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DIRECT INTEGRATION OF FIELD EQUATIONS

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1. INTRODUCTION

The direct integration of field equations is first treated in this work with the aid of a scalar-vector Green's theorem. In such a treatment, potential functions are not needed, but the result can be interpreted in terms of these functions. The advantage of this approach is to eliminate several concepts that are often difficult for students to digest, such as the double layer of charges to characterize an aperture electrostatic field, the fictitious magnetic surface current to model a tangential electric field distribution, and the gauge conditions for the potential functions in magnetostatics and electrodynamics.

Our method follows closely the original work of Stratton and Chu [1]. We first derive a scalar-vector Green's theorem with no restriction placed on the two functions involved in the theorem. The theorem is then applied separately to electrostatics, magnetostatics, and finally to electrodynamics. The result of the last case is identical to the one originally formulated by Stratton and Chu. They are also equivalent to the expressions obtained by Schelkunoff [2] with the aid of equivalence principle.

When the tangential components of the electromagnetic field are discontinuous on the diffracting surface, Stratton and Chu deliberately added a line integral to the expression for the electric field and another one to the expression for the magnetic field to make them satisfy Maxwell's equations. This amendment was criticized by Schelkunoff, who claimed that his formulas were stronger. In fact, as we show in this work, Schelkunoff's expressions are equivalent to Stratton and Chu's; hence, his formulas are also restricted to continuous surface field distributions. A superior formulation is due to Franz [3], who derived his formulas with the aid of the free-space magnetic dyadic Green function. We present a detailed comparison between Franz's formulas and Stratton and Chu's, and show that the line integrals added by Stratton and Chu are contained inherently in Franz's formulas.

In this article the new differential operational notations for the divergence, $\dot{\nabla}$, and the curl, $\overset{\times}{\nabla}$, together with the long-established operational notation for the gradient, ∇, will be used. The rationale for adopting the new notation for the divergence and the curl is explained thoroughly in [4], particularly from the historical point of view as discussed in Chapter 8.

2. SCALAR-VECTOR GREEN'S THEOREM

The basic theorem needed in the first part of this work is a scalar-vector Green's theorem. The theorem involves one scale function and one vector function, hence its name. The derivation of this theorem from a vector-dyadic Green's theorem is found in Appendix 1. The theorem states

$$\iiint_V [(\nabla\cdot\mathbf{F})\nabla f + \mathbf{F}\cdot\nabla\nabla f + f\nabla\times\nabla\times\mathbf{F}]dV$$
$$= \oiint_S [(\hat{n}\cdot\mathbf{F})\nabla f + (\hat{n}\times\mathbf{F})\times\nabla f + (\hat{n}\times\nabla\times\mathbf{F})f]dS \qquad (1)$$

where f is a scalar function and $\mathbf{F}$ is a vector function, and all the functions are considered to be integrable in V and on S, including the generalized functions. We can now apply this theorem to integrate the field equations for various cases.

3. ELECTROSTATICS

We consider a region with a volume charge density ρ and an imaginary closed surface S_i in which there may be another charge distribution ρ_i and/or a scattering body B as shown in Fig. 1. The basic equations in

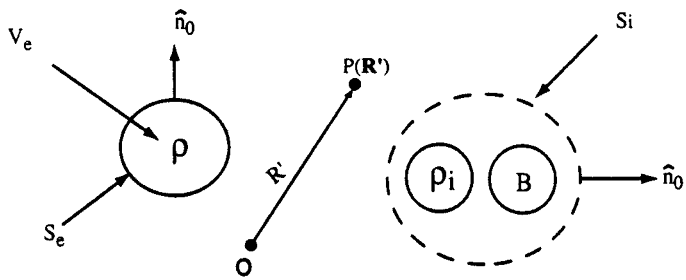

Figure 1. A charge distribution ρ inside a surface S_e in the neighborhood of an imaginary surface S_i.

electrostatics for the $\mathbf{E}$ field in an air medium are

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad (2)$$
$$\nabla\times\mathbf{E} = 0. \qquad (3)$$

Now we apply (1) to integrate (2) under the constraint of (3) with $\mathbf{F} = \mathbf{E}$ and $f = g_0$ where g_0 denotes the free-space scalar Green's function for a scalar Poisson's equation known to be

$$g_0 = g_0(\mathbf{R}, \mathbf{R}') = \frac{1}{4\pi|\mathbf{R} - \mathbf{R}'|}. \tag{4}$$

The function g_0 satisfies the equation

$$\nabla\nabla g_0 = -\delta(\mathbf{R} - \mathbf{R}'), \tag{5}$$

where $\delta(\mathbf{R} - \mathbf{R}')$ denotes a three-dimensional delta function. The function g_0 is singular at $\mathbf{R} = \mathbf{R}'$. It is the only location where the function and its derivatives do not exist, but the integral of $-\mathbf{E}\nabla\nabla g_0$ or $\mathbf{E}\delta(\mathbf{R} - \mathbf{R}')$ exists and is given by

$$\iiint_V \mathbf{E}\delta(\mathbf{R} - \mathbf{R}')\, dV = \begin{cases} \mathbf{E}(\mathbf{R}'), & V \in \mathbf{R}' \\ 0, & V \notin \mathbf{R}'. \end{cases} \tag{6}$$

By applying (1) to the region occupied by ρ with the aid of (2), (3), and (5) we obtain

$$\iiint_{V_e} \frac{\rho}{\epsilon_0}\nabla g_0\, dV = \oint\!\!\!\oint_{S_e} [(\hat{n}_0 \cdot \mathbf{E})\nabla g_0 + (\hat{n}_0 \times \mathbf{E}) \times \nabla g_0]\, dS. \tag{7}$$

The surface S_e encloses the volume V_e occupied by ρ. By applying (1) to the region exterior to S_e and S_i and bounded by a surface at infinity S_∞ we obtain

$$\begin{aligned} -\mathbf{E}(\mathbf{R}') = &- \oint\!\!\!\oint_{S_e} [(\hat{n}_0 \cdot \mathbf{E})\nabla g_0 + (\hat{n}_0 \times \mathbf{E}) \times \nabla g_0]\, dS \\ &- \oint\!\!\!\oint_{S_i} [(\hat{n}_0 \cdot \mathbf{E})\nabla g_0 + (\hat{n}_0 \times \mathbf{E}) \times \nabla g_0]\, dS. \end{aligned} \tag{8}$$

