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FIELDS OF PARTICLES AND BEAMS EXITING A CONDUCTOR

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1. INTRODUCTION

Increased research in relativistic electron beams has followed interest in their propagation in gas, in their use as sources of intense microwave radiation, or their possible use in controlled fusion devices. Their associated self fields are, of course, similarly of interest.

Somewhat surprisingly, most work to date has studied Cerenkov radiation of beams in air [1–7], appropriate roughly for energies $\gtrsim 22$ MeV (STP air), even though less energetic beams have been of at least as much interest, and the conventional (non-Cerenkov) fields can be enormous for any energy and are not so sharply confined to small radiation angles. It appears most attention has been directed to frequency domain parameters such as intensity distributions, whereas there has been relatively little analysis of the explicit time domain nature of the fields.

Here we study the conventional (non-Cerenkov) electromagnetic fields of relativistic particles and beams entering air or vacuum. The small difference of the index of refraction from unity in air plays no role. Special attention is given to electrons or beams exiting a conducting foil into vacuum.

These transient fields of emerging particles are further of general interest, since the fields of many particle configurations, especially those passing through accelerator gaps or emerging from accelerators, originated under similar circumstances.

While field calculation for these circumstances is a relatively-straightforward exercise in classical electromagnetic theory, certain features make the fields of especial interest.

The particles can safely be assumed to move at constant velocity. Since neither a perfect conductor nor constant velocity particles can separately emit radiation, it is somewhat surprising that there should be any radiation at all when a particle exits a perfect conductor at constant velocity. Thus the "mechanism" by which the radiation is produced is of interest itself, and of wider generality.

Further, in this and many electromagnetic problems, perfect conductor boundary conditions at the metal surface are used to calculate the fields. But a point electron cannot exit a perfect conductor; infinite energy is required. Thus the boundary conditions employed preclude the effect from happening at all, and it is not clear that the calculational procedure is correct or self-consistent. Indeed, related treatments [8–9] have introduced an artificial finite radius hole to sidestep the problem. These subtleties are related to the point nature of an electron and the classical abstraction of a perfect conductor as a continuous medium. It is useful to know when these elementary calculations are valid and when one must use a more realistic model of the material.

One has the intuitive feeling that, if the result is reasonable, these apparent paradoxes can safely be ignored. Here we show under what circumstances the usual boundary conditions ($E_{\text{tangential}} = 0$), and standard electromagnetic calculation procedures, can be used. The necessary mathematics is not complicated.

It is further demonstrated here that the radiation produced when a particle exits a conductor at constant velocity is equivalent to the time averaged fields of a charge undergoing finite acceleration. This allows one to reconcile the angular dependence $\sin\theta/(1-\beta\cos\theta)$ with the well-known angular dependence $\sin\theta/\left(1-\beta(t')\cos\theta\right)^3$ of finite acceleration, where t' is retarded time.

We also show that the fields produced by an electron exiting a conductor at constant velocity are equivalent to those of a suddenly accelerated charge or to beta-decay fields, the fields in those cases exhibiting no additional bremsstrahlung due to sudden acceleration over and above the radiation in the conductor problem.

Radiation peaks at the angle $1/\gamma$, similar to the case of a longitudinally accelerated electron. This angle is shown to have a simple geometric origin; it is where the Lorentz-compressed Coulomb fields intercept the causality sphere, and their flux joins the radiation flux, causing $1/\gamma$ to be the angle where the field both peaks and significantly changes its angular dependence. With minor modification, this is also true for a particle undergoing any motion.

If the conductor has finite conductivity the problem can be considered one of transition radiation, and this is calculated. In this case the transition radiation formation length reduces to the usual conductor skin depth. The above subtleties can be clarified, and it is shown that perfect conductor boundary conditions can be used so long as one observes frequencies less than $4\pi\sigma/\epsilon$, where σ is the conductivity, and ϵ the dielectric constant, of the metal.

Since the fields are a straightforward consequence of Maxwell's Equations, we may be accused of belaboring the obvious. However, in addition to there being radiation with no accelerated charge, certain other features of beam fields are very unfamiliar. Field structure is unusual in that Coulomb fields blend with radiation fields for an extended length of time. Also, radiated fields are proportional to beam current I, rather than the usual dI/dt. The fields evolve through an "immersion phase" where the beam is still immersed in its own radiation fields, to a "separation phase" in which the radiation field has broken away from the beam. The immersion phase obtains for almost all practical times for realistic beams. Then the Coulomb fields drop off like $1/R$, not $1/R^2$, wherever they are non-zero, i.e., everywhere inside the causality sphere. It is worth clarifying these features.

No matter how distant, the Coulomb fields alone and the radiation fields alone do not have zero divergence, but the sum of course does. We explain how the radiation fields break away from the beam, with field lines always remaining connected. Total energy carried off in radiation is computed.

It is of value to understand a phenomenon from several viewpoints, as has been emphasized before [10]. We show that beam radiation is "equivalent" to radiation from accelerated charges, and to dipole radiation, and to transition radiation, and, for beams exiting an open ended pipe, to scattering of fields off conductors of finite size, and to similar field structures just outside a terminated transmission line. The existence and general behavior of radiation follows as well merely from the general principles of flux conservation and field line continuity.

2. FIELDS OF A POINT CHARGE

When a beam of charged particles is injected into vacuum from, say, an accelerator, the fields of the newly exposed charge must develop in time, eventually taking on the character of the Lorentz-transformed Coulomb fields riding with the beam at velocity $v < c$, together with any radiation fields advancing outward at speed c.

We start by studying the fields of an individual point charge in the two cases of 1) its being suddenly accelerated from rest to a velocity v, and 2) its exiting from a perfect conductor at constant velocity. Fields are obtained by directly solving Maxwell's Equations via the retarded potentials. Potentials for a beam are then constructed by superposing those of elementary charges, and beam fields obtained by differentiation. Again two cases are considered: 1) a beam of total charge Q and finite length ℓ emerges from the origin at constant velocity while $-Q$ accumulates at the origin, and 2) the beam exits a perfectly conducting plane at constant velocity.

A point particle of charge q starts from rest at the origin of coordinates $x = y = z = 0$ at time $t = 0$ and suddenly receives an infinite pulsed acceleration. It subsequently moves with speed v in the positive z direction, leaving an equal opposite charge $-q$ at the origin. We shall also use spherical coordinates $(R,\ \theta,\ \varphi)$, with $R = (x^2 + y^2 + z^2)^{1/2}$, the polar angle measured from the $+z$ axis, and φ the azimuth angle, as well as cylindrical coordinates $(r,\ \varphi,\ z)$, with $r = (x^2 + y^2)^{1/2} = R \sin\theta$, and $z = R\cos\theta$. We imagine some external agent supplying the force necessary to move the charge.

In Gaussian units the charge density for $t > 0$ is

$$\rho(\vec{R}, t) = -q\delta^3(\vec{R}) + q\delta(x)\delta(y)\delta(z - vt), \tag{1}$$

and the current density is

$$\vec{J}(\vec{R}, t) = vq\delta(x)\delta(y)\delta(z - vt)\hat{z}. \tag{2}$$

2.1 Potentials for Pulsed Acceleration

In the absence of all conducting boundaries, and in the Lorentz gauge, the scalar and vector potentials Φ and $\vec{A}$ are given by

$$\Phi(\vec{R}, t) = \int d^3R' \frac{\rho(\vec{R}', t')}{R_1} \tag{3}$$

$$A(\vec{R}, t) = \frac{1}{c} \int d^3R' \frac{J(\vec{R}', t')}{R_1} \tag{4}$$

in terms of the free space retarded Green's function $1/R_1$. Here

$$R_1 = |\vec{R} - \vec{R}'|_{\text{ret}} = \sqrt{r^2 + [z - z'(t')]^2} \tag{5}$$

and $t' = t - R_1/c$ is the retarded time. J and $\vec{A}$ have only a z component which we denote by J and A as in Eq. (4). Figure 1 sketches the geometry.

Evaluating A first, the integrals over x and y are trivial, leaving

$$A = \beta q \int_0^{ct} dz' \frac{\delta(z' - vt + \beta R_1)}{R_1} \tag{6}$$

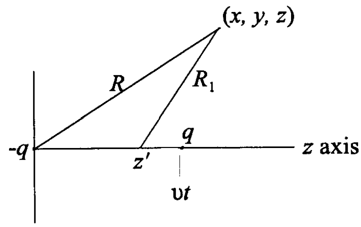


Figure 1. Charge q at $z = vt$, with $-q$ left at origin. Observer is at $R = (x, y, z)$.

substituting for t', where $\beta = v/c$. Since the charge is never beyond $z' = ct$, we may stop the integral at that upper limit. The δ function contributes at $z' = vt - \beta R_1 < vt$, and doing the integral in Eq. (6), we obtain for $R < ct$,

$$A = \frac{\beta q}{R_1 \dfrac{\partial}{\partial z'}(z' - vt + \beta R_1)} = \frac{\beta q}{R_1 + \beta(z' - z)}, \tag{7}$$

where we have used $\partial R_1/\partial z' = (z' - z)/R_1$, and where z' is to be evaluated where the δ function makes its contribution. Using Eq. (5), this is where

$$z' - vt + \beta\sqrt{r^2 + (z - z')^2} = 0. \tag{8}$$

Further using

$$R_1 + \beta(z' - z) = \sqrt{(vt - z)^2 + r^2/\gamma^2}, \tag{9}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ is the usual relativistic factor, we finally have

$$A(\vec{R}, t) = \frac{\beta q}{\sqrt{(vt - z)^2 + r^2/\gamma^2}} u(ct - R). \tag{10}$$

$u(x)$ is the step function, $u(x) = 0$ for $x < 0$, and $u(x) = 1$ for $x > 0$. We shall not always explicitly write this causality factor; fields and potentials will always vanish for $R > ct$.

