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A NOTE ON DIMENSION REDUCTION AND FINITE ENERGY LOCALIZED WAVE SOLUTIONS TO THE 3-D KLEIN-GORDON AND SCALAR WAVE EQUATIONS. PART II. X WAVE-TYPE

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1. THE SUPERLUMINAL BOOST SPECTRAL REPRESENTATION

Consider the axisymmetric (with respect to the coordinate z), homogeneous, three-dimensional Klein-Gordon equation modelling the propagation of waves in a collisionless plasma, viz.,

$$\left(\nabla_\rho^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2}{c^2}\right)\psi(\rho, z, t) = 0. \tag{1}$$

Here, c denotes the speed of light *in vacuo* and ω_p is the (radian) plasma frequency. A Fourier-Hankel representation of a general solution to this equation is given as

$$\begin{aligned}\psi(\rho, z, t) = &\int_{-\infty}^{\infty} dk_z \int_0^{\infty} d\omega \int_0^{\infty} d\kappa\kappa J_0(\kappa\rho) e^{ik_z z} e^{i\omega t} \\ &\times \delta\left(\frac{\omega^2}{c^2} - k_z^2 - \kappa^2 - \frac{\omega_p^2}{c^2}\right)\tilde{\psi}_0(\kappa, k_z, \omega),\end{aligned} \tag{2}$$

where $\delta(\cdot)$ denotes the Dirac Delta function and $J_0(\cdot)$ is the ordinary Bessel function of order zero. Upon integration with respect to ω, we obtain the Whittaker spectral representation

$$\psi(\rho, z, t) = \int_{-\infty}^{\infty} dk_z \int_0^{\infty} d\kappa\kappa J_0(\kappa\rho) e^{ik_z z} e^{ict\sqrt{\kappa^2+k_z^2+\omega_p^2/c^2}}\tilde{\psi}_1(\kappa, k_z). \tag{3}$$

The Klein-Gordon equation [cf. Eq. (1)] is invariant under the superluminal Lorentz transformation

$$\begin{aligned} z \to \sigma = &\gamma\frac{v}{c}\left(z - \frac{c^2}{v}t\right), \quad ct \to \tau = -\gamma(z - vt); \\ \gamma = &\frac{1}{\sqrt{(v^2/c^2) - 1}}; \quad v > c. \end{aligned} \tag{4}$$

It follows, then, from Eqs. (3) and (4) that

$$\psi(\rho, z, t) = \int_{-\infty}^{\infty} dk_z \int_0^{\infty} d\kappa\kappa J_0(\kappa\rho) e^{ik_z \sigma} e^{i\tau\sqrt{\kappa^2+k_z^2+\omega_p^2/c^2}}\tilde{\psi}_1(\kappa, k_z), \tag{5}$$

a relationship referred to as the *superluminal boost spectral representation* [2]. This representation involves a superposition of products of two plane waves: one moving along the positive z-direction with speed $v > c$ and the other traveling in the same direction at the subluminal speed c^2/v. Although these two plane waves appear to be individually unidirectional, the superluminal boost representation in Eq. (5) generally consists of forward and backward (with respect to z) traveling components. In practical applications, one has to be careful to isolate the backward traveling components in order to to be able to generate a completely causal forward traveling wavefield.

2. DIMENSION-REDUCTION APPROACH TO X WAVE-TYPE FINITE ENERGY SOLUTIONS TO THE KLEIN-GORDON EQUATION

With the variable change $\kappa' = \sqrt{\kappa^2 + k_z^2}$, Eq. (5) assumes the following form:

$$\psi(\rho, z, t) = \int_{-\infty}^{\infty} dk_z \int_{0}^{\infty} d\kappa' \kappa' J_0 \left(\rho\sqrt{\kappa'^2 - k_z^2 - \omega_p^2/c^2}\right) e^{ik_z\sigma} e^{i\kappa'\tau} \widetilde{\psi}_2(\kappa', k_z). \tag{6}$$

The choice of the spectrum

$$\widetilde{\psi}_2(\kappa', k_z) = \widetilde{F}(k_z) \left(e^{-a_1\kappa'}/\kappa'\right) H\left(\kappa' - \sqrt{k_z^2 + \omega_p^2/c^2}\right), \quad a_1 > 0, \tag{7}$$

where $H(\cdot)$ denotes the Heaviside unit step function, allows the integration over κ' to be carried out explicitly [3], with the result

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} \int_{-\infty}^{\infty} dk_z \widetilde{F}(k_z) e^{ik_z\sigma} e^{-\sqrt{k_z^2 + \omega_p^2/c^2}\sqrt{\rho^2 + (a_1 - i\tau)^2}}. \tag{8}$$

For $\widetilde{F}(k_z) = \delta(k_z)$, we obtain the zero-order X wave solution for the 3-D Klein-Gordon equation, viz.,

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} e^{-(\omega_p/c)\sqrt{\rho^2 + (a_1 - i\tau)^2}}. \tag{9}$$