The integral evaluated at S_∞ vanishes if we assume that $\mathbf{E}$ varies as $1/R^2$ when R approaches infinity. Because of the continuity of the normal and tangential components of the field across S_e, an elimination of the surface integral on S_e between (7) and (8) yields

$$\mathbf{E}(\mathbf{R}') = \iiint_{V_e} \frac{\rho}{\epsilon_0}\nabla g_0 dV + \oint\!\!\!\oint_{S_i} [(\hat{n}_0 \cdot \mathbf{E})\nabla g_0 + (\hat{n} \times \mathbf{E}) \times \nabla g_0]\, dS. \tag{9}$$

Now,

$$\nabla g_0 = -\nabla' g_0, \tag{10}$$

and

$$\nabla' g_0 \times (\hat{n}_0 \times \mathbf{E}) = \nabla\!\!\!\!\times' g_0 (\hat{n}_0 \times \mathbf{E}), \tag{11}$$

where ∇' and $\nabla\!\!\!\!\times'$ denote the gradient and the curl operators defined with respect to the primed variable pertaining to $\mathbf{R}'$, the site of the field under consideration, (9) can therefore be written in the form

$$\begin{aligned}\mathbf{E}(\mathbf{R}') = &- \nabla' \iiint_{V_e} \frac{1}{\epsilon_0} \rho g_0 \, dV - \nabla' \oiint_{S_i} (\hat{n}_0 \cdot \mathbf{E}) g_0 \, dS \\ &+ \nabla\!\!\!\!\times' \oiint_{S_i} (\hat{n}_0 \times \mathbf{E}) g_0 \, dS.\end{aligned} \tag{12}$$

This is the integral expression of the electric field we are seeking. If we denote

$$\frac{1}{\epsilon_0} \iiint_{V_e} \rho g_0 \, dV + \oiint_{S_i} (\hat{n}_0 \cdot \mathbf{E}) g_0 \, dS = \Phi_{es}(\mathbf{R}'), \tag{13}$$

$$\oiint_{S_i} (\hat{n}_0 \times \mathbf{E}) g_0 \, dS = \mathbf{A}_{es}(\mathbf{R}'), \tag{14}$$

then

$$\mathbf{E}(\mathbf{R}') = -\nabla' \Phi_{es}(\mathbf{R}') + \nabla\!\!\!\!\times' \mathbf{A}_{es}(\mathbf{R}'). \tag{15}$$

Equation (15) shows $\mathbf{E}(\mathbf{R}')$ can be evaluated from the differentials of two potential functions, one representing an electrostatic potential, Φ_{es}, and another representing an electrostatic vector potential, $\mathbf{A}_{es}$. For those who prefer to derive integral expressions by applying the equivalence principle, a topic to be mentioned in a later section, the term $\hat{n} \times \mathbf{E}$ in (12) would be interpreted as the equivalent surface magnetic current density. To introduce a magnetic current in electrostatics would be very awkward indeed.

The classical treatment of this problem was usually done with the aid of a scalar potential theory. According to this approach, if a scalar potential function ϕ is defined at the very beginning such that

$$\mathbf{E} = -\nabla \phi, \tag{16}$$

then ϕ satisfies the scalar Poisson's equation

$$\nabla \nabla \phi = -\frac{\rho}{\epsilon_0}. \tag{17}$$

The integral expression of (17) for the problem under consideration is found in many books, for example [1, Sec. 3.4] and [5]. The result is

$$\begin{aligned}\phi(\mathbf{R}') = \frac{1}{\epsilon_0} \iiint\limits_{V_e} \rho g_0 \, dV - \oint\!\!\!\oint\limits_{S_i} (\hat{n}_0 \cdot \nabla \phi) g_0 \, dS \\ + \oint\!\!\!\oint\limits_{S_i} (\hat{n}_0 \cdot \nabla g_0) \phi \, dS.\end{aligned} \tag{18}$$

The last term in (18) is commonly interpreted as the potential due to a double layer of charge. Since such a layer does not exist in the actual problem, this interpretation appears to be very difficult for students to appreciate. By taking a gradient of (18) in the primed system and with the aid of the identity

$$\nabla' \times \oint\!\!\!\oint\limits_{S_i} (\hat{n}_0 \times \mathbf{E}) g_0 \, dS = -\nabla' \oint\!\!\!\oint\limits_{S_i} (\hat{n}_0 \cdot \nabla g_0) \phi \, dS, \tag{19}$$

our (12) can be recovered. The main feature of the present treatment is a more direct approach of finding $\mathbf{E}$ without introducing the potential function. Equation (19) also indicated that the potential functions are not unique. In our presentation, an electrostatic vector potential appears quite naturally in the formulation that is absent in the conventional theory using ϕ.

4. MAGNETOSTATICS

We consider a similar problem shown in Fig. 1 with ρ replaced by $\mathbf{J}$, a solenoidal electric current distribution. The basic equations for the magnetostatic $\mathbf{H}$ field with $\mathbf{J}$ placed in air are given by

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{20}$$

$$\nabla \mathbf{H} = 0. \tag{21}$$

We now apply (1) to this problem with $\mathbf{F} = \mathbf{H}$ and $f = g_0$. By following steps similar to those of the previous case we obtain

$$\mathbf{H}(\mathbf{R}') = \iiint\limits_{V_e} g_0 \nabla\times \mathbf{J}\, dV - \oiint\limits_{S_e} (\hat{n}_0 \times \mathbf{J}) g_0\, dS + \oiint\limits_{S_i} [(\hat{n}_0 \cdot \mathbf{H})\nabla g_0 + (\hat{n}_0 \times \mathbf{H}) \times \nabla g_0]\, dS. \tag{22}$$

