APPLICATION-ORIENTED RELATIVISTIC ELECTRODYNAMICS (2)

D. Censor

Ben Gurion University of the Negev
Department of Electrical and Computer Engineering
Beer Sheva, Israel 84105

Abstract–This article is a revised and upgraded edition of a previous one published in this journal, hence the label (2), see the General Remarks section below.

Relativistic Electrodynamics, for many years a purely academic subject from the point of view of the applied physicist and electromagnetic radiation engineer, is nowadays recognized as pertinent to many practical applications. We therefore need to define a syllabus and explore the best methods for educating future generations of such users. Such an attempt is presented here, and is of course biased by personal preferences. What emerges as general guidelines are the facts that Relativistic Electrodynamics should be presented axiomatically, without trying to "explain the physical meaning" of Special Relativity, that four-vectors and their mathematical properties should be emphasized, and that the field tensors, an elegant formalism, albeit of limited practical use, should be avoided. Use of four-fold Fourier transforms not only greatly simplifies the relevant manipulations, it is also of paramount importance for discussion of dispersive media. This approach yields many concepts as mathematical results, e.g., the Relativistic Doppler effect, which therefore do not require a long phenomenological discussion with many "explanations". Introducing this approach as early as possible opens new vistas for the student and the educator, indeed some of the new results here do not appear in textbooks on Special Relativity. One of the main results shown here is the fact that the generalized Fermat principle states that the ray will propagate in such a manner that the proper time will be minimized (or extremized, in general). It also strips the mystique of this principle, showing that it is in fact equivalent to a modest mathematical condition on the smoothness of the phase function. The presentation is constructed in a way

that allows the student to gradually overcome difficulties in assimilating new concepts and applying them. In that too it is different from many conventional presentations.

1. GENERAL REMARKS

The present paper is a revision and upgrading of an article previously published in this journal [1]. During almost a decade since its publication, the paper [1] served as the backbone for a course by the same name, offered numerous times to graduate students, mainly from electrical engineering departments, in numerous universities in many countries. Experience, new ideas for explaining certain points, new understanding of the fundamentals, awareness of students' needs — all these suggested that the time has come for an upgraded version.

The general thrust of the first version is retained. The topsy-turvy approach to Special Relativity [2] is used, heavy mathematical tools are avoided, and only Euclidean space, mostly in Cartesian coordinates is used. Without relenting on the basic ideas and their consequences, but

also avoiding being bogged down in too much detail, the theoretical subjects and some applications are presented to the students. Present day too practical curricula call for such conciliation with the cornerstones of our fundamental understanding of the physical world. The graduate program, after the chase for grades in the undergraduate heyday is over, and before the short-term short-sighted demands of R&D and industrial practicalities dominate the agenda, is the student's last chance for understanding the scientific basics of his engineering pursuits.

Some new material has been added, to cover a diverse range of examples. The velocity-dependent problem of scattering by a moving cylinder found its way into the lecture material, and new aspects regarding nonlinear wave propagation is presented. Although not purely a relativistic problem, its presentation using Minkowskian four-vectors is a boon. It also helps in understanding Minkowski's methodology for constitutive relations in moving media. Finally, also included is the recent subject of Volterra differential operators for constitutive relations, in media at rest and in motion.

2. INTRODUCTION AND RATIONALE

The intimate relationship between Maxwell's theory for the electromagnetic field and Einstein's Special Relativity theory [3] is generally recognized nowadays. Throughout the present century many educators found it necessary to include a chapter on Special Relativity in textbooks devoted to electromagnetic field theory, e.g., in the book by Becker, edited by Sauter [4] (a book that has its roots in the last century and appeared practically in sixteen editions!), see also Stratton [5], Fano Chu and Adler [6], Sommerfeld [7], Jordan and Balmain [8], Panofsky and Phillips [9], Shadowitz [10], Jackson [11], Portis [12], Lorrain and Corson [13], Wangness [14], Griffiths [15], Frankl [16], Chen [17], Kong [18], Plonus [19], Eringen and Maugin [20], Schwartz [21]. This list is representative, rather than exhaustive.

Last but not least, the pioneering book by Van Bladel [22] must be mentioned. In an attempt to serve the needs of the engineering community, the book compiles results of many relevant studies. The topics chosen are more or less of practical nature, related to Relativistic Electrodynamics. It is hoped that experiments like Van Bladel's book and the present article contribute to clarify the question of how to present an application-oriented course of this kind to students.

By scrutinizing the above mentioned and other textbooks, it becomes apparent that a specific approach suitable for educating applied physicists and electrical engineers, especially in the area of electromagnetic radiation engineering, is lacking. Some authors introduce Special Relativity theory in the traditional "Gedanken experiment" approach, and by the time the reader finishes with the moving trains, flashing torchlights, and rods and clocks, the relevance to practical electromagnetic problems is obscured. Others move along more formalistic lines and derive the field tensors, mostly by using general coordinate systems and the heavy machinery of differential geometry, i.e., covariant and contravariant coordinate systems. Experience shows that the mathematical elegance hardly provides an incentive for the engineering student to move on in this field. On the other hand, we are nowadays aware of some real-life problems, e.g., design of satellite supported global navigation and positioning systems (GPS), which involve special (and sometimes even general) relativistic considerations related to precision of time and frequency bases and errors incurred during propagation through complicated inhomogeneous and time varying media, and everything in the presence of relative motion between objects. It is therefore mandatory to devise the methodological tools and suitable representations for teaching Relativistic Electrodynamics to applied physics and electrical engineering students. In the course of such a pedagogical experiment with electrical engineering graduates, it became clear that the rudiments of Special Relativity should be presented axiomatically, with as little phenomenological "explanations" as feasible, working on the assumption that this aspect has been covered at least to some extent in "Baby physics" courses. To repeat this part of the story means that in a one quarter (or semester) course there might not be sufficient time for effectively discussing the more advanced topics presented here. It also became clear that four-dimensional Fourier transforms should be introduced right from the beginning, an unorthodox approach as far as this author is aware. This facilitates the work in an algebraic, rather than differential equations environment, thus simplifying mathematical manipulations. It also became clear that four-vectors, which are easily handled, almost as easily as the classical three-vectors, should be extensively used. Most of the students met had a fair to good grasp of vector analysis and linear algebra, and the introduction of four-vectors and dyadics did not pose a problem. However, only Euclidean systems are considered, and even in this context,

the elegance of the electromagnetic field tensors and the associated representation of Maxwell's equations has been avoided. Within these limits, it is then the personal preference of the teacher that will guide him to emphasize certain classes of problems. From this point of view the specific material described here serves merely as an example. But carefully choosing the examples also serves to get some new insight into supposedly old problems. For example, the section on the Fermat principle shows that the generalized principle, for inhomogeneous and time dependent media, acquires a new meaning that can only be stated in the context of Special Relativity: Verbally stated, it says that the ray propagates along a path that minimizes (or in general extremizes) the proper time. It is also shown that the Fermat principle is equivalent to a simple mathematical condition on the smoothness of the phase function.