On the spherical shell $ct = R$, the square root in Eq. (10) becomes, after a little algebra,

$$\sqrt{(\beta R - z)^2 + r^2/\gamma^2} = R - \beta z = R(1 - \beta\cos\theta) \tag{11}$$

so that A jumps from 0 to

$$A = \frac{\beta q}{R(1 - \beta \cos \theta)} \tag{12}$$

just behind the spherical wave front.

A similar calculation of Φ shows

$$\Phi(\vec{R}, t) = q \left[\frac{1}{\sqrt{(vt - z)^2 + r^2/\gamma^2}} - \frac{1}{R} \right] u(ct - R) = \frac{1}{\beta} A - \frac{q}{R} u(ct - R). \tag{13}$$

where A is that of Eq. (10). Equations (10) and (13) are the potentials of a point charge starting at $t = 0$ and moving with constant velocity.

2.2 Potentials for Particle Exiting a Conducting Plane

Turning now to the case of a particle passing out of a conductor, an elemental beam is a particle exiting an infinite perfectly conducting plane at $z = 0$ at constant velocity v. An image charge $-q$ guarantees the boundary conditions, and the same free space Green's function can be used with

$$\rho(\vec{R}, t) = q[\delta(z - vt) - \delta(z + vt)]\delta(x)\delta(y), \tag{14}$$

$$J(\vec{R}, t) = vq[\delta(z - vt) + \delta(z + vt)]\delta(x)\delta(y), \tag{15}$$

for $t > 0$. Then one finds by a similar evaluation of integrals,

$$A(\vec{R}, t) = \beta q \left[\frac{1}{S_-} + \frac{1}{S_+} \right] u(ct - R), \tag{16}$$

$$\Phi(\vec{R}, t) = q \left[\frac{1}{S_-} - \frac{1}{S_+} \right] u(ct - R). \tag{17}$$

The first term in each equation here is the potential of the charge q, and the second term is that of the image charge $-q$. We have introduced

$$\begin{aligned} S_- &\equiv \sqrt{(vt - z)^2 + r^2/\gamma^2}, \\ S_+ &\equiv \sqrt{(vt + z)^2 + r^2/\gamma^2}. \end{aligned} \tag{18}$$

Equations (16) and (17) are the potentials for $t > 0$ of a particle exiting a conducting plane at constant velocity.

On the shell $ct = R$, S_+ is $[(\beta R + z)^2 + r^2/\gamma^2]^{1/2} = R(1 + \beta\cos\theta)$, and S_- is given by Eq. (11). When they are combined, the potentials jump from zero to

$$A = \frac{2\beta q}{R(1 - \beta^2\cos^2\theta)}, \tag{19}$$

$$\Phi = \frac{2\beta q\cos\theta}{R(1 - \beta^2\cos^2\theta)} \tag{20}$$

just behind the spherical wave front. A, Eq. (19), is just $2/(1+\beta\cos\theta)$ times the vector potential of Eq. (12). Just behind the wave front the two vector potentials are not very different; the image charge does not radiate much in its backward hemisphere. The radiation fields likewise will be not very different in the two cases of either a pulsed acceleration with $-q$ left at the origin or a charge q emerging from a conducting plane at constant velocity.

2.3 Fields for Pulsed Acceleration

The non-zero cylindrical field components, $B = B_\varphi$, E_r, and E_z, are readily obtained from potentials Eq. (10) and (13),

$$B = \frac{\beta q}{\gamma^2}\frac{r}{S_-^3}u(ct - R) + \frac{\beta q\sin\theta}{R(1 - \beta\cos\theta)}\delta(ct - R), \tag{21}$$

$$E_r = q\left[\frac{r}{\gamma^2 S_-^3} - \frac{\sin\theta}{R^2}\right]u(ct - R) + \frac{\beta q\sin\theta\cos\theta}{R(1 - \beta\cos\theta)}\delta(ct - R), \tag{22}$$

and

$$E_z = q\left[\frac{z - vt}{\gamma^2 S_-^3} - \frac{\cos\theta}{R^2}\right]u(ct - R) - \frac{\beta q\sin^2\theta}{R(1 - \beta\cos\theta)}\delta(ct - R). \tag{23}$$

These are the cylindrical field components when charge $-q$ is left at the origin. See Figure 2. The first terms, proportional to $u(ct - R)$, are the Lorentz-transformed Coulomb fields of the moving charge and the Coulomb fields of the stationary charge. They are confined to the spherical volume $R < ct$, and drop suddenly to zero at $R = ct$. They clearly decay as $1/r^2$ from axis, or as $1/R^2$ from the origin.

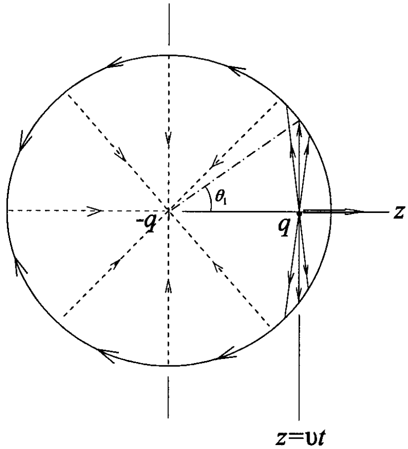


Figure 2. Fields of charge q moving with velocity v, and $-q$ left at origin.

The second terms in Equations (21)–(23) are the radiation fields, confined to the infinitely thin shell at $R = ct$. The spherical components of these fields are $E_R = 0$, and

$$E_\theta = B = \frac{\beta q \sin\theta}{R(1-\beta\cos\theta)}\delta(ct-R). \tag{24}$$

These radiation fields, and those discussed later, share a factor $(1-\beta\cos\theta)^{-1}$. Its origin is field compression arising from the coherent superposition of radiations from a moving charge due to the component of its motion toward the observer [11].

2.4 Peak Radiation Angle

Before obtaining fields of a particle exiting a conductor, observe that by direct calculation the radiated field, Eq(24), is found to peak at $\sin\theta = 1/\gamma, \cos\theta = \beta$.

From Figure 2, the Coulomb field intersects the radiation sphere at angle θ_1 given by

$$\begin{aligned} \cos\theta_1 &= \frac{vt}{ct} = \beta, \\ \sin\theta_1 &= \sqrt{1-\beta^2} = 1/\gamma \sim \theta_1. \end{aligned} \tag{25}$$

This is precisely the peak radiation angle, and is caused by maximum flux density where the Coulomb field joins the radiation field. Instantaneous field patterns of particles undergoing continuous finite acceleration show the same merging of fields, but at a somewhat smaller angle due to a $(1-\beta\cos\theta)^3$ denominator instead of the first power as in Eq. (24); see Section 3. This demonstrates the physical reason why radiation from accelerated charges peaks at $\sim 1/2\gamma$. This result is often derived, but its physical origin is seldom made clear. At angles greater than $1/\gamma$, the same flux is spread over an increasing area, making radiated fields smaller. For angles less than $1/\gamma$ most of the flux has already been taken by the Coulomb fields.

Even though the radiation fields, Eq. (24), decay as $1/R$ while the Coulomb field decays as $1/R^2$, no matter how distant the observer the radiation field itself violates $\vec{\nabla}\cdot\vec{E}=0$ since E_θ is a function of θ not equal to $1/\sin\theta$. Its non-zero divergence ($\sim 1/R^2$) is balanced by the contribution from the radial Coulomb field suddenly dropping to zero at $R=ct$, so the total fields are divergence free. The Coulomb field picks up the electric flux from the radiation field.

The energy flux radiated per unit solid angle in frequency interval $d\omega$ is

$$I(\omega)d\omega = \frac{cR^2}{8\pi^2}|\tilde{E}(\omega)|^2 d\omega \tag{26}$$

where $\tilde{E}(\omega)=\int dte^{-i\omega t}\vec{E}(\vec{R},t)$ is the Fourier transform of the radiation field. From Eq. (24) find

$$I(\omega) = \frac{q^2\beta^2\sin^2\theta}{8\pi^2 c(1-\beta\cos\theta)^2} \tag{27}$$

independent of ω. The zero acceleration time sets no time scale; all frequencies are present with equal strength, and the total energy radiated is infinite.

2.5 Fields for a Particle Exiting a Conducting Plane

For this case, from the potentials Eqs. (16) and (17), one finds

$$B = \frac{\beta q}{\gamma^2}\left[\frac{r}{S_-^3}+\frac{r}{S_+^3}\right]u(ct-R) + \frac{2\beta q\sin\theta}{R(1-\beta^2\cos^2\theta)}\delta(ct-R), \tag{28}$$

$$E_r = \frac{q}{\gamma^2}\left[\frac{r}{S_-^3}-\frac{r}{S_+^3}\right]u(ct-R) + \frac{2\beta q\sin\theta\cos\theta}{R(1-\beta^2\cos^2\theta)}\delta(ct-R), \tag{29}$$

$$E_z = \frac{q}{\gamma^2}\left[\frac{z-vt}{S_-^3} - \frac{z+vt}{S_+^3}\right] u(ct-R) - \frac{2\beta q \sin^2\theta}{R(1-\beta^2\cos^2\theta)}\delta(ct-R), \quad (30)$$

likewise the sum of Coulomb and radiation fields. See Figure 3.