The volume integral in (22) can be changed to

$$\begin{aligned} \iiint\limits_{V_e} g_0 \nabla\times \mathbf{J}\, dV &= \iiint\limits_{V_e} [\nabla\times (g_0 \mathbf{J}) - \nabla g_0 \times \mathbf{J}]\, dV \\ &= \oiint\limits_{S_e} (\hat{n}_0 \times \mathbf{J}) g_0\, dS - \iiint\limits_{V_e} \nabla g_0 \times \mathbf{J}\, dV \\ &= \oiint\limits_{S_e} (\hat{n}_0 \times \mathbf{J}) g_0\, dS + \nabla'\times \iiint\limits_{V_e} g_o \mathbf{J}\, dV. \end{aligned} \tag{23}$$

Substituting (23) into (22) and changing ∇g_0 to $-\nabla' g_0$ we obtain

$$\mathbf{H}(\mathbf{R}') = \nabla'\times \iiint\limits_{V_e} g_0 \mathbf{J}\, dV + \nabla'\times \oiint\limits_{S_i} g_0 (\hat{n}_0 \times \mathbf{H})\, dS - \nabla' \oiint\limits_{S_i} g_0 (\hat{n}_0 \cdot \mathbf{H})\, dS. \tag{24}$$

This is the integral expression for $\mathbf{H}(\mathbf{R}')$ that we are seeking. If we denote

$$\iiint\limits_{V_e} g_0 \mathbf{J}\, dV + \oiint\limits_{S_i} g_0 (\hat{n}_0 \times \mathbf{H})\, dS = \mathbf{A}_{ms}, \tag{25}$$

$$\oiint\limits_{S_i} g_0 (\hat{n}_0 \cdot \mathbf{H})\, dS = \Phi_{ms}, \tag{26}$$

then

$$\mathbf{H}(\mathbf{R}') = \nabla'\times \mathbf{A}_{ms} - \nabla \Phi_{ms}. \tag{27}$$

From the point of view of potential theory A_{ms} represents a magnetostatic vector potential function and Φ_{ms} a magnetostatic scalar potential function. It is seen that a vector potential function alone is not sufficient to formulate this problem from the point of view of potential theory. The existence of Φ_{ms} also explains why the scattering of a D.C. magnetic field by a magnetic body can be treated by a scalar potential theory [1, Sec. 4.2]. The conventional method of finding $\mathbf{H}(\mathbf{R}')$ is to define a vector potential $\mathbf{A}$ first and then find the integral expression for $\mathbf{A}$. Such a presentation is found in [1, Sec. 4.15].

5. ELECTROMAGNETICS

For simplicity, we consider a harmonically oscillating electromagnetic field for this case. The basic equations in Maxwell's theory are

$$\nabla\times\mathbf{E} = i\omega\mu_0\mathbf{H}, \tag{28}$$
$$\nabla\times\mathbf{H} = \mathbf{J} - i\omega\epsilon_0\mathbf{E}, \tag{29}$$
$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{30}$$
$$\nabla\cdot\mathbf{H} = 0, \tag{31}$$
$$\nabla\cdot\mathbf{J} = i\omega\rho. \tag{32}$$

A time factor $e^{-i\omega t}$ is used in this work. By eliminating $\mathbf{E}$ or $\mathbf{H}$ between (28) and (29) we obtain

$$\nabla\times\nabla\times\mathbf{E} - k^2\mathbf{E} = i\omega\mu_0\mathbf{J}, \tag{33}$$

and

$$\nabla\times\nabla\times\mathbf{H} - k^2\mathbf{H} = \nabla\times\mathbf{J}. \tag{34}$$

The wave number k in (33) and (34) is equal to $\omega(\mu_0\epsilon_0)^{1/2}$. The static sources in Fig. 1 are now replaced by dynamic current sources $\mathbf{J}$ and $\mathbf{J}_i$ accompanied by charge densities ρ and ρ_i.

To find the integral expression for $\mathbf{E}$ we let

$$\mathbf{F} = \mathbf{E}, \tag{35}$$
$$f = G_0 = \frac{e^{ik|\mathbf{R}-\mathbf{R}'|}}{4\pi|\mathbf{R}-\mathbf{R}'|}, \tag{36}$$

where G_0 denotes the free-space scalar Green's function that satisfies the equation

$$\nabla\cdot\nabla G_0 + k^2 G_0 = -\delta(\mathbf{R}-\mathbf{R}'). \tag{37}$$

Substituting (35) and (36) into (1) and making use of (28) to (34) we obtain

$$\begin{aligned}\mathbf{E}(\mathbf{R}') = {} & i\omega\mu_0 \iiint\limits_{V_e} G_0 \mathbf{J}\, dV - \frac{1}{\epsilon_0}\nabla' \iiint\limits_{V_e} G_0 \rho\, dV \\ & + i\omega\mu_0 \oiint\limits_{S_i} G_0(\hat{n}_0 \times \mathbf{H})\, dS - \nabla' \oiint\limits_{S_i} G_0(\hat{n}_0 \cdot \mathbf{E})\, dS \\ & + \nabla\times' \oiint\limits_{S_i} G_0(\hat{n}_0 \times \mathbf{E})\, dS, \end{aligned} \tag{38}$$

$$\begin{aligned}\mathbf{H}(\mathbf{R}') = {} & \nabla\times' \iiint\limits_{V_e} G_0 \mathbf{J}\, dV + \nabla\times' \oiint\limits_{S_i} G_0(\hat{n}_0 \times \mathbf{H})\, dS \\ & - \nabla' \oiint\limits_{S_i} G_0(\hat{n}_0 \cdot \mathbf{H})\, dS - i\omega\epsilon_0 \oiint\limits_{S_i} G_0(\hat{n}_0 \times \mathbf{E})\, dS. \end{aligned} \tag{39}$$

These are the same equations obtained previously by Stratton and Chu except that we have presented them in a slightly different form with the differential operators ∇' and $\nabla\times'$ placed outside the integrals. For convenience, the surface integrals in (38) and (39) will be designated as the *diffraction integrals.*

