Progress In Electromagnetics Research, **PIER 28, 43–95, 2000**

TABLES OF THE SECOND RANK CONSTITUTIVE TENSORS FOR LINEAR HOMOGENEOUS MEDIA DESCRIBED BY THE POINT MAGNETIC GROUPS OF SYMMETRY

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1. INTRODUCTION

Continuous and discrete space and discrete time symmetry of Maxwell's equations is defined by symmetry of the physical media, i.e., by the symmetry of constitutive tensors. We will consider symmetry of a tensor as its property to be invariant under proper and improper rotations, time-reversal operation and combinations of the space and time operations. This symmetry is also in the corollaries of Maxwell's equations, for example in the wave equations for plane waves. The resultant symmetry of the media, of the sources, of the geometry of a given problem and of the boundary conditions must be present in all the solutions of this problem, i.e., in eigenvalues and eigenwaves, eigenvectors, Green's functions, etc.

In this paper, we will investigate symmetry of the constitutive tensors. Constitutive equations establish relations among field vectors completing macroscopic Maxwell's equations to full selfconsistent system. The constitutive tensors in the constitutive equations which can describe electromagnetic properties of media in a wide frequency range from low frequencies to optical region, are also called material or property tensors.

Constitutive relations may be written in different approximations. In some cases, they may contain integral-differential operators [1]. A general form of the relations which takes into account possible multiple effects that result from the interaction of electromagnetic field with media is [2–5]:

$$D_i = \varepsilon_{ij} E_j + \gamma^E_{ij} \partial_t E_j + \xi^E_{ijk} \Delta_j E_k + \beta^E_{ij} B_j + \alpha^E_{ij} \partial_t B_j + \zeta^E_{ijk} \Delta_j B_k + \ldots \tag{1}$$

$$H_i = \mu^{-1}_{ij} B_j + \gamma^B_{ij} \partial_t B_j + \zeta^B_{ijk} \Delta_j B_k + \beta^B_{ij} E_j + \alpha^B_{ij} \partial_t E_j + \zeta^B_{ijk} \Delta_j E_k + \ldots \tag{2}$$

In these equations, E_k and H_i are the electric and magnetic field components; D_i and B_k are electric and magnetic induction components; ε_{ij} , γ^B_{ij} , ζ^E_{ijk} , etc. are constitutive tensors of different ranks. The space Δ_j and time ∂_t derivatives take into account spatial and temporal variations of fields (i.e., the spatial and time dispersion of media).

The multipole expansions (1) and (2) are used in plane time-harmonic wave description of electromagnetic effects in media. This approach is valid in long-wave approximation, when the wavelength is much greater than the linear dimensions of the repetitive unit of the medium.

It is assumed usually, that the relative magnitudes of the multipole contributions to a physical effect are ordered as follows [6]:

$$\begin{array}{ccccc} \text{electric diplole} \gg & \left\{ \begin{array}{l} \text{electric quadrupole} \\ \text{magnetic diploe} \end{array} \right. & \gg & \left\{ \begin{array}{l} \text{electric octopole} \\ \text{magnetic quadrupole} \end{array} \right. & \gg \ldots \\ & & & & \\ 1 & 10^{-2} & & 10^{-4} & \end{array} \tag{3}$$

where the numbers below correspond to the relative magnitudes of the multipole contributions.

Depending on the nature of the medium and phenomena in question, different approximations to equations (1) and (2) are used in electromagnetic calculations. In accordance with (3), most of the electromagnetic phenomena may be satisfactorily described within the limiting number of the terms in (1) and (2).

The electromagnetic community dealing with the problems of propagation, radiation, scattering and diffraction of electromagnetic waves in different (bi)anisotropic media utilizes usually the constitutive tensors as given ones. The problem of determination of the constitutive tensors is a classical one and discussed in many papers, books and textbooks on crystallophysics, physics and electromagnetic theory. A macroscopic theory of the constitutive relations satisfying the principle of frame independence and the second law of thermodynamics for example is developed in [7]. Some general properties of the constitutive relations in the time domain are discussed in [57]. The most detailed information of electromagnetic properties of a given medium may be obtained using microscopic models and theories, such as methods of the quantum physical kinetics. Sometimes phenomenological tensors are used for the medium description, for example, Drude-Born-Fedorov [8], Tellegen [9], Post [10], Landau [11], Ziolkowski [5]. In other cases, the constitutive tensors may be obtained as a result of electrodynamic calculations, in particular for moving media [12, 13]. In practical engineering work, the constitutive tensors are often obtained by experimentation.

The great interest in new artificial bianisotropic materials possessing different physical properties and an enormous amount of recent research in this field can be explained by the wide range of their potential applications. Indeed, a large number of possible tensor structures promises many new interesting electromagnetic solutions. In most practical cases, those media are used which have certain symmetry. This is because the symmetrical media make it possible to choose and control physical effects used in electromagnetic components and devices.

In the synthesis of new electromagnetic materials, the problem may be formulated as an inverse one, which is what should be the symmetry of the medium and also physical properties of the constitutive particles and the host medium to achieve the desired material tensors?

In this paper we will deal only with geometry and will not concern with the physical properties of medium particles, and consequently, with the numerical values of the tensor parameters. Nevertheless the information which is obtained from the symmetry can be significant and may be used as a first step in the synthesis of new artificial media for electromagnetic applications.

In what follows, we will be interested in the structure of constitutive tensors. It means equality of some of their elements to each other, equality of the element modulus but with opposite sign, or equality of the elements to zero. Such a structure is dictated by space-time symmetry of the medium. Symmetry may reduce significantly the number of independent parameters of the tensors and simplify the following analysis of the medium. It is important to note that the tensor structure obtained from symmetry consideration is frequency independent. It is also independent of any microscopic theory used for calculations of tensor element values, i.e., model independent.

The natural and elegant mathematical tool for symmetry analysis is group theory. Geometrical and time-reversal symmetry of a medium can be described by point magnetic groups. The number of these groups is limited, hence one can calculate all the possible material tensor structures.

The existing classifications of bianisotropic media, for example in [1, 14–16] are based on electromagnetic properties and physical effects in the media, and reduced on the whole to the structure of the constitutive tensors. But the structure of the tensors is defined by symmetry of the medium. Therefore, a natural way of bianisotropic medium classification is the group-theoretical one.

To our knowledge, no full tables of the constitutive tensors of the second rank for the media comprising all the 122 discrete point magnetic groups and 21 continuous magnetic groups of symmetry have yet been published in literature. The relevant information dispersed in different papers and books on crystallophysics and electromagnetics is incomplete and contain a large number of mistakes. Some examples of constitutive tensors for specific media are given in many publications, for example in [1, 16–25, 39, 40]. The structure of the permittivity and permeability tensors for nonmagnetic crystals is defined in the books [8, 26].

In physics, the magnetoelectric effect is considered usually in *magnetic* crystals [20, 27, 28], and Tables of the magnetoelectric polarizability tensors for magnetic crystals can be found for example in [20, 27, 28, 41, 58]. However, most of our results for the magnetoelectric tensors for 58 groups of the third category are in disagreement with those in Table 9 of [27, Table 9, pp. 137, 138] and in Table 2 of [28] because we use a more general approach. Besides, the magnetoelectric tensors can be not null-valued in some *nonmagnetic* media. An example of this is omega-media [29]. Hence to complete the picture of possible constitutive tensors describing magnetoelectric effect, it is necessary to include the nonmagnetic groups into consideration as well.

As the main result, we will present the full Tables of the calculated constitutive tensors for bianisotropic media. Some of these results have been obtained by the author in [30–35]. We will also show that the tensor structures described by some magnetic groups coincide completely hence we can reduce the number of symmetry groups under consideration more than twice and present the Tables in a compact form. In order to simplify reading the paper, we will give in Appendices a short description of the magnetic groups and their notations, of the group theoretical method used for tensor calculations, Curie's principle of symmetry superposition which is used for determination of the complex structure symmetry. Therefore, no previous knowledge of group theory is needed.

Thus this paper is an effort to present the whole picture of possible constitutive tensors of the second rank for bianisotropic media.

2. THE PROBLEM FORMULATION

We will consider unbounded linear, stationary and in general dissipative bianisotropic media in the frequency domain. The media under consideration are assumed to be homogeneous, that is the constitutive tensors are not functions of space variables (long-wave approximation), but electromagnetic properties of the media depend in general on direction in space (for anisotropic cases). The elements of the constitutive tensors are in general complex due to complex electromagnetic field consideration and due to possible losses in media. The values of the tensor elements depend on frequency; it means that the media are time dispersive and obey the Kramers-Kronig relations [11, 48] which are a consequence of causality.

For bianisotropic media, $\mathbf{D}$ and $\mathbf{B}$ are related to both $\mathbf{E}$ and $\mathbf{H}$. The ensuing discussion will be restricted by consideration of the macroscopic constitutive relations in the following form:

$$\mathbf{D} = [\varepsilon]\mathbf{E} + [\xi]\mathbf{H} \tag{4}$$
$$\mathbf{B} = [\zeta]\mathbf{E} + [\mu]\mathbf{H} \tag{5}$$

where the 3×3 tensors of the second rank $[\varepsilon]$ and $[\mu]$ are the tensors of the permittivity and permeability, $[\xi]$ and $[\zeta]$ describe the magnetoelectric crosscoupling. The tensors $[\xi]$ and $[\zeta]$ are usually referred to as magnetoelectric tensors. The approximation (4) and (5) is good enough for many practical purposes. The four 3×3 tensors in the most general form contain 36 independent parameters. Notice that the functional dependence in (4) and (5) may include also a spatial dispersion.

The constitutive relations in another form, in the so called Boys-Post representation are used by some authors [3, 10, 20]:

$$\mathbf{D} = [\varepsilon']\mathbf{E} + [\alpha']\mathbf{B} \tag{6}$$
$$\mathbf{H} = [\beta']\mathbf{E} + [\mu']^{-1}\mathbf{B} \tag{7}$$

which is more natural for various reasons. For example, the fields $\mathbf{E}$ and $\mathbf{B}$ are well-defined, unique and origin-independent [4]. These fields are considered as field vectors (primitive fields), while $\mathbf{D}$ and $\mathbf{H}$ as excitation vectors (induction fields). Each pair of the vectors $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{D}$, $\mathbf{H}$ form a single tensor in four-dimensional space, and (6) and (7) are Lorentz-covariant [1]. However, the tensors in (6) and (7) can be expressed in terms of the tensors of (4) and (5):

$$[\varepsilon'] = [\varepsilon] - [\xi][\mu]^{-1}[\zeta], \tag{8}$$
$$[\alpha'] = [\xi][\mu]^{-1} \tag{9}$$
$$[\beta'] = -[\mu]^{-1}[\zeta] \tag{10}$$
$$[\mu'] = [\mu] \tag{11}$$

Besides, the symmetry structure of the pairs of the constitutive tensors $[\varepsilon]$ and $[\varepsilon']$, $[\xi]$ and $[\alpha']$, $[\zeta]$ and $[\beta']$, $[\mu]$ and $[\mu']$ in two sets in (6), (7) and (4), (5) is one and the same because these pairs possess equal space-time transformation properties. Among other things, it is an important argument in behalf of classification of bianisotropic media on symmetry principles. In what follows, we will consider the constitutive relations in the representation (4) and (5) which is more suitable for example, in the solution of boundary value problems, where the boundary conditions are written in terms $\mathbf{E}$ and $\mathbf{H}$, and in some other applications.

Analysis of the magnetoelectric effect based on density of stored energy (or on thermodynamic potential) [11, 20, 41] shows that the tensors $[\xi]$ and $[\zeta]$ in (4), (5) satisfies the condition $[\xi] = [\zeta]^t$. But for dispersive media, a definition of electromagnetic energy as a thermodynamic quantity is impossible [11]. Therefore, we can not use in general the above condition for tensor calculation. This problem will be discussed in more detail in Section 5.

Notice that the medium description in the form (4) and (5) for plane time-harmonic waves may correspond to any order in multipole expansions (1) and (2) [36] so that the constitutive relations (4) and (5) may comprise electric dipole, electric quadrupole, magnetic dipole etc. contributions.

Our main task is to describe a method of tensor calculation and to calculate the structure of the tensors $[\varepsilon]$, $[\mu]$, $[\xi]$ and $[\zeta]$, which is dictated by space-time symmetry of media for all the 122 discrete crystallographic and 21 continuous magnetic groups.