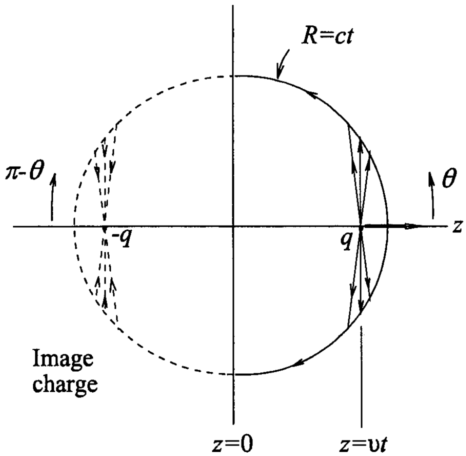


Figure 3. Fields of charge q moving with velocity v away from perfectly conducting plane at $z = 0$.

The spherical components of only the radiation fields are $E_R = 0$, and

$$E_\theta = B = \frac{2\beta q \sin\theta}{R(1-\beta^2\cos^2\theta)}\delta(ct-R). \quad (31)$$

The denominator $(1-\beta^2\cos^2\theta)$ in Equations (28)–(31) comes from combining the term from the radiating charge $(1-\beta\cos\theta)^{-1}$ with that from the image charge $(1+\beta\cos\theta)^{-1}$. By factoring $2/(1+\beta\cos\theta)$ out of Eq. (31), one sees

$$E(Eq.\ 31) = \frac{2}{1+\beta\cos\theta}\cdot E(Eq.\ 24), \quad (32)$$

so these radiated fields are not very different from the case of pulsed acceleration, as mentioned after Eq. (20). When $\beta = 1$, the factor $(1+\cos\theta)/2$ is the fractional solid angle at angles greater than θ, and the difference between (24) and (31) is whether flux is or is not bled out of the radiation sphere into $-q$ at the origin.

The radiated field peaks at $\theta = \sin^{-1}(1/\gamma\beta)$ if $\gamma > \sqrt{2}$, and $\pi/2$ if $\gamma < \sqrt{2}$, reducing to the former case (Equation (25)) when $\gamma \gg 1$. Non-relativistically both cases peak near $\pi/2$, as does non-relativistic dipole radiation.

The radiated frequency spectrum is

$$I(\omega) = \frac{q^2\beta^2 \sin^2\theta}{2\pi^2 c(1 - \beta^2\cos^2\theta)^2}, \tag{33}$$

to be compared with Equation (27). It takes zero time for a point charge to cross the surface $z = 0$, so again there is no physically occurring time to set a characteristic time scale or frequency scale. For a related problem, Maresca and Liboff [8] obtained fields Eqs. (28)–(30) by much more elaborate mathematical machinery.

2.6 Divergent Radiated Energy

The infinite radiated energy is associated with the point nature of q and the mathematical plane interface of the conductor. Infinite work is required to extract a particle from inside this medium. The work required to move q from z_1 to z_2 off the surface is $W = q^2(1/4z_1 - 1/4z_2)$, and diverges as $z_1 \to 0$. This is also true for a moving charge, for its Lorentz compressed fields still behave as $1/z^2$.

A line charge (beam of zero radius) also requires infinite work to extract from a perfect conductor, but it diverges only as $\ln(z_1)$. A beam of finite radius (of continuous charge density) requires only finite work.

Just outside a real material, when z approaches the inter-atomic spacing $d \sim 10^{-8}$ cm, the functional form for W breaks down, and it settles in, to order of magnitude, near the value $W \approx q^2/4d \approx 3.5\text{eV}$, which is the correct order of magnitude of typical work functions.

The only reason the beam can exit the metal at all is precisely because a real metal differs from an idealized perfect conductor. Nevertheless, as shown in Section 5, it is justified to use perfect conductor boundary conditions, at least for not too high frequencies.

3. RADIATION FROM FINITE ACCELERATION

The particle radiation fields, Eq. (24) or (31), have an angular dependence $\sin\theta/(1-\beta\cos\theta)$, whereas conventional acceleration fields have a dependence $\sin\theta/(1-\beta\cos\theta)^3$ evaluated at retarded time. It is worth reconciling these two expressions.

The Lienard-Wiechert fields for a particle in free space with prescribed velocity $c\beta(t)$ show [12] it produces radiation

$$E = \frac{q}{c}\left[\frac{\dot{\beta}\sin\theta}{(1-\hat{n}\cdot\vec{\beta})^3 R_2}\right]_{\text{ret}} \tag{34}$$

where $R_2 = |\vec{R}-\vec{R}_p(t)|$ is the distance between the observer at $\vec{R}$ and the particle at $\vec{R}_p(t)$, $\hat{n}$ is the unit vector from particle to observer, $\dot{\beta} = d\beta/dt$, and the subscript ret means the right hand side is evaluated at the earlier time $t' = t - R_2(t')/c$.

Let the particle undergo constant acceleration $c\dot{\beta} = c\beta_0/\Delta t$ for a short time Δt, carrying it from velocity $v = 0$ to $v = c\beta_0 < c$, and from $z = 0$ to $z = (1/2)c\dot{\beta}\Delta t^2 = z_1$. Figure 4 sketches the radiation pattern for $R_2 \gg z_1$.

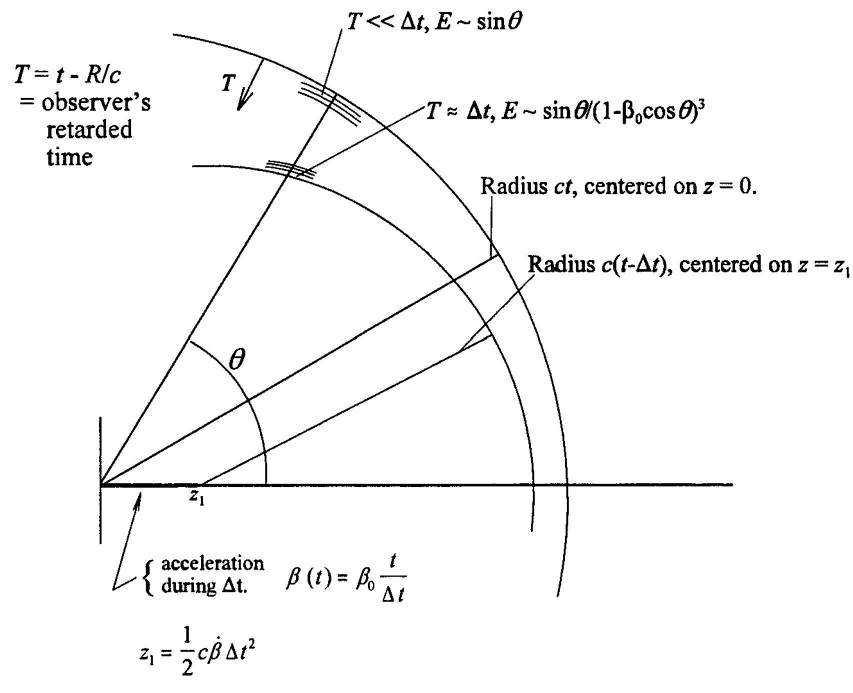


Figure 4. Field of a charge undergoing finite constant acceleration.

For small retarded times t', $\beta(t') \ll 1$, and the fields vary as $\sin\theta$, being ordinary non-relativistic dipole radiation. For late retarded times, $\beta \approx \beta_0 \approx 1$, and the angular dependence is $\sin\theta/(1-\beta_0\cos\theta)^3$, sharply peaked forward. The angular dependence quickly shifts forward within the pulse as β increases from 0 to β_0.

If Δt is short compared to measuring instrument resolution time, the observer can measure only the time average field; it is the only observable when the maximum measured frequency is less than $2\pi/\Delta t$. This, of course, is always the case when $\Delta t \to 0$.

For $R_2 \gg z_1$, the time average field is

$$\overline{E} = \frac{q\sin\theta}{cR\Delta t}\int\left[\frac{\dot{\beta}}{(1-\beta\cos\theta)^3}\right]_{\mathrm{ret}} dt. \tag{35}$$

From Eq. (5) and the definition of t', one has $dt = (1-\beta(t')\cos\theta)dt'$, so the integral here is

$$\int\frac{\dot{\beta}dt'}{(1-\beta(t')\cos\theta)^2} = \frac{\beta_0}{1-\beta_0\cos\theta}, \tag{36}$$

so that

$$\overline{E} = \frac{q}{c}\frac{\dot{\beta}\sin\theta}{R(1-\beta_0\cos\theta)}, \tag{37}$$

with an angular dependence intermediate between the two extremes. In the limit $\Delta t \to 0$, $\dot{\beta} \to \beta_0\delta(t)$, and $\overline{E}$ is what is observed and is as calculated previously [Eq. (24) with $\dot{\beta}/c = \beta\delta(ct-R)$]. $\overline{E}$, Eq. (37), is also the field, and I, Eq. (27), the intensity spectrum, due to beta decay (with heuristic quantum corrections) [13].

Likewise, radiation from a charge q exiting an infinite plane conductor at constant velocity is the same as charges q and $-q$ moving together in space without conductors, and charge $-q$ suddenly reversing its motion and continuing in the opposite direction, or the same as pair production in the rest frame.