From the point of view of potential theory if we denote

$$\mathbf{A}_e = \iiint\limits_{V_e} G_0 \mathbf{J}\, dV + \oiint\limits_{S_i} G_0(\hat{n}_0 \times \mathbf{H})\, dS, \tag{40}$$

$$\phi_e = \frac{1}{\epsilon_0} \iiint G_0 \rho\, dV + \oiint\limits_{S_i} G_0(\hat{n}_0 \cdot \mathbf{E})\, dS, \tag{41}$$

$$\mathbf{A}_m = \oiint\limits_{S_i} G_0(\hat{n}_0 \times \mathbf{E})\, dS, \tag{42}$$

$$\phi_m = \oiint\limits_{S_i} G_0(\hat{n}_0 \cdot \mathbf{H})\, dS, \tag{43}$$

then

$$\mathbf{E}(\mathbf{R}') = i\omega\mu_0 \mathbf{A}_e - \nabla'\phi_e + \nabla\times' \mathbf{A}_m, \tag{44}$$

$$\mathbf{H}(\mathbf{R}') = -i\omega\epsilon_0 \mathbf{A}_m - \nabla'\phi_m + \nabla\times' \mathbf{A}_e. \tag{45}$$

It can be shown that when the fields are *continuous* on S_i we have the gauge condition

$$\nabla' \cdot \mathbf{A}_e = i\omega\epsilon_0\phi_e, \tag{46}$$

$$\nabla' \cdot \mathbf{A}_m = -i\omega\mu_0\phi_m, \tag{47}$$

Then (44) and (45) can be written in the form

$$\mathbf{E}(\mathbf{R}') = i\omega\mu_0\mathbf{A}_e + \frac{i}{\omega\epsilon_0}\nabla'\nabla' \cdot \mathbf{A}_e + \nabla' \times \mathbf{A}_m, \tag{48}$$

$$\mathbf{H}(\mathbf{R}') = -i\omega\epsilon_0\mathbf{A}_m - \frac{i}{\omega\mu_0}\nabla'\nabla' \cdot \mathbf{A}_m + \nabla' \times \mathbf{A}_e. \tag{49}$$

The surface integrals in (48) and (49) are the same expressions obtained by Schelkunoff [2] using the method of potentials with the aid of the equivalence principle [6]. The function $\mathbf{F}$ in his expressions, (18) and (19) of [2], corresponds to our $-\mathbf{A}_m$, and his $\mathbf{A}$ corresponds to the surface integral in our $\mathbf{A}_e$. Schelkunoff uses $e^{i\omega t}$ as the time factor in contrast to our $e^{-i\omega t}$, which is also used by Stratton and Chu. When the electromagnetic fields, $\mathbf{E}$ and $\mathbf{H}$, are discontinuous on S_i, (46) and (47) are no longer applicable. Schelkunoff's formulas also become inapplicable, since they are derived under these conditions.

For discontinuous $\mathbf{H}$ field on S_i let us consider the surface integral in (40). Its divergence is

$$\begin{aligned}\nabla' \cdot \oiint_{S_i} G_0(\hat{n}_0 \times \mathbf{H})\,dS &= \oiint_{S_i} \hat{n}_0 \cdot (\nabla G_0 \times \mathbf{H})\,dS \\ &= \oiint_{S_i} \hat{n}_0 \cdot [\nabla \times (G_0\mathbf{H}) - G_0\nabla \times \mathbf{H}]\,dS. \end{aligned} \tag{50}$$

Let C be the line contour separating the closed surface S_i into two open surfaces S_1 and S_2 across which $\mathbf{H}$ is discontinuous. Then

$$\oiint_{S_i} \hat{n}_0 \cdot \nabla \times (G_0\mathbf{H}) = \oint_C G_0(\mathbf{H}_1 - \mathbf{H}_2) \cdot d\ell,$$

and

$$\oiint_{S_i} \hat{n}_0 \cdot (-G_0\nabla \times \mathbf{H})\,dS = i\omega\epsilon_0 \oiint_{S_i} G_0(\hat{n}_0 \cdot \mathbf{E})\,dS,$$

hence

$$
\begin{aligned}
\nabla' \cdot \mathbf{A}_e &= i\omega \iiint_V G_0 \rho \, dv + i\omega\epsilon_0 \oiint_{S_i} G_0 (\hat{n}_0 \cdot \mathbf{E}) \, dS \\
&\quad + \oint_C G_0 (\mathbf{H}_1 - \mathbf{H}_2) \cdot d\ell_1 \\
&= i\omega\epsilon_0 \phi_e + \oint_C G_0 (\mathbf{H}_1 - \mathbf{H}_2) \cdot d\ell_1. \qquad (51)
\end{aligned}
$$

Similarly, when $\mathbf{E}$ is discontinuous on S_i

$$
\nabla' \cdot \mathbf{A}_m = -i\omega\mu_0 \phi_m + \oint_C G_0 (\mathbf{E}_1 - \mathbf{E}_2) \cdot d\ell_1. \qquad (52)
$$

Substituting the functions ϕ_e and ϕ_m from (51) and (52) into (44) and (45), we obtain

$$
\begin{aligned}
\mathbf{E}(\mathbf{R}') &= i\omega\mu_0 \mathbf{A}_e + \frac{i}{\omega\epsilon_0} \nabla' \nabla' \cdot \mathbf{A}_e + \nabla' \times \mathbf{A}_m \\
&\quad + \frac{i}{\omega\epsilon_0} \nabla' \oint_C G_0 (\mathbf{H}_1 - \mathbf{H}_2) \cdot d\ell_1, \qquad (53) \\
\mathbf{H}(\mathbf{R}') &= -i\omega\epsilon_0 \mathbf{A}_m - \frac{i}{\omega\mu_0} \nabla' \nabla' \cdot \mathbf{A}_m + \nabla' \times \mathbf{A}_e \\
&\quad - \frac{i}{\omega\mu_0} \nabla' \oint_C G_0 (\mathbf{E}_1 - \mathbf{E}_2) \cdot d\ell_1. \qquad (54)
\end{aligned}
$$