3. METHOD OF TENSOR CALCULATIONS

The structure of the tensors describing a symmetrical medium depends on the mutual orientation of the chosen Cartesian coordinate system x, y, z and the symmetry axes and planes of the medium. We will hold the rule for choosing the symmetry element orientations with respect to the coordinate axes given in Table 1. Notice that using the presented below method, one can easily calculate the constitutive tensors for other orientations of the (anti)axes and/or (anti)planes.

The constitutive tensors for any group can be calculated using the relations (D17)–(D20) with corresponding generators of the group. In order to reduce the volume of calculations, we will use several artifices. First, in accordance with the Herman-Hermann theorem [37, 38], some of the groups lead to identical tensor structure (we will show below how to find such groups). The Herman-Hermann theorem for our case reads as follows: "If C_n is an axis of symmetry for a constitutive tensor (of rank 2) and $n > 2$, then the axes C_3, C_4, $\cdots$, and C_∞ are also the axes of symmetry for this tensor". In other words, all the axes of geometrical symmetry higher than 2, are converted into the axes of infinite order for the tensors of the second rank; or equivalently, the

Table 1. 32 nonmagnetic crystallographic groups in international, Shubnikov and Schoenflies notations, generating matrices of the groups and orientations of the axes and the planes.

N	Notations			Generating matrices	Orientation of axes and planes
	International	Shubnikov	Schoenflies		
1	m3m	$\tilde{6}/4$	O_h	$\begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_4 \vert\vert z$
2	$\bar{4}3m$	$3/\tilde{4}$	T_d	$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$S_4 \vert\vert z$
3	$m\bar{3}$	$\tilde{6}/2$	T_h	$\begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_2 \vert\vert z$
4	432	3/4	O	$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_4 \vert\vert z$
5	23	3/2	T	$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_2 \vert\vert z$
6	6/mmm	m·6:m	D_{6h}	$\begin{vmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_6 \vert\vert z$
7	6mm	6·m	C_{6v}	$\begin{vmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_6 \vert\vert z$
8	6/m	6:m	C_{6h}	$\begin{vmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_6 \vert\vert z$
9	622	6:2	D_6	$\begin{vmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_6 \vert\vert z$
10	$\bar{6}m2$	$\tilde{3}\cdot m = m\cdot 3:m$	D_{3h}	$\begin{vmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_3 \vert\vert z$

Table 1. (Continued)

11	$\bar{6}$	3:m	C_{3h}	$\begin{vmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_3 \vert\vert z$
12	6	6	C_6	$\begin{vmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_6 \vert\vert z$
13	$\bar{3}$m	$\tilde{6}\cdot$m	D_{3d}	$\begin{vmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_3 \vert\vert z$
14	3m	3·m	C_{3v}	$\begin{vmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_3 \vert\vert z$
15	32	3:2	D_3	$\begin{vmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_3 \vert\vert z$
16	$\bar{3}$	$\tilde{6}$	$S_6=C_{3i}$	$\begin{vmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_3 \vert\vert z$
17	3	3	C_3	$\begin{vmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_3 \vert\vert z$
18	4/mmm	m·4:m	D_{4h}	$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_4 \vert\vert z$
19	$\bar{4}$2m	$\tilde{4}\cdot$m	$V_d=D_{2d}$	$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$S_4 \vert\vert z$ $C_s \vert\vert x$
20	4mm	4·m	C_{4v}	$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_4 \vert\vert z$
21	4/m	4:m	C_{4h}	$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_4 \vert\vert z$
22	422	4:2	D_4	$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_4 \vert\vert z$

Table 1. (Continued)

23	$\bar{4}$	$\tilde{4}$	S_4	$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$S_4 \| z$
24	4	4	C_4	$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_4 \| z$
25	mmm	m·2:m	$V_h = D_{2h}$	$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_2 \| z$ $C_s \perp x,y,z$
26	mm2	2·m	C_{2v}	$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_2 \| z$ $C_s \perp x,y$
27	222	2:2	$V = D_2$	$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_2 \| x,y,z$
28	2/m	2:m	C_{2h}	$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_2 \| z$
29	m	m	$C_{1h} = C_s$	$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	$C_s \perp z$
30	2	2	C_2	$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	$C_2 \| z$
31	$\bar{1}$	$\tilde{2}$	$S_2 = C_i$	$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$	any
32	1	1	C_1	$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$	any

tensors of the second rank can not have the axes of symmetry higher than 2, but can have the axis of infinite order. As a result, with the axis of infinite order and with a plane of symmetry, which is perpendicular to this axis, the corresponding tensor may acquire a "virtual" center of symmetry, which leads to the null magnetoelectric effect.

Therefore the symmetry of a medium and the symmetry of the tensor which describes the medium may not coincide. The tensor symmetry may be higher than the symmetry of the medium, which the tensor describes. This peculiarity is stipulated by the transformation formulas of the tensors and may be explained as follows. With $n \geq 3$ one and the same transformation of the tensor components must be valid for different values of the transformation coefficients (for different angles of rotation), and it is possible only if the tensor has the axis of the infinite order. This is reflected for example, in the fact that the cubic crystals do not have isotropic symmetry of their unit sells, nevertheless their tensors are degenerate to the scalars.

Second, at the beginning we will calculate the tensors corresponding to the media of the second category (Appendix A) and this will be the basis for other calculations. The tensors for the media described by the groups of the third category will be calculated using the results for the media described by the groups of the second category as follows. Generators for any group $G(H)$ of the third category are the generators of H (they correspond to the unitary elements) and any antiunitary element of the group $G(H)$. Hence, in order to calculate tensors for the group $G(H)$, we take the calculated tensors for the group H of the second category and apply only one transformation equation ((D19) for $[\varepsilon]$ and $[\mu]$ and (D20) for $[\zeta]$ and $[\xi]$) with a generator corresponding to the chosen antiunitary element. As a result, the tensors $[\varepsilon]$ and $[\mu]$ in general acquire a more simple structure, but because of the special form of the relations (D20), the tensors $[\zeta]$ and $[\xi]$ turn out to be coupled, i.e., $[\zeta]$ is expressed in terms of $[\xi]$ or vice versa. That is, the tensors $[\zeta]$ and $[\xi]$ do not change their structure under such transformations. It means that the structure of the magnetoelectric tensors is defined completely by the corresponding unitary subgroup. The principal distinction of the tensors for the groups of the second and the third categories is as follows: for the groups of the second category the tensors $[\zeta]$ and $[\xi]$ are always independent from each other, but for the groups of the third category, these tensors are always coupled.

The tensors for the media described by the groups of the first category (nonmagnetic ones) are obtained from the results for the media described by the groups of the second category using the condition of reciprocity [1]:

$$[\mu] = [\mu]^t, \quad [\varepsilon] = [\varepsilon]^t, \quad [\xi] = -[\zeta]^t \tag{12}$$

From (12) we see that the tensors $[\zeta]$ and $[\xi]$ for nonmagnetic media are always coupled.

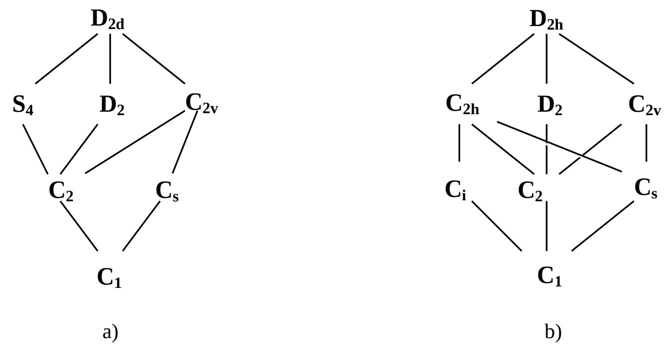

Figure 1. The group tree a) for the group D_{2d} , b) for the group D_{2h} .

Third, we begin our calculations with the lowest group and calculate the tensors for a higher group using the previous results with the help of a new generator for the higher group under consideration.

Now, let us return to the problem of determination of those groups which give the identical tensor structures. In order to use the Herman-Hermann theorem we should find the highest possible discrete groups which contain axes not higher than 2. In the case of the second category groups, there are two such groups: D_{2d} and D_{2h} . All the other groups containing axes of the order not higher than 2, are subgroups of these ones. The corresponding group trees are presented in Fig. 1a and Fig. 1b. Some of the groups in these Figures coincide. In all, there are 10 different discrete groups in these group trees:

$$\mathbf{D_{2d}},\ \mathbf{D_{2h}}\ \mathbf{S_4},\ \mathbf{D_2},\ \mathbf{C_{2h}},\ \mathbf{C_{2v}},\ \mathbf{C_i},\ \mathbf{C_2},\ \mathbf{C_s},\ \mathbf{C_1} \tag{13}$$

Therefore, the 7 continuous groups

$$\mathbf{K_h},\ \mathbf{K},\ \mathbf{D_{\infty h}},\ \mathbf{D_\infty},\ \mathbf{C_{\infty v}},\ \mathbf{C_{\infty,h}},\ \mathbf{C_\infty} \tag{14}$$

and 10 groups (13) exhaust all the possible variants of symmetries for the groups of the second category. The symmetries higher than D_{2d} and D_{2h} will give the tensors coinciding with those for the corresponding continuous groups. This consideration is also valid for the groups of the first category, i.e., in nonmagnetic cases.

The situation with the groups of the third category is more complicated and we have to consider much more possible variants of the tensor structure. We will not discuss it here. The tensors for the lowest

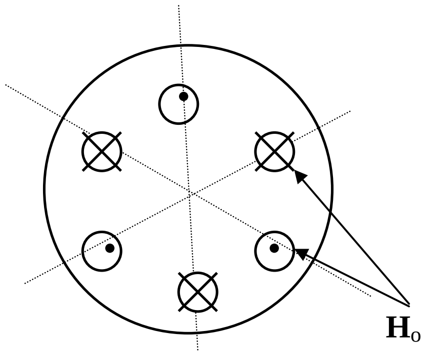

Figure 2. An example of the magnetic structure described by the group $C_6(C_3)$.

groups are given in Table 7, and for continuous groups and for some higher discrete groups are in Table 6.

Another peculiarity in calculations of the tensors exists for some groups of the third category. Let us consider as an example the group $C_6(C_3)$. This group describes for example the magnetic structure in Fig. 2. The group contains three unitary elements E, C_3 and C_3^2 and three antiunitary elements TC_6, TC_6^3 and TC_6^5. In accordance with the Herman-Hermann theorem, the symmetry C_3 of the medium corresponds to the continuous group C_∞ for the tensor. Combining (D18) and (D20), it is easy to show that the unitary elements E, C_3 and C_3^2 together with the antiunitary ones TC_6, TC_6^3 and TC_6^5 lead to the constitutive tensors satisfying the conditions of reciprocity (12). Therefore, it is logically to place the tensors for the group $C_6(C_3)$ in Table 2 for nonmagnetic groups in the line corresponding to the group C_∞. Analogously we can consider the groups $D_6(D_3)$, $C_{6v}(C_{3v})$, $C_{6h}(C_{3i})$ and some others which are in Table 2.

4. TABLES OF CONSTITUTIVE TENSORS FOR MEDIA DESCRIBED BY MAGNETIC POINT GROUPS

The calculated constitutive tensors are presented in Tables 2–7. These tensors have been found in the most general, canonical form which is dictated by space-time symmetry. The structure of these tensors take into account all the possible physical effects in the media (in the

Table 2. Constitutive tensors for media described by the continuous groups of the first category, some discrete groups of the first category and some discrete groups of the third category.

N^o	Magnetic group	$[\mu]$	$[\varepsilon]$	$[\xi]$	$[\zeta]$	Number of independent parameters
1	$\underline{K_h}$ T_h, T_d, O_h, $O_h(T_h)$, $O_h(T_d)$	μ	ε	0	0	2
2	$\underline{K}$ T, O, O(T)	μ	ε	ξ	$-\xi$	3
3	$\underline{D_{\infty h}}$ D_{3d}, D_{3h}, D_{4h}, D_{6h}, $D_{6h}(D_{3d})$, $D_{6h}(D_{3h})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	4
4	$\underline{D_\infty}$ D_3, D_4, D_6, $D_6(D_3)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & 0 & 0 \\ 0 & -\xi_{11} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	6
5	$\underline{C_{\infty v}}$ C_{3v}, C_{4v}, C_{6v}, $C_{6v}(C_{3v})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	5
6	$\underline{C_{\infty h}}$ C_{3h}, C_{3i}, C_{4h}, C_{6h}, $C_{6h}(C_{3i})$, $C_{6h}(C_{3h})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	4
7	$\underline{C_\infty}$ C_3, C_4, C_6, $C_6(C_3)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	7

framework of these second rank tensors description). The structure of the tensors is frequency independent. This very important property is a consequence of the symmetry approach used for the tensor calculations. The values of their elements are, of course, frequency dependent.