3.1 Physical Origin of Factor $1/(1-\beta\cos\theta)$

The potentials of a particle at rest are

$$\Phi = \frac{q}{R}, \qquad A = 0.$$

For a moving particle the finite speed of light has two effects, retardation and compression. Its potentials are

$$\Phi(\vec{R},t) = \frac{q}{R(t')[1-\hat{n}(t')\cdot\vec{\beta}(t')]}, \quad A = \beta(t')\Phi. \tag{38}$$

The retarded time argument t' expresses retardation. The factor $1/(1-\hat{n}\cdot\vec{\beta}) = 1/(1-\beta\cos\theta)$ expresses field compression in the direction of the observer [11].

The compression factor arises from the finiteness of c, not its constancy. It is of first order in v/c and would also occur for acoustic waves. Purists may therefore not refer to it as a relativistic effect.

Although Φ and A behave as $1/(1-\hat{n}\cdot\vec{\beta})$, the instantaneous field

$$E \sim \frac{\partial A}{\partial t} = \frac{1}{1-\hat{n}\cdot\vec{\beta}}\frac{\partial A}{\partial t'} \sim \frac{1}{(1-\hat{n}\cdot\vec{\beta})^3} \tag{39}$$

has the denominator cubed. The time averaged field regains the potentials' behavior as the inverse first power.

The time average fields $[\sim (1-\beta\cos\theta)^{-1}]$ radiate most at $\theta \sim 1/\gamma$, due to the merging of Coulomb and radiation fields as discussed following Eq. (25). The more sharply peaked acceleration fields, Eq. (34), for $\beta \to 1$ radiate most at the somewhat smaller angle [14] $\theta \sim 1/2\gamma$, but the physical reason is the same.

4. FIELDS OF A BEAM

Fields of a beam are obtained by superposition of point particle expressions. The beam contains total charge Q and is taken to have constant current $I = Q/\tau$ that turns on and off suddenly with pulse duration τ.

4.1 Opposite Charge Left at Origin

In the case that $-Q$ accumulates at the origin while the beam emerges at constant velocity, the superposition of Equation (10) gives for the vector potential

$$A = \beta\int_0 \frac{Iu(c(t-t_1)-R)}{\sqrt{(v(t-t_1)-z)^2+r^2/\gamma^2}}dt_1. \tag{40}$$

When the observer's retarded time $T = t - R/c$ is less than the beam pulse length τ (sec) then the upper limit in Eq. (40) is T, and

$$A = \frac{I}{c} \ln \left[\frac{\sqrt{(\beta R - z)^2 + r^2/\gamma^2} - (\beta R - z)}{\sqrt{(vt - z)^2 + r^2/\gamma^2} - (vt - z)} \right] u(ct - R). \tag{41}$$

Using Eq. (11), the numerator here can be simplified, obtaining

$$A = \frac{I}{c} \ln \left[\frac{(1-\beta)(R+z)}{S_- - (vt - z)} \right]. \quad (0 < T < \tau) \tag{42}$$

A parallel calculation for the scalar potential shows

$$\Phi = \frac{1}{\beta} A - \frac{Q' vT}{R}. \quad (0 < T < \tau) \tag{43}$$

Here A is from Eq. (42), $Q' = I/v = Q/\ell$ is the charge per unit length, and $\ell = v\tau$ is the beam pulse length.

For retarded times after the beam has exited, $T > \tau$, the upper limit in Eq. (40) is τ, and one finds

$$A = \frac{I}{c} \ln \left[\frac{\sqrt{vt - \ell - z)^2 + r^2/\gamma^2} - (vt - \ell - z)}{S_- - (vt - z)} \right] \; (T > \tau) \tag{44}$$

$$\Phi = \frac{1}{\beta} A - \frac{Q}{R}, \quad (T > \tau) \tag{45}$$

where A is from Eq. (44). Equations (42)–(45) are the potentials of a thin beam of length ℓ emerging from the origin at constant velocity with the opposite charge $-Q$ accumulating at the origin.

For the fields, one finds, for $0 < T < \tau$,

$$\begin{aligned} B &= \frac{I}{cr} \left(\frac{vt - z}{S_-} + \cos\theta \right), \\ E_r &= \frac{1}{\beta} B - \frac{I \sin\theta}{cR} \left(1 + \frac{cT}{R} \right), \\ E_z &= Q' \left(\frac{1}{\gamma^2 S_-} - \frac{1}{R} \right) - \frac{I \cos\theta}{cR} \left(1 + \frac{cT}{R} \right). \end{aligned} \tag{46}$$

For later retarded times it is convenient to define

$$S_{\ell-} \equiv \sqrt{(vt-\ell-z)^2 + r^2/\gamma^2},$$
$$S_{\ell+} \equiv \sqrt{(vt-\ell+z)^2 + r^2/\gamma^2}, \tag{47}$$

and obtain for $T > \tau$,

$$B = \frac{I}{cr}\left(\frac{vt-z}{S_-} - \frac{vt-\ell-z}{S_{\ell-}}\right),$$
$$E_r = \frac{1}{\beta}B - \frac{Q\sin\theta}{R^2}, \tag{48}$$
$$E_z = \frac{Q'}{\gamma^2}\left(\frac{1}{S_-} - \frac{1}{S_{\ell-}}\right) - \frac{Q\cos\theta}{R^2}.$$

The terms in Q/R^2 are the fields due to charge $-Q$ left at the origin.

In the shell $0 < T < \tau$, the fields (46) behave as $1/R$ for large R. In the limit $R \gg vT$, the terms of order $1/R$, after converting to spherical components of E, are $E_R = 0$, and

$$E_\theta = B = \frac{\beta I\sin\theta}{cR(1-\beta\cos\theta)}, \tag{49}$$

having the same form as the point particle fields, but finite and stretched over a time τ. There are no radiation fields for $T > \tau$. The radiation fields are a pulsed step function in time, as is the beam current itself, and are proportional to I.

Fields for the case $\beta \to 1$ are sketched in Figure 5, adapted from Longmire [15], who studied this limit.

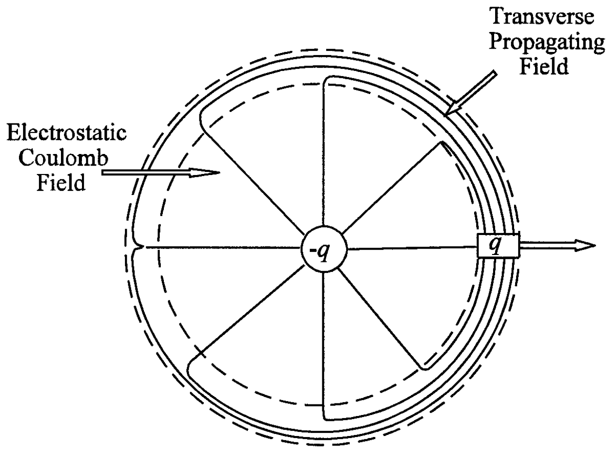
For a fixed observer, the time integral of the radiation field, Eq. (49),

$$\int E_\theta dt = \frac{Q\beta}{c}\frac{\sin\theta}{R(1-\beta\cos\theta)} \tag{50}$$

is the same as the time integral of the point particle field, Eq. (24), and depends only on the amount of charge in the beam.

The frequency spectrum of radiated energy is

$$I(\omega)d\omega = \frac{Q^2\beta^2\sin^2\theta}{8\pi^2 c(1-\beta\cos\theta)^2}\left(\frac{\sin\omega t/2}{\omega t/2}\right)^2 d\omega, \tag{51}$$



$$E_\theta = \frac{\beta I \sin\theta}{cR(1-\beta\cos\theta)} \xrightarrow{\beta=1} \frac{I\sin\theta}{cR(1-\cos\theta)}$$

$$= \frac{2I}{cR\sin\theta}\cdot\frac{1+\cos\theta}{2}$$

$$= \frac{2I}{cr}\cdot\underbrace{\frac{1+\cos\theta}{2}}_{\text{flux bleed off into Coulomb field}}$$

Figure 5. Field for beam moving near the speed of light.

differing from the point particle case by the modulating factor $\mathrm{sinc}^2(\omega\tau/2) \equiv [\sin(\omega\tau/2)/(\omega\tau/2)]^2$, arising from the square pulse wave form. Radiated energy is discussed later.

4.2 Beam Exiting a Conducting Plane

For this case superposition of Equations (16) and (17) yields

$$A(\vec{R},t) = \frac{I}{c}\left[\ln\left(\frac{(1-\beta)(R+z)}{S_- - (vt-z)}\right) + \ln\left(\frac{(1-\beta)(R-z)}{S_+ - (vt+z)}\right)\right],$$
$$(0 < T < \tau) \tag{52}$$

$$\Phi(\vec{R},t) = Q'\left[\ln\left(\frac{(1-\beta)(R+z)}{S_- - (vt-z)}\right) - \ln\left(\frac{(1-\beta)(R-z)}{S_+ - (vt+z)}\right)\right],$$

$$(0 < T < \tau) \tag{53}$$

These potentials again vanish for $T < 0$, and start to rise linearly in T for $T > 0$. Similarly, for later retarded times

$$A = \frac{I}{c}\left[\ln\left(\frac{S_{\ell-} - (vt - \ell - z)}{S_- - (vt - z)}\right) + \ln\left(\frac{S_{\ell+} - (vt - \ell + z)}{S_+ - (vt + z)}\right)\right], \quad (T > \tau) \tag{54}$$

$$\Phi = Q'\left[\ln\left(\frac{S_{\ell-} - (vt - \ell - z)}{S_- - (vt - z)}\right) - \ln\left(\frac{S_{\ell+} - (vt - \ell + z)}{S_+ - (vt + z)}\right)\right]. \quad (T > \tau) \tag{55}$$

The second terms in these four equations are the potentials of the image beam.