Because $\mathbf{A}_e$ and $\mathbf{A}_m$ satisfy the differential equations

$$
\begin{aligned}
\nabla' \cdot \nabla' \mathbf{A}_e + k^2 \mathbf{A}_e &= 0, \\
\nabla' \cdot \nabla' \mathbf{A}_m + k^2 \mathbf{A}_m &= 0,
\end{aligned}
$$

when $\mathbf{R}'$ is not located in S_i, the first two terms in (53) and (54) can be changed to

$$
\begin{aligned}
i\omega\mu_0 \mathbf{A}_e + \frac{i}{\omega\epsilon_0} \nabla' \nabla' \cdot \mathbf{A}_e &= -\frac{i\omega\mu_0}{k^2} \nabla' \cdot \nabla' \mathbf{A}_e + \frac{i}{\omega\epsilon_0} \nabla' \nabla' \cdot \mathbf{A}_e \\
&= \frac{i}{\omega\epsilon_0} (\nabla' \nabla' \cdot \mathbf{A}_e - \nabla' \cdot \nabla' \mathbf{A}_e) \\
&= \frac{i}{\omega\epsilon_0} \nabla' \times \nabla' \times \mathbf{A}_e, \qquad (55)
\end{aligned}
$$

and

$$-i\omega\epsilon_0\mathbf{A}_m - \frac{i}{\omega\mu_0}\nabla'\nabla'\mathbf{A}_m = \frac{-i}{\omega\mu_0}\not\nabla'\not\nabla'\mathbf{A}_m. \tag{56}$$

Hence

$$\mathbf{E}(\mathbf{R}') = \not\nabla'\mathbf{A}_m + \frac{i}{\omega\epsilon_0}\not\nabla'\not\nabla'\mathbf{A}_e - \frac{i}{\omega\epsilon_0}\nabla'\oint_C G_0(\mathbf{H}_1 - \mathbf{H}_2)\cdot d\ell_1, \tag{57}$$

$$\mathbf{H}(\mathbf{R}') = \not\nabla'\mathbf{A}_e - \frac{i}{\omega\mu_0}\not\nabla'\not\nabla'\mathbf{A}_m + \frac{i}{\omega\mu_0}\nabla'\oint_C G_0(\mathbf{E}_1 - \mathbf{E}_2)\cdot d\ell_1. \tag{58}$$

Equations (57) and (58) are the expressions for $\mathbf{E}$ and $\mathbf{H}$ derived from Stratton and Chu's formulas when the fields are discontinuous on S_i. Since it is obvious that $\nabla'\mathbf{E}(\mathbf{R}')$ and $\nabla'\mathbf{H}(\mathbf{R}')$ do not vanish, the two authors deliberately add a line integral

$$\frac{i}{\omega\epsilon_0}\nabla'\oint_C G_0(\mathbf{H}_1 - \mathbf{H}_2)\cdot d\ell_1 \tag{59}$$

to (57) and another line integral

$$-\frac{i}{\omega\mu_0}\nabla'\oint_C G_0(\mathbf{E}_1 - \mathbf{E}_2)\cdot d\ell_1 \tag{60}$$

to (58) so the resultant expressions would satisfy Maxwell's equations. Actually they consider S_1 to be an aperture and S_2 an opaque screen with $\mathbf{E}_2$ and $\mathbf{H}_2$ equal to zero on the screen. The inclusion of the line integrals so added was criticized by Schelkunoff [2, p. 51], and he stated that

> "More than the Green's formula is needed to identify $\hat{n}\times\mathbf{H}$ and $\mathbf{E}\times\hat{n}$ with equivalent sources on (S). Thus, in practice, we really need the stronger form of the theorem given by (18) and (19)."

He apparently did not realize that his formulas are restricted to continuous fields on S_i obtained under the gauge conditions specified by (46) and (47). Under these conditions his formulas are equivalent to Stratton and Chu's diffraction integrals. His criticism has therefore no foundation. A superior formulation without the defects contained in Stratton and Chu's and Schelkunoff's formulations is due to Franz [3]. This is described in the following section.

6. FRANZ'S ORIGINAL FORMULAS AND THEIR DERIVATION INCLUDING THE SOURCE TERMS

In Franz's original work he considers the integration of Maxwell's equations in a source-free region outside of S_i in Fig. 1 with $\mathbf{J} = 0$, and $\rho = 0$ in (29) and (30). He then applies the method of dyadic Green function to integrate the homogeneous Maxwell's equations, namely

$$\nabla\!\!\!\nabla \mathbf{E} = i\omega\mu_0 \mathbf{H} \tag{61}$$

and

$$\nabla\!\!\!\nabla \mathbf{H} = -i\omega\epsilon_0 \mathbf{E}. \tag{62}$$

The dyadic Green function that he used is what we designate nowadays as the free-space magnetic dyadic Green function [7, p. 60] defined by

$$\bar{\bar{G}}_{m0}(\mathbf{R}, \mathbf{R}') = \nabla\!\!\!\nabla \left[\bar{\bar{I}} G_0 \right] = \nabla G_0 \times \bar{\bar{I}}, \tag{63}$$

where G_0 is the same scalar Green function defined by (36) and $\bar{\bar{I}}$ is the idem factor. He then applies the vector-dyadic Green's theorem of the form

$$\begin{aligned} &\iiint_V \left[\mathbf{F} \cdot \nabla\!\!\!\nabla\, \nabla\!\!\!\nabla \bar{\bar{G}} - (\nabla\!\!\!\nabla\, \nabla\!\!\!\nabla \mathbf{F}) \cdot \bar{\bar{G}} \right] dV \\ &\qquad = \oiint_{S_i} \hat{n}_0 \cdot \left[\mathbf{F} \times \nabla\!\!\!\nabla \bar{\bar{G}} + (\nabla\!\!\!\nabla \mathbf{F}) \times \bar{\bar{G}} \right] dS \end{aligned} \tag{64}$$

to integrate (61) and (62) with $\mathbf{F} = \mathbf{E}$ or $\mathbf{H}$ and $\bar{\bar{G}} = \bar{\bar{G}}_{m0}$. He did not provide many details in his article but merely presented his result as follows:

$$\mathbf{E}_f(\mathbf{R}') = \nabla\!\!\!\nabla' \mathbf{A}_m + \frac{i}{\omega\epsilon_0} \nabla\!\!\!\nabla' \nabla\!\!\!\nabla' \mathbf{A}_{se}, \tag{65}$$

$$\mathbf{H}_f(\mathbf{R}') = \nabla\!\!\!\nabla' \mathbf{A}_{se} - \frac{i}{\omega\mu_0} \nabla\!\!\!\nabla' \nabla\!\!\!\nabla' \mathbf{A}_m, \tag{66}$$

where $\mathbf{A}_m$ is the same function defined by (42) and $\mathbf{A}_{se}$ is the surface integral in (40), that is,

$$\mathbf{A}_{se} = \oiint_{S_i} G_0(\hat{n}_0 \times \mathbf{H})\, dS. \tag{67}$$

