boldsymbol{\rho}) = j^n H_n^{(2)}(k\rho) e^{jn\varphi}. \quad (126)$$

Let us define two circularly polarized complex vectors

$$\mathbf{a}_{\pm} = \mathbf{u}_x \pm j\mathbf{u}_y = e^{\pm j\varphi}(\mathbf{u}_{\rho} \pm j\mathbf{u}_{\varphi}) \quad (127)$$

with the properties

$$\mathbf{a}_{\pm} \cdot \mathbf{a}_{\pm} = 0, \quad \mathbf{a}_{\pm} \cdot \mathbf{a}_{\mp} = 2, \quad (128)$$

$$\mathbf{a}_{\pm} \cdot \mathbf{u}_{\rho} = e^{\pm j\varphi}, \quad \mathbf{a}_{\pm} \cdot \mathbf{u}_{\varphi} = \pm j e^{\pm j\varphi}. \quad (129)$$

With the help of these, the basis functions are seen to satisfy

$$\begin{aligned} \mathbf{a}_{\pm} \cdot \nabla w_n(\boldsymbol{\rho}) &= j^n \mathbf{a}_{\pm} \cdot \left[\mathbf{u}_{\rho} k H_n^{(2)'}(k\rho) + \frac{jn}{\rho} \mathbf{u}_{\varphi} H_n^{(2)}(k\rho) \right] e^{jn\varphi} \\ &= \frac{j^n}{\rho} \left[k\rho H_n^{(2)'}(k\rho) \mp n H_n^{(2)}(k\rho) \right] e^{j(n\pm 1)\varphi}, \end{aligned} \quad (130)$$

where prime denotes the derivative with respect to the argument. Applying recursion formulas of the Hankel function we arrive at the simple relation

$$\mathbf{a}_{\pm} \cdot \nabla w_n(\boldsymbol{\rho}) = jk w_{n\pm 1}(\boldsymbol{\rho}). \quad (131)$$

Thus, the basis functions of different orders can be expressed in terms of that of order zero through powers of certain operators:

$$w_{\pm n}(\boldsymbol{\rho}) = Q_{\pm}^n(\nabla_t) w_0(\boldsymbol{\rho}), \quad Q_{\pm}(\nabla_t) = \frac{\mathbf{a}_{\pm} \cdot \nabla}{jk}. \quad (132)$$

Physically, this corresponds to expressing multipoles through differentiations of the monopole, which was shown already by Maxwell for the electrostatic case [36].

⁵ It is not proper to call the functions $w_n(\boldsymbol{\rho})$ eigenfunctions of the Helmholtz operator $\nabla_t^2 + k^2$ because they have a source at the z axis. The functions $J_n(k\rho) \cos n\varphi$ are true eigenfunctions of the Helmholtz operator.

In terms of the inverse mapping we can write the relation

$$w_{\mp n}(\boldsymbol{\rho}) = Q_{\pm}^{-n}(\nabla_t)w_0(\boldsymbol{\rho}), \quad \Rightarrow \quad Q_{\pm}(\nabla_t) = Q_{\mp}^{-1}(\nabla_t). \quad (133)$$

This would imply $Q_+(\nabla_t)Q_-(\nabla_t) = 1$, but actually we have

$$Q_+(\nabla_t)Q_-(\nabla_t) = -\frac{1}{k^2}(\mathbf{a}_+ \cdot \nabla)(\mathbf{a}_- \cdot \nabla) = -\frac{\nabla_t^2}{k^2}. \quad (134)$$

However, when operating on functions w_n and using the two-dimensional Green function equation [35]

$$(\nabla_t^2 + k^2)w_0(\boldsymbol{\rho}) = (\nabla_t^2 + k^2)H_0^{(2)}(k\rho) = -4j\delta(\boldsymbol{\rho}), \quad (135)$$

we obtain

$$\begin{aligned} Q_+(\nabla_t)Q_-(\nabla_t)w_n(\boldsymbol{\rho}) &= -Q_+^n(\nabla_t)\frac{\nabla_t^2}{k^2}w_0(\boldsymbol{\rho}) = Q_+^n(\nabla_t)\left[w_0(\boldsymbol{\rho}) + \frac{4j}{k^2}\delta(\boldsymbol{\rho})\right] \\ &= w_n(\boldsymbol{\rho}) + \frac{4j}{k^2}Q_+^n(\nabla_t)\delta(\boldsymbol{\rho}), \end{aligned} \quad (136)$$

which means that, indeed, for $\rho \neq 0$ we can replace ∇_t^2 by $-k^2$ and $Q_+(\nabla_t) = Q_-^{-1}(\nabla_t)$. The difference in fields only at $\rho = 0$ is of no concern for us. The source of such a difference field is what has been called a nonradiating source [34] and can be omitted in the following.