The present article is organized as follows: First, Relativistic Electrodynamics is introduced axiomatically, using the topsy-turvy approach [2]. The introduction of compact notation conventions facilitates exploring properties of relevant Minkowski four-vectors, and a discussion of the Lorentz transformation, the associated differential operators, and some important conclusions of the theory, exemplified by the ruler, and twins, paradoxes. Then the four-dimensional Fourier transformations are introduced, providing the technique of algebraization for the Maxwell equations. The associated spectral domain four-vector is identified as the relativistic Doppler effect. These four-fold integrals bring up the important and nontrivial question of the validity of transformed spatiotemporal transformations formulas when stated in the spectral domain. Further exploration of four-vectors follows. Next, four-potentials are introduced. This is followed by a discussion of the cross multiplication operation and the related curl operation. It is mentioned, without going into too much detail, that we are dealing with tensor operations, and the general advice is to work with the various components (this is done without mentioning the anti-symmetric tensors and their properties, which would encumber the presentation without contributing to application-oriented problems). A section on the proper time and related concepts in mechanics follows. At this point a section with the provocative title "The breakdown of Special Relativity" is introduced. It is emphasized that for varying velocities, Special Relativity becomes a heuristic approximation, holding only for slowly accelerated objects. This observation, noted by Einstein [3],

somehow faded out from many later tractates and textbooks.

We now have enough tools for discussing specific problems. What might sometimes appear as a melange of unrelated subjects is actually an attempt to lead the student gradually from the less complicated to the more sophisticated subjects. As a first example, the Minkowski methodology for the constitutive relations for moving media is discussed, and the derivation for homogeneous, dispersive, and anisotropic media is demonstrated. Dispersion equations and their relativistic invariance are discussed. This provides the basis for discussing Hamiltonian ray propagation for inhomogeneous and time varying dispersive media. This is followed by a section discussing the generalized Fermat principle, and its associated Euler-Lagrange equations, which are once again the Hamiltonian ray equations. As a further application, which is of course biased by the author's personal preferences, the question of ray propagation in lossy media is discussed, in the context of the ray equations, their generalization to lossy media and the questions of Lorentz transformations and mathematical complex analyticity involved. The advantage of using Minkowskian four-vectors even for non-relativistic problems is further demonstrated by the application of the Volterra functional series to wave propagation in nonlinear systems. The novel concept of the associated Volterra differential operators is also introduced, for homogeneous and inhomogeneous media. Finally Relativistic Electrodynamics is applied to the classical problem of scattering by a cylinder, in order to derive the formulas for scattering by a moving object. The interaction of multipolity in the result is an interesting consequence. Accordingly multipoles in motion acquire higher multipole modes — a monopole becomes a monopole plus a dipole, etc. Squeezing all this into a single semester course requires considerable sleight, and the extent of using these or different applications depends on the teacher, the available span of time, and the audience.

3. TRADITIONAL AND TOPSY-TURVY SPECIAL RELATIVITY

In this section Relativistic Electrodynamics is introduced. The formalism needed by the applied physicist and engineer is stipulated in an axiomatic manner. The introduction of the field tensors and the ensuing elegant representation of the field equations by means of operations on these tensors, a cornerstone of relativistic formalism, is obviated. Four-vectors in Minkowski space are introduced along the way as a no-

tational and operational tool, rather than a metaphysical-conceptual generalization of the space-time manifold idea, as sometimes implied in books specializing in relativity theory. This somewhat peremptory methodology follows the realization that we do not have the time to thoroughly plough the background knowledge, lest no time will remain to teach the pertinent engineering aspects.

Maxwell's equations for the electromagnetic field (in the "unprimed" frame of reference denoted by Γ) are given by

$$
\begin{aligned}
\partial_{\mathbf{x}} \times \mathbf{E} &= -\partial_t \mathbf{B} - \mathbf{j}_m \\
\partial_{\mathbf{x}} \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{j}_e \\
\partial_{\mathbf{x}} \cdot \mathbf{D} &= \rho_e \\
\partial_{\mathbf{x}} \cdot \mathbf{B} &= \rho_m
\end{aligned}
\tag{1}
$$

where $\partial_{\mathbf{x}}$ (often symbolized by ∇ and called "Nabla", or sometimes "Del") and ∂_t denote the space and time derivative operators, respectively. In general all the fields are space and time dependent, e.g., $\mathbf{E} = \mathbf{E}(\mathbf{X})$. Here

$$
\mathbf{X} = (\mathbf{x}, ict)
\tag{2}
$$

symbolizes the space-time dependence, actually $\mathbf{X}$ denotes the event (world point) in the sense of a Minkowski-space location vector, as discussed below, where c is the universal constant of the speed of light, and i is the unit imaginary complex number $i^2 = -1$. For symmetry and completeness, in the present representation, the Maxwell equations include the usual electric (index e), as well as the fictitious magnetic (index m), current and charge density sources. To date, the existence of the magnetic current and charge densities in (1) has not been empirically established. Therefore at this time they should be considered as fictitious, in the sense that they are auxiliary and not intrinsic physical entities. Magnetic currents and charges are amply used in a variety of practical problems to emulate equivalent sources, e.g., when surface sources are designed to satisfy certain discontinuities in the fields. See Lindell [23], Stratton [5], or Kong [18], as well as many other textbooks cited above. However, we should always be aware of the fact that physicists have not given up the quest for magnetic charges and currents, e.g., see Jackson [11].