It is an infinite energy LW pulse propagating without distortion along the z-direction with the superluminal speed v. For $\widetilde{F}(k_z) = \delta(k_z - k_{zo})$, $k_{zo} > 0$, we find from Eq. (8) another infinity energy LW solution; specifically,

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} e^{ik_{zo}\sigma} e^{-\sqrt{k_{zo}^2 + \omega_p^2/c^2}\sqrt{\rho^2 + (a_1 - i\tau)^2}}. \quad (10)$$

The formal introduction of the two new variables

$$Z = \sigma = \gamma\frac{v}{c}\left(z - \frac{c^2}{v}t\right), \quad T = i\frac{1}{c}\sqrt{\rho^2 + (a_1 - i\tau)^2} \quad (11)$$

amounts to a reduction in dimensionality (or number of coordinates); it brings Eq. (10) into the form

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} e^{ik_{zo}Z} e^{i\sqrt{k_{zo}^2 + \omega_p^2/c^2}\,cT}, \quad (12)$$

which is recognized as a product of the zero-order X wave solution to the 3-D scalar wave equation (cf. Sec. 2) and a monochromatic (wavenumber $k_{zo} = \omega_0/c$) solution to the 1-D Klein-Gordon equation with variables Z and T.

Finite energy X wave-type LW solutions to the 3-D Klein-Gordon equation can be derived by means of weighted superpositions over the free parameter k_{zo} in Eq. (12). However, because of the special structure of Eq. (12), such solutions assume the form

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} \psi_{1D}(Z, T), \quad (13)$$

where $\psi_{1D}(Z, T)$ is an analytic solution to the 1-D Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial T^2} - \frac{\omega_p^2}{c^2}\right)\psi_{1D}(Z, T) = 0. \quad (15)$$

Analyticity stems from the fact that $\psi_{1D}(Z, T)$ is formed from Eq. (12) by means of a superposition over positive wavenumbers k_{zo}.

As an illustration of the dimension-reduction approach, consider the following specific analytic solution to the 1-D Klein-Gordon equation:

$$\psi_{1D}(Z,T) = \frac{1}{\sqrt{a_2 - i(Z+cT)}} e^{-\frac{\omega_p}{c}\sqrt{[a_3+i(Z-cT)][a_2-i(Z+cT)]}}, \ a_{2,3} > 0. \tag{16}$$

When it is used in conjunction with the *ansatz* (13), and the coordinates Z and T are expressed in terms of z, t and ρ using the definitions in Eq. (11), the following novel X wave-type finite energy LW solution to the 3-D Klein-Gordon equation results:

$$\psi(\rho,z,t) = \frac{1}{\sqrt{\rho^2+(a_1-i\tau)^2}} \frac{1}{\sqrt{a_2 - i\left(\sigma + i\sqrt{\rho^2+(a_1-i\tau)^2}\right)}} \times e^{-\frac{\omega_p}{c}\sqrt{\left[a_3+i\left(\sigma-i\sqrt{\rho^2+(a_1-i\tau)^2}\right)\right]\left[a_2-i\left(\sigma+i\sqrt{\rho^2+(a_1-i\tau)^2}\right)\right]}}. \tag{17}$$

3. DIMENSION-REDUCTION APPROACH TO X WAVE-TYPE FINITE ENERGY SOLUTIONS TO THE SCALAR WAVE EQUATION

The 3-D scalar wave is invariant under the superluminal Lorentz transformation given in Eq. (4). Starting, then, from the superluminal boost representation [cf. Eq. (5)], with the constraint $\omega_p = 0$, a procedure paralell to that followed in the previous section leads to the spectral representation

$$\psi(\rho,z,t) = \frac{1}{\sqrt{\rho^2+(a_1-i\tau)^2}} \int_{-\infty}^{\infty} dk_z \widetilde{F}(k_z) e^{ik_z\sigma} e^{-|k_z|\sqrt{\rho^2+(a_1-i\tau)^2}} \tag{18}$$

corresponding to Eq. (8). For $\widetilde{F}(k_z) = \delta(k_z)$, we obtain the zero-order X wave solution for the 3-D scalar wave equation

$$\psi(\rho,z,t) = \frac{1}{\sqrt{\rho^2+(a_1-i\tau)^2}} \tag{19}$$

which was introduced independently by Lu and Greenleaf [4] and Ziokowski, Besieris and Shaarawi [5]. (In the latter reference, this solution was referred to as a *slingshot superluminal pulse*.) It is an infinite energy LW pulse propagating without distortion along the z-direction with the superluminal speed v. Choosing $\widetilde{F}(k_z) = \delta(k_z - k_{zo})$, $k_{zo} > 0$, in Eq. (18), we obtain another infinite energy LW solution; specifically,

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} e^{ik_{zo}\sigma} e^{-k_{zo}\sqrt{\rho^2+(a_1-i\tau)^2}}. \tag{20}$$

This wavepacket combines features present in both the zero-order X wave [cf. Eq. (19)] and the focus wave mode (FWM) [6, 7]. For this reason it has been called focused X wave (FXW) [2]. It resembles the zero-order X wave, except that its highly focused central region has a tight exponential localization, in contrast to the loose algebraic transverse dependence of the zero-order X wave.