Thus, we have arrived at the following recursion relation for the basis functions

$$\begin{aligned} H_n^{(2)}(k\rho)e^{\pm jn\varphi} &= j^{-n}(\mathbf{a}_{\pm} \cdot \nabla/jk)^n H_0^{(2)}(k\rho) \\ &= j^{-n}(\mathbf{a}_{\mp} \cdot \nabla/jk)^{-n} H_0^{(2)}(k\rho), \end{aligned} \quad (137)$$

which also works when $H^{(2)}$ is replaced by $H^{(1)}$, J or Y functions on both sides of the equation. Taking the average of the two relations (137) we can further write

$$\begin{aligned} H_n^{(2)}(k\rho)\cos n\varphi &= j^n \frac{1}{2} [(\mathbf{a}_{\pm} \cdot \nabla/jk)^n + (\mathbf{a}_{\pm} \cdot \nabla/jk)^{-n}] H_0^{(2)}(k\rho) \\ &= j^n T_n(\partial_x/jk) H_0(k\rho), \end{aligned} \quad (138)$$

by applying the definition of the Chebyshev polynomial:

$$T_n(x) = \frac{1}{2} \left[\left(x + j\sqrt{1-x^2} \right)^n + \left(x - j\sqrt{1-x^2} \right)^n \right]. \quad (139)$$

(137) is a generalization of

$$J_n(k\rho) \cos n\varphi = j^{-n} T_n(\partial_x/jk) J_0(k\rho), \quad (140)$$

given originally by van der Pol [37]. Similar operator relations in three dimensions have also been discussed in [38–40].

Now the field in terms of the basis functions, also called the multipole field, can be expressed in operator form in terms of the function $w_0(\boldsymbol{\rho})$ as

$$E(\boldsymbol{\rho}) = EQ(\nabla_t)w_0(\boldsymbol{\rho}), \quad (141)$$

with the multipole operator defined by

$$\begin{aligned} Q(\nabla_t) &= \alpha_0 + \sum_{n=1}^{\infty} [\alpha_n Q_+^n(\nabla_t) + \alpha_{-n} Q_-^n(\nabla_t)] \\ &= \alpha_0 + \sum_{n=1}^{\infty} [\alpha_n (\mathbf{a}_+ \cdot \nabla / jk)^n + \alpha_{-n} (\mathbf{a}_- \cdot \nabla / jk)^n] \\ &= \sum_{n=-\infty}^{\infty} \alpha_n Q_+^n(\nabla_t). \end{aligned} \quad (142)$$

Its source, the multipole source, is defined by

$$\begin{aligned} J(\boldsymbol{\rho}) &= \frac{1}{jk\eta} (\nabla_t^2 + k^2) E(\boldsymbol{\rho}) = \frac{E}{jk\eta} Q(\nabla_t) (\nabla_t^2 + k^2) w_0(\boldsymbol{\rho}) \\ &= -\frac{4E}{k\eta} Q(\nabla_t) \delta(\boldsymbol{\rho}). \end{aligned} \quad (143)$$

Obviously, since only delta function and its derivatives are involved, $J(\boldsymbol{\rho})$ is a collection of multipole sources. However, in some cases it may be possible to express the infinite sum of operators as a function of ∇_t which can be interpreted as a continuous source.

4.2 Synthesis of Radiation Patterns

The previous operator expressions can be applied, for example, if we wish to find the source giving rise to a certain radiation pattern. Starting from the far-field approximation $\rho \rightarrow \infty$ of (141), allowing us to set $\nabla_t \rightarrow -jk\mathbf{u}_\rho$, we have [35]

$$E(\rho, \varphi) \approx EQ(-jk\mathbf{u}_\rho) \sqrt{\frac{2j}{\pi k\rho}} e^{-jk\rho}, \quad Q_\pm(-jk\mathbf{u}_\rho) = -e^{\pm j\varphi}. \quad (144)$$

$Q(-jk\mathbf{u}_\rho)$ is a function of the radial unit vector \mathbf{u}_ρ and it determines the radiation pattern. Since its expression is of the form of the Fourier-series,

$$Q(-jk\mathbf{u}_\rho) = \sum_{n=-\infty}^{\infty} (-1)^n \alpha_n e^{jn\varphi}, \quad (145)$$

when inserted in (144), the coefficients α_n can be determined from a given far field pattern through a well-known procedure. This leads to the possibility to determine, in multipole expansion, a source radiating the desired pattern.

As a simplifying restriction, let us assume that the radiation pattern is symmetric in the φ coordinate. In this case we have $\alpha_n = \alpha_{-n}$ and, consequently, $J_n = J_{-n}$. The far-field approximation (144) is, then, of the Fourier cosine-series form

$$E(\rho, \varphi) \approx E \sqrt{\frac{2j}{\pi k \rho}} e^{-jk\rho} F(\varphi), \quad (146)$$

$$F(\varphi) = Q(-jk\mathbf{u}_\rho) = \alpha_0 + 2 \sum_{n=1}^{\infty} (-1)^n \alpha_n \cos n\varphi. \quad (147)$$

If the radiation-pattern function $F(\varphi)$ is given, the coefficients are obtained from orthogonality as

$$\alpha_n = \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} F(\varphi) \cos n\varphi d\varphi. \quad (148)$$

After finding the coefficients, the multipole operator defining the field with the given radiation pattern (141) and the corresponding source (143) can be expressed as

$$Q(\nabla_t) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n [Q_+^n(\nabla_t) + Q_+^{-n}(\nabla_t)] = \alpha_0 + 2 \sum_{n=1}^{\infty} \alpha_n T_n(\partial_x/jk). \quad (149)$$

In this symmetric case it is seen to contain only the differential operator ∂_x .