The statement of Maxwell's equations (1) is incomplete in the sense that it is unrelated to the rest of physics. For example, we need a way of linking Electrodynamics to familiar concepts like force and energy

introduced in the context of mechanics. One way of achieving this goal is by stating a force formula. Thus the presence of a conventional charge q_e can be detected through the forces exerted on it according to the Lorentz force formula

$$\mathbf{f}_e = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3}$$

The teaching of electromagnetic theory in a phenomenological-historical way, as evolving from crucial experiments and the consequent "laws" that are added into the model, tends to obscure the fact that (3) is extrinsic and does not follow from Maxwell's equations. This important fact should be stressed at this point. Actually (3) is an extension of the simple Coulomb force formula $\mathbf{f}_e = q_e\mathbf{E}$ which should be considered not as a "law" but as a link between mechanics and Electrodynamics. Inasmuch as $\mathbf{f}_e$ is supposedly already known from mechanics, $\mathbf{f}_e = q_e\mathbf{E}$ can be considered as a definition of the proportionality coefficient q_e. Once it is defined we have at our disposal the "rationalized" Giorgi MKSQ system of units. On introducing Special Relativity axiomatically, (3) can be derived from $\mathbf{f}_e = q_e\mathbf{E}$. Inspired by symmetry considerations, the analog of (3) for magnetic sources is assumed to be

$$\mathbf{f}_m = q_m(\mathbf{H} - \mathbf{v} \times \mathbf{D}) \tag{4}$$

again an extension derived by means of relativistic transformation formulas given below from a magnetic force formula $\mathbf{f}_m = q_m\mathbf{H}$. Although this example is far fetched, it demonstrates the symmetry introduced into Maxwell's equations by stipulating magnetic sources, and the stimulus it provides for looking at things in a new way.

Special Relativity theory, dealing with observations performed in inertial systems, i.e., frames of reference in relative uniform motion, has been announced by Einstein [3], however he considers there only free space ("vacuum") electrodynamics.

Einstein's Special Relativity theory (relevant statements are denoted by S for "Special") postulates:

(S-1). Light speed c is a universal constant observed in all inertial frames.

(S-2). Maxwell's equations provide the model or "law of nature" for describing the electromagnetic field. I.e., the theory recognizes (1) above.

(S-3). Maxwell's equations existing for all observers in inertial frames of reference have the same functional structure (henceforth: Covariance). This means that if (1) exists for an observer in one frame of

reference, in another inertial frame (the "primed" frame of reference Γ'), Maxwell's equations have the form:

$$
\begin{aligned}
\partial_{\mathbf{x}'} \times \mathbf{E}' &= -\partial_{t'}\mathbf{B}' - \mathbf{j}'_m \\
\partial_{\mathbf{x}'} \times \mathbf{H}' &= \partial_{t'}\mathbf{D}' + \mathbf{j}'_e \\
\partial_{\mathbf{x}'} \cdot \mathbf{D}' &= \rho'_e \\
\partial_{\mathbf{x}'} \cdot \mathbf{B}' &= \rho'_m
\end{aligned}
\tag{5}
$$

where now $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$, and the native, or proper, space-time coordinates in the Γ' system are denoted by $\mathbf{X}' = (\mathbf{x}', ict')$, which is also a Minkowski locational four-vector.

The consequences of the above three postulates follow:

(S-i). From (S-1), i.e., the constancy of the speed of light, the Lorentz space-time transformations $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$, mediating between spatiotemporal coordinates in Γ and Γ' are developed in the form:

$$
\begin{aligned}
\mathbf{x}' &= \widetilde{\mathbf{U}} \cdot (\mathbf{x} - \mathbf{v}t) \\
t' &= \gamma \left(t - \mathbf{v} \cdot \mathbf{x}/c^2\right)
\end{aligned}
\tag{6}
$$

Here $\mathbf{v}$ is the velocity by which Γ' is moving, as observed from Γ, and we define

$$
\gamma = (1 - \beta^2)^{-1/2}, \qquad \beta = \nu/c, \quad \nu = |\mathbf{v}|,
$$
$$
\widetilde{\mathbf{U}} = \widetilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}}, \quad \hat{\mathbf{v}} = \mathbf{v}/\nu
\tag{7}
$$

where the tilde denotes dyadics. Briefly, a dyadic is just a different notation for a matrix, or in general, a tensor, and very convenient to use in conjunction with vectors. E.g., a dyadic can be created by juxtaposed vectors (or a linear combination of such) without a dot or cross multiplication sign between them, amounting to an external product of matrices. Here $\widetilde{\mathbf{I}}$ is the idemfactor or unit Dyadic (same as unit matrix). From (6) and the chain rule of calculus follows the transformation $\partial_{\mathbf{X}'} = \partial_{\mathbf{X}'}[\partial_{\mathbf{X}}]$ for the space-time differential operators,

$$
\begin{aligned}
\partial_{\mathbf{x}'} &= \widetilde{\mathbf{U}} \cdot \left(\partial_{\mathbf{x}} + \mathbf{v}\partial_t/c^2\right) \\
\partial_{t'} &= \gamma(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})
\end{aligned}
\tag{8}
$$

and compacted in a four-vector *form*

$$
\partial_{\mathbf{X}} = \left(\partial_{\mathbf{x}}, -\frac{i}{c}\partial_t\right)
\tag{9}
$$

is the four-gradient operator. How a quadruplet like (9) is tested to qualify as a proper Minkowski four-vector is discussed below.

(S-ii) From the axioms (S-2), (S-3) above, the transformation formulas for the fields in Γ and Γ' are derived in the following form:

$$\begin{aligned}
\mathbf{E}' &= \widetilde{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
\mathbf{B}' &= \widetilde{\mathbf{V}} \cdot (\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2) \\
\mathbf{D}' &= \widetilde{\mathbf{V}} \cdot (\mathbf{D} + \mathbf{v} \times \mathbf{H}/c^2) \\
\mathbf{H}' &= \widetilde{\mathbf{V}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \\
\widetilde{\mathbf{V}} &= \gamma \widetilde{\mathbf{I}} + (1 - \gamma)\hat{\mathbf{v}}\hat{\mathbf{v}}
\end{aligned} \tag{10}$$

where $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$ and $\mathbf{E} = \mathbf{E}(\mathbf{X})$, etc., and the Lorentz transformation (6) is identically satisfied $\mathbf{X}' \equiv \mathbf{X}'[\mathbf{X}]$ by the spatiotemporal coordinates indicated in (10). Similarly, for the sources we derive the transformation formulas:

$$\begin{aligned}
\mathbf{j}'_{e,m} &= \widetilde{\mathbf{U}} \cdot (\mathbf{j}_{e,m} - \mathbf{v}\rho_{e,m}) \\
\rho'_{e,m} &= \gamma \left(\rho_{e,m} - \mathbf{v} \cdot \mathbf{j}_{e,m}/c^2 \right)
\end{aligned} \tag{11}$$

for the corresponding e-, or m-, indexed sources.

Topsy-turvy (T) Special Relativity is stated in inverse order:

(T-1) Instead of assuming the constancy of the speed of light (S-l above), we assume the validity of the Lorentz transformation (S-i), i.e., (6).