Some of the tensors for the discrete groups are collected with the corresponding tensors for continuous groups having the same structure of the tensors. It has allowed us to form compact Tables. The continuous groups in Tables 2, 4, 6 are underlined.

The structure of the tensors $[\mu]$ and $[\varepsilon]$ and also $[\xi]$ and $[\zeta]$ coincide in all Tables because they have the same transformation properties and are calculated by the analogous expressions (D17), (D19) and (D18), (D20), respectively.

The tensors $[\xi]$ and $[\zeta]$ in the groups of the second category are coupled in no way. For the groups of the first and the third category,

Table 3. Constitutive tensors for media described by discrete groups of the first category.

N^{o}	Magnetic group	$[\mu]$	$[\varepsilon]$	$[\xi]$	$[\zeta]$	Number of independent parameters
1	D_{2d}	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & -\xi_{12} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	5
2	D_{2h}	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	6
3	S_4	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & -\xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix}$	6
4	D_2	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & 0 & 0 \\ 0 & -\xi_{22} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	9
5	C_{2h}	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	8
6	C_{2v}	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{21} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & -\xi_{21} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	8
7	C_i	$\begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ \mu_{13} & \mu_{23} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	12
8	C_2	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & -\xi_{21} & 0 \\ -\xi_{12} & -\xi_{22} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	13
9	C_s	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & 0 & \xi_{13} \\ 0 & 0 & \xi_{23} \\ \xi_{31} & \xi_{32} & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 & -\xi_{31} \\ 0 & 0 & -\xi_{32} \\ -\xi_{13} & -\xi_{23} & 0 \end{vmatrix}$	12
10	C_1	$\begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ \mu_{13} & \mu_{23} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & -\xi_{21} & -\xi_{31} \\ -\xi_{12} & -\xi_{22} & -\xi_{32} \\ -\xi_{13} & -\xi_{23} & -\xi_{33} \end{vmatrix}$	21

the tensors $[\zeta]$ are expressed in terms of $[\xi]$. In the groups of the first category, these tensors are coupled by the condition of reciprocity (12). For the groups of the third category, the relation between the tensors $[\zeta]$ and $[\xi]$ is defined by the condition (D20) corresponding to an antiunitary element of the group. The tensors $[\mu]$ and $[\varepsilon]$ however, transformed into themselves for both cases of unitary and antiunitary

Table 4. Constitutive tensors for media described by continuous and some discrete groups of the second category.

N°	Magnetic group	[μ]	[ε]	[ξ]	[ζ]	Number of independent parameters
1	$\underline{\mathbf{K}}_{\mathbf{h}}$ $\mathbf{T_h, T_d, O_h}$	μ	ε	0	0	2
2	$\underline{\mathbf{K}}$ $\mathbf{T, O}$	μ	ε	ξ	ζ	4
3	$\underline{\mathbf{D}}_{\infty\mathbf{h}}$ $\mathbf{D_{3d}, D_{3h}, D_{4h}, D_{6h}}$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	4
4	$\underline{\mathbf{D}}_{\infty}$ $\mathbf{D_3, D_4, D_6}$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \zeta_{11} & 0 & 0 \\ 0 & \zeta_{11} & 0 \\ 0 & 0 & \zeta_{33} \end{vmatrix}$	8
5	$\underline{\mathbf{C}}_{\infty\mathbf{v}}$ $\mathbf{C_{3v}, C_{4v}, C_{6v}}$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & \zeta_{12} & 0 \\ -\zeta_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	6
6	$\underline{\mathbf{C}}_{\infty\mathbf{h}}$ $\mathbf{C_{3h}, C_{3i}, C_{4h}, C_{6h}}$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	6
7	$\underline{\mathbf{C}}_{\infty}$ $\mathbf{C_3, C_4, C_6}$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \zeta_{11} & \zeta_{12} & 0 \\ -\zeta_{12} & \zeta_{11} & 0 \\ 0 & 0 & \zeta_{33} \end{vmatrix}$	12

elements because these tensors are on the main diagonal of the composed constitutive tensor $[K]$ (Appendices D3, D4).

Notice that in several cases, there exists "accidental" degeneracy of the tensor structures, i.e., coincidence of them for different groups of symmetry. For example, the constitutive tensors coincide for the groups K_h of the first and the second category, and also for the groups $D_{\infty h}$ of the first and the second category and $C_{\infty h}$ of the first category.

In particular cases of concrete media, the number of independent parameters of the tensors may decrease. For example, if the medium described by the group C_∞ is gyromagnetic and nongyroelectric, the tensor $[\varepsilon]$ becomes diagonal. If the medium described by D_∞ has magnetoelectric properties only in z-direction, $[\xi]$ and $[\zeta]$ have only z-components ξ_{33} and ζ_{33} and for reciprocal case $\xi_{33} = -\zeta_{33}$ [56], etc. In all the magnetic cubic groups, the constitutive tensors degenerate in scalars. It means that the corresponding media are isotropic.

Notice that the presented results are valid also for the tensors in the relations (6)–(7), as was mentioned in Section 2.

Table 5. Constitutive tensors for media described by discrete groups of the second category.

N°	Magnetic group	[μ]	[ε]	[ξ]	[ζ]	Number of independent parameters
1	$\mathbf{D_{2d}}$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & \zeta_{12} & 0 \\ \zeta_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	6
2	$\mathbf{D_{2h}}$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	6
3	$\mathbf{S_4}$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} \zeta_{11} & \zeta_{12} & 0 \\ \zeta_{12} & -\zeta_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix}$	10
4	$\mathbf{D_2}$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \zeta_{11} & 0 & 0 \\ 0 & \zeta_{22} & 0 \\ 0 & 0 & \zeta_{33} \end{vmatrix}$	12
5	$\mathbf{C_{2h}}$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	10
6	$\mathbf{C_{2v}}$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{21} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & \zeta_{12} & 0 \\ \zeta_{21} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	10
7	$\mathbf{C_i}$	$\begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	18
8	$\mathbf{C_2}$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \zeta_{11} & \zeta_{12} & 0 \\ \zeta_{21} & \zeta_{22} & 0 \\ 0 & 0 & \zeta_{33} \end{vmatrix}$	20
9	$\mathbf{C_s}$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & 0 & \xi_{13} \\ 0 & 0 & \xi_{23} \\ \xi_{31} & \xi_{32} & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 & \zeta_{13} \\ 0 & 0 & \zeta_{23} \\ \zeta_{31} & \zeta_{32} & 0 \end{vmatrix}$	18
10	$\mathbf{C_1}$	$\begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} \\ \zeta_{31} & \zeta_{32} & \zeta_{33} \end{vmatrix}$	36

If we are given a medium, in order to find the constitutive tensors we need only the magnetic group of symmetry of the medium. Then using Tables 2–7 we obtain immediately the desired tensors. As an example [31], let us consider the uniaxial omega medium which consists of two metallic Ω-shaped wire elements in the form of a hat oriented at 90° to each other (Fig. 3). The hat-elements are disposed periodically in an isotropic host material forming a square array and preserving the

Table 6. Constitutive tensors for media described by continuous and some discrete groups of the third category.

N^o	Magnetic group	$[\mu]$	$[\varepsilon]$	$[\xi]$	$[\zeta]$	Number of independent parameters
1	$\underline{K_h(K)}$ $T_h(T)$, $T_d(T)$, $O_h(O)$	μ	ε	ξ	ξ	2
2	$\underline{D_{\infty h}(D_\infty)}$ $D_{4h}(D_4)$, $D_{3d}(D_3)$, $D_{3h}(D_3)$, $D_{6h}(D_6)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	6
3	$\underline{D_{\infty h}(C_{\infty v})}$ $D_{4h}(C_{4v})$, $D_{3d}(C_{3v})$, $D_{3h}(C_{3v})$, $D_{6h}(C_{6v})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & -\xi_{12} & 0 \\ \xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	5
4	$\underline{D_{\infty h}(C_{\infty h})}$ $D_{4h}(C_{4h})$, $D_{3d}(C_{3i})$, $D_{3h}(C_{3h})$, $D_{6h}(C_{6h})$,	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	6
5	$\underline{D_\infty(C_\infty)}$ $D_3(C_3)$, $D_4(C_4)$, $D_6(C_6)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & -\xi_{12} & 0 \\ \xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	9
6	$\underline{C_{\infty v}(C_\infty)}$ $C_{3v}(C_3)$, $C_{4v}(C_4)$, $C_{6v}(C_6)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	9
7	$\underline{C_{\infty h}(C_\infty)}$ $C_{3h}(C_3)$, $C_{3i}(C_3)$, $C_{4h}(C_4)$, $C_{6h}(C_6)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & -\xi_{12} & 0 \\ \xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	7

axis of the hat along the z-axis. This medium has the z-axis of the fourth order C_4 and four planes of symmetry C_s passing through this axis. These elements of symmetry define the group of the first category (nonmagnetic) C_{4v}. From Table 2, we obtain the desired tensors of the Ω-medium which coincide with the known ones obtained phenomenologically [21]:

$$
\begin{aligned}
&[\mu] = \begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}, \quad [\varepsilon] = \begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}, \\
&[\xi] = \begin{vmatrix} 0 & \xi_{12} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad [\zeta] = \begin{vmatrix} 0 & \xi_{12} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}
\end{aligned}
\tag{15}
$$

5. DISCUSSION OF THE RESULTS

In this Section, we will discuss briefly some differences between our results and those published in the previous literature.

Table 7. Constitutive tensors for media described by discrete groups of the third category.

N°	Magnetic group	[μ]	[ε]	[ξ]	[ζ]	Number of independent parameters
1	$D_{4h}(D_{2d})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	5
2	$D_{4h}(D_{2h})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	4
3	$C_{4h}(S_4)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix}$	6
4	$C_{4h}(C_{2h})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	4
5	$D_4(D_2)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{22} & 0 & 0 \\ 0 & -\xi_{11} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	7
6	$D_{2d}(S_4)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & -\xi_{12} & 0 \\ -\xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix}$	8
7	$D_{2d}(D_2)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	6
8	$D_{2d}(C_{2v})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{21} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & -\xi_{12} & 0 \\ -\xi_{21} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	6
9	$C_{4v}(C_{2v})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{21} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{21} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	6
10	$D_{2h}(D_2)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	9
11	$D_{2h}(C_{2h})$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	8
12	$D_{2h}(C_{2v})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ \xi_{21} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{21} & 0 \\ \xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	8

Table 7. (Continued)

13	$S_4(C_2)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{22} & -\xi_{12} & 0 \\ -\xi_{21} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	9
14	$C_4(C_2)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{22} & \xi_{12} & 0 \\ \xi_{21} & -\xi_{11} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	9
15	$D_2(C_2)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & \xi_{21} & 0 \\ \xi_{12} & -\xi_{22} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	13
16	$C_{2h}(C_i)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ -\mu_{13} & -\mu_{23} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	12
17	$C_{2h}(C_2)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{21} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	13
18	$C_{2h}(C_s)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & 0 & \xi_{13} \\ 0 & 0 & \xi_{23} \\ \xi_{31} & \xi_{32} & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 & \xi_{31} \\ 0 & 0 & \xi_{32} \\ \xi_{13} & \xi_{23} & 0 \end{vmatrix}$	12
19	$C_{2v}(C_2)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & -\xi_{21} & 0 \\ -\xi_{12} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	13
20	$C_{2v}(C_s)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & \mu_{23} \\ 0 & -\mu_{23} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & -\varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & \xi_{13} \\ \xi_{21} & 0 & 0 \\ \xi_{31} & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & -\xi_{21} & \xi_{31} \\ -\xi_{12} & 0 & 0 \\ \xi_{13} & 0 & 0 \end{vmatrix}$	12
21	$C_i(C_1)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ \mu_{13} & \mu_{23} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{21} & \xi_{31} \\ \xi_{12} & \xi_{22} & \xi_{32} \\ \xi_{13} & \xi_{23} & \xi_{33} \end{vmatrix}$	21
22	$C_2(C_1)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ -\mu_{13} & -\mu_{23} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & -\xi_{21} & \xi_{31} \\ -\xi_{12} & -\xi_{22} & \xi_{32} \\ \xi_{13} & \xi_{23} & -\xi_{33} \end{vmatrix}$	21
23	$C_s(C_1)$	$\begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ -\mu_{13} & -\mu_{23} & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{21} & -\xi_{31} \\ \xi_{12} & \xi_{22} & -\xi_{32} \\ -\xi_{13} & -\xi_{23} & \xi_{33} \end{vmatrix}$	21

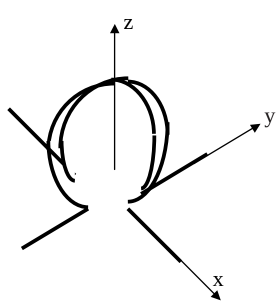

Figure 3. Uniaxial Ω-element in the form of a hat, the group C_4.