The complete fields are, for $0 < T < \tau$,

$$\begin{aligned} B &= \frac{I}{cr}\left(\frac{vt - z}{S_-} + \frac{vt + z}{S_+}\right), \\ E_r &= \frac{Q'}{r}\left(\frac{vt - z}{S_-} - \frac{vt + z}{S_+} + 2\cos\theta\right), \\ E_z &= \frac{Q'}{\gamma^2}\left(\frac{1}{S_-} + \frac{1}{S_+}\right) - \frac{2Q'}{R}, \end{aligned} \tag{56}$$

and, for $T > \tau$,

$$\begin{aligned} B &= \frac{I}{cr}\left[\frac{vt - z}{S_-} - \frac{vt - \ell - z}{S_{\ell-}} + \frac{vt + z}{S_+} - \frac{vt - \ell + z}{S_{\ell+}}\right], \\ E_r &= \frac{Q'}{r}\left(\frac{vt - z}{S_-} - \frac{vt - \ell - z}{S_{\ell-}} - \frac{vt + z}{S_+} + \frac{vt - \ell + z}{S_{\ell+}}\right), \\ E_z &= \frac{Q'}{\gamma^2}\left(\frac{1}{S_-} - \frac{1}{S_{\ell-}} + \frac{1}{S_+} - \frac{1}{S_{\ell+}}\right), \end{aligned} \tag{57}$$

Equations (56) and (57) are the complete exterior fields of a thin beam of length ℓ emerging normally at constant velocity from an infinite perfectly conducting plane. (They are also the fields of a beam exiting a long, thin pipe, whose image charge races back on the pipe's outer surface. See Section 7).

As written, Equations (56) are not the simple manifest sum of beam fields plus image fields. The fields due to the beam alone are, for $0 < T < \tau$,

$$\begin{aligned} B &= \frac{I}{cr}\left(\frac{vt - z}{S_-} + \cos\theta\right), \\ E_r &= \frac{1}{\beta} B, \\ E_z &= Q'\left(\frac{1}{\gamma^2 S_-} - \frac{1}{R}\right). \end{aligned} \tag{58}$$

For $T > \tau$, Equations (57) are the manifest sum of beam and image fields, the first two terms (in S_- and $S_{\ell -}$) being the beam fields and the last two terms (in S_+ and $S_{\ell +}$) being the fields due to the image beam.

For $T > \tau$ there are no radiation fields and the exact fields in Eqs. (48) or (57) are Lorentz-transformed Coulomb fields. For $T < \tau$ both Coulomb and radiation fields are present. These fields have been combined algebraically, and the individual terms in Eqs. (46) or (56) do not separately correspond to either Coulomb or radiation fields. The expressions are simpler when so combined.

4.3 Relativistic Beam

For large γ fields are very small behind the shell $R < c(t - \tau)$. Radiation fields propagate at speed c and the Coulomb fields move with the beam at speed v. We imagine the beam remaining of constant radius. Field structure is as follows.

The beam tip vt lags slowly behind the field tip ct. So long as $ct - vt < c\tau = \ell/\beta$, or

$$ct < \frac{\ell}{\beta(1-\beta)} = \frac{1+\beta}{\beta}\gamma^2 \ell \approx 2\gamma^2 \ell, \tag{59}$$

the beam is still immersed in the shell $0 < T < \tau$. The beam and its fields are then as depicted in Figure 6a (“Immersion” phase). When $ct = \ell/\beta(1-\beta)$ the beam has lagged the full distance $c\tau$ behind the field front and the fields are as shown in Figure 6b. For $ct \gg 2\gamma^2 \ell$, the beam is widely separated from the shell defined by $0 < T < \tau$, and the fields are as depicted in Figure 6c (“Separation” phase).

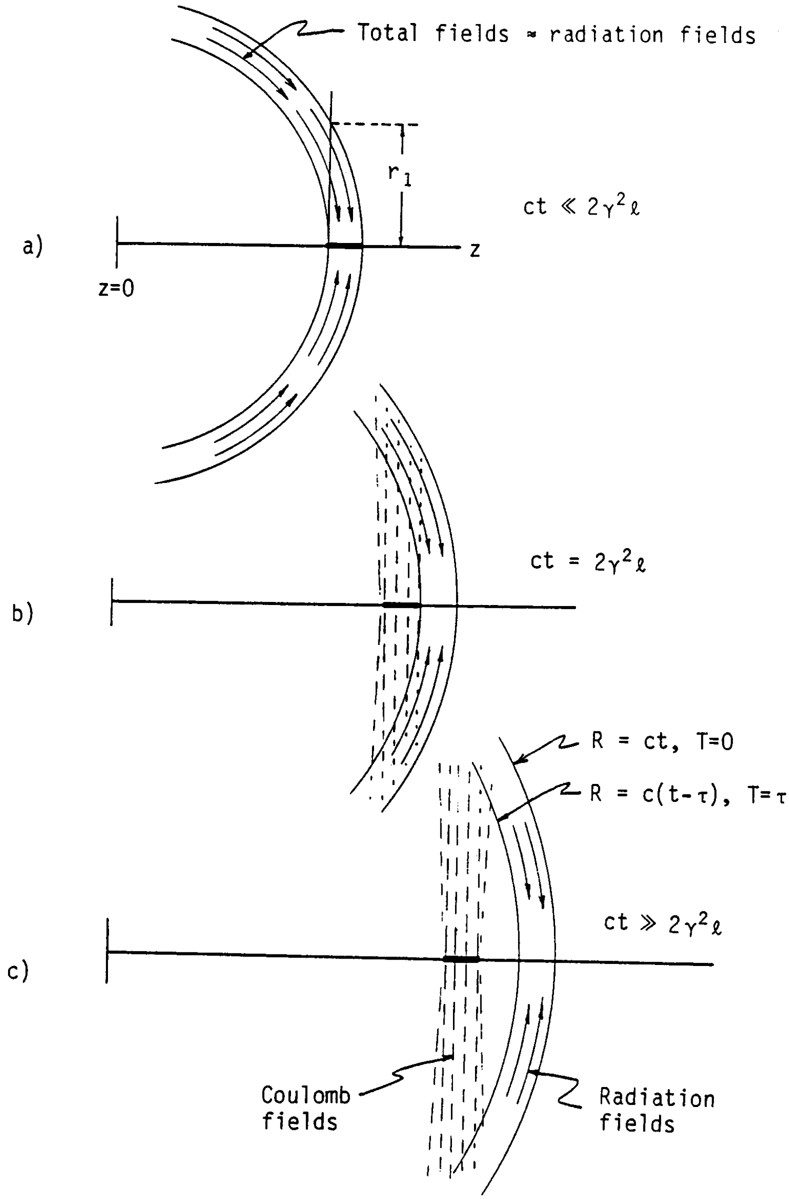


Figure 6. Developing field structure of a relativistic beam.

The very late time fields, Figure 6c, after separation, look like those of a small bunch of charge as in Figure 3, but with the thin radiation shell spread out to thickness $c\tau = \ell/\beta$, and the point particle Coulomb field replaced by that of a line charge of length ℓ.

Practical beams are almost always in the immersion phase. For a nominal pulse length, say $\ell = 10\,\mathrm{m}$, and a moderate γ, say $\gamma = 20$, $2\gamma^2\ell = 8\,\mathrm{km}$, much farther than endoatmospheric propagation distances. [The minimum energy loss rate of relativistic electrons in full density air is ~ 0.2 MeV/m, so an electron's range is always less than $R_e = (\gamma-1)mc^2/0.2 = 2.5(\gamma-1)$ meters. Thus the critical distance $2\gamma^2\ell$ exceeds R_e by a factor

$$\frac{2\gamma^2\ell}{2.5(\gamma-1)} \approx \gamma\ell \gg 1,$$

with ℓ measured in meters]. During propagation, the field structure of almost all endoatmospheric beams (of constant radius) would be as shown in Figure 6a. The ℓ used in $2\gamma^2\ell$ should be that appropriate to the mean forward beam velocity, somewhat less than the individual particle γ when betatron oscillations are accounted for.

4.4 Field Structure and Radiation

As was the case for a point particle, the plane of the beam tip, $z = vt$, intersects the field front $R = ct$ at polar angle θ given by $\sin\theta = 1/\gamma$, and radiation likewise peaks there.

After separation has occurred the field lines are roughly as drawn in Figure 7. In this figure field lines are shown dashed, the length of each dash roughly corresponding to field strength. All field lines terminate either on the beam or the image charge at $z = 0$.

For $\theta > 1/\gamma$, significant fields are confined almost entirely to the shell $0 < T < \tau$. The θ component of electric flux in this region is

$$F = 2\pi r \frac{\ell}{\beta} E_\theta = 4\pi Q, \tag{60}$$

the last equality following from $\int d\vec{S}\cdot\vec{E} = 4\pi Q$. Thus for $\theta \gg 1/\gamma$, Equation (60) shows

$$E_\theta = \frac{2\beta Q}{\ell r} \approx \frac{2Q'}{r}. \tag{61}$$

This field is the usual static cylindrical field from a line charge, swept back into a spherical shell.