No volume integral occurs in his formulation. For completeness we have filled in the details of Franz's work in Appendix 2. In order to compare Franz's with Stratton and Chu's formulation we also include the sources $\mathbf{J}$ and ρ. The proper dyadic Green function to integrate the equation for $\mathbf{E}$ is the free-space electric dyadic Green function defined by

$$\bar{\bar{G}}_{e0}(\mathbf{R}, \mathbf{R}') = \left(\bar{\bar{I}} + \frac{1}{k^2} \nabla\nabla \right) G_0. \tag{68}$$

The expression for $\mathbf{E}$ can also be obtained by integrating the equation for $\mathbf{H}$ using $\bar{\bar{G}}_{m0}$. The integration of Maxwell's equations using $\bar{\bar{G}}_{e0}$ was briefly introduced by the present author on two previous occasions [8, 9]. The same method was used by Sancer [10], whose results will be quoted later. The detailed analysis of Franz's formulation and ours are given in Appendix 2. We are using the notation $\mathbf{E}_f$ to denote Franz's diffraction integrals as in (65) and (66) and the complete formulas including the contributions by sources $\mathbf{J}$ and ρ as $\mathbf{E}_F$ and $\mathbf{H}_F$, then one finds

$$\mathbf{E}_F(\mathbf{R}') = \nabla' \mathbf{A}_m + \frac{i}{\omega\epsilon_0} \nabla' \nabla' \mathbf{A}_e, \tag{69}$$

$$\mathbf{H}_F(\mathbf{R}') = \nabla' \mathbf{A}_e - \frac{i}{\omega\mu_0} \nabla' \nabla' \mathbf{A}_m, \tag{70}$$

where $\mathbf{A}_e$ and $\mathbf{A}_m$ are the functions defined previously by (40) and (41). Comparing (69) and (70) with Stratton and Chu's formulas we find

$$\mathbf{E}_F = \mathbf{E}_{sc} + \frac{i}{\omega\epsilon_0} \nabla' \oint_C G_0(\mathbf{H}_1 - \mathbf{H}_2) \cdot d\ell, \tag{71}$$

$$\mathbf{H}_F = \mathbf{H}_{sc} - \frac{i}{\omega\mu_0} \nabla' \oint_C G_0(\mathbf{E}_1 - \mathbf{E}_2) \cdot d\ell. \tag{72}$$

For clarity, we have used the subscript "*sc*" to denote Stratton and Chu's expressions. It is evident that Franz's formulas thus derived are superior because they are applicable to continuous or discontinuous distribution of the fields on S_i and they do satisfy Maxwell's equations. It should be mentioned that (71) and (72) are identical to the formulas obtained independently by Sancer [10, Eqs. (2.24) and (2.25)] based also on the method of $\bar{\bar{G}}_{e0}$. He did not convert them to a compact form as described by (69) and (70). His article contains much more useful information not covered here. As a whole we must give credit to Franz for his elegant formulation of the vectorial Huygens's principle.

The reader may be interested to note that by taking the curl of (71) and (72) we find

$$\nabla \times \mathbf{E}_{sc} = \nabla \times \mathbf{E}_F = i\omega\mu_0 \mathbf{H}_F \tag{73}$$

and

$$\nabla \times \mathbf{H}_{sc} = \nabla \times \mathbf{H}_F = \mathbf{J} - i\omega\epsilon_0 \mathbf{E}_F. \tag{74}$$

In other words, it is possible to find $\mathbf{E}_F$ and $\mathbf{H}_F$ from $\mathbf{H}_{sc}$ and $\mathbf{E}_{sc}$, respectively. This feature was pointed out in a previous communication [9] without the detailed analysis as found in the present article.

7. CONCLUSION

In this work we have studied quite thoroughly the integration of field equations in electrostatics, magnetostatics, and electromagnetics, first based on a scalar-vector Green's theorem and then by a vector-dyadic Green's theorem for the electromagnetic equations. Some deficiencies in past works of Stratton and Chu and of Schelkunoff have been pointed out. The superior nature of Franz's formulation is emphasized. This investigation demonstrates clearly that the so-called equivalence principle does not serve a useful purpose in the teaching of electromagnetics.

It is worth mentioning that for physically realizable problems the line integrals added by Stratton and Chu are of no concern to us nowadays as a result of Meixner's edge condition [11]. Collin [12, Sec. 1.5] has examined very carefully a canonical problem using cylindrical waves to demonstrate the continuity of $\mathbf{E}$ and $\mathbf{H}$ across the boundary of a conducting wedge. When approximate distribution is used, such as the one that occurred in physical optics approximation where S_1 corresponds to a lit region and S_2 to a completely shadowed region, then the line integrals must be added to both Stratton and Chu's and Schelkunoff's formulas while Franz's formulas cover them automatically. A remark made by the master physicist Arnold Sommerfeld in his recapitulation of Franz's formulation [13, p. 328] may be of interest in this regard:

> "*The vectorial Huygen's principle is no magic wand for the solution of boundary value problems,* but it is of interest as a generalization of the time-honored idea of Christian Huyens."

The classical work of Levine and Schwinger [14] to treat the diffraction of a plane electromagnetic wave by a circular hole in a conducting screen as a boundary-value problem attests the master's view.