Simple Pattern

As a simple example, let us find a source with the radiation pattern $F(\varphi) = \cos^2(\varphi/2)$, possessing a null in the direction $\varphi = \pm\pi$ and

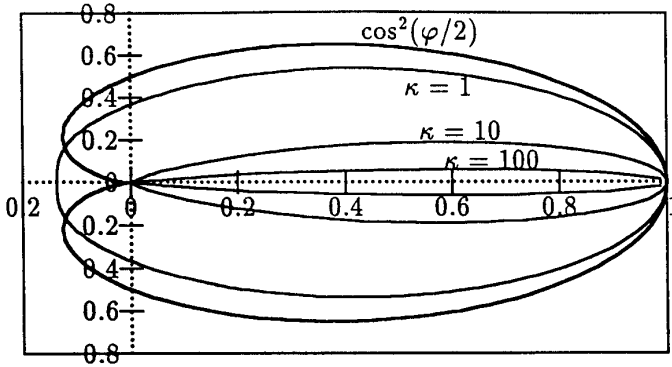


Figure 1. Polar plots of the radiation pattern $\cos^2(\varphi/2)$ and the normalized Gaussian radiation pattern $\exp[-2\kappa \sin^2(\varphi/2)]$ for three parameter values $\kappa = 1, 10$ and 100 .

maximum at $\varphi = 0$, see Figure 1. Because $F(\varphi) = (1 + \cos \varphi)/2$, we can easily find from (148)

$$\alpha_0 = 1/2, \quad \alpha_1 = -\frac{1}{4}, \quad \alpha_{n>1} = 0. \tag{150}$$

In this special case the multipole operator has only two terms,

$$Q(\nabla_t) = \frac{1}{2} - \frac{\partial_x}{2jk}, \tag{151}$$

and the multipole source has the form

$$J(\boldsymbol{\rho}) = -\frac{2E}{k\eta} \left[\delta(x) - \frac{1}{jk} \delta'(x) \right] \delta(y), \tag{152}$$

i.e., consisting of a monopole and a dipole source.

Let us check this result by inserting the source in the field expression

$$E(\boldsymbol{\rho}) = -jk\eta \int \frac{1}{4j} H_o^{(2)}(kD) J(\boldsymbol{\rho}') dx' dy'. \tag{153}$$

Inserting the far-field approximation to the distance function

$$D = \sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2} \approx \rho \sqrt{1 - 2\frac{\boldsymbol{\rho} \cdot \boldsymbol{\rho}'}{\rho^2}} \approx \rho - \boldsymbol{\rho}' \cdot \mathbf{u}_\rho \tag{154}$$

we have

$$H_0^{(2)}(kD) \approx \sqrt{\frac{2j}{\pi D}} e^{-jkD} \approx \sqrt{\frac{2j}{\pi \rho}} e^{-jk\rho} e^{jk\boldsymbol{\rho}' \cdot \mathbf{u}_\rho}. \quad (155)$$

The field integral in the far region then becomes

$$E(\boldsymbol{\rho}) \approx -\frac{k\eta}{4} \sqrt{\frac{2j}{\pi \rho}} e^{-jk\rho} \int e^{jk\boldsymbol{\rho}' \cdot \mathbf{u}_\rho} J(\boldsymbol{\rho}') dx' dy'. \quad (156)$$

Inserting (152) we obtain

$$\begin{aligned} E(\boldsymbol{\rho}) &\approx E \sqrt{\frac{2j}{\pi \rho}} e^{-jk\rho} \int \frac{1}{2} e^{jkx' \cos \varphi} \left[\delta(x') - \frac{1}{jk} \delta'(x') \right] \delta(y) dx' \\ &= E \sqrt{\frac{2j}{\pi \rho}} e^{-jk\rho} \frac{1}{2} (1 + \cos \varphi), \end{aligned} \quad (157)$$

which is the far field with the correct radiation pattern.

Gaussian Pattern

As a second example let us consider a radiation pattern of Gaussian type:

$$F(\varphi) = Q(-jk\mathbf{u}_\rho) = A \exp[-2\kappa \sin^2(\varphi/2)] = A e^{-\kappa} e^{\kappa \cos \varphi}, \quad (158)$$

where A is chosen so that the integral from $-\pi$ to π gives unity. Large κ makes the beamwidth small, see Figure 1, in which case the function can be approximated by $A \exp[-\kappa\varphi^2/2]$. Equating (147) with $F(\varphi)$ we can solve

$$\alpha_0 = \frac{A}{2\pi} e^{-\kappa} \int_{-\pi}^{\pi} e^{\kappa \cos \varphi} d\varphi = A e^{-\kappa} I_0(\kappa) = \frac{1}{2\pi}, \quad \Rightarrow \quad A = \frac{e^\kappa}{2\pi I_0(\kappa)}. \quad (159)$$

The expression for the general coefficient is

$$\alpha_n = \alpha_{-n} = \frac{A}{2\pi} (-1)^n e^{-\kappa} \int_{-\pi}^{\pi} e^{\kappa \cos \varphi} \cos n\varphi d\varphi = (-1)^n \frac{I_n(\kappa)}{2\pi I_0(\kappa)}. \quad (160)$$

For $\kappa \rightarrow \infty$ the coefficients are finite: $\alpha_n \rightarrow (-1)^n/2\pi$. From (143) the multipole source can now be expressed in the operational form

$$J(\boldsymbol{\rho}) = -\frac{4E}{2\pi k\eta I_0(\kappa)} \left[I_0(\kappa) + 2 \sum_{n=1}^{\infty} (-1)^n I_n(\kappa) T_n(\partial_x/jk) \right] \delta(\boldsymbol{\rho}). \quad (161)$$