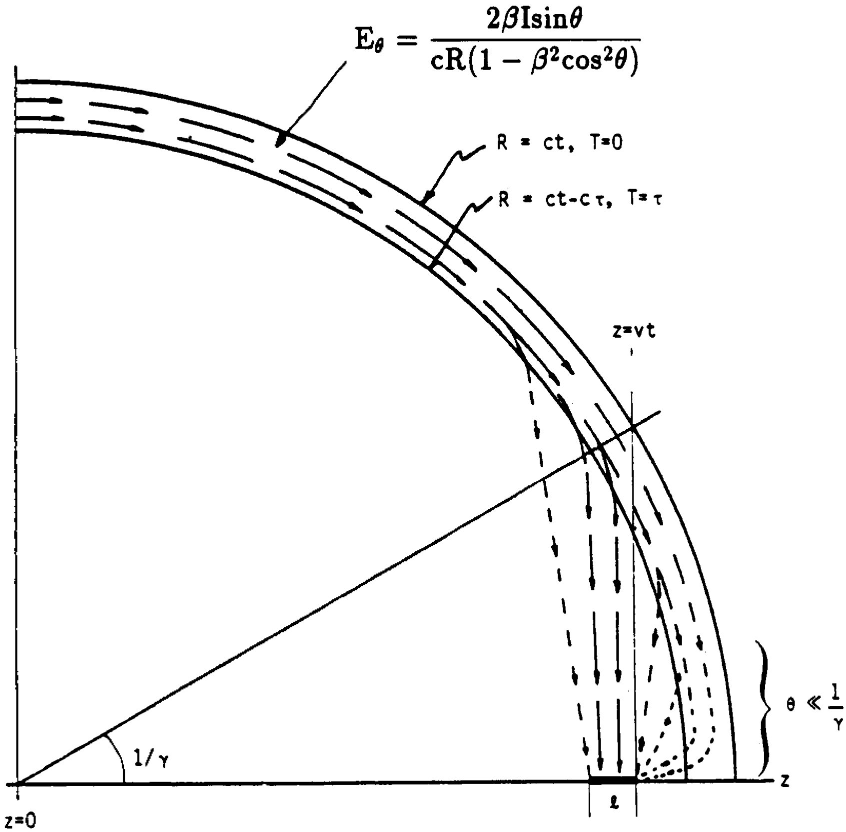


Figure 7. Field lines when $ct \gg 2\gamma^2\ell$. Field strength is indicated by length of line segment, not by line density.

In the radiation shell $0 < T < \tau$, one may use $vt - z = vT + (\beta - \cos\theta)R$ to expand the exact fields in powers of vT/R. The terms behaving as $1/R$, when converted to spherical components, are

$$E_\theta = B = \frac{2\beta I \sin\theta}{cR(1-\beta^2\cos^2\theta)}. \tag{62}$$

These are the radiation fields from a relativistic beam exiting a conducting plane. This field is just $2/(1+\beta\cos\theta)$ times the field when the charge $-Q$ is left at the origin, Equation (49).

For $\gamma \gg 1$, in the separation phase at all angles, or in any phase at $\theta \gg 1/\gamma$ and $R \gg \ell$, Coulomb fields E_R are small compared with radiation fields E_θ. Forming E_R from Eq. (56) and using Eq. (62), one finds, neglecting angle factors,

$$\frac{E_R}{E_\theta} \sim \frac{\ell}{\beta^2 R} \sim \frac{\ell}{R}.$$

For a very distant observer E_θ will be the only sensibly large field. However its divergence is not zero. It is balanced by the Coulomb field dropping to zero in the radiation shell, contributing a counterbalancing divergence.

4.5 The Immersion Phase

When $ct \ll 2\gamma^2\ell$ fields are almost entirely confined to the radiation shell. In the plane of the beam tail, $z = vt - \ell$, the largest cylindrical radius available in the causality sphere is r_1, indicated in Figure 6a, given by $r_1 = ct\sin\theta_1$, where $\cos\theta_1 = z/ct = \beta - \ell/ct$, or

$$r_1 = ct\sqrt{1 - (\beta - \ell/ct)^2}. \tag{63}$$

The quantity inside the radical is, after some algebra,

$$1 - \left(\beta - \frac{\ell}{ct}\right)^2 = \frac{2\beta\ell}{ct}\left(1 - \frac{\ell}{2\beta ct} + \frac{ct}{2\beta\gamma^2\ell}\right) \approx \frac{2\beta\ell}{ct},$$

so that

$$r_1 \sim \sqrt{2\beta\ell ct} = \gamma\ell\sqrt{\frac{2\beta ct}{\gamma^2\ell}} \ll \gamma\ell \tag{64}$$

The Lorentz transformed Coulomb field of a beam is packed into a disc containing the beam and, for $r > \gamma\ell$, fans out with an angle $\sim 1/\gamma$ about the disc's plane. (In the plane of the beam center, $z = vt - \ell/2$, it is $E_r = (Q/r)[(\ell/2)^2 + r^2/\gamma^2]^{-1/2}$.) This field falls off as $1/r$ for $r < \gamma\ell$, and as $1/r^2$ for $r > \gamma\ell$. Equation (64) shows that the available radii are too small to reach the radius $\gamma\ell$. Relativity and causality conspire to cut off the Coulomb fields before they have a chance to decay as $1/r^2$. Both the Coulomb fields and the total fields everywhere behave as $1/r$, or, for fixed θ, as $1/R$.

When $ct \gg 2\gamma^2\ell$, radii greater than $\gamma\ell$ are available, and the Coulomb field will exhibit its $1/R^2$ limiting form. Then the Coulomb fields are spatially separated from the radiation fields.

5. TRANSITION RADIATION FROM A GOOD CONDUCTOR

The fields of an electron exiting a conductor are the same as those of an impulsively accelerated electron (plus image field). We shall also see that they are the same as those of an electron exiting an open-ended tube, in which there is only one medium, and so clearly are not transition radiation. The fields are the same, but the physical mechanism of their origin is necessarily different; the equality of fields by themselves cannot be used to ascertain the physical origin of the radiation. It is therefore a meaningful question to ask whether a particle exiting a "perfect" conductor can be considered a limiting case of transition radiation.

5.1 Single Particle Transition Radiation

Transition radiation when a particle of constant velocity passes from a medium of relative dielectric constant ϵ_1 into one with relative dielectric constant ϵ_2 is generally discussed in the frequency domain and intended for common dielectric materials [16].

The complex dielectric constant of a medium with conductivity σ and relative dielectric constant ϵ_1' is

$$\epsilon_1 = \epsilon_1' + 4\pi i\sigma/\omega. \tag{65}$$

We investigate the transition radiation when an electron exits such a material into vacuum $(\epsilon_2 = 1)$. For typical metals, the conductivity term is of order $10^{17}/\omega$, completely dominating the ordinary dielectric constant except at extremely high frequencies.

The quantity usually presented is the power spectrum

$$I(\omega) = \frac{q^2 v^2 \sqrt{\epsilon_2}\sin^2\theta\cos^2\theta}{\pi^2 c^3}|D|^2, \tag{66}$$

where

$$D = \frac{(\epsilon_1 - \epsilon_2)(1 - \beta^2\epsilon_2 - \beta\sqrt{\epsilon_1 - \epsilon_2\sin^2\theta})}{(1 - \beta^2\epsilon_2\cos^2\theta)(1 - \beta\sqrt{\epsilon_1 - \epsilon_2\sin^2\theta})(\epsilon_1\cos\theta + \sqrt{\epsilon_1\epsilon_2 - \epsilon_2^2\sin^2\theta})}. \tag{67}$$

We are interested in the limit $\epsilon_2 \to 1$, and $\epsilon_1 \gg 1$. In this limit, D approaches

$$D \to \frac{1}{(1-\beta^2\cos^2\theta)\cos\theta}, \tag{68}$$

and the spectrum

$$I(\omega) \to \frac{q^2\beta^2\sin^2\theta}{\pi^2 c(1-\beta^2\cos^2\theta)^2}, \tag{69}$$

the same as calculated previously with perfect conductor boundary conditions, Eq. (33). (The factor of 2 difference is due to the convention of folding negative frequencies into positive frequencies in the present formulas). Thus, radiation due to particles or beams exiting a conductor can be considered the limit of transition radiation from a good conductor as the conductivity becomes arbitrarily large.

The above discussion shows conventional electromagnetic methods with perfect conductor boundary conditions can safely be used so long as we only observe frequencies

$$\omega < \frac{4\pi\sigma}{\epsilon_1'} \approx \frac{10^{17}}{\epsilon_1'}. \tag{70}$$

The numerical value makes conservative use of typical (DC) metal conductivities. Thus models of a real material are needed only for the extremely high frequency part of the spectrum. As we concentrate on the basic electrodynamics of the process, the perfect conductor formalism will suffice.

Transition radiation accumulates over a formation length $L = v/(\omega\,|1-\beta\sqrt{\epsilon_1}\cos\theta|)$ when observed at angle θ from the particle's trajectory. We note that for frequencies obeying Eq. (70), $\epsilon_1 \sim 4\pi i\sigma/\omega \gg 1$, and, not too near $\theta = \pi/2$, L reduces to

$$L \sim \frac{\delta}{\sqrt{2}\cos\theta}, \quad \delta = \frac{c}{\sqrt{2\pi\sigma\omega}}, \tag{71}$$

essentially the usual conductor skin depth δ, as it must.

5.2 Beam Transition Radiation

It takes some 10^{-8} cm for an electron to emerge from a surface. A relativistic 1kA beam contains 2×10^{11} electrons/cm. Therefore typical beams appear to be continuous charge distributions as they exit. Shot noise, always present, would be discernable only for weak beams or sensitive detectors. The individual electrons' low frequency (wavelength comparable with or larger than beam radius) transition radiations add coherently, and the observer sees the coherent superposition over the entire beam. The dominant wavelength is therefore on the order of the beam length. Beam radiation can be thought of as low frequency ($\sim$ RF) coherent transition radiation from all electrons.