The main difference of our approach for tensor calculations from that given for example by Birss [27] and O'Dell [20], consists in the following. In order to calculate the magnetoelectric tensors $[\xi]$ and $[\zeta]$ in the cases of the 58 magnetic groups of the third category groups which contain both unitary and antiunitary operators, we use two different types of the transformation formulas. The relations (D20) correspond to an antiunitary element of the group. These relations are different from (D18) corresponding to unitary elements. The relations (D18) define the transformation properties of $[\xi]$ and $[\zeta]$ separately whilst (D20) couple the tensors $[\xi]$ and $[\zeta]$ with each other.

In the books [20, 27] however, this circumstance is not taken into account, the relations of the type (D20) are not used at all, and as a result our tensors in Tables 6, 7 for most of the groups differ from the corresponding tensors presented in previous publications. Some of the lower classes where we have different from Birss's results are:

$$\begin{aligned}&C_s(C_1),\ C_2(C_1),\ C_{2v}(C_2),\ D_2(C_2),\ C_{2v}(C_s),\ S_4(C_2),\ C_4(C_2),\\ &C_{4v}(C_4),\ D_3(C_3),\ C_{3v}(C_3), D_{3h}(C_{3h}),\ C_{4v}(C_{2v}),\ D_{2d}(S_4),\ D_4(D_2),\\ &D_4(C_4),\ D_6(C_6),\ \text{etc.}\end{aligned} \tag{16}$$

There are also many discrepancies for higher groups.

The tensors $[\xi]$ and $[\zeta]$ correspond to magnetoelectric coupling through the polarization in media. In accordance with Onsager's principle [42], these tensors under time reversal operator are transformed into each other (Appendix D3):

$$[\xi] = -[\zeta]^t \tag{17}$$

but not into themselves as in the case of the tensors $[\varepsilon]$ and $[\mu]$. Under a combined space-time operator $T[R]$, the tensors $[\xi]$ and $[\zeta]$ are also transformed into each other according to (D20). This property of the magnetoelectric tensors becomes obvious with six-vector description of the problem (Appendix D). The results of Indenbom [28] and Birss [27] have been obtained probably for the transformation relations (D18) where $[\xi]$ and $[\zeta]$ are transformed into themselves, which is to our opinion, a global mistake.

Let us clarify this point of view in more detail using two simplest examples: the groups of the third category $C_s(C_1)$ and $C_2(C_1)$ (they are $\underline{m}$ and $\underline{2}$ in the International notations, respectively). In order to find the structure of constitutive tensors, we can use the relations (D18, D20) where the matrices $[R]$ are three-dimensional representations of the corresponding unitary group elements and of the geometrical part of the antiunitary elements.

First, we will consider the group $C_s(C_1)$. If the plane of symmetry C_s is perpendicular to the z-axis, the symmetry matrix $[R]$ of this geometrical operation is

$$[R] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \tag{18}$$

We have only two elements in the group $C_s(C_1)$: e (identity) and the antiunitary element TC_s (T is the time reversal operator). In order to find the magnetoelectric tensors $[\xi]$ and $[\zeta]$ in the constitutive equations (4), (5), we must use the relations (D20) with the matrix (18) which defines the geometrical part C_s of the antiunitary element TC_s (notice that the identity e does not give any information). The result of calculations is as follows:

$$[\xi] = \begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}, \qquad [\zeta] = \begin{vmatrix} \xi_{11} & \xi_{21} & -\xi_{31} \\ \xi_{12} & \xi_{22} & -\xi_{32} \\ -\xi_{13} & -\xi_{23} & \xi_{33} \end{vmatrix} \tag{19}$$

We see that both tensors contain all 9 parameters and these tensors are coupled, i.e., $[\zeta]$ is expressed in terms of $[\xi]$. These tensors differ from those in Table 9 of Birss where the structure of them is given by

$$\begin{vmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{vmatrix} \tag{20}$$

with four nul-elements. This structure can be obtained from (19) using an additional constraint

$$[\xi] = [\zeta]^t \tag{21}$$

which follows from the theory of thermodynamic potential [11]. This condition has been discussed also in [20, 41] and used probably in [20, 27] for tensor calculations. However, the constraint (21) is not valid in general case of dispersive media [21] where losses exist always and a definition of the thermodynamic potential is impossible [11] (notice that the condition of losslessness $[\xi] = [\zeta^*]^t$ also couples the magnetoelectric tensors, but this coupling differs from (21)).

We will demonstrate a defect of (21) on a simple example. The constitutive tensors for isotropic chiral media (which are spatially dispersive) degenerate in scalars and (4), (5) are written in the form [14]:

$$\mathbf{D} = \varepsilon\mathbf{E} - jk\sqrt{\varepsilon_0\mu_0}\,\mathbf{H} \tag{22}$$

$$\mathbf{B} = jk\sqrt{\varepsilon_0\mu_0}\,\mathbf{E} + \mu\mathbf{H} \tag{23}$$

Application of the condition (21) to (22) and (23) gives $k = 0$ which is of course incorrect. Besides, the condition (21) together with the condition of reciprocity from (12)

$$[\xi] = -[\zeta]^t \tag{24}$$

leads to an erroneous conclusion that reciprocal bianisotropic media can not exist at all.

A more realistic symmetry approach has been recently suggested in [4] where quantum mechanical perturbation theory allowed the authors to obtain the constitutive relations in the electric quadrupole-magnetic dipole approximation. The electromagnetic tensors $[\alpha']$ and $[\beta']$ in (6), (7) are expressed as a sum of two parts:

$$[\alpha'] = [\alpha]_m + [\alpha]_n \tag{25}$$

$$[\beta'] = [\beta]_m + [\beta]_n \tag{26}$$

where $[\alpha]_m$ and $[\beta]_m$ denote a "magnetic" part, and $[\alpha]_n$ and $[\beta]_n$ indicate a "nonmagnetic" part ot the corresponding tensors $[\alpha']$ and $[\beta']$. Relations which couple the tensors $[\alpha']$ and $[\beta']$ are as follows:

$$[\alpha]_m = -[\beta]_m^t \tag{27}$$

$$[\alpha]_n = [\beta]_n^t \tag{28}$$

However, the perturbation theory with limiting number of members in multipole expansion is also not free from approximations. Therefore, we can not accept the condition (21) and also (25) and (26) as universal. In order to obtain the constitutive tensors in a more general form, we have refused of any additional constraints in our calculations and used only two fundamental principles: space-time symmetry and Onsager principle.

Now let us apply to the group $C_2(C_1)$. In this case, the symmetry matrix for the rotation about z-axis by π is

$$[R] = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{29}$$

The relation (D20) gives

$$[\xi] = \begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}, \qquad [\zeta] = \begin{vmatrix} -\xi_{11} & -\xi_{21} & \xi_{31} \\ -\xi_{12} & -\xi_{22} & \xi_{32} \\ \xi_{13} & \xi_{23} & -\xi_{33} \end{vmatrix}, \tag{30}$$

which are also different from that in Birss' Table 9:

$$\begin{vmatrix} 0 & 0 & Q_{13} \\ 0 & 0 & Q_{23} \\ Q_{31} & Q_{32} & 0 \end{vmatrix} \tag{31}$$

Again, (31) can be obtained from (30) using (21). Therefore for the 58 crystallographic magnetic point groups of the third category which contain both unitary and antiunitary operators, it has been necessary to revise all the Indenbom', Birss', and O'Dell' information.

The second important discrepancy of our results from those in the previous publications is that in our case, the magnetoelectric tensors $[\xi]$ and $[\zeta]$ for the magnetic groups of the second and the third category have different structure. The tensors for the groups of the second category are uncoupled because coupling between them can appear only at the expense of the operator of time reversal T, but this operator is not present in the groups of the second category. The tensors $[\xi]$ and $[\zeta]$ for the groups of the third category are expressed in terms of each other because antiunitary elements of these groups consist of the combined space-time operators. The authors of the previous publications did not make difference of them.

Thirdly, we consider the structure of the magnetoelectric tensors $[\xi]$ and $[\zeta]$ for the nonmagnetic groups. We present also the tensors for 21 continuous magnetic groups. Earlier the nonmagnetic and continuous groups were excluded from consideration.

And finally, we present our Tables in compact form uniting the groups in accordance with the tensor transformation properties.

6. CONCLUSIONS

This work has been stimulated by two reasons. First, the previously published Tables of the constitutive tensors for bianisotropic media [20, 27, 28, 41] are not complete and contain to our opinion, many principal mistakes. Second, existing methods of tensor calculations are rather involved and require a good knowledge of group theory, but this theory is not a common mathematical tool for electromagnetic community. This paper therefore is written for those dealing with complex media in electromagnetic problems but who are not specialists in group theory.

Some remarks should be made in conclusion. The approach used in this paper is based on very general grounds, namely on symmetry principles. We may consider Tables 2–7 as a classification of bianisotropic media. This classification based on the laws of space-time transformations of the constitutive tensors (space-time symmetry) is universal, because it comprises all the possible natural and artificial media described by tensors of the second rank. This classification does not depend on physical properties of a medium (in particular, on possible effects of the mutual interactions between the particles of the medium) because it is a geometrical approach. On the other hand, symmetry defines some possible physical effects in the medium and in accordance with Neuman's principle [27], the effects which are "forbidden" completely.

In deriving the possible structure of the constitutive tensors, we have applied in this paper a pure geometrical approach without using a detailed physical knowledge of media. We abstain here from discussing a physical interpretation of the obtained results. Obviously, with the new results, we must reconsider all the general properties of media described by the tensors in Tables 2–7, such as enantiomorphism, reciprocity, magnetoelectric effect etc. It will be presented elsewhere.

Perhaps, not all the solutions in Tables 2–7 are physically realizable, perhaps, some elements of the tensors may be relatively small in values, but these problems can not be solved by group-theoretical methods. On this stage, we can only say that the tensors in these Tables are mathematically permissible and any physical theory can not give more complicated structure of the tensors.

Besides space-time symmetry constraints considered in this paper, some other restrictions when imposed on the constitutive tensors can simplify them. For example, the idealization of losslessness [1]:

$$[\mu] = [\mu^*]^t, \qquad [\varepsilon] = [\varepsilon^*]^t, \qquad [\xi] = [\zeta^*]^t \tag{32}$$

leads to further reduction of the number of independent parameters.

The results of this paper may serve as a basis for synthesis of new artificial materials for electromagnetic applications, as a test for the tensor calculations by other methods or measurements and for theoretical electrodynamical investigations.

APPENDIX A. ELEMENTS OF GROUP THEORY AND THE THEORY OF REPRESENTATIONS

We will present here only minimum relevant information from the magnetic group theory and the theory of representations [43–46], which is used in the paper. An excellent reference to the magnetic group theory and its application to physical problems is a book of Koptsik [47].

A.1 Group

A group G is a set of elements $u_i \in G$ which

1) satisfies the following condition: if u_i and u_j are elements of G, their product $u_i u_j$ is also an element of the same group,
2) possesses an associative law of combination, that is, $u_i(u_j u_k) = (u_i u_j)u_k$,
3) contains a unit element e such that $eu_i = u_i e = u_i$,
4) for every u_i contains an inverse element u_i such that $u_i^{-1}u_i = u_i u_i^{-1} = e$.