However, this can be drastically simplified by expanding the multipole operator as

$$\begin{aligned} Q(\nabla_t) &= \sum_{n=-\infty}^{\infty} \alpha_n [Q_+^n(\nabla_t) + Q_-^n(\nabla_t)] \\ &= \frac{1}{2\pi I_0(\kappa)} \sum_{n=-\infty}^{\infty} (-1)^n [Q_+^n(\nabla_t) + Q_-^n(\nabla_t)] I_n(\kappa) \\ &= \frac{1}{2\pi I_0(\kappa)} \exp \left\{ -\frac{\kappa}{2} [Q_+(\nabla_t) + Q_+^{-1}(\nabla_t)] \right\} \\ &\quad \exp \left\{ -\frac{\kappa}{2} [Q_-(\nabla_t) + Q_-^{-1}(\nabla_t)] \right\}. \end{aligned} \quad (162)$$

The last form comes from the summing formula [32] (5. 8. 3. 2)

$$\sum_{n=-\infty}^{\infty} t^n I_n(\kappa) = \exp \left\{ \frac{\kappa}{2} (t + t^{-1}) \right\}. \quad (163)$$

Because of $Q_{\pm}(\nabla_t) = Q_{\mp}^{-1}(\nabla_t)$, we end up in the simple result

$$Q(\nabla_t) = \frac{e^{-\kappa\partial_x/jk}}{2\pi I_0(\kappa)}. \quad (164)$$

Thus, the multipole source with the Gaussian radiation field can be written simply as

$$\begin{aligned} J(\boldsymbol{\rho}) &= -\frac{4E}{k\eta} Q(\nabla_t) \delta(\boldsymbol{\rho}) = -\frac{4E e^{-\kappa\partial_x/jk}}{2\pi k\eta I_0(\kappa)} \delta(\boldsymbol{\rho}) \\ &= -\frac{4E}{2\pi k\eta I_0(\kappa)} \delta \left(x - \frac{\kappa}{jk} \right) \delta(y). \end{aligned} \quad (165)$$

This means that we have found a source giving the Gaussian beam of radiation as a point source in complex space. This idea was first

expressed by Deschamps [41]. By adding nonradiating sources we can find an infinity of other such sources.

Again it is a simple thing to check the result by inserting (165) in the far-field formula (156)

$$\begin{aligned}
 E(\boldsymbol{\rho}) &\approx -\frac{k\eta}{4} \sqrt{\frac{2j}{\pi\rho}} e^{-jk\rho} \int e^{jk\boldsymbol{\rho}' \cdot \mathbf{u}_\rho} \left(-\frac{4E}{2\pi k\eta I_0(\kappa)} \delta\left(x' - \frac{\kappa}{jk}\right) \delta(y') \right) dx' dy' \\
 &= \frac{E}{2\pi I_0(\kappa)} \sqrt{\frac{2j}{\pi\rho}} e^{-jk\rho} e^{\kappa \cos \varphi} = E \sqrt{\frac{2j}{\pi\rho}} e^{-jk\rho} A e^{-\kappa} e^{\kappa \cos \varphi}, \quad (166)
 \end{aligned}$$

which is seen to reproduce the original radiation pattern (158).

5 CONCLUSION

Heaviside operator formalism was reviewed and different methods for expressing the operational form $F(\partial_z)\delta(z)$, $F(\nabla_t)\delta(\boldsymbol{\rho})$ or $F(\nabla)\delta(\mathbf{r})$ in terms of computable functions were presented. Such expressions often occur in electromagnetic problems when studying the response of a impulse input through a linear system. For future convenience, a table of operational rules was compiled for one-dimensional operators in the form $F(\partial_z)\delta(z) = f(z)$. As an application, operator formulation of two-dimensional multipole expansions was considered, in which forming the multipole out of a monopole source (delta-function source) can be expressed through a linear operator.

Appendix: Table of Operation Rules

Applying a table of Laplace transforms in, e.g., [27], and tables of integral identities as [31, 32], a table of pseudo-differential operators $F(\partial_z)$ operating on the delta function⁶ has been collected for future convenience in working with operators. The items in the table are of the form

$$F(\partial_z)\delta(z) = f(z)$$

and arranged according to the operator function $F(\partial_z)$. Reference to the source of some of the more uncommon rules are given in the form [reference]p‘pagenumber’(equation number) at the end of the rule.

⁶ $n = 1, 2, 3, \dots$ is an integer while ν may be any number.

Integer Powers

$$\partial_z^{-n} \delta(z) = \frac{z^{n-1}}{(n-1)!} \theta(z) \quad (167)$$

$$\frac{1}{\partial_z + \beta} \delta(z) = e^{-\beta z} \theta(z) \quad (168)$$

$$\frac{\partial_z + \beta_1}{\partial_z + \beta_2} \delta(z) = \delta(z) + (\beta_1 - \beta_2) e^{-\beta_2 z} \theta(z) \quad (169)$$

$$\frac{1}{(\partial_z + \beta)^2} \delta(z) = z e^{-\beta z} \theta(z) \quad (170)$$

$$\frac{1}{(\partial_z + \beta_1)(\partial_z + \beta_2)} \delta(z) = -\frac{e^{-\beta_1 z} - e^{-\beta_2 z}}{\beta_1 - \beta_2} \theta(z) \quad (171)$$

$$\frac{1}{\partial_z^2 + \beta^2} \delta(z) = \frac{1}{\beta} \sin(\beta z) \theta(z) \quad (172)$$

$$\frac{1}{\partial_z^2 - \beta^2} \delta(z) = \frac{1}{\beta} \sinh(\beta z) \theta(z) \quad (173)$$