APPENDIX 1: A SCALR-VECTOR GREEN'S THEOREM

This theorem can be derived in several different ways. The most efficient one is to use a vector-dyadic Green's theorem of the second kind [4, p. 125] in the form

$$\iiint\limits_V \left[\mathbf{F}\cdot\nabla\times\nabla\times\bar{\bar{G}}-(\nabla\times\nabla\times\mathbf{F})\cdot\bar{\bar{G}}\right]dV$$

$$=-\oiint\limits_S \hat{n}\cdot\left[\mathbf{F}\times\nabla\times\bar{\bar{G}}+(\nabla\times\mathbf{F})\times\bar{\bar{G}}\right]dS. \tag{A.1}$$

This theorem is also needed in explaining Franz's theory in Appendix 2. It is partly for this reason we adopt this approach so that some rudimentary dyadic analysis can be introduced. In (A.1), $\mathbf{F}$ is a vector function and $\bar{\bar{G}}$ is a dyadic function. Let

$$\bar{\bar{G}}=f\bar{\bar{I}}, \tag{A.2}$$

where f is a scalar function and where $\bar{\bar{I}}$ denotes an idemfactor defined by

$$\bar{\bar{I}}=\sum_i \hat{x}_i\hat{x}_i, \tag{A.3}$$

in a rectangular coordinate system or any orthogonal curvilinear system. The idemfactor has the property

$$\mathbf{F}\cdot\bar{\bar{I}}=\bar{\bar{I}}\cdot\mathbf{F}=\mathbf{F}. \tag{A.4}$$

In the following analysis we also need a dyadic identity

$$\mathbf{a}\cdot(\mathbf{b}\times\bar{\bar{c}})=(\mathbf{a}\times\mathbf{b})\cdot\bar{\bar{c}}=-\mathbf{b}\cdot(\mathbf{a}\times\bar{\bar{c}}). \tag{A.5}$$

In this identity the dyadic function $\bar{\bar{c}}$ must be kept at the posterior position unless it is a symmetric dyadic function. The gradient, divergence, and curl of $\bar{\bar{G}}$ are then given by

$$\nabla\bar{\bar{G}}=\nabla\left(f\bar{\bar{I}}\right)=(\nabla f)\,\bar{\bar{I}}, \tag{A.6}$$

$$\nabla\cdot\bar{\bar{G}}=\nabla\cdot\left(f\bar{\bar{I}}\right)=\nabla f\cdot\bar{\bar{I}}=\nabla f, \tag{A.7}$$

$$\nabla\times\bar{\bar{G}}=\nabla\times\left(f\bar{\bar{I}}\right)=\nabla f\times\bar{\bar{I}}, \tag{A.8}$$

and

$$\hat{n}\cdot\left(\mathbf{F}\times\nabla\times\bar{\bar{G}}\right)=(\hat{n}\times\mathbf{F})\cdot\nabla\times\bar{\bar{G}}=(\hat{n}\times\mathbf{F})\cdot\left(\nabla f\times\bar{\bar{I}}\right)=(\hat{n}\times\mathbf{F})\times\nabla f, \tag{A.9}$$

$$\begin{aligned}\mathbf{F}\cdot\nabla\times\nabla\times\bar{\bar{G}}&=\mathbf{F}\cdot\left(\nabla\nabla\bar{\bar{G}}-\nabla\cdot\nabla\bar{\bar{G}}\right)\\&=\mathbf{F}\cdot\nabla\nabla f-\mathbf{F}\nabla\cdot\nabla f=\nabla\cdot(\mathbf{F}\nabla f)-\nabla\cdot\mathbf{F}\nabla f-\mathbf{F}\nabla\cdot\nabla f.\end{aligned} \tag{A.10}$$

One can easily recognize that these identities are the dyadic version of similar identities in vector analysis. The volume integral of the term $\nabla\cdot(\mathbf{F}\nabla f)$ in (A.10) can be converted into a surface integral by means of the dyadic divergence theorem, that is,

$$\iiint\limits_V \nabla\cdot(\mathbf{F}\nabla f)\,dV=\oiint\limits_n(\hat{n}\cdot\mathbf{F})\nabla f\,dS. \tag{A.11}$$

Now substituting all the relevant functions into (A.1) we obtain the desired scalar-vector Green's theorem:

$$\begin{aligned}&\iiint\limits_V[(\nabla\cdot\mathbf{F})\nabla f+\mathbf{F}\nabla\cdot\nabla f+f\nabla\times\nabla\times\mathbf{F}]\,dV\\&\qquad=\oiint\limits_S[(\hat{n}\cdot\mathbf{F})\nabla f+(\hat{n}\times\mathbf{F})\times\nabla f+(\hat{n}\times\nabla\times\mathbf{F})f]\,dS.\end{aligned} \tag{A.12}$$

APPENDIX 2: FRANZ'S DIFFRACTION INTEGRALS AND THE COMPLETE EXPRESSIONS OF THE ELECTROMAGNETIC FIELD

In Franz's original work he considers only the homogeneous Maxwell's equations without the sources in the region of integration bounded by S_i and S_∞ (Fig. 1). They are

$$\nabla\times\mathbf{E}=i\omega\mu_0\mathbf{H}, \tag{A.13}$$

$$\nabla\times\mathbf{H}=-i\omega\epsilon_0\mathbf{E}. \tag{A.14}$$

By eliminating $\mathbf{E}$ or $\mathbf{H}$ we obtain

$$\nabla\times\nabla\times\mathbf{E}-k^2\mathbf{E}=0, \tag{A.15}$$

$$\nabla\times\nabla\times\mathbf{H}-k^2\mathbf{H}=0. \tag{A.16}$$

The vector-dyadic Green's function of the second kind stated by (A.1) with $\hat{n} = -\hat{n}_0$ on S_i becomes

$$\iiint\limits_V \left[\mathbf{F} \cdot \nabla\times\nabla\times \bar{\bar{G}} - (\nabla\times\nabla\times \mathbf{F}) \cdot \bar{\bar{G}}\right] dV$$
$$= \oiint\limits_{S_i} \left[(\hat{n}_0 \times \nabla\times \mathbf{F}) \cdot \bar{\bar{G}} + (\hat{n}_0 \times \mathbf{F}) \cdot \nabla\times \bar{\bar{G}}\right] dS. \qquad \text{(A.17)}$$

We have already deleted the surface integral at infinity because the functions $\mathbf{F}$ and $\bar{\bar{G}}$ to be considered later satisfy the radiation condition.