(T-2) Here too we start with the same postulate (S-2) on the validity of equation (1).

(T-3) We postulate the validity of the formulas for the transformations of fields as given by (9), (10), i.e., what above constituted (S-ii).

The consequences are:

(T-i) From (T-1) we derive the constancy of c, the speed of light, i.e., (S-1) in the first model.

(T-ii) From (T-1), i.e., (6), (8) and (T-2), (T-3) we derive the covariance of Maxwell's equations, i.e., (S-3) of the previous model.

One might argue that the present model loses the motivation for universality and simplicity, displayed in the S-model. While this might be a valid argument, nevertheless it is compensated by the fact that the T-model is much easier to handle in the classroom, and the S-model can be mentioned in retrospect, showing how the two models are equivalent.

It is easily verified that the inverse transformation is obtained from e.g., (6), (8), (10), (11) by exchanging primed and unprimed symbols and inverting the sign of $\mathbf{v}$. This is also valid for other transformations given below.

This, in a nutshell, is the basis of Special Relativistic Electrodynamics. Some brief references will be made below regarding Relativistic Mechanics.

Note that $\widetilde{\mathbf{U}} \cdot \widetilde{\mathbf{V}} = \gamma \widetilde{\mathbf{I}}$. Also interesting are the roles of the dyadics $\widetilde{\mathbf{U}}$, $\widetilde{\mathbf{V}}$, in sorting out the components of the three-dimensional vectors into parallel and perpendicular components with respect to the velocity $\mathbf{v}$ and multiplying by γ. The product of $\widetilde{\mathbf{U}}$, $\widetilde{\mathbf{V}}$ and a vector attaches a factor γ to the parallel, perpendicular, component of the vector, respectively. Of course we know that the reason for a three dimensional vector to be associated with either $\widetilde{\mathbf{U}}$, or $\widetilde{\mathbf{V}}$ depends whether it is a "true" vector, i.e., the spatial part of a four-vector, like $\mathbf{x}$ or $\mathbf{j}$, or a component of an antysymmetrical tensor, like $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$, respectively. Exactly this is the part of the story that we should avoid discussing with students novices, to the subject of Relativistic Electrodynamics, and present the theory axiomatically. The finer details can wait for a later encounter with this material.

Minkowski [24] introduced the four-vector concept which will enable us to compact our notation and simplify the algebraic and differential manipulations. To the three components x_j, $j = 1, 2, 3$ we add $x_4 = ict$, thus for real t we have now an imaginary coordinate x_4. Henceforth four-vectors will be denoted by capital boldface characters, like in (2) and (9). It is not necessary at this stage to introduce the geometrical concepts pertaining to the Minkowski space, i.e., to describe the Lorentz transformation as a rotation in this space. What is important for the student to know is the fact that the length of a four-vector is invariant with respect to the Lorentz transformation (6). It can be verified as an exercise that subject to (6)

$$\mathbf{X} \cdot \mathbf{X} = \mathbf{x} \cdot \mathbf{x} - c^2 t^2 = \mathbf{X}' \cdot \mathbf{X}' = \mathbf{x}' \cdot \mathbf{x}' - c^2 t'^2 \qquad (12)$$

This also explains how a four-vector scalar product is obtained. In the specific case that in (12) a constant value is chosen, it must be a zero. The reason is simple: Our specific choice of the Lorentz transformation in the form (6) prescribes that at $t = t' = 0$ also $\mathbf{x} = \mathbf{x}' = 0$, hence for this specific case

$$\mathbf{X} \cdot \mathbf{X} = \mathbf{X}' \cdot \mathbf{X}' = 0 \qquad (13)$$

The "null vector" $\mathbf{X}$, as it is called, implied in (13), defines the celebrated "light cone". If only one space coordinate x is considered, then (13) amounts to the pair of lines $x = \pm ct$. If two space coordinates are employed, then $x^2 + y^2 = (ct)^2$ implies a cone whose axis is along the t-coordinate. Thus (13) formally defines a cone in the corresponding Minkowski space. From (6), (7), it is clear that the proviso for real values for spatiotemporal coordinates is $\nu \leq c$. Thus

$$\mathbf{X} \cdot \mathbf{X} \leq 0 \tag{14}$$

defines the light cone and its interior domain. The condition (14) states the relativistic causal relation between events. Writing (14) for one spatial coordinate in the form $|x| \leq c|t|$ reveals that the "light cone" concept amounts to a statement that physical velocity (velocity of objects, energy packets, and signals bearing information) cannot exceed c.

To qualify as a Minkowski space four-vector, a quadruplet like (9) must satisfy a Lorentz transformation similar to (6). In general, a four-vector qualifies as such if its scalar (inner) multiplication with another four-vector is an invariant, identical for all inertial systems, such as (13). In fact such products are used to derive invariants, which often are recognized as some conservation property of a system.

Thus by showing that subject to (6) and (8) $\partial_{\mathbf{X}} \cdot \mathbf{X} = \partial_{\mathbf{X}'} \cdot \mathbf{X}'$, it is established that (9) is a four-vector. Alternatively, by inspection of (6) and (8) it becomes clear that a duality exists, in the sense that replacing $\partial_{\mathbf{x}} \Leftrightarrow \mathbf{x}$, $ct \Leftrightarrow -c^{-1}\partial_t$, consistently leads from one transformation to the other. This convenient device will be amply used throughout to test and define four-vectors. Inasmuch as current and charge sources in (11) follow the same transformation formulas as the Lorentz transformation (6), we also identify as four-vectors

$$\mathbf{J}_{e,m} = (\mathbf{j}_{e,m}, ic\rho_{e,m}) \tag{15}$$

for the e, m, indices. It then follows from Maxwell's equations (1) that

$$\begin{aligned} (\partial_{\mathbf{x}} \times \mathbf{H} - \partial_t \mathbf{D}, \ ic\partial_{\mathbf{x}} \cdot \mathbf{D}) \\ (-\partial_{\mathbf{x}} \times \mathbf{E} - \partial_t \mathbf{B}, \ ic\partial_{\mathbf{x}} \cdot \mathbf{B}) \end{aligned} \tag{16}$$

are also four-vectors, therefore their spatial parts (first expression in parentheses) transform like $\mathbf{x}$, and their temporal coordinates (expression in parentheses multiplying ic) transform like t, according to

(6). This provides another example for constructing a four-vector, and from the covariance of the Maxwell equations (S-3) or (T-ii), (16) in terms of Γ' fields is valid too.