Fractional Powers

$$\frac{1}{\partial_z^\nu} \delta(z) = \frac{z^{\nu-1}}{\Gamma(\nu)} \theta(z) \quad (174)$$

$$\left(\sqrt{\partial_z + \beta_1} - \sqrt{\partial_z + \beta_2} \right) \delta(z) = -\frac{1}{2z\sqrt{\pi z}} \left(e^{-\beta_1 z} - e^{-\beta_2 z} \right) \theta(z) \quad (175)$$

$$\frac{1}{\sqrt{\partial_z + \beta}} \delta(z) = \frac{e^{\beta z}}{\sqrt{\pi z}} \theta(z) \quad (176)$$

$$\begin{aligned} \sqrt{\sqrt{\partial_z^2 + \beta^2} - \partial_z} \delta(z) &= \frac{1}{j\sqrt{2}} \left[\sqrt{\partial_z + j\beta} - \sqrt{\partial_z - j\beta} \right] \delta(z) \\ &= \frac{\sin \beta z}{z\sqrt{2\pi z}} \theta(z), \quad [26](139) \end{aligned} \quad (177)$$

$$\sqrt{\frac{\sqrt{\partial_z^2 + a^2} \pm \partial_z}{\partial_z^2 + a^2}} \delta(z) = \sqrt{\frac{2}{\pi z}} \begin{pmatrix} \cos az \\ \sin az \end{pmatrix} \theta(z), \quad [26](140, 141) \quad (178)$$

$$\sqrt{\frac{\sqrt{\partial_z^2 - a^2} \pm \partial_z}{\partial_z^2 - a^2}} \delta(z) = \sqrt{\frac{2}{\pi z}} \begin{pmatrix} \cosh az \\ \sinh az \end{pmatrix} \theta(z), \quad [26], \text{p.236, (142, 143)} \quad (179)$$

$$\frac{1}{\sqrt{\partial_z^2 + \beta^2}} \delta(z) = J_0(\beta z) \theta(z) \quad (180)$$

$$\frac{1}{(\partial_z^2 + \beta^2)^{\nu+1/2}} \delta(z) = \frac{\sqrt{\pi} z^\nu}{(2\beta)^\nu \Gamma(\nu + 1/2)} J_\nu(\beta z) \theta(z),$$

$(\nu > -1/2)$ [26](146) (181)

$$\frac{(\sqrt{\partial_z^2 + \beta^2} - \partial_z)^\nu}{\sqrt{\partial_z^2 + \beta^2}} \delta(z) = \beta^\nu J_\nu(\beta z) \theta(z) \tag{182}$$

$$\begin{aligned} \left(\frac{\sqrt{\partial_z^2 + \beta^2} - \partial_z}{\sqrt{\partial_z^2 + \beta^2} + \partial_z}\right)^{n/2} \delta(z) &= \frac{1}{\beta^n} \left(\sqrt{\partial_z^2 + \beta^2} - \partial_z\right)^n \delta(z) \\ &= \frac{n}{z} J_n(\beta z) \theta(z) \end{aligned} \tag{183}$$

$$\begin{aligned} \left(\frac{\alpha \partial_z - \sqrt{\partial_z^2 + \beta^2}}{\alpha \partial_z + \sqrt{\partial_z^2 + \beta^2}}\right) \delta(z) &= \frac{\alpha - 1}{\alpha + 1} \delta(z) - \frac{8\alpha}{\alpha^2 - 1} \sum_{n=1}^{\infty} n \\ &\quad \left(\frac{\alpha - 1}{\alpha + 1}\right)^n \frac{J_{2n}(\beta z)}{z} \theta(z) \end{aligned} \tag{184}$$

$$\begin{aligned} &\frac{1}{\sqrt{\partial_z^2 + \beta^2}} \frac{\alpha \partial_z - \sqrt{\partial_z^2 + \beta^2}}{\alpha \partial_z + \sqrt{\partial_z^2 + \beta^2}} \delta(z) \\ &= \left(\frac{\alpha - 1}{\alpha + 1} J_0(\beta z) - \frac{4\alpha}{\alpha^2 - 1} \sum_{n=1}^{\infty} \left(\frac{\alpha - 1}{\alpha + 1}\right)^n J_{2n}(\beta z)\right) \theta(z) \end{aligned} \tag{185}$$