An observer would not see this coherence in optical frequencies emitted in transition radiation as beams exit foils. There is no phase coherence among the optical photons emitted from individual electrons across the beam diameter. Optical transition radiation is sometimes used as a beam diagnostic. Since the term "transition radiation" is commonly employed in this latter sense, the term must be used with caution when referring to the low frequency electromagnetic pulse from beams.

6. RELATION TO DIPOLE RADIATION

Beam radiation may be thought of as ordinary dipole radiation. A beam has a dipole moment $P = \int \rho z dz = Q' v^2 t^2/2$ which increases quadratically in t while the beam is emerging, with $\ddot{P} = vI$, and linearly in t after the beam has fully emerged, with $\ddot{P} = 0$. The non-relativistic radiated field is

$$E_{\rm rad} = \frac{\ddot{P}\sin\theta}{c^2 R} = \frac{\beta I \sin\theta}{cR}. \tag{72}$$

To this one may heuristically append the field compression factor to account for the charge motion,

$$E_{\rm rad} \to \frac{\beta I \sin\theta}{cR(1-\beta\cos\theta)}, \tag{73}$$

the same as obtained in Eq. (49). The emerging charge exposes fields that must build up in the new medium (vacuum); the radiated field structure is just that of the dipole moment of the rapidly moving charge.

7. BEAM EXITING OPEN-ENDED PIPE

Figure 8 shows a beam pulse moving down a drift tube. Its self fields terminate on the inside walls on the image charge. As the beam exits,

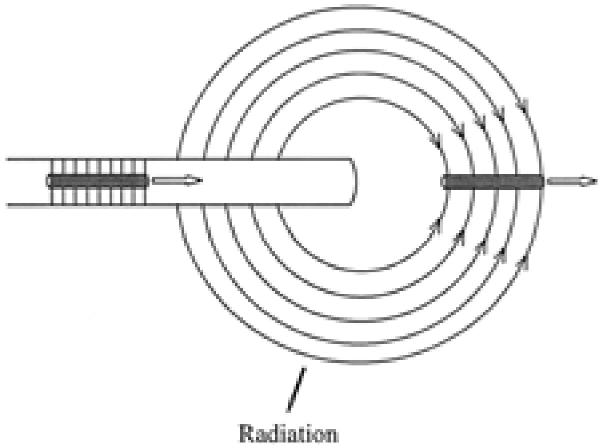


Figure 8. Beam exiting an open-ended pipe.

the image charge turns the corner and moves in the opposite direction on the outside pipe wall. The field structure is as shown, always terminating on the image charge.

For an open ended pipe, the beam electrons never transition from one medium to another; there is no transition radiation. Nevertheless the effect is similar to beams exiting a conducting medium. In the pipe case, before launch, the pipe shorts out the beam self fields for $r > r_{\text{pipe}}$. In the perfect conductor case, the conductor shorts out fields for $r > 0$. In both cases fields are "released" upon launch. The fields differ in the two cases only for wavelengths shorter than or comparable to the pipe radius. Thus even for open ended pipes the radiation is similar to the coherently superposed transition radiation of Section 5.

7.1 Ordinary Scattering

Consider a charge Q that has been moving at constant velocity $v < c$ through empty space for a long time. It is accompanied only by its Lorentz compacted Coulomb field and circumferential magnetic field, each of which falls off like $1/R^2$ far from the charge. Far ahead, off in the distance, sits a conducting object (sphere). cf. Figure 9a. There is no radiation and nothing is happening.

Eventually, as the charge passes, the field will sweep over the sphere, set it ringing, and scatter off that conducting object, Figure 9b. This

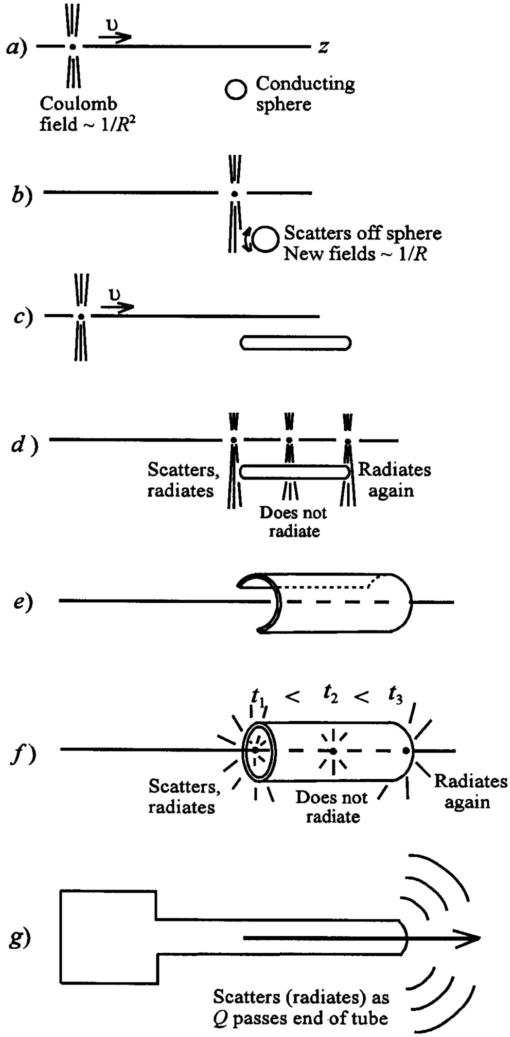


Figure 9. Showing how radiation of a beam exiting an open-ended pipe is equivalent to field scattering off a conductor.

scattering gives rise to radiation fields that drop off as $1/R$ from the sphere. The energy radiated comes from the energy that was in the charge's fields. Eventually it will come from the charge's kinetic energy.

Repeat the experiment with the sphere replaced by an elongated cigar (Figure 9c). Scattering and radiation are produced when the fields sweep over either end, Figure 9d. No radiation is produced when the field is between ends.

Now extend the cigar azimuthally into an annular band partway around the particle's trajectory (Fig. 9e). Again radiation is created only when the field passes over either end.

Continue the annular band, closing it into a cylinder surrounding the trajectory (Fig. 9f). Scattering occurs and radiation is produced only at the ends, when the particle enters and exits the cylinder.

Now extend the left end of the cylinder far back to where the particle was originally accelerated, or back to where we do not care about the radiation created when Q entered the long, narrow cylinder, and enclose the region in a large conducting building (Fig. 9g). Scattering and radiation are now produced only as Q exits the right end. It is the radiation from the launch of a charged particle beam from an open ended pipe, and its origin may be considered to be ordinary scattering.

7.2 Transmission Line Fields

The same radiation can be understood in terms of a transmission line terminated in a certain way. Consider a coaxial transmission line propagating a short TEM monopulse. The only energy in the system is in the fields. Extend the center conductor, and splay back the shield into a perpendicular plane, Figures 10a and 10b. Let the pulse emerge.

The extended wire permits the pulse to continue past the shield plane. A reflected pulse is possible; TEM modes are still supported. The part not reflected continues out the wire, with field structure shown, Fig. 10c. These are radiation fields, behaving as $1/R$. The hardware is an elementary stub antenna, with perhaps a long stub, and imperfect impedance match.

Now replace the center conductor with a pencil of charge moving down the axis of the shield cylinder, Fig. 10d. There is kinetic energy as well as field energy. The fields are still TEM, terminating on the charge instead of a wire. Let the charge emerge.

The entire field exits; TEM reflection is now forbidden, Figure 9e. The exterior field structure is the same as in Figure 10c. Thus beam radiation may be understood as ordinary antenna radiation.

The preceding discussion has made it clear, perhaps painfully clear, that radiation of a particle or beam exiting a conductor may be understood from different points of view, as a stand alone solution to Maxwell's Equations, as dipole radiation, accelerated particle radia-

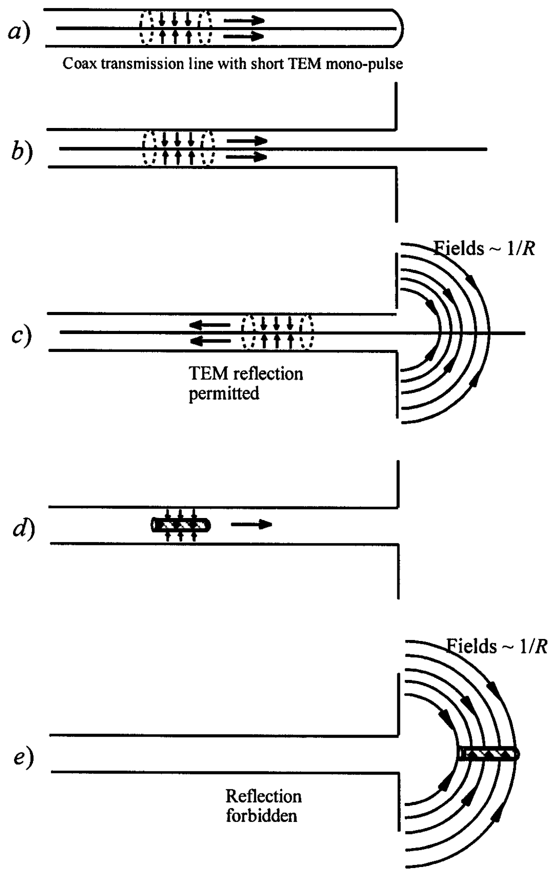


Figure 10. Beam radiation as extended transmission line fields.

tion, transition radiation, antenna radiation, scattering, etc.