In general, any element of a group commutes only with the unit element and with the inverse one. If this is fulfilled for all the elements of the group, the group is called abelian.

The number of elements of a group is its order M. Using a small number of elements called generators one can get all the other elements of the group. Any subset of G which by itself forms a group H is called a subgroup. That is, all the four above group axioms are valid for a subgroup H as well.

A.2 Point Groups

We consider symmetry operations which bring an object (a particle, a field, a medium) into self-coincidence. These operations are usual space rotations and reflections. All the symmetry operations of an object form a point group.

Classes. Any group G may be divided into classes such that the elements in each class are the conjugates of one another. Two group elements u_i and u_j are said to be conjugate with respect to G if there exists an element u_k in G such that

$$u_k u_i u_k^{-1} = u_j \tag{A1}$$

A.3 Magnetic Groups

For magnetic structures, it is necessary to include into consideration the operation of time reversal T which changes the sign of time $(t) \rightarrow (-t)$, and combinations of space symmetry operations with T. The Wigner time reversal operator T [44] (see a description of this operator in Appendix D.3) commutes with all space operators and $T^2 = 1$.

There exist three categories of 122 discrete point crystalligraphic magnetic groups and 21 continuous magnetic groups of symmetry. In the nonmagnetic state, media are described by 32 point magnetic groups of the first category G. Sometimes, these groups are called nonmagnetic ones. The group G consists of a unitary subgroup G (it contains the usual rotation-reflection operations) and products of the operator T with all the elements of H. The full magnetic group describing the media is then $G + TH$ including $T = Te$ where e is the unit element. The media are reciprocal.

In the case of 32 magnetic groups of the second category $\mathbf{G}$, there is no elements with time reversal operator T, and T is not an element of the groups, therefore the media are in general nonreciprocal. The nomenclature and the notations of the groups of the first (nonmagnetic) category and that of the second (magnetic) category coincide. In order to distinguish them, we will use boldface type for the groups of the second category.

The 58 magnetic groups of the third category $G(H)$ contain in addition to the unitary operators of the unitary subgroup H, an equal number of antiunitary operators which are the product of the usual geometrical symmetry operators and the operator T. These combined operators form a conjugate class TH' of the subgroup H and lead to the existence of antiaxes, antiplanes and anticenter of symmetry. Notice that the elements of H' are distinguished from that of H. The unitary operators of a magnetic group of the third category form a unitary subgroup of index 2. The operator T itself is not an element

Table 8. The content of magnetic groups of symmetry.

First category	Second category	Third category
G=H+TH including T	**G** without T	G(H)=H+TH′ (H≠H′) T only in combination with rotations-reflections

Table 9. Subgroup relations of a nomagnetic group $G = H + TH$ and its magnetic subgroup $G'(H')$.

	Nonuitary group		Unitary group
Nonmagnetic group	G	⟶	H
	↓		↓
Magnetic group	G′	⟶	H′

of the magnetic groups of the third category, therefore the media described by these groups are in general nonreciprocal. Moving dielectric and gyrotropic media and chiroferrites for example, are described by the groups of this category [33]. The content of the three categories of magnetic groups is presented in Table 8.

In Russian literature [47], the groups of the first category are called neutral (or gray), the groups of the second category are polar (or one-colored), and the groups of the third category are two-colored.

If a magnetic group of the third category $G'(H')$ is a subgroup of the nonmagnetic one G which has the structure $G = H + TH$, the subgroup relations are given in Table 9. The arrows in Table 9 demonstrate that H and H' are the unitary subgroups of G and G' respectively, and G' and H' are subgroups of G and H respectively. Such relations may exist for example for a medium which is described by a group G in nonmagnetic state and by a group $G'(H')$ after application of a dc magnetic field.

In general, generators of a symmetry group can be chosen in different ways. In the case of magnetic group of symmetry of the third category $G(H)$ one can define as generators for example, generators of the unitary subgroup H and any antiunitary element. Such a choice of generators is an exceedingly useful device in the problem of tensor calculations (Section 3).

A.4 Representations

The theory of representations deals with mapping of groups on groups of linear operators (matrices). If for any product $u_i u_j$ of a group G, the product $D(u_i)D(u_j)$ corresponds where $D(u_i)$ and $D(u_j)$ are matrices of the group G_m, they say that the group G_m is a matrix representation of the group G. These square matrices are unitary and nonsingular and the order of them is called the dimension of the representation. In the case of our present interest, these matrices are the symmetry matrices $[R]$ of a given media (Table 1). The dimensionality of these matrices is three.

The matrices $[R]$ for rotation through an angle α (about the x, y, and z-axes for example are

$$[R]_x = \begin{vmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{vmatrix},$$
$$[R]_y = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & \pm 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix}, \qquad \text{(A2)}$$
$$[R]_z = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{vmatrix}$$

where the sign $+$ in the units of the matrices corresponds to proper (pure) rotation, the sign $-$ to improper rotation (pure rotation about an axis followed by reflection in a plane perpendicular to the axis). Analogous matrices can be written for any direction of the rotation axis. The matrices $[R]$ for proper rotations about the axes x, y, z through (are respectively:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}; \quad \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}; \quad \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}; \qquad \text{(A3)}$$

The matrices for reflections in the planes $x = 0$, $y = 0$, $z = 0$ and for inversion are respectively:

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}; \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}; \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}; \quad \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}; \tag{A4}$$

The determinant of $[R]$ for pure rotation is $+1$, for improper rotation, reflection and inversion is equal to -1.

The combined symmetry operator $T[R]$ in the groups of the third category means that after rotation or reflection $[R]$ we come to the same geometry but with opposite sign of magnetization $\mathbf{M}$ (or external dc magnetic field $\mathbf{H}_o$) which would correspond to physical symmetry, and the operator T changes the sign of $\mathbf{M}$ (or $\mathbf{H}_o$) restoring thus the symmetry completely.

The following simple examples illustrate the concept of unitary (space) and antiunitary (space-time) operators in magnetic symmetry. A vector $\mathbf{H}_o$ normal to a plane of symmetry preserves its direction after reflection (Fig. 4a), but it changes the direction to the opposite one if it is parallel to a plane of symmetry (Fig. 4b), because it is an axial vector. These two cases correspond to the magnetic symmetry operator C_s. After application of the operator T to the reflected magnetic field $\mathbf{H}_o$ in the Fig. 4b, we come to the case of antiplane of symmetry TC_s (Fig. 4c). Thus, if we have a geometrical plane of symmetry in a medium, and a dc magnetic field $\mathbf{H}_o$, applied parallel to this plane, the plane is converted to an antiplane (Fig. 4c) which is described by the antiunitary element TC_s.

APPENDIX B. NOTATIONS OF ELEMENTS OF SYMMETRY, OF SYMMETRY OPERATIONS AND OF CRYSTALLOGRAPHIC POINT GROUPS

B.1 General Information

Different systems of crystallographic group notations are used in practice. The relevant information can be found for example in the book by M. Lax [46]. The most popular are three of the notations [43]:

1) the International (devised by Hermann and Mauguin),
2) the Schoenflies,
3) the Shubnikov.

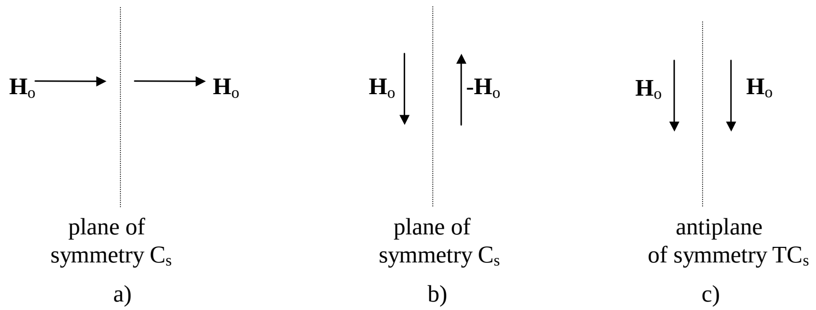

Figure 4. Examples of magnetic symmetry: a), b) plane of symmetry, c) antiplane of symmetry.

Below, we will describe briefly these three systems (Table 1). Notice that the notations of group elements corresponding to symmetry operations, the symmetry operations themselves and the notations of the groups may coincide. For example, the symbol C_2 in the Schoenflies notations denotes the operation of rotation about an axis by π and also it may denote the group C_2 consisting of the two elements: the identity e and the rotation C_2. The meaning of the notations will be clear from the context.

In all three systems of notations, the n-fold proper rotations are considered as elementary operators. In order to obtain the remaining operators, we can form products of the rotations with the inversion operator, or alternatively with the reflection in a plane perpendicular to the n-fold axis. We will consider first the notations of the symmetry elements.

B.2 The International System

In this system, n denotes the n-fold proper rotation, $\overline{n}$ with the bar above the symbol denotes improper rotation, i.e., rotation followed by reflection in a plane perpendicular to the axis of rotation. The symbol m corresponds to reflection in a plane, n/m denotes an n-fold axis with a mirror plane perpendicular to it. The symbol nm designates an n-fold axis with a plane of symmetry containing this axis, and $n2$ an n-fold axis with a two-fold axis perpendicular to it. Antiunitary elements are denoted by the bar below a symbol.

There exist two variants of the International systems, namely full and abbreviated one. For example, abbreviated mmm corresponds to $(2/m)(2/m)(2/m)$ in full notations. In some cases, the notations in these systems coincide. In Table 1, abbreviated symbols are used.

B.3 The Shubnikov System

In the Shubnikov system, a rotation by $2\pi/n$ (n is integer) is denoted by the symbol n. Rotation-reflection axis which corresponds to an improper rotation, is indicated by a tilde over the corresponding symbol, so that the Schoenflies S_2 (see below) is $\tilde{2}$, S_4 is $\tilde{4}$ etc. The presence of a plane of symmetry is indicated by the letter m. A single dot between two symmetry elements in a notation denotes that the two elements are parallel, for example $2 \cdot m$ corresponds to the group C_{2v} in Schoenflies system. A double dot (colon) is used to indicate that the symmetry elements are perpendicular. A diagonal line is used to indicate that the group contains two axis which are not at right angles to one another so that for example $3/2$ denotes the group which has a three-fold axis and a two-fold axis at some angle other than $\pi/2$. The bar below a symbol denotes that this is the antiunitary element.

B.4 The Schoenflies Notations

In this notations, a proper rotation through $2\pi/n$ about a certain axis is denoted by the symbol C_n. The symbol σ (and also C_s) defines reflection in a plane. Reflection in a plane perpendicular to the principal axis is denoted by σ_h (the subscript h for horizontal), while σ_v (the subscript v for vertical) is used for reflection in a plane passing through the axis, and σ_d (d for diagonal) designates a mirror plane containing the axis but diagonal to an already existing σ_v. A combined operation C_n and σ_h is denoted by S_n which is improper rotation. Therefore, the inversion i which presents a rotation C_2 (rotation by π) and reflection σ_h may be denoted as S_2.

Apply now to the group notations in the Schoenflies system. The groups with one axis of symmetry are denoted by C_n. Joining σ_h to C_n gives the groups C_{nh}. The groups having an n-fold axis and a system of two-fold axes at right angles to it are denoted by D_n (dihedral groups). D_{nd} and D_{nh} contain in addition the planes σ_d and σ_h, respectively. The higher groups T and O contain only pure rotations, but T_d has also planes of symmetry, and T_h and O_h contain the center of symmetry [45].

The Shubnikov system contains generators of the corresponding groups. The International notations also show the generators but the number of generators is often redundant. Apparently, this is the reason that Table 3 of [27] contains in many cases more number of generators that it is necessary for determination of the constitutive tensors. It concerns for example, the groups D_{3d}, C_6, C_{3h}, and some others.

The correspondence of the three systems of the group notations is given in Tables 1, 10, 11, and 12. In what follows we will use mainly the Schoenflies system which is particularly suitable for notation of the groups of the third category. In this case, the Schoenflies notation shows explicitly the structure of the groups, i.e., the unitary subgroup and the antiunitary elements.