Exponentials and Trigonometric Functions

$$e^{a\partial_z} \delta(z) = \delta(z + a) \tag{186}$$

$$\frac{1}{1 \pm e^{-a\partial_z}} \delta(z) = \sum_{n=0}^{\infty} (\mp 1)^n \delta(z - na) \tag{187}$$

$$\left(\frac{\cosh}{\sinh}\right)(a\partial_z) \delta(z) = \frac{1}{2} [\delta(z + a) \pm \delta(z - a)] \tag{188}$$

$$\frac{\sinh a\partial_z}{a\partial_z} \delta(z) = \frac{1}{2a} \theta(a^2 - z^2) \tag{189}$$

$$\left(\frac{\sinh a\partial_z}{a\partial_z}\right)^2 \delta(z) = \frac{1}{a^2} (a - |z|) \theta(a^2 - z^2) \tag{190}$$

$$\frac{\tanh a\partial_z}{\partial_z} \delta(z) = \theta(z) + 2 \sum_{n=1}^{\infty} (-1)^n \theta(z - 2na) \tag{191}$$

$$\begin{pmatrix} \cos \\ \sin \end{pmatrix} (a\partial_z)\delta(z) = \frac{1}{2} [\delta(z + ja) \pm \delta(z - ja)] \quad (192)$$

$$\frac{1}{\cos(a\partial_z)}\delta(z) = \frac{1}{2a \cosh(\pi z/2a)} \quad (193)$$

$$\begin{aligned} \frac{1}{\sin(a\partial_z)}\delta(z) &= \frac{e^{\pi z/2a}}{2a \cosh(\pi z/2a)} \\ &= \frac{1}{a(1 + e^{-\pi z/a})}, \quad [30](3.311.9) \end{aligned} \quad (194)$$

$$\begin{aligned} \cot(a\partial_z)\delta(z) &= \frac{e^{\pi z/2a}}{2a \sin(\pi z/2a)} \\ &= \frac{1}{a(1 - e^{-\pi z/a})}, \quad [30](3.311.8) \end{aligned} \quad (195)$$

$$e^{(a\partial_z)^2}\delta(z) = \frac{e^{-(z/2a)^2}}{\sqrt{4\pi a}}, \quad [8]p.387 \quad (196)$$

$$\frac{1}{\partial_z} \cos(\beta/\partial_z)\delta(z) = \text{ber}\left(2\sqrt{\beta z}\right)\theta(z), \quad [42]p.5 \quad (197)$$

$$\frac{1}{\partial_z} \sin(\beta/\partial_z)\delta(z) = \text{bei}\left(2\sqrt{\beta z}\right)\theta(z), \quad [42]p.5 \quad (198)$$

$$\frac{1}{\partial_z} e^{-\beta/\partial_z}\delta(z) = J_0\left(2\sqrt{\beta z}\right)\theta(z), \quad [26](161) \quad (199)$$

$$\frac{1}{\partial_z^{\nu+1}} e^{-\beta/\partial_z}\delta(z) = (z/\beta)^{\nu/2} J_\nu\left(2\sqrt{\beta z}\right)\theta(z), \quad [26](162) \quad (200)$$

$$\frac{1}{\sqrt{\beta\partial_z}} e^{\beta/\partial_z}\delta(z) = \frac{1}{\sqrt{\pi\beta z}} \cosh\left(2\sqrt{\beta z}\right)\theta(z), \quad [26](165) \quad (201)$$

$$\frac{1}{\partial_z} e^{-\sqrt{a\partial_z}}\delta(z) = \text{erfc}\left(\sqrt{a/4z}\right)\theta(z), \quad [26](168) \quad (202)$$

$$\begin{aligned} & \frac{e^{-\sqrt{\partial_z^2 + \beta^2}a}}{\sqrt{\partial_z^2 + \beta^2} \left(\partial_z + \sqrt{\partial_z^2 + \beta^2}\right)^\nu} \delta(z) \\ &= \frac{1}{\beta^\nu} \left(\frac{z-a}{z+a}\right)^{\nu/2} J_\nu\left(\beta\sqrt{z^2 - a^2}\right)\theta(z-a) \end{aligned} \quad (203)$$

$$\begin{aligned} & \frac{e^{-\sqrt{\partial_z^2 - \beta^2}a}}{\sqrt{\partial_z^2 - \beta^2} \left(\partial_z + \sqrt{\partial_z^2 - \beta^2}\right)^\nu} \delta(z) \\ &= \left(\frac{z-a}{z+a}\right)^\nu I_{2\nu}\left(\beta\sqrt{z^2 - a^2}\right)\theta(z-a) \end{aligned} \quad (204)$$

Logarithms

$$\frac{\ln \partial_z}{\partial_z} \delta(z) = - (0.5772.. + \ln z) \theta(z), \quad [29](865.901) \quad (205)$$

$$\frac{\ln \partial_z}{\sqrt{\partial_z}} \delta(z) = - \frac{1}{\sqrt{\pi z}} (0.5772.. + 2 \ln 2 + \ln z) \theta(z), \quad [29](865.904) \quad (206)$$

$$\ln \frac{\partial_z + \beta_1}{\partial_z + \beta_2} \delta(z) = \frac{1}{z} \left(e^{-\beta_2 z} - e^{-\beta_1 z} \right) \theta(z), \quad [26](155) \quad (207)$$

$$\ln \sqrt{\frac{\partial_z + \beta}{\partial_z - \beta}} \delta(z) = \frac{1}{z} \sinh(\beta z) \theta(z) \quad (208)$$

$$\ln \frac{\partial_z^2 + \beta_1^2}{\partial_z^2 + \beta_2^2} \delta(z) = \frac{2}{z} (\cos \beta_2 z - \cos \beta_1 z) \theta(z), \quad [26](158) \quad (209)$$

$$\frac{1}{\partial_z} \ln(a \partial_z \pm 1) \delta(z) = - \text{Ei}(\mp z/a) \theta(z). \quad [32](2.5.3.1, .8) \quad (210)$$