The Lorentz transformation $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$ (6) can be written in a mixed tensor-matrix form as $\mathbf{X}' = \widetilde{\mathbf{W}} \cdot \mathbf{X}$, revealing the anti-symmetry with respect to the imaginary off-diagonal terms

$$
\begin{bmatrix} \mathbf{x}' \\ ict' \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{U}} \cdot & \dfrac{i\gamma\mathbf{v}\cdot}{c} \\ -\dfrac{i\gamma\mathbf{v}\cdot}{c} & \gamma \end{bmatrix} \begin{bmatrix} \mathbf{x}' \\ ict' \end{bmatrix} \tag{17}
$$

i.e., $\widetilde{\mathbf{W}}$ in (17) is an Hermitian matrix. Or we can represent $\widetilde{\mathbf{W}}$ in matrix forms with scalar entries W_{ij}, $i, j = 1, \ldots, 4$. For example, take $\mathbf{v}$ in the $i, j = 1$ direction, this yields

$$
\begin{bmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \tag{18}
$$

once again (18) is Hermitian. It is easily verified that the

$$
\det[W_{ij}] = 1 \tag{19}
$$

It is interesting and useful for the sequel to show that the unity determinant $\det[W_{ij}]$ is the Jacobian of four-dimensional integrations. Working in Cartesian components in four-space, and using the Einstein summation convention, i.e., that indices indicate rows and columns of a matrix, and an index appearing twice in a term is a dummy index on which summation is to be performed, we have

$$
\left[\frac{\partial X'_i}{\partial X_j} \right] = \left[\frac{\partial}{\partial X_j} W_{ik} X_k \right] = \left[W_{ik} \frac{\partial X_k}{\partial X_j} \right] = [W_{ik} \delta_{kj}] = [W_{ij}] \tag{20}
$$

and from (19), the Jacobian is unity. Or in terms of dyadics, we have

$$
\partial_{\mathbf{X}} \mathbf{X}' = \partial_{\mathbf{x}} \widetilde{\mathbf{W}} \cdot \mathbf{X}' = \widetilde{\mathbf{W}} \cdot \partial_X \mathbf{X}' = \widetilde{\mathbf{W}} \cdot \widetilde{\mathbf{I}} = \widetilde{\mathbf{W}} \tag{21}
$$

and (19) follows.

4. RULER AND TWINS PARADOXES

These subjects provide some deeper insight into the implications of the Lorentz transformation, as well as some exercise in the manipulation of the formulas.

For the ruler paradox, we assume a ruler of length L, which we would like to pass through a slit of length l. If $l < L$ and no tilting is allowed the mission cannot be accomplished. So here Special Relativity comes to the rescue: Let us move back, and set the ruler in motion at a velocity v parallel to the ruler and the slot. The ruler will now be of length L as observed in the co-moving frame of reference Γ'. But in the slot's system of reference Γ, length contraction occurs, due to the factor γ in the Lorentz transformation (6). The observer in Γ waits until the seemingly shortened ruler is above the slot, and jerks it down. The mission is accomplished, or is it? The paradox stems from the fact that the observer in Γ' predicts a similar shortening, of the relatively moving slot, which makes the situation even worse! The relevant one-dimensional form of (6) is now given by

$$
\begin{aligned}
x' &= \gamma(x - \nu t) \\
t' &= \gamma(t - x\nu/c^2)
\end{aligned}
\tag{22}
$$

At time $t' = t = 0$ the two systems coincide at $x' = x = 0$. Let us assume $x' = x = 0$ to be aligned to the trailing edge of the ruler. At this time $t = 0$ other points are related through $x' = \gamma x$. If at $t = 0$ the leading edge is in Γ' at $x' = L$, this then corresponds to $x = L/\gamma$ in Γ. If $x = l \geq L/\gamma$, then the ruler can pass through the slot. However, from the second equation (22) it follows that for the observer this happens at an earlier time $t' = -\gamma x\nu/c^2$. Specifically at $t' = -L\nu/c^2$. So, from the point of view of the Γ' observer, the leading edge goes under the table at $t' = -L\nu/c^2$, the trailing edge follows at $t' = 0$. If the thickness of the table and the ruler are zero, there is no tilting. So much for the ruler paradox.

The twins paradox is even more exciting: Assume two twins identical in every respect, which for our story means that they have the same expected life span. While one brother stays at home, the other travels, and thus undergoes a time dilatation (i.e., his clock is slower). Returning home, the traveling brother is younger (his aging was slower), compared to the brother that stayed at home. Since Special Relativity teaches us that all inertial systems are equivalent, why is it one twin

and not the other that underwent slower aging? The paradox is explained by a careful analysis of the Lorentz transformation. To (22) we now add a third reference system Γ'' governed by

$$x'' = \gamma(x + \nu t)$$
$$t'' = \gamma(t + x\nu/c^2) \tag{23}$$

The new system moves in the opposite direction and possesses the same reference location and time, i.e., at time $t'' = t = 0$ the two systems coincide at $x'' = x = 0$. For simplicity we assume the same velocity (in opposite directions) in (22), (23).

The travelling twin jumps on the "train" Γ' at $t = t' = 0$. We will discuss the question of acceleration that "jump" implies later. He moves, as seen by the twin at home, according to $x = \nu t$ for a time duration T, hence he covers a distance

$$X = \nu T \tag{24}$$

The traveling twin does not move in the train, so his coordinate is $x' = 0$, which according to (22) is consistent with (24). Substituting T, T' and (24) into (22), we get

$$T' = \gamma(T - \nu^2 T/c^2) = T/\gamma \tag{25}$$

Now the traveling twin jumps on the train Γ''. From (23) we see that he does that at

$$t'' = \gamma(T + \nu^2 T/c^2) = T_1'' = \gamma T \left(1 + \nu^2/c^2\right) \tag{26}$$

at the position in the train Γ'' given by

$$x'' = \gamma(x + \nu t) = X_1'' = \gamma(\nu T + \nu T) = 2\gamma\nu T \tag{27}$$

He keeps traveling on Γ'' until he returns to the point of departure $x = 0$. This happens at time $t = 2T$. The corresponding time in Γ'' is found from (23)

$$t'' = \gamma(2T + \nu 0/c^2) = T_2'' = 2\gamma T \tag{28}$$

The total time the travelling twin spent on trains is given by

$$T_{total} = T' + T_2'' - T_1'' \tag{29}$$

which by substitution from the relevant expressions above becomes

$$T_{total} = 2T/\gamma \tag{30}$$

Therefore the difference of the two twins' time lapse, in terms of the time of the brother at home in Γ is

$$\Delta T = 2T - T_{total} = 2T(1 - 1/\gamma) \tag{31}$$

Note that the result (30) would also be obtained by a simplistic resort to the time dilatation phenomenon, but would raise, as above, the paradox of symmetry, i.e., why does it happen to one twin and not the other. The systematic approach above leaves no loopholes.