APPENDIX C. CRYSTALLOGRAPHIC AND CONTINUOUS MAGNETIC POINT GROUPS AND THEIR DECOMPOSITION DIAGRAMS

C.1 General Information

It is proved in crystallophysics that the permitted axes of rotational symmetry for crystals are C_1, C_2, C_3, C_4 and C_6. Such symmetry as C_5, C_7 etc. can not exist in crystals. This restriction is defined by spatial structure of crystal lattices. We may imagine an artificial medium which consists of particles with the symmetry for example C_5 but if these particles form a regular periodical structure (array), the resultant symmetry of the material (and the constitutive tensors) will differ from C_5 and depend also on the symmetry of the array. The symmetry of the medium can be determined using Curie's principle (Appendix E).

In Table 1 the International, the Shubnikov, and the Schoenflies notations, the number of group elements, generating matrices for 32 crystallographic point groups of the first category are presented. The structure of generating matrices depends on the mutual orientation of the symmetry elements (planes, axes) and the axes of the chosen coordinate system. We will use the rectangular coordinate system. The orientation of the symmetry elements about the coordinate system is given in Table 1. The symbol $||$ means parallel, $\perp$ denotes perpendicular.

Table 10. 58 crystallographic magnetic point groups of the third category in the international, Shubnikov and Schoenflies notations and number of the group elements.

International	Shubnikov	Schoenflies	Number of elements	International	Shubnikov	Schoenflies	Number of elements
$\underline{m}3m$	$\tilde{\underline{6}}/\underline{4}$	$O_h\,(T_d)$	48	$\bar{\underline{3}}m$	$\tilde{\underline{6}}\cdot m$	$D_{3d}\,(C_{3v})$	12
$\underline{m}3\underline{m}$	$\tilde{\underline{6}}/4$	$O_h\,(O)$	48	$\bar{\underline{3}}\underline{m}$	$\tilde{\underline{6}}\cdot\underline{m}$	$D_{3d}\,(D_3)$	12
$m3\underline{m}$	$\tilde{6}/\underline{4}$	$O_h\,(T_h)$	48	$\bar{3}\underline{m}$	$\tilde{6}\cdot\underline{m}$	$D_{3d}\,(C_{3i})$	12
$\underline{4}3\underline{2}$	$3/\underline{4}$	$O(T)$	24	$\bar{\underline{3}}$	$\tilde{\underline{6}}$	$C_{3i}\,(C_3)$	6
$\bar{\underline{4}}3\underline{m}$	$3/\bar{\underline{4}}$	$T_d\,(T)$	24	$\bar{\underline{6}}$	$3:\underline{m}$	$C_{3h}\,(C_3)$	6
$\underline{m}3$	$\tilde{\underline{6}}/2$	$T_h\,(T)$	24	$3\underline{m}$	$3\cdot\underline{m}$	$C_{3v}\,(C_3)$	6
$\underline{6}/mm\underline{m}$	$m\cdot\underline{6}:m$	$D_{6h}\,(D_{3h})$	24	$3\underline{2}$	$3:\underline{2}$	$D_3\,(C_3)$	6
$6/\underline{m}mm$	$m\cdot 6:\underline{m}$	$D_{6h}\,(C_{6v})$	24	$\underline{4}/\underline{mm}m$	$m\cdot\underline{4}:\underline{m}$	$D_{4h}\,(D_{2d})$	16
$6/\underline{mmm}$	$\underline{m}\cdot 6:\underline{m}$	$D_{6h}\,(D_6)$	24	$4/\underline{m}mm$	$m\cdot 4:\underline{m}$	$D_{4h}\,(C_{4v})$	16
$\underline{6}/\underline{m}m\underline{m}$	$m\cdot\underline{6}:\underline{m}$	$D_{6h}\,(D_{3d})$	24	$4/\underline{mmm}$	$\underline{m}\cdot 4:\underline{m}$	$D_{4h}\,(D_4)$	16
$\underline{6}/m\underline{mm}$	$\underline{m}\cdot 6:m$	$D_{6h}\,(C_{6h})$	24	$\underline{4}/mm\underline{m}$	$m\cdot\underline{4}:m$	$D_{4h}\,(D_{2h})$	16
$\underline{6}/m$	$\underline{6}:m$	$C_{6h}\,(C_{3h})$	12	$4/m\underline{mm}$	$\underline{m}\cdot 4:m$	$D_{4h}\,(C_{4h})$	16
$6/\underline{m}$	$6:\underline{m}$	$C_{6h}\,(C_6)$	12	$\bar{\underline{4}}\underline{2}m$	$\tilde{\underline{4}}\cdot m$	$D_{2d}\,(C_{2v})$	8
$\underline{6}/\underline{m}$	$\underline{6}:\underline{m}$	$C_{6h}\,(C_{3i})$	12	$\bar{\underline{4}}2\underline{m}$	$\tilde{\underline{4}}\cdot\underline{m}$	$D_{2d}\,(D_2)$	8
$\underline{6}m\underline{m}$	$\underline{6}\cdot m$	$C_{6v}\,(C_{3v})$	12	$\bar{4}\underline{2m}$	$\tilde{4}\cdot\underline{m}$	$D_{2d}\,(S_4)$	8
$6\underline{mm}$	$6\cdot\underline{m}$	$C_{6v}\,(C_6)$	12	$\underline{4}m\underline{m}$	$\underline{4}\cdot m$	$C_{4v}\,(C_{2v})$	8
$\underline{6}2\underline{2}$	$\underline{6}:2$	$D_6\,(D_3)$	12	$4\underline{mm}$	$4\cdot\underline{m}$	$C_{4v}\,(C_4)$	8
$6\underline{22}$	$6:\underline{2}$	$D_6\,(C_6)$	12	$\underline{4}2\underline{2}$	$\underline{4}:2$	$D_4\,(D_2)$	8
$\underline{6}$	$\underline{6}$	$C_6\,(C_3)$	6	$4\underline{22}$	$4:\underline{2}$	$D_4\,(C_4)$	8
$\bar{\underline{6}}m\underline{2}$	$m\cdot 3:\underline{m}$	$D_{3h}\,(C_{3v})$	12	$\underline{4}/\underline{m}$	$\underline{4}:\underline{m}$	$C_{4h}\,(S_4)$	8
$\bar{6}\underline{m2}$	$\underline{m}\cdot 3:\underline{m}$	$D_{3h}\,(D_3)$	12	$4/\underline{m}$	$4:\underline{m}$	$C_{4h}\,(C_4)$	8
$\bar{\underline{6}}\underline{m}2$	$\underline{m}\cdot 3:m$	$D_{3h}\,(C_{3h})$	12	$\underline{4}/m$	$\underline{4}:m$	$C_{4h}\,(C_{2h})$	8

Table 10. (Continued)

International	Shubnikov	Schoenflies	Number of elements
$\bar{\underline{4}}$	$\tilde{\underline{4}}$	$S_4\,(C_2)$	4
$\underline{4}$	$\underline{4}$	$S_4\,(C_2)$	4
$\underline{m}mm$	$m\cdot 2:\underline{m}$	$D_{2h}\,(C_{2v})$	8
$\underline{mmm}$	$\underline{m}\cdot 2:\underline{m}$	$D_{2h}\,(D_2)$	8
$m\underline{mm}$	$\underline{m}\cdot 2:m$	$D_{2h}\,(C_{2h})$	8
$\underline{22}2$	$2:\underline{2}$	$D_2\,(C_2)$	4
$\underline{m}m\underline{2}$	$\underline{2}\cdot m$	$C_{2v}\,(C_s)$	4
$\underline{mm}2$	$2\cdot\underline{m}$	$C_{2v}\,(C_2)$	4
$\underline{2}/m$	$\underline{2}:m$	$C_{2h}\,(C_s)$	4
$2/\underline{m}$	$2:\underline{m}$	$C_{2h}\,(C_2)$	4
$\underline{2}/\underline{m}$	$\underline{2}:\underline{m}$	$C_{2h}\,(C_i)$	4
$\underline{m}$	$\underline{m}$	$C_s\,(C_1)$	2
$\underline{2}$	$\underline{2}$	$C_2\,(C_1)$	2
$\bar{\underline{1}}$	$\tilde{\underline{2}}$	$C_i\,(C_1)$	2

For most of the groups, the number of generating matrices is one or two, and only for the groups D_{2h}, D_{4h} and D_{6h} three generators are written. In reality, in order to calculate the constitutive tensors for the last two groups, we need only two generators, because the axes C_4 and C_6 in the constitutive tensors correspond to the axes of infinite order (Section 3). The matrices given in Table 1 are used for tensor calculations in Section 3.

The group subordination (the group tree) for 32 crystallographic groups is given in Fig. 5 [55]. In this group tree and in the trees in

Table 11. 7 continuous magnetic groups of the second category in the international, Shubnikov and Schoenflies notations.

N	Notations		
	International	Shubnikov	Schoenflies
1	$\infty/\infty/\mathbf{mmm}$	$\infty/\infty\cdot\mathbf{m}$	$\mathbf{K_h}$
2	$\infty/\infty\mathbf{2}$	∞/∞	$\mathbf{K}$
3	$\infty/\mathbf{mmm}$	$\mathbf{m}\cdot\infty\mathbf{:m}$	$\mathbf{D}_{\infty\mathbf{h}}$
4	$\infty\mathbf{2}$	$\infty\mathbf{:2}$	$\mathbf{D}_{\infty}$
5	$\infty\mathbf{mm}$	$\infty\cdot\mathbf{m}$	$\mathbf{C}_{\infty\mathbf{v}}$
6	$\infty/\mathbf{m}$	$\infty\mathbf{:m}$	$\mathbf{C}_{\infty\mathbf{h}}$
7	∞	∞	$\mathbf{C}_{\infty}$

Table 12. 7 continuous magnetic groups of the third category in the international, Shubnikov and Schoenflies notations.

N	Notations		
	International	Shubnikov	Schoenflies
1	$\infty/\infty/\underline{\mathrm{mmm}}$	$\infty/\infty\cdot\underline{\mathrm{m}}$	$K_h(K)$
2	$\infty/\underline{\mathrm{mmm}}$	$\underline{\mathrm{m}}\cdot\infty:\underline{\mathrm{m}}$	$D_{\infty h}(D_{\infty})$
3	$\infty/\underline{\mathrm{m}}\mathrm{mm}$	$\mathrm{m}\cdot\infty:\underline{\mathrm{m}}$	$D_{\infty h}(C_{\infty v})$
4	$\infty/\mathrm{m}\underline{\mathrm{mm}}$	$\underline{\mathrm{m}}\cdot\infty:\mathrm{m}$	$D_{\infty h}(C_{\infty h})$
5	$\infty\underline{2}$	$\infty:\underline{2}$	$D_{\infty}(C_{\infty})$
6	$\infty\underline{\mathrm{mm}}$	$\infty\cdot\underline{\mathrm{m}}$	$C_{\infty v}(C_{\infty})$
7	$\infty/\underline{\mathrm{m}}$	$\infty:\underline{\mathrm{m}}$	$C_{\infty h}(C_{\infty})$

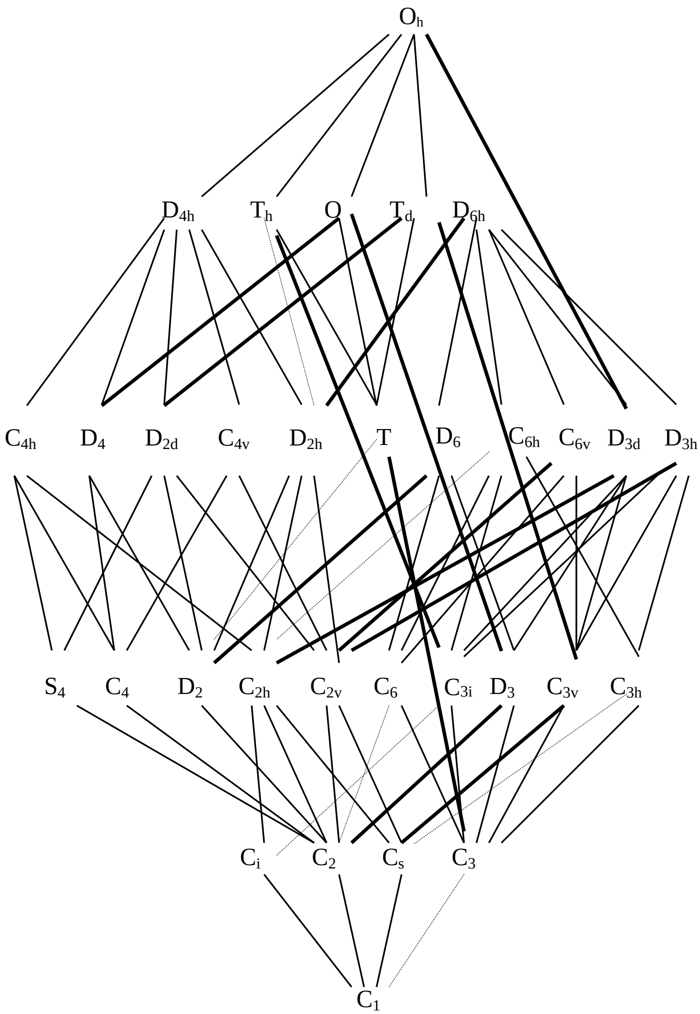

Figure 5. The group tree for the 32 crystallographic point groups of the first category.