Bessel Functions

$$I_n(a \partial_z) \delta(z) = \frac{(-1)^n T_n(z/a)}{\pi \sqrt{a^2 - z^2}} \theta(a^2 - z^2), \quad [32](2.18.1.10) \quad (211)$$

$$K_n(a \partial_z) \delta(z) = \frac{T_n(z/a)}{\sqrt{z^2 - a^2}} \theta(z - a), \quad [32](2.18.1.12) \quad (212)$$

$$\begin{aligned} \partial_z^\nu I_\nu(a \partial_z) \delta(z) &= \frac{\Gamma(\nu + \frac{1}{2})}{\pi^{3/2}} \left(\frac{\cos 2\nu\pi}{(a^2 - z^2)^{\nu+1/2}} \theta(a^2 - z^2) \right. \\ &\quad \left. - \frac{\sin 2\nu\pi}{(z^2 - a^2)^{\nu+1/2}} \theta(z - a) \right) \\ \nu < 1/2, \quad [25]p.277(4) \quad (213) \end{aligned}$$

$$\begin{aligned} \frac{1}{\partial_z^\nu} I_\nu(a \partial_z) \delta(z) &= \frac{(a^2 - z^2)^{\nu-1/2}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2}) (2a)^\nu} \theta(a^2 - z^2), \\ \nu > -1/2, \quad [25]p.277(5) \quad (214) \end{aligned}$$

$$\begin{aligned} \frac{1}{\partial_z^\nu} K_\nu(a \partial_z) \delta(z) &= \frac{(z^2 - a^2)^{\nu-1/2}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2}) (2a)^\nu} \theta(z - a), \\ \nu > -1/2, \quad [25]p.278(14) \quad (215) \end{aligned}$$

$$\frac{1}{\partial_z^n} e^{\mp ja\partial_z} H_n^{(1,2)}(a\partial_z) \delta(z) = \mp j \frac{2[z(z \mp 2ja)]^{n-1/2}}{\sqrt{\pi}(2a)^n \Gamma(n+1/2)} \theta(z),$$

[25]p.273, 274(9, 10) (216)

$$I_0\left(a\sqrt{\partial_z^2 - \beta^2}\right) \delta(z) = \frac{\cos\left(\beta\sqrt{a^2 - z^2}\right)}{\pi\sqrt{a^2 - z^2}} \theta(a^2 - z^2),$$

[25]p.282(27) (217)

$$\frac{1}{\sqrt{\partial_z}} I_{n+1/2}(a\partial_z) \delta(z) = \frac{(-1)^n}{\sqrt{2\pi a}} P_n(z/a) \theta(a^2 - z^2),$$

[32](2.17.5.2) (218)

$$\frac{1}{\sqrt{\partial_z}} K_{n+1/2}(a\partial_z) \delta(z) = \sqrt{\frac{\pi}{2a}} P_n(z/a) \theta(z - a),$$

[32](2.17.5.4) (219)

$$K_{\partial_z}(a) \delta(z) = \frac{1}{2} e^{-a \cosh z},$$

[8]p.31 (220)

$$j_n(ja\partial_z) \delta(z) = \frac{1}{2aj^n} P_n(z/a) \theta(a^2 - z^2) \quad [8]p.394$$

(221)

Other Functions

$$\frac{1}{(\partial_z^2 + \beta^2)^{n/2}} T_n\left(\frac{\partial_z}{\sqrt{\partial_z^2 + \beta^2}}\right) \delta(z) = \frac{z^{n-1}}{(n-1)!} \cos(\beta z) \theta(z),$$

[29](860.99) (222)

$$e^{a^2 \partial_z^2} \operatorname{erfc}(a\partial_z) \delta(z) = \frac{1}{2a\sqrt{\pi}} e^{-z^2/4a^2} \theta(z),$$

[26](189) (223)

$$\arctan \frac{\beta}{\partial_z} \delta(z) = \frac{1}{z} \sin(\beta z) \theta(z),$$

[26](181), [29](861.01) (224)

$$\frac{1}{\partial_z} \operatorname{arccot}(a\partial_z) \delta(z) = \operatorname{Si}(z/a) \theta(z), \quad [42]p.5$$

(225)

$$\delta(j\partial_z) \delta(z) = \frac{1}{2\pi}, \quad [8]p.385$$

(226)

$$\theta(j\partial_z) \delta(z) = \frac{1}{2} \operatorname{sgn}(j\partial_z) \delta(z) = \frac{1}{2\pi j z}$$

(227)