Now we let

$$\mathbf{F} = \mathbf{E}, \qquad \text{(A.18)}$$
$$\bar{\bar{G}} = \bar{\bar{G}}_{m0} = \nabla\times \left(G_0 \bar{\bar{I}}\right) = -\nabla'\times \left(G_0 \bar{\bar{I}}\right) = -\nabla' G_0 \times \bar{\bar{I}}, \qquad \text{(A.19)}$$

where $\bar{\bar{I}}$ denotes an idemfactor and the function $\bar{\bar{G}}_{m0}$ satisfies the equation

$$\nabla\times\nabla\times \bar{\bar{G}}_{m0} - k^2 \bar{\bar{G}}_{m0} = \nabla\times \left[\bar{\bar{I}} \delta(\mathbf{R} - \mathbf{R}')\right]. \qquad \text{(A.20)}$$

Then

$$\begin{aligned} \nabla\times \bar{\bar{G}}_{m0} = \nabla\times\nabla\times \left(G_0 \bar{\bar{I}}\right) &= \nabla\nabla\cdot \left(G_0 \bar{\bar{I}}\right) - \nabla\cdot\nabla \left(G_0 \bar{\bar{I}}\right) \\ &= \nabla\nabla G_0 - \nabla\cdot\nabla G \bar{\bar{I}} \\ &= \nabla'\nabla' G_0 - \nabla'\cdot\nabla' G_0 \bar{\bar{I}}. \end{aligned} \qquad \text{(A.21)}$$

Substituting these functions into (A.17) we obtain

$$\begin{aligned} &\iiint\limits_V \mathbf{E} \cdot \nabla\times \left(\bar{\bar{I}} \delta\right) dV \\ &= \oiint\limits_{S_i} \Big[-i\omega\mu_0 (\hat{n}_0 \times \mathbf{H}) \cdot \left(\nabla' G_0 \times \bar{\bar{I}}\right) \\ &\qquad + \nabla'\nabla' G_0 \cdot (\hat{n}_0 \times \mathbf{E}) - \nabla'\cdot\nabla' G_0 (\hat{n}_0 \times \mathbf{E})\Big] dS \\ &= \oiint\limits_{S_i} [i\omega\mu_0 \nabla' G_0 \times (\hat{n}_0 \times \mathbf{H}) \\ &\qquad + \nabla'\nabla'\cdot G_0 (\hat{n}_0 \times \mathbf{E}) - \nabla'\cdot\nabla' G_0 (\hat{n}_0 \times \mathbf{E})] dS \\ &= \oiint\limits_{S_i} [i\omega\mu_0 \nabla'\times G_0 (\hat{n}_0 \times \mathbf{H}) + \nabla'\times\nabla'\times G_0 (\hat{n}_0 \times \mathbf{E})] dS. \end{aligned} \qquad \text{(A.22)}$$

The volume integral in (A.22) can be changed to

$$\iiint_V \mathbf{E}\cdot\nabla\left(\bar{\bar{I}}\delta\right)\,dV = \iiint_V \left[\nabla\left(\mathbf{E}\times\bar{\bar{I}}\delta\right)+\bar{\bar{I}}\delta\cdot\nabla\mathbf{E}\right]\,dS$$
$$= \nabla'\mathbf{E} = i\omega\mu_0\mathbf{H}(\mathbf{R}'). \tag{A.23}$$

The integral involving the divergence term in (A.23) vanishes as a result of the dyadic divergence theorem. We have thus obtained the celebrated formula of Franz for the $\mathbf{H}$ field:

$$\mathbf{H}(\mathbf{R}') = \nabla'\oiint_{S_i} G_0(\hat{n}_0\times\mathbf{H})\,dS - \frac{i}{\omega\mu_0}\nabla'\nabla'\oiint_{S_i} G_0(\hat{n}_0\times\mathbf{E})\,dS. \tag{A.24}$$

By integrating the equation for $\mathbf{H}$ also using $\bar{\bar{G}}_{m0}$ we obtain

$$\mathbf{E}(\mathbf{R}') = \nabla'\oiint_{S_i} G_0(\hat{n}_0\times\mathbf{E})\,dS + \frac{i}{\omega\epsilon_0}\nabla'\nabla'\oiint_{S_i} G_0(\hat{n}_0\times\mathbf{H})\,dS. \tag{A.25}$$

In the text, these expressions are denoted by $\mathbf{H}_f$ and $\mathbf{E}_f$, (65) and (66). We are not certain whether Franz carried out the analysis we do here because he did not supply the details in his paper. We have used very freely the vector and dyadic identities given in [4, Appendix B] omitting the derivations.

In order to compare Stratton and Chu's or Schelkunoff's formula with Franz's we need the complete expressions for $\mathbf{E}$ and $\mathbf{H}$ including the sources located outside of S_i. The equations for $\mathbf{E}$ and $\mathbf{H}$ are then given by (33) and (34). We can still use $\bar{\bar{G}}_{m0}$ to integrate the equation for $\mathbf{H}$, which is now given by

$$\nabla\nabla\mathbf{H} - k^2\mathbf{H} = \nabla\mathbf{J}. \tag{A.26}$$

The result yields

$$\mathbf{E}(\mathbf{R}') = \iiint\left(i\omega\mu_0 G_0\mathbf{J} - \frac{1}{\epsilon_0}\nabla' G_0\rho\right)dV + \nabla'\oiint_{S_i} G_0(\hat{n}_0\times\mathbf{E})\,dS$$
$$+\frac{i}{\omega\epsilon_0}\nabla'\nabla'\oiint_{S_i} G_0(\hat{n}_0\times\mathbf{H})\,dS. \tag{A.27}$$

This expression can also be found by integrating the equation for $\mathbf{E}$ given by

$$\nabla\times\nabla\times\mathbf{E} - k^2\mathbf{E} = i\omega\mu_0\mathbf{J}, \tag{A.28}$$

with the aid of the free-space electric dyadic Green function defined by

$$\bar{\bar{G}}_{e0} = \left(\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla\right) G_0. \tag{A.29}$$

In fact, this is what Sancer [10] did. The expression for $\mathbf{H}(\mathbf{R}')$ obtained by integrating (A.26) with $\bar{\bar{G}}_{e0}$ yields

$$\begin{aligned}\mathbf{H}(\mathbf{R}') = &\nabla'\times \iiint G_0 \mathbf{J}\, dV + \nabla'\times \oiint_{S_i} G_0(\hat{n}_0 \times \mathbf{H})\, dS \\ &- \frac{i}{\omega\epsilon_0}\nabla'\times\nabla'\times \oiint_{S_i} G_0(\hat{n}_0 \times \mathbf{E})\, dS.\end{aligned} \tag{A.30}$$

In the text, we denote (A.27) and (A.30) by $\mathbf{E}_F$ and $\mathbf{H}_F$ respectively.

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