Figs. 1, 5, 6, a given group of a lower row is a subgroup of those groups from which the lines come up to the given group. A heavy line in Fig. 5 indicates that the subgroup is not invariant (a definition of an invariant subgroup see for example in [43, p. 28]. A dotted line shows that the subgroup is not of index 2 under the above group, that is the number of the group elements is not twice as much as in the subgroup. It means

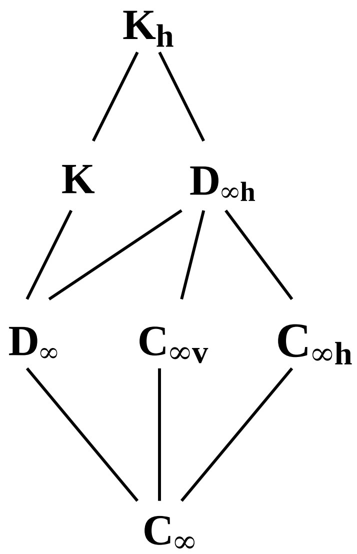

Figure 6. The group tree for the 7 continuous magnetic groups of the second category.

that the corresponding two groups connected by the heavy line or by the dotted one can not form a group of the third category.

Notice, that the 32 point groups of Table 1 form 7 different crystallographic systems:

1, 2, 3, 4, 5 -cubic (regular) system,
6, 7, 8, 9, 10, 11, 12 - hexagonal,
13, 14, 15, 16, 17 - trigonal,
18, 19, 20, 21, 22, 23, 24 - tetragonal,
25, 26, 27 -orthorhombic,
28, 29, 30 - monoclinic,
31, 32 - triclinic.

The 58 crystallographic magnetic point groups of the third category are presented in Table 10, where their International, Shubnikov and Schoenflies notations are given and also the number of the group elements are written.

C.2 The Continuous Groups of the Second Category

Now we turn our attention to the continuous groups of symmetry [53]. These groups contain at least one axis of symmetry of infinite order C_∞. The continuous groups occupy a special place in the whole system of magnetic groups because

1) they describe a large number of media,
2) more than a half discrete groups result in the same structure of the constitutive tensors as continuous ones,
3) some physical effects are exhibited in the media described by these groups in a "pure" form.

Therefore we will give a short description of symmetry elements for these groups.

Let us consider first the continuous groups of the second category (Table 11). There is 7 group of this type. The corresponding group tree is given in Fig. 6.

The group $\mathbf{K_h}$. This group has

-an infinite number of axes of symmetry passing through the origin of the coordinates,

-an infinite number of planes,

-the center of symmetry i.

The group $\mathbf{K}$. This group has

-an infinite number of axes passing through the origin, and no planes and no center.

Below without loss of generality, the principal axis C_∞ will be along the z-axis for all the cases of continuous groups.

The group $\mathbf{D_{\infty h}}$. It has

-one principal axis C_∞ (which is along the z-axis),

-the plane of symmetry $z = 0$ (the plane is denoted by σ_h),

-an infinite number of the two-fold axes C_2 lying in the plane $z = 0$,

-an infinite number of the planes σ_v passing through the z-axis,

-the center of symmetry i.

The group $\mathbf{D_\infty}$. This group is characterized by

-one principal axis C_∞,

-an infinite number of the two-fold axes C_2 lying in the plane $z = 0$.

The group $\mathbf{C_{\infty v}}$. It has

-the principal axis,

-an infinite number of the planes σ_v passing through this axis.

The group $\mathbf{C_{\infty h}}$. It possesses
-the principal axis C_∞,
-the plane σ_h,
-the center of symmetry i.

The group $\mathbf{C_\infty}$. The group has only one axis of symmetry C_∞.

C.3 The Continuous Groups of the First Category

The group subordination for the 7 groups of the first category coincides with that given in Fig. 6. The nomenclature of these groups does not differ from that for the groups of the second category (Table 11). But the groups of the second category describe magnetic media while the groups of the first category describe nonmagnetic ones. The difference in the content of these groups is clear from Table 8.

C.4 The Continuous Groups of the Third Category

The structure of the 7 magnetic groups of the third category $G(H)$ (Table 12) is clear from their Schoenflies notations. As it was described in Appendix A.3, the unitary subgroup H of the group G which contains the elements without time reversal operator is given in the parenthesis of $G(H)$. All the other elements of the group G are with T. For example, $C_{\infty h}(C_\infty)$ consists of the axis C_∞, the antiplane $T_{\sigma h}$, and the anticenter Ti.

Let us consider the content of these groups in detail.

The group $K_h(K)$. This group contains
-an infinite number of axes of symmetry, and
-an infinite number of antiplanes of symmetry passing through the coordinate origin.

The group $D_{\infty h}(D_\infty)$. It has the following elements of symmetry:
-one principal axis C_∞ of an infinite order,
-an antiplane of symmetry $T\sigma_h$ which is perpendicular to the axis C_∞,
-an infinite number of antiplanes $T\sigma_v$ passing through the axis C_∞,
-an infinite number of the two-fold axes C_2 lying in the antiplane $T\sigma_h$,
-the anticenter of symmetry Ti.

The group $D_{\infty h}(C_{\infty v})$. The group has the following elements of symmetry:

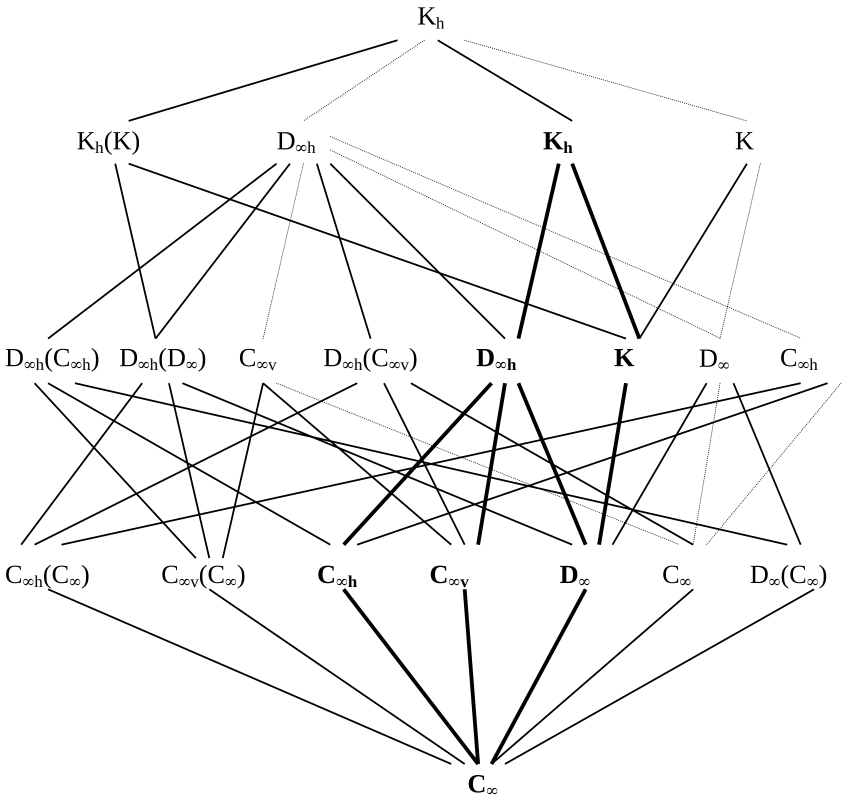

Figure 7. The group tree for the 21 continuous magnetic groups of the first, second and third category.

-one principal axis C_∞ of an infinite order,

-an infinite number of planes σ_v passing through the axis C_∞ ,

-an antiplane of symmetry $T_{\infty h}$ which is perpendicular to the axis C_∞ ,

-an infinite number of the two-fold antiaxes TC_2 lying in the antiplane $T_{\infty h}$,

-the anticenter of symmetry Ti .

Notice, that such symmetry has for example, a ring dc magnetic field produced by a linear electric current.

<u>The group</u> $D_{\infty h}(C_{\infty h})$. The group contains:

-one principal axis C_∞ ,

-the plane of symmetry σ_h,

-the center of symmetry i,

-an infinite number of the two-fold antiaxes TC_2 lying in the plane $z = 0$,

-an infinite number of antiplanes $T_{\infty v}$ passing through z-axis.

The group $D_\infty(C_\infty)$. The group consists of:

-the principal axis C_∞,

-an infinite number of the two-fold antiaxes TC_2, lying in the plane $z = 0$.

The group $C_{\infty v}(C_\infty)$. The group contains:

-the principal axis C_∞,

-an infinite number of antiplanes $T_{\infty v}$, passing through z-axis.

The group $C_{\infty h}(C_\infty)$. The group contains:

-the principal axis C_∞,

-an antiplane of symmetry $T\sigma_h$.

-the anticenter of symmetry T_i.

The subordination of the 21 continuous groups of the first, second and the third categories is given in Fig. 7. The group tree for non-magnetic groups is given by dotted lines, for the magnetic groups of the second category by heavy lines. These two group trees are formally identical (see also Fig. 6), but the content of the group is different. The magnetic groups of the second category (they are denoted by bold type) are subgroups of the first category groups, because the latter contain an additional element of symmetry T, the time-reversal operator.

APPENDIX D. SYMMETRY PROPERTIES OF THE SECOND-RANK TENSORS

D.1 General Information

Tensors of the second rank are directional quantities. Here, we must distinguish between the polar and the axial tensors. The polar tensors define a linear relation between two polar vectors or between two axial ones, and the axial tensors determine the relation between an axial vector and a polar one. For example, from (4) and (5) we see that $[\varepsilon]$ and $[\mu]$ are the polar tensors and $[\xi]$, $[\zeta]$ are the axial ones because $\mathbf{E}$ and $\mathbf{D}$ are polar vectors, but $\mathbf{H}$ and $\mathbf{B}$ are axial ones.

The laws of spatial transformations are different for the polar and for the axial tensors [48]:

$$[a'] = [R][a][R]^{-1} \tag{D1}$$

for polar tensors, and

$$[A]' = (\det[R])[R][A][R]^{-1} \tag{D2}$$

for axial ones. In (D1) and (D2), $[a]$ denotes $[\varepsilon]$ or $[\mu]$, $[A]$ corresponds to $[\xi]$ or $[\zeta]$, $[R]$ is a rotation-reflection matrix described in Appendix A.4. Taking into account that $\det[R] = +1$ for proper rotation and $\det[R] = -1$ for improper rotation, from (D1) and (D2) we see that the axial and polar tensors transform under rotations equally but under reflection and inversion, they transform in a different way. In other words, the axial tensors are sensitive to the change of the coordinate system sign.

Any tensor may be decomposed into the sum of the symmetric and anti-symmetric parts. In a certain coordinate system, the symmetric part may be brought out to the diagonal form and the anti-symmetric part to the simplest form

$$\begin{vmatrix} 0 & -a_{12} & 0 \\ a_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{D3}$$

in another coordinate system (in some cases, these two coordinate systems may coincide). Notice that the symmetric and anti-symmetric parts of $[\xi]$ and $[\zeta]$ describe different physical processes in media. The symmetric part transforms the vectors of electric and magnetic field one into another so that they lie on one straight line. It may be called a longitudinal effect. The anti-symmetric part of the tensors $[\xi]$ and $[\zeta]$ transforms two orthogonal components of the electric and the magnetic vectors one into another and describes a transverse effect which is analogous to the Hall effect.