Once a charge Q is released from a conducting enclosure, open ended or not, its Coulomb fields must build up anew. At the outer extent of these field lines, at $R = ct$, the radial Coulomb field lines cannot terminate; they must connect with a transverse field. Only E_θ is permitted, the azimuthal component E_φ being ruled out by symmetry. The total flux in E_θ must equal $4\pi Q$. As the area available to E_θ increases linearly with R, E_θ must drop off as $1/R$. Thus flux conservation alone implies the existence of radiation when a charge is suddenly exposed.

8. GENERAL RADIATED FIELDS AND ENERGY

When beam current is not constant, expressions are more complicated, but tractable formulae can still be obtained for the radiation fields, if not the total fields. Non-constant beam currents can arise from time dependent launched beam currents themselves or from net current changing in time as the beam propagates, as would be the case for a beam launched into a gas that can be ionized. Fields can then be expressed as the (time-dependent) field due to beam launch, plus a term corresponding to additional radiation arising from changes in current waveform during beam propagation after launch. Approximate treatments including air plasma currents have been given by Briggs [17] and by Longmire [15].

8.1 General Radiation Formula

Let I_b be the beam current as launched, and I_p be any plasma conduction currents that may be produced in the ambient gas. The applicable current for field production, that occurs in Maxwell's Equations, is the net current $I_N = I_b + I_p$, the rate of total charge transport out of the accelerator. In practice I_N is unknown apriori but can be measured.

When I_N is not constant, radiated fields are best obtained from the radiated vector potential

$$\vec{A}_{\text{rad}}(\vec{R}, t) = \frac{1}{cR} \int d^3R' \vec{J}_N(\vec{R}', t'), \tag{74}$$

and $\vec{E}_{\text{rad}} = \vec{B}_{\text{rad}} \times \hat{n} = (\nabla \times \vec{A}_{\text{rad}}) \times \hat{n}$. Here $\hat{n}$ is the unit vector toward the observer, $\vec{J}_N$ is the net current density, and $t' = t - |\vec{R} - \vec{R}'|/c$ is retarded time. For a thin beam propagating along z the integral reduces to $\int dz' I_N(z', t')$,where $I_N(z, t) = \int J_N dx'dy'$ is the net current. The local retarded time reduces to $t' = T + z' \cos\theta/c$, where $T = t - R/c$ is the observer's retarded time defined before.

The distance $\xi = vt - z$ behind beam tip is a more convenient independent variable than t, and writing $I = I(z, \xi)$ we have

$$A_{\text{rad}} = \frac{1}{cR} \int dz' I_N(z', \xi' = vT - (1 - \beta\cos\theta)z'). \tag{75}$$

Radiations originating from z' at t' come from a distance $\xi' = vt' - z' = vT - (1 - \beta\cos\theta)z'$ behind beam tip. This permits switching the integration variable to ξ',

$$A_{\text{rad}} = \frac{1}{cR(1-\beta\cos\theta)} \int_0^{vT} d\xi' I_N(z', \xi'), \tag{76}$$

where, during the integration,

$$z' = (vT - \xi')/(1-\beta\cos\theta). \tag{77}$$

These formulae apply to a beam emerging from a point, not for a beam emerging from a plane conductor. In either case, however, the basic "physical mechanism" of radiation is the sudden appearance of a current, i.e., the rapid charge separation and its newly exposed fields, being loose terminology for the time rate of change of the volume integral of J_N. As the beam emerges, the volume integral is proportional to t (for constant I_N), and the radiated field $\sim \partial A/\partial t$ is proportional to I_N.

Since J_N and A_{rad} have only a z component, the radiated E has only a polar theta component

$$E_\theta = \frac{\sin\theta}{c}\frac{\partial A_{\text{rad}}}{\partial T} = \frac{\beta\sin\theta}{(1-\beta\cos\theta)cR} \left[I_N(z=0, \xi=vT) + \frac{1}{(1-\beta\cos\theta)} \int_0^{vT} d\xi' \frac{\partial I_N(z', \xi')}{\partial z'} \right] \tag{78}$$

In the integrand here, after the $\partial/\partial z'$ is taken, z' is replaced by Equation (77) and the integral on ξ' performed.

Equation (78) is a general thin beam radiation formula. The first term is the conventional radiation due to launch, being proportional to $I_N(z=0,\xi)$, the net current at the launch point as a function of time as the beam exits. It generalizes Eq. (49) to currents a function of ξ. A similar generalization holds for Eq. (62) for a beam exiting a conducting plane; in this case Eq. (78) is to be multiplied by $2/(1+\beta\cos\theta)$.

The second term in (78) contributes additional radiation whenever the net current time history changes as the beam propagates downstream. Its angular dependence is not manifest from Eq. (78) as it stands, for one of the factors $1/(1-\beta\cos\theta)$ disappears if one changes integration variables by replacing $d\xi'$ by $-(1-\beta\cos\theta)dz'$ from Eq. (77).

8.2 Radiated Energy

Let the current neutralization fraction be f_m, so that $I_N = (1 - f_m)I_b$. For radiated energy calculations we again take the current to be constant for a time τ.

If $\vec{S} = (c/4\pi)\vec{E} \times \vec{H}$ is the Poynting vector, the energy radiated is, for fields Eq. (49), in the case that the opposite charge accumulates at the origin,

$$\begin{aligned} W_1 &= R^2 \int dt \int d\Omega S = \frac{\tau}{2c} \int_{-1}^{1} \left(\frac{\beta I_N \sin\theta}{1 - \beta\cos\theta} \right)^2 d\cos\theta \\ &= \frac{2\ell I_N^2}{c^2\beta^2} \left(\ln[(1+\beta)\gamma] - \beta \right), \end{aligned} \tag{79}$$

and, for the case of a beam emerging from a conductor, Eq. (62),

$$W_2 = \frac{\ell I_N^2}{c^2\beta^2} \left[(1+\beta^2) \ln[(1+\beta)\gamma] - \beta \right], \tag{80}$$

integrating in this case over only the forward hemisphere. β and γ here are those appropriate to the mean forward velocity of the current waveform, and, for a self-pinched beam in gas, can differ from individual particle quantities, β_p, γ_p, because of betatron oscillations. However β is still near unity for relativistic beams, and γ enters only logarithmically.

Comparing these with the kinetic energy $K = (\gamma_p - 1)mc^2\ell I_b/e\beta c$ in an electron beam one obtains

$$\frac{W_1}{K} = \frac{2(1-f_m)^2 I_b}{mc^3/e} \frac{\ln[(1+\beta)\gamma] - \beta}{(\gamma_p - 1)\beta}, \tag{81}$$

and

$$\frac{W_2}{K} = \frac{2(1-f_m)^2 I_b}{mc^3/e} \frac{(1+\beta^2)\ln[(1+\beta)\gamma] - \beta}{2(\gamma_p - 1)\beta}. \tag{82}$$

Here $mc^3/e = 17\,\text{kA}$. The extreme relativistic limit of both is

$$\frac{W_{1,2}}{K} = \frac{2(1-f_m)^2 I_b}{mc^3/e} \frac{\ln(2\gamma)}{\gamma_p}, \quad (\gamma, \gamma_p \gg 1) \tag{83}$$

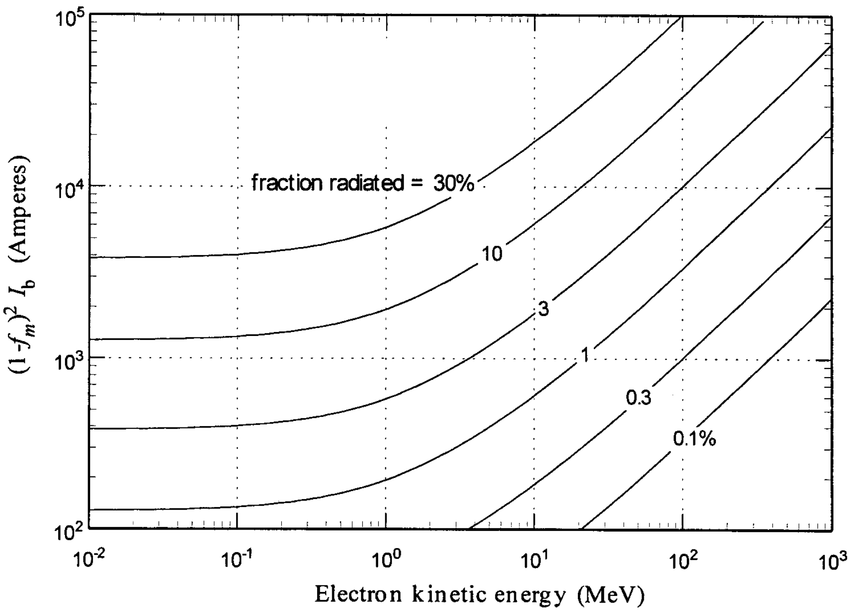


Figure 11. Contours of constant fraction of radiated energy, Eq(81).

and non-relativistic limits are

$$\frac{W_2}{K} = \frac{2W_1}{K} = \frac{8(1-f_m)^2 I_b}{3mc^3/e}, \quad (\beta \ll 1). \tag{84}$$

Equation (81), appropriate when $-Q$ is left near the origin, is shown in Figure 11 ($\gamma = \gamma_p$ has been used in the figure). A beam with energy less than a few MeV and current greater than a few kiloamperes radiates an appreciable fraction of its energy. The calculation assumed particle velocity v constant, and is unreliable when radiated fractions $W_{1,2}/K$ approach unity.

Calculation of total field energy and retarding force on the beam requires treating a beam of finite radius.

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