Finally, let us consider the "jumping" problem in the special-relativistic discussion. The Lorentz transformation assumes constant relative velocity $\mathbf{v}$ between all inertial frames of reference. Moreover, General Relativity (which is outside the scope of the present discussion) predicts time dilatation effects due to acceleration. Whatever the effect of acceleration might be, we assume here that the "jump" is done instantaneously, taking zero time, and therefore does not affect the time budget computed above.

5. FOURIER TRANSFORMS AND THE DOPPLER EFFECT

Consider the four-fold Fourier transformation, which for brevity the writing of four integration signs and their limits from $-\infty$ to $+\infty$ is compacted in the form

$$f(x, y, z, ict) = q \int f(k_x, k_y, k_z, i\omega/c)$$
$$\cdot e^{i(k_x x + K_y y + K_z z + (i\omega/c)ict)} dk_x dk_y dk_z di\omega/c \tag{32}$$
$$q = (2\pi)^{-4}$$

Note that we use the same notation f for the function and its transform. To avoid ambiguity the arguments are shown too. For brevity of notation (32) will now be denoted as

$$f(\mathbf{X}) = q \int (d^4\mathbf{K}) f(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} \tag{33}$$

where $\mathbf{X}$ is given in (2) and

$$\mathbf{K} = (\mathbf{k}, i\omega/c) \tag{34}$$

is *formally* written as a four-vector, although at this stage we still need to show that it actually is a Minkowski four-vector. The integration four-volume element $(d^4\mathbf{K})$ becomes clear upon comparing (33) and (32). In an obvious manner, the associated inverse transformation is given by

$$f(\mathbf{K}) = \int (d^4\mathbf{X}) f(\mathbf{X}) e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{35}$$

Now apply the four-dimensional gradient operation (9) to (33). We then obtain

$$\partial_{\mathbf{X}} f(\mathbf{X}) = q \int (d^4\mathbf{K}) f(\mathbf{K}) i\mathbf{K} e^{i\mathbf{K}\cdot\mathbf{X}} \tag{36}$$

In view of the four-gradient operator on the left, (36) constitutes a four-vector expression. This implies that $\mathbf{K}$, (34), is indeed a four-vector. By inspection of (2), (6) and (34), one derives the transformation formulas $\mathbf{K}' = \mathbf{K}'[\mathbf{K}]$

$$\begin{aligned} \mathbf{k}' &= \widetilde{\mathbf{U}} \cdot (\mathbf{k} - \mathbf{v}\omega/c^2) \\ \omega' &= \gamma(\omega - \mathbf{v} \cdot \mathbf{k}) \end{aligned} \tag{37}$$

This is the relativistic Doppler effect first announced by Einstein [3]. How many rivers of ink have flown in order to "explain" the relativistic Doppler effect and the concept of "Phase invariance"! All this becomes superfluous when the present systematic approach is adopted. From the associated inverse Fourier transformation (35) one is led to construct the analog of (8), (9) in $\mathbf{K}$ space, thus obtaining another four-vector differential operator

$$\partial_{\mathbf{K}} = (\partial_{\mathbf{k}}, -ic\partial_{\omega}) \tag{38}$$

and the associated transformation formulas

$$\begin{aligned} \partial_{\mathbf{k}'} &= \widetilde{\mathbf{U}} \cdot (\partial_{\mathbf{k}} + \mathbf{v}\partial_{\omega}) \\ \partial_{\omega'} &= \gamma \left(\partial_{\omega} + \frac{1}{c^2}\mathbf{v} \cdot \partial_{\mathbf{k}} \right) \end{aligned} \tag{39}$$

which we could of course derive directly from (37) by using the chain rule of calculus.

Obviously this is a two way street: We could have started from the Doppler effect (37), and through the Fourier transformation arrive at the Lorentz transformation (6). Therefore, without losing its general properties, the theory of Special Relativity could have been started in the spectral domain (actually, the question of the roles of the spatiotemporal and spectral domains is much broader, and quite loaded with philosophical questions, see [25, 26]).

At this point it is worthwhile to realize that indeed this was the case, in a sense: Abraham [27], see also Pauli [28], before the advent of Einstein's theory [3], already derived the relativistically correct results for reflection by a moving mirror.

Subjecting Maxwell's equations (1), (5) to the Fourier transformation (33) yields algebraic equations which are often easier to manipulate. This is achieved by replacing components of $\partial_{\mathbf{X}}$ by the corresponding components of $i\mathbf{K}$. Thus applying the Fourier transformation (33) to (1) yields

$$
\begin{aligned}
i\mathbf{k} \times \mathbf{E} &= i\omega\mathbf{B} - \mathbf{j}_m \\
i\mathbf{k} \times \mathbf{H} &= -i\omega\mathbf{D} + \mathbf{j}_e \\
i\mathbf{k} \cdot \mathbf{D} &= \rho_e \\
i\mathbf{k} \cdot \mathbf{B} &= \rho_m
\end{aligned}
\tag{40}
$$

where the transformed fields $\mathbf{E} = \mathbf{E}(\mathbf{K})$ etc. are understood. We can of course apply the Fourier transformation also to (5) and obtain in a consistent manner

$$
\begin{aligned}
i\mathbf{k}' \times \mathbf{E}' &= i\omega'\mathbf{B}' - \mathbf{j}'_m \\
i\mathbf{k}' \times \mathbf{H}' &= -i\omega'\mathbf{D}' + \mathbf{j}'_e \\
i\mathbf{k}' \cdot \mathbf{D}' &= \rho'_e \\
i\mathbf{k}' \cdot \mathbf{B}' &= \rho'_m
\end{aligned}
\tag{41}
$$

and here $\mathbf{E}' = \mathbf{E}'(\mathbf{K}')$ etc. A cardinal question arising at this point is whether the field transformation formulas (10), (11) hold in the spectral domain $\mathbf{K}$ too, and in what sense? Transforming the two sides of the first equation (10), we now get,

$$
\mathbf{E}'(\mathbf{X}') = \widetilde{\mathbf{V}} \cdot (\mathbf{E}(\mathbf{X}) + \mathbf{v} \times \mathbf{B}(\mathbf{X})) = q \int (d^4\mathbf{K}')\mathbf{E}'(\mathbf{K}')e^{i\mathbf{K}'\cdot\mathbf{X}'}
$$