REFERENCES

1. Whittaker, E., "Oliver Heaviside," *Bull. Calcutta Math. Soc.*, Vol. 20, 199–220, Sep. 1928. Reprint in reference [10].
2. Nahin, P. J., *Oliver Heaviside, Sage in Solitude*, IEEE Press, New York, 1988.
3. Heaviside, O., "On operators in physical mathematics, Part 1," *Proc. Roy. Soc.*, Vol. 52, 504–529, February 1893. In condensed form, [6], Vol. 2, 434–443.
4. Heaviside, O., "On operators in physical mathematics, Part 2," *Proc. Roy. Soc.*, Vol. 54, 105–143, June 1893. In condensed form, [6], Vol. 2, 443–457.
5. Heaviside, O., "On operators in physical mathematics, Part 3," submitted to *Proc. Roy. Soc.*, not published. A condensed form appeared in [6], Vol. 2, 457–492.
6. Heaviside, O., *Electromagnetic Theory*, Volumes 1, 2, 3, Ernest Benn Ltd., London, second reprint, 1925.
7. Carson, J. R., *Electric Circuit Theory and the Operational Calculus*, McGraw-Hill, New York, 1926.
8. van der Pol, B., and H. Bremmer, *Operational Calculus Based on the Two-Sided Laplace Integral*, 2nd ed., Cambridge University Press, 1959.
9. Mikusinski, J., *Operational Calculus*, Pergamon Press, Warsaw, 1959.
10. Moore, D. H., *Heaviside Operational Calculus, an Elementary Foundation*, American Elsevier, New York, 1971.
11. Krabbe, G., *Operational Calculus*, Plenum, New York, 1975.
12. Berg, E. J., *Heaviside's Operational Calculus*, 2nd ed., McGraw-Hill, New York, 1936.
13. Stephens, E., *The Elementary Theory of Operational Mathematics*, McGraw-Hill, New York, 1937.
14. Bush, V., *Operational Circuit Analysis*, Wiley, New York, 1948.
15. Josephs, H. J., *Heaviside's Electric Circuit Theory*, 2nd ed., Methuen, London, 1950.
16. Hallen, E., *Tvungna Svängningar. Operationskalkyl*, (Forced Oscillations, Operational Calculus, in Swedish), Erik Hallen, Stockholm, 1965.
17. Kullstam, P. A., "Heaviside's operational calculus: Oliver's revenge," *IEEE Trans. Education*, Vol. 34, No. 2, 155–166, May 1991.
18. Kullstam, P. A., "Heaviside's operational calculus applied to electrical circuit problems," *IEEE Trans. Education*, Vol. 35, No. 4, 266–277, November 1992.

19. Lindell, I. V., "Heaviside operational calculus in electromagnetic image theory," *J. Electro. Waves Appl.*, Vol. 11, No. 1, 119–132, 1997.
20. Lindell, I. V., K. I. Nikoskinen, and A. Viljanen, "Electrostatic image method for the anisotropic half space," *IEE Proceedings, Science, Measurement & Technology*, Vol. 144, No. 4, 156–163, July 1997.
21. Pincherle, S., "Funktionaloperationen und-Gleichungen," *Encyclopädie der Mathematischen Wissenschaften*, Vol. 2, Part 1, 761–817, Teubner, Leipzig, 1916.
22. Cooper, J. L. B., "Heaviside and the operational calculus," *The Math. Gazette*, Vol. 36, 5–19, 1952.
23. Lützen, J., "Heaviside's operational calculus and the attempts to rigorize it," *Arch. Hist. Exact Sci.*, Vol. 21, 161–200, 1979.
24. Petrova, S. S., "Heaviside and the development of symbolic calculus," *Arch. Hist. Exact Sci.*, Vol. 37, 1–23, 1987.
25. Erdelyi, A. ed., *Tables of Integral Transforms*, Vol. 1, Bateman Manuscript Project, McGraw-Hill, New York, 1954.
26. Doetsch, G., *Anleitung zum Praktischen Gebrauch der Laplace-Transformation*, 225–253, Oldenbourg, Munich, 1961.
27. Abramowitz, M., and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1964.
28. Hladik, J., *La Transformation de Laplace a Plusieurs Variables*, Masson, Paris, 1969.
29. Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, 4th ed., Macmillan, New York, 1961.
30. Gradshteyn, I. S., and I. M. Ryzhik, *Tables of Integrals, Sums, Series and Products*, 4th ed., Academic Press, New York, 1980.
31. Prudnikov, A. P., Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series, Elementary Functions*, Gordon and Breach, New York, 1986.
32. Prudnikov, A. P., Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series, Special Functions*, Gordon and Breach, New York, 1986.
33. Lindell, I. V., "Heaviside operational method for delta function sequences," *J. Electro. Waves Appl.*, Vol. 12, 913–927, 1998.
34. Lindell, I. V., *Methods for Electromagnetic Field Analysis*, 2nd ed., Oxford University Press and IEEE Press, 1995.
35. Harrington, R. F., *Time-Harmonic Electromagnetic Fields*, McGraw-Hill, New York, 1961.
36. Maxwell, J. C., *A Treatise on Electricity and Magnetism*, Vols. 1, 2, 3rd ed., Clarendon Press, Oxford, 1904. Original edition, 1873.

37. van der Pol, B., "A generalization of Maxwell's definition of solid harmonics to waves in n dimensions," *Physica*, Vol. 3, No. 6, 393–397, June 1936.
38. Erdelyi, A., "Zur Theorie der Kugelwellen," *Physica*, Vol. 4, No. 2, 107–120, February 1937.
39. Bouwkamp, C. J., and H. B. G. Casimir, "On multipole expansions in the theory of electromagnetic radiation," *Physica*, Vol. 20, 539–554, 1954.
40. Schwinger, J., L. L. DeRaad, K. A. Milton, and W. Tsai, *Classical Electrodynamics*, Perseus Books, Reading MA, 1998.
41. Deschamps, G. A., "The Gaussian beam as a bundle of complex rays," *Electron. Lett.*, Vol. 7, No. 23, 684–685, 1971.
42. Rybner, J., "Om Heaviside's operatorregning," (in Danish), *Ingeniören*, No. 64, 1940.