The FXW given in Eq. (20) can be rewritten as

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} e^{ik_{zo}\left(\frac{v}{c}\gamma z + i\sqrt{\rho^2+(a_1-i\tau)^2}\right)} e^{-ik_{zo}\gamma ct}. \tag{21}$$

The introduction of the two new variables

$$Z = \frac{v}{c}\gamma z + i\sqrt{\rho^2 + (a_1 - i\tau)^2},\ T = \gamma t \tag{22}$$

brings Eq. (21) into the form

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} e^{ik_{zo}Z} e^{-ik_{zo}cT}, \tag{23}$$

which is recognized as a product of the zero-order X wave solution to the 3-D scalar wave equation and a monochromatic (wavenumber $k_{zo} = \omega_0/c$) solution to the 1-D scalar wave equation with variables Z and T.

Finite energy X wave-type LW solutions can be derived by means of weighted superpositions over the free parameter k_{zo} in Eq. (23). However, because of the special structure of Eq. (23), such solutions assume the form

$$\psi(\rho, z, t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} \psi_{1D}(Z, T), \tag{24}$$

where $\psi_{1D}(Z, T)$ is a solution to the 1-D scalar wave equation

$$\left(\frac{\partial^2}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial T^2}\right)\psi_{1D}(Z, T) = 0. \tag{25}$$

Of the two possible general solutions to this equation, viz., $g^+(Z - cT)$ and $g^-(Z + cT)$, it is clear from Eq. (23) that only analytic solutions of the form $\psi_{1D}(Z,T) = g^+(Z - cT)$ are allowed to be used in connection with the *ansatz* (24). In other words, $\psi_{1D}(Z,T)$ is equal to an analytic signal

$$\hat{g}(z) \equiv \frac{1}{\pi}\int_0^\infty dk_z e^{ik_z z}\widetilde{G}(k_z),\ \mathrm{Im}\{z\} \geq 0, \tag{26}$$

with the replacement $z \to Z - cT$. It should be noted, furthermore, that because of the dependence of $\psi_{1D}(Z,T)$ on $Z - cT$, the definitions of the variables Z and T are not unique. In the place of the definitions given in Eq. (22), one could assign, for example, variables Z and T as in Eq. (11) without affecting the final solution.

As an illustration, consider the following specific analytic solution to Eq. (25):

$$\psi_{1D}(Z,T) = \frac{e^{i\frac{b}{p}(Z-cT)}}{\left[a_2 - i\frac{1}{p}(Z - cT)\right]^q}. \tag{27}$$

Here, b, p, q and a_2 are arbitrary real positive quantities. Using this solution together with Eq. (24), and introducing the definitions of the coordinates Z, T, we obtain

$$\psi_{3D}(\rho,z,t) = \frac{1}{\sqrt{\rho^2 + (a_1 - i\tau)^2}} \frac{e^{-\frac{b}{p}\sqrt{\rho^2+(a_1-i\tau)^2}} e^{i\frac{b}{p}\sigma}}{\left[a_2 - i\frac{1}{p}\left(\sigma + i\sqrt{\rho^2 + (a_1 - i\tau)^2}\right)\right]^q}. \tag{28}$$

For $p = 1$, this is precisely the modified focused X wave (MFXW) finite energy pulse derived by Besieris *et al.*, [2]. As a check, it should be noted that in the limit $\omega_p \to 0$, the solution given in Eq. (17) coincides with that in Eq. (28) if $b = 0$, $p = 1$, and $q = 1/2$.

4. CONCLUDING REMARKS

The dimension-reduction technique expounded in this paper is essentially a method of *descent,* whereby solutions to the 3-D Klein-Gordon and scalar wave equations can be found from complex analytic solutions of the 1-D Klein-Gordon and scalar wave equations, respectively. A key factor underlying the dimension-reduction method is the choice of the spectrum $\widetilde{\psi}_2(\kappa', k_z) = \widetilde{F}(k_z)\left(e^{-a_1\kappa'}/\kappa'\right)H\left(\kappa' - \sqrt{k_z^2 + \omega_p^2/c^2}\right)$, $a_1 > 0$, in Eq. (7). This choice provides a delicate balance that allows the formulation of the dimension-reduction method, which, in turn, facilitates the derivation of a large, but restricted, class of X wave-type finite energy LW solutions to the 3-D Klein-Gordon and scalar wave equations. The aforementioned balance is disturbed for other choices of the spectrum $\widetilde{\psi}_2(\kappa', k_z)$. In such cases, one can obtain finite energy LW solutions directly from the superluminal boost representation [cf. Eq. (6)].

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