$$
= q \int (d^4\mathbf{K})\widetilde{\mathbf{V}} \cdot [\mathbf{E}(\mathbf{K}) + \mathbf{v} \times \mathbf{B}(\mathbf{K})]e^{i\mathbf{K}\cdot\mathbf{X}}
\tag{42}
$$

and the question before us is whether the integrands are identical, which is not obvious, [29]. By identifying the dummy integration variables as the proper spectral domain variables, obeying $\mathbf{K}' = \mathbf{K}'[\mathbf{K}]$ as above in (37), the exponentials become identical, because the scalar product $\mathbf{K}' \cdot \mathbf{X}' = \mathbf{K} \cdot \mathbf{X}$ is a Minkowski space invariant. This is sometimes referred to as the "phase invariance principle", although in the present context of Minkowskian four-vectors it is quite trivial. Furthermore, it is easily shown that the change of variable in the integrals (42) involves a Jacobian whose value is unity, just like in (20), (21), $\mathbf{K}' = \mathbf{K}'[\mathbf{K}] = \widetilde{\mathbf{W}} \cdot [\mathbf{K}]$ and therefore

$$d^4\mathbf{K}' = \det\left[\partial_{\mathbf{K}}\mathbf{K}'\right] d^4\mathbf{K} = d^4\mathbf{K} \tag{43}$$

Consequently (42) can be recast as

$$\int (d^4\mathbf{K}) \left[\mathbf{E}'(\mathbf{K}') - \widetilde{\mathbf{V}} \cdot (\mathbf{E}(\mathbf{K}) + \mathbf{v} \times \mathbf{B}(\mathbf{K}))\right] e^{i\mathbf{K}\cdot\mathbf{X}} = 0 \tag{44}$$

implying that the expression in brackets in (44) vanishes, hence it is established that the transformation formulas (10), (11) hold in $\mathbf{K}$ space too.

Corresponding to (43) we also derive

$$d^4\mathbf{X}' = \det\left[\partial_{\mathbf{x}}\mathbf{X}'\right] d^4\mathbf{X} = d^4\mathbf{X} \tag{45}$$

The results (43), (45) are usually phrased in the Special Relativity jargon as saying that "the four-dimensional volume element is a relativistic invariant". This is of course true only in the strict sense of performing the change of variables as above. These statement holds for any four-vector space, e.g., a representation space can be assigned to $\mathbf{J}_e$, $\mathbf{J}_m$, and volume elements be defined, $(d^4\mathbf{J}_e)$, $(d^4\mathbf{J}_m)$, which will also be relativistic invariants in this sense. All this is of course well known, e.g., see Pauli [28], the difficulty is in explaining it to our application-oriented students in a simple and coherent manner. Once Maxwell's equations and the field transformation formulas are available in algebraic form, it becomes much easier to manipulate the expression, e.g., to verify the Maxwell's equations covariance, i.e., showing that by substitution of the $\mathbf{K}$ space field transformation formulas (10), (11) into the unprimed set of Maxwell's equation (40), the primed set (41) is obtained.

6. INVARIANTS GALORE

In a sense, all physical laws and models are declarations about the invariance of certain quantities. Conservation laws are obviously in this category, but many other properties, e.g., symmetry in whatever sense, is also a declaration that something is unaffected, or conserved, or invariant, subject to some operation. Even writing a mathematical (algebraic, differential, integral etc.) equation for a physical law, such that everything appears on the left and is equal to zero or a constant or a unity dyadic, etc., on the right, is a declaration that "something" (the expression on the left) possesses some immutable properties.

   The scalar product of two four-vectors is one way of deriving Lorentz invariants, some of them have been recognized as fundamental "laws", others are less important, but stand by for whenever they might be used. Thus (13) is a cornerstone of Special Relativity theory. Not less important is the fact that the D'Alembert operator

$$\partial_{\mathbf{X}} \cdot \partial_{\mathbf{X}} = \partial_{\mathbf{x}} \cdot \partial_{\mathbf{x}} - c^{-2}(\partial_t)^2 \tag{46}$$

is a Lorentz invariant. Another invariant that has been elevated to the status of "law" is the equation of continuity

$$\partial_{\mathbf{X}} \cdot \mathbf{J}(\mathbf{X}) = \partial_{\mathbf{x}} \cdot \mathbf{j}(\mathbf{X}) + \partial_t \rho(\mathbf{X}) = O \tag{47}$$

resulting from a divergence $\partial_{\mathbf{x}}\cdot$ operation on the first, second equation of (1) and substitution of the fourth, third equations, respectively. In (47) the corresponding electric or magnetic sources are understood. Clearly if (47) vanishes in one inertial system, it vanishes in all, because of the invariant nature of (47). The same conclusion is reached by applying the divergence $\partial_{\mathbf{x}'}\cdot$ to (5). It follows of course that in the spectral domain $\mathbf{K}$ the corresponding relations

$$\mathbf{K} \cdot \mathbf{J}(\mathbf{K}) = \mathbf{k} \cdot \mathbf{j}(\mathbf{K}) - \omega\rho(\mathbf{K}) = \mathbf{0} \tag{48}$$

hold in all inertial systems of reference.

   Although the following invariant (note that it is not zero as in (47), (48))

$$\partial_{\mathbf{K}} \cdot \mathbf{J}(\mathbf{K}) = \partial_{\mathbf{k}} \cdot \mathbf{j}(\mathbf{K}) + c^2 \partial_\omega \rho(\mathbf{K}) \tag{49}$$

is not recognized as a "law", I would like my students to be able to see that (49) follows from (35) by identifying f with $\mathbf{J}$ and multiplying

both sides by $-i\mathbf{X}$, which is equivalent to taking the $\mathbf{K}$ space four-gradient derivative $\partial_{\mathbf{K}}$ on both sides of (35).