D.2 Space Symmetry of Tensors

If a medium is described by a group of symmetry, the constitutive tensors are invariant under the operators of this group. In order to define the structure of the constitutive tensors, we may use only generators of the group. Thus, using six-vector formalism [49], the tensors may be calculated by the following relations:

$$[K] = [P][K][P] \tag{D4}$$

where the 6×6 constitutive tensor $[K]$ is

$$[K] = \begin{vmatrix} [\varepsilon] & [\xi] \\ [\zeta] & [\mu] \end{vmatrix} \tag{D5}$$

and a 6×6 mapping operator:

$$[P] = \begin{vmatrix} [R] & 0 \\ 0 & (\det[R])[R] \end{vmatrix} \tag{D6}$$

where $\det$ stands for determinant. $\mathrm{Det}[R]$ is included in the operator P in order to take into account the axial nature of the vector $\mathbf{H}$.

D.3 Time-reversal Symmetry of Tensors

Under time symmetry, we will consider the symmetry under reversal in the direction of time, that is under the transformation of changing the sign of time $(t) \to -(t)$. This transformation is denoted by the symbol T. Under time reversal, electric charge, electric flux and electric field are even, but current, magnetic flux and magnetic field and Poynting vector are odd. The tensors $[\mu]$ and $[\varepsilon]$ do not change their signs under time reversal, i.e., they are even in time. The situation with transformation properties of the tensors $[\xi]$ and $[\zeta]$ is much complicated. Firstly, in accordance with Onsager's principle [42,54] (see below), under time reversal they are transformed into each other. Secondly, they are odd in time because for example in equation (4) the tensor $[\xi]$ must transform the odd under time reversal vector $\mathbf{H}$ into the even in time vector $\mathbf{D}$. In order to adjust these different transformation properties of $\mathbf{D}$ and $\mathbf{H}$, the tensor $[\xi]$ must be odd in time.

Strictly speaking there no exists time-reversal symmetry, because for example, the time-reversal operator converts a damping wave into a growing one and vice versa, that is the dissipative processes are not time reversible. In physics, it is considered as a violation of the second law of thermodynamics: the entropy of a system may only increase [20]. In order to overcome this difficulty, an artifice has been suggested in [50]: to use the so-called restricted time reversal operator. This operator does not apply to the imaginary dissipative terms of the constitutive parameters and to that of the wave vector. This preserves the damping or the growing character of the wave under time reversal.

In the case of determination of the constitutive tensor structure, we however do not consider the time-reversal operator T literally. In our consideration, the operation T means changing the signs of all the velocities $\mathbf{v}$, including the signs of currents, the signs of magnetization $\mathbf{M}$ and magnetic field $\mathbf{H}$, including external dc magnetic field $\mathbf{H}_o$. Hence we consider one and the same medium in two possible states, with $\mathbf{v}$ and with $-\mathbf{v}$, that corresponds to the state with $\mathbf{M}$ and $\mathbf{H}$ and with $-\mathbf{M}$ and $-\mathbf{H}$ respectively.

In order to find transformation properties of the constitutive tensors under the operator T in the presence of dc magnetic field $\mathbf{H}_o$, we must turn to the Onsager's theorem of irreversible thermodynamics [42]: if a force $\mathbf{F}$ induces a response $\mathbf{R}$, and they are connected by a susceptibility χ:

$$\mathbf{R} = \chi\mathbf{F}, \tag{D8}$$

then in the presence of a dc magnetic field, χ has a property

$$\chi(-\mathbf{H}_o) = \chi^t(\mathbf{H}_o) \tag{D9}$$

where the superscript t denotes transposition. A discussion of application of the Ousager principle to the constitutive tensors is in [54].

In our case, the force is the six-vector $\mathbf{e}$:

$$\mathbf{e}(\mathbf{r}) = \begin{vmatrix} \mathbf{E} \\ \mathbf{H} \end{vmatrix}, \tag{D10}$$

the response is the six-vector combined from $\mathbf{D}$ and $\mathbf{B}$, and the susceptibility is the constitutive tensor $[K]$:

$$\begin{vmatrix} \mathbf{D} \\ \mathbf{B} \end{vmatrix} = [K] \begin{vmatrix} \mathbf{E} \\ \mathbf{H} \end{vmatrix} \tag{D11}$$

The time-reversal operator T acts on the time-harmonic quantities and on the operators as follows:

1) complex conjugates all the quantities,
2) reverses all the velocities, a current vector and changes the sign of magnetic field $\mathbf{H}$ ($\mathbf{H}$ is odd under time inversion) and changes also the sign of an external dc magnetic field $\mathbf{H}_o$;
3) transposes the tensors (according to the Onsager's principle);

We may write:

$$T\begin{vmatrix}\mathbf{D}\\ \mathbf{B}\end{vmatrix} = T[K]\begin{vmatrix}\mathbf{E}\\ \mathbf{H}\end{vmatrix} \tag{D12}$$

or

$$\begin{vmatrix}\mathbf{D}^*\\ -\mathbf{B}^*\end{vmatrix} = \begin{vmatrix}[\varepsilon^*]^t & [\zeta^*]^t\\ [\xi^*]^t & [\mu^*]^t\end{vmatrix}\begin{vmatrix}\mathbf{E}^*\\ -\mathbf{H}^*\end{vmatrix} \tag{D13}$$

Complex conjugation of (D13) gives:

$$\mathbf{D} = [\varepsilon]^t\mathbf{E} - [\zeta]^{\mathbf{t}}\mathbf{H} \tag{D14}$$

$$\mathbf{B} = -[\xi]^t\mathbf{E} + [\mu]^{\mathbf{t}}\mathbf{H} \tag{D15}$$

Comparing these relations with (4) and (5), we see that invariantness under time reversal corresponds to reciprocity of the medium (12).

For nonmagnetic media described by the magnetic groups of the first category, the tensors $[\mu]$ and $[\varepsilon]$ are symmetrical about the main diagonal, but the tensors $[\xi]$ and $[\zeta]$ turn out to be coupled by the relation $[\xi] = -[\zeta]^t$ and it is a consequence of the time reversal symmetry. For the magnetic groups of the second and the third category, the tensors $[\mu]$ and $[\varepsilon]$ are in general nonsymmetrical about the main diagonal. According to Onsager principle, these tensors are transposed with changing the sign of $\mathbf{H}_o$.

D.4 Combined Space-time Symmetry of Tensors

If a medium is described by the magnetic groups of the third category, in order to obtain the constitutive tensors we can use along with unitary operators an antiunitary one as well because the medium is invariant under all operators of the group. An antiunitary operator is a combined operator $T[P]$. Application of this operator to the constitutive relations (D11) gives:

$$[K] = [P][K]^t[P] \tag{D16}$$

In contrast with (D4), we obtain here a relation which couples $[K]$ and its transposed $[K]^t$. The tensors $[\xi]$ and $[\zeta]$ are out of the main diagonal of $[K]$ so that (D16) couples these two tensors.

Thus from invariantness of a medium under coordinate transformations (D4) and (D16) we obtain the following expressions: for the case of unitary operators (they correspond to the space transformations (D4))

$$[R][\varepsilon] = [\varepsilon][R], \quad [R][\mu] = [\mu][R] \tag{D17}$$

$$[R][\xi] = \det([R])[\xi][R], \quad [R][\zeta] = \det([R])[\zeta][R] \tag{D18}$$

and for the case of antiunitary operators (they correspond to the space-time transformations (D16)):

$$[R][\varepsilon] = [\varepsilon]^t[R], \quad [R][\mu] = [\mu]^t[R] \tag{D19}$$

$$[R][\xi] = -\det([R])[\zeta]^t[R], \quad [R][\zeta] = -\det([R])[\xi]^t[R] \tag{D20}$$

Notice that in accordance with theory of magnetic groups (Appendix A), we can use in (D17) and (D18) generators of the corresponding unitary subgroup and in (D19) and (D20) only one (any) antiunitary element. The relations (D17)–(D20) coincide with those obtained in [51] by application of symmetry operators to Maxwell's equations.

We say that a tensor has a certain element of symmetry if this tensor is not changed after applying the corresponding symmetry operator. Let us clarify this by an example. We will investigate symmetry of the magnetic permeability tensor $[\mu]$ of a ferrite magnetized in z-direction by dc magnetic field:

$$[\mu] = \begin{vmatrix} \mu & -jk & 0 \\ jk & \mu & 0 \\ 0 & 0 & \mu_z \end{vmatrix} \tag{D21}$$

Using for example $[R]_z$ of (A2) in the relation (D17) for $[\mu]$, it is easy to show that the tensor $[\mu]$ is not changed under rotation by any angle about the axis z. It means that this tensor has the axis z of infinite order as an element of its symmetry. Analogously, we can demonstrate also that this tensor possesses an infinite number of antiplanes of symmetry $T\sigma_v$ passing through the axis z, an infinite number of the two-fold antiaxes TC_2 lying in the plane $z = 0$, a plane of symmetry σ_h which is perpendicular to z, and the center of symmetry i. These elements of symmetry form the group $D_{\infty h}(C_{\infty h})$. Hence, the tensor of magnetic permeability (D21) has the symmetry $D_{\infty h}(C_{\infty h})$ (Appendix C.4), (Table 6).

APPENDIX E. CURIE'S PRINCIPLE

Artificial composites may consist of a host material, some inclusions (particles) and may be under external fields and forces (perturbations). The host material may have certain symmetry, the inclusions and their spatial arrangements may also be described by certain groups of symmetry. External perturbation may be of different nature (for example, electric and magnetic fields, mechanical forces, temperature fields and their combinations) and of different symmetry. In this case, the problem of determination of symmetry group of the medium is solved using Curie's principle [47]. In a mathematical language, Curie's principle can be written as intersection of the symmetry groups of all the constitutive symmetry elements of a given medium: the host material, the shape of the particles and their arrangements, the external perturbations, etc.:

$$G_{res} = G_1 \cap G_2 \cap G_3 \dots \tag{E1}$$

that expresses the principle of symmetry superposition, namely the symmetry of a complex object is defined by the highest common subgroup of the groups which describe the constituents of the object. This principle allows one to find the resultant symmetry G_{res}.

In many cases natural and artificial media may be considered as continuous, homogeneous and isotropic, when external fields and forces (perturbations) are not applied. External perturbations described by the tensors of the second, first and zero rank may change electromagnetic properties of the media. In particular, a medium under perturbation may become anisotropic, gyrotropic or bianisotropic one. The external perturbations may be both being created deliberately (for example, magnetization of ferrites by dc magnetic field), and undesirable (heating in powerful em devices). Notice that insertion of particles in a host material and even a movement of the medium (for moving dielectric and gyrotropic media [12, 13, 33]) may also be considered as a perturbation.

In the absence of external perturbations, the isotropic homogeneous media are described by two point continuous symmetry groups [30], K_h and K. The group K_h is the highest possible one and it corresponds to the symmetry of sphere with the planes of symmetry. The group K, on the other hand, may be represented by a sphere without planes of symmetry with all the diameters twisted through an equal angle in one direction. The isotropic chiral medium has such symmetry. Scalar perturbations obviously can not change the initial symmetry of the isotropic medium K_h and K. A medium with the symmetry K_h under an external perturbation acquires the symmetry of this perturbation.

As examples of application of Curie's principle, we consider two cases:

(1) static electric field E and dc magnetic field H intersecting at right angle are applied to an isotropic magnetic medium and
(2) dc magnetic field applied to a chiral isotropic magnetic medium.

In case (1), the resultant magnetic symmetry of the vectors $\mathbf{E}$ and $\mathbf{H}$ (and consequently, of the medium, because the original medium was isotropic) is $C_{2v}(C_s)$. This group has one plane, one antiplane of symmetry and the antiaxis coinciding with the vector $\mathbf{E}$ direction. The corresponding constitutive tensors are presented in Table 7. In case (2), the chiral medium under dc magnetic field acquires the symmetry $D_\infty(C_\infty)$, because the group K describing chiral medium and the group $D_{\infty h}(C_{\infty h})$ describing dc magnetic field have one common element, namely the axis C_∞ and besides, an infinite number of the axes of the second order perpendicular to the axis C_∞ are converted under dc magnetic field into the antiaxes. The constitutive tensors for this medium are in Table 6.

Notice that the constitutive particles may be asymmetric, i.e., may not have symmetry at all and may be distributed in the host material randomly, but nevertheless the medium composed from such particles will be described by the highest possible groups of symmetry K_h or K because we consider homogeneous media in a long-wave approximation.

ACKNOWLEDGMENT

This work was supported by the Brazilian Agencies CAPES and CNP'q.

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