We have already introduced many four-vectors, e.g.,

$$\mathbf{X}, \partial_{\mathbf{X}}, \mathbf{J}_e, \partial_{\mathbf{J}_e}, \mathbf{J}_m, \partial_{\mathbf{J}_m}, \mathbf{K}, \partial_{\mathbf{K}} \tag{50}$$

including (16), and many more that are introduced below or elsewhere. Needless to say that linear combinations of invariants, operations like (49) acting on four-vectors, and so on, also yield invariants, hence we are dealing with an infinite group. Another way of deriving invariants is through the field transformation formulas in (10). Of course this is related to the properties of the field tensors, but can be easily verified directly. Thus we have [5] the following expressions

$$c^2\mathbf{B}\cdot\mathbf{B} - \mathbf{E}\cdot\mathbf{E} = c^2\mathbf{B}'\cdot\mathbf{B}' - \mathbf{E}'\cdot\mathbf{E}'$$
$$\mathbf{H}\cdot\mathbf{H} - c^2\mathbf{D}\cdot\mathbf{D} = \mathbf{H}'\cdot\mathbf{H}' - c^2\mathbf{D}'\cdot\mathbf{D}'$$
$$\mathbf{B}\cdot\mathbf{E} = \mathbf{B}'\cdot\mathbf{E}'$$
$$\mathbf{H}\cdot\mathbf{D} = \mathbf{H}'\cdot\mathbf{D}' \tag{51}$$
$$\mathbf{B}\cdot\mathbf{H} - \mathbf{E}\cdot\mathbf{D} = \mathbf{B}'\cdot\mathbf{H}' - \mathbf{E}'\cdot\mathbf{D}'$$
$$c^2\mathbf{B}\cdot\mathbf{D} + \mathbf{E}\cdot\mathbf{H} = c^2\mathbf{B}'\cdot\mathbf{D}' + \mathbf{E}'\cdot\mathbf{H}'$$

Still another way for deriving invariants through the Jacobian determinant is shown above (43), (45).

7. POTENTIALS

As a variation on the theme, the potentials will be discussed in the context of the Fourier transformed algebraic Maxwell equations. The original equations are split into two sets of fields one driven by $\mathbf{j}_e$, ρ_e, the other by $\mathbf{j}_e$, ρ_m. This yields

$$
\begin{array}{l|l}
i\mathbf{k}\times\mathbf{E}_e = i\omega\mathbf{B}_e & i\mathbf{k}\times\mathbf{E}_m = i\omega\mathbf{B}_m - \mathbf{j}_m \\
i\mathbf{k}\times\mathbf{H}_e = -i\omega\mathbf{D}_e + \mathbf{j}_e & i\mathbf{k}\times\mathbf{H}_m = -i\omega\mathbf{D}_m \\
i\mathbf{k}\cdot\mathbf{D}_e = \rho_e & i\mathbf{k}\cdot\mathbf{D}_m = 0 \\
i\mathbf{k}\cdot\mathbf{B}_e = 0 & i\mathbf{k}\cdot\mathbf{B}_m = \rho_m
\end{array} \tag{52}
$$

respectively, where $\mathbf{E}_e = \mathbf{E}_e(\mathbf{K})$ etc. Corresponding to (52) there exists in the primed frame of reference Γ' a set of Maxwell's equations

with primed symbols. The transformation formulas relating $\mathbf{K}$ and the fields in both frames are given above. The students are more acquainted with the e-indexed set in (52). The relation between the two sets follows from the formal similarity and leads to the following duality "dictionary". By substitution according to this dictionary we obtain the e-indexed set of Maxwell's equations from the m-indexed one, and vice-versa:

$$
\begin{aligned}
\mathbf{j}_e &\Leftrightarrow -\mathbf{j}_m \\
\rho_e &\Leftrightarrow -\rho_m \\
\mathbf{E}_e &\Leftrightarrow \mathbf{H}_m \\
\mathbf{H}_e &\Leftrightarrow \mathbf{E}_m \\
\mathbf{B}_e &\Leftrightarrow -\mathbf{D}_m \\
\mathbf{D}_e &\Leftrightarrow -\mathbf{B}_m \\
\mathbf{A}_e &\Leftrightarrow -\mathbf{A}_m \\
\phi_e &\Leftrightarrow -\phi_m \\
\Phi_e &\Leftrightarrow -\Phi_m
\end{aligned}
\tag{53}
$$

In (53) the potentials have been included, defined according to

$$
\begin{array}{l|l}
\mathbf{B}_e = i\mathbf{k} \times \mathbf{A}_e & \mathbf{D}_m = i\mathbf{k} \times \mathbf{A}_m \\
\mathbf{E}_e = -i\mathbf{k}\phi_e + i\omega\mathbf{A}_e & \mathbf{H}_m = i\mathbf{k}\phi_m - i\omega\mathbf{A}_m \\
\Phi_e = \left(\mathbf{A}_e, \dfrac{i}{c}\phi_e\right) & \Phi_m = \left(\mathbf{A}_m, \dfrac{i}{c}\phi_m\right)
\end{array}
\tag{54}
$$

In (54) the potentials have been formally grouped into two four-vectors, essentially having the same structure as $\mathbf{K}$, (34). Note that dimensionally $\{\mathbf{A}\} = \{\phi/c\}$ hence there exists no other alternative for grouping these terms. It therefore follows from (37) that the associated transformation formulas should be

$$
\begin{aligned}
\mathbf{A}'_{e,m} &= \widetilde{\mathbf{U}} \cdot \left(\mathbf{A}_{e,m} - \mathbf{v}\phi_{e,m}/c^2\right) \\
\phi'_{e,m} &= \gamma(\phi_{e,m} - \mathbf{v} \cdot \mathbf{A}_{e,m})
\end{aligned}
\tag{55}
$$

for the e, m indices correspondingly.

It should be emphasized that the way the four-potentials (54) are introduced is a definition, rather than a consequence. These definitions imply (55) and guarantee that Φ_e, Φ_m are indeed four-vectors. Therefore $\Phi_e \cdot \Phi_e$, $\Phi_m \cdot \Phi_m$ and $\Phi_e \cdot \Phi_m$ as well as any product with

any other four-vectors constitute new Lorentz invariants. As before, some are more interesting, others do not seem to have an immediate application. Noteworthy is the invariant

$$\mathbf{K} \cdot \mathbf{\Phi_e} = \mathbf{k} \cdot \mathbf{A}_e - \frac{\omega}{c^2}\phi_e \tag{56}$$

and the m-indexed analog. In free space $c^{-2} = \mu_o \varepsilon_o$ hence if the value of the invariant (56) is set to zero, it becomes the well-known Lorentz condition (in $\mathbf{K}$ space). However, in material media (56) ceases to be the Lorentz condition. This is a point that might cause some confusion, especially in view of the fact that the Lorentz condition is a gauge transformation invariant, as explained in many of the textbooks cited above.

8. THE CROSS MULTIPLICATION AND CURL OPERATIONS

Teachers of a first course in electromagnetic field theory at sophomore or junior level are aware of the fact that vector analysis, in particular the Curl operation, are a major stumbling block for most students. Witness the long introductory chapters or detailed appendices in most textbooks. Suddenly, after some assimilation of the new concepts took place, they are told in the context of Relativistic Electrodynamics that the Curl operation is "not really a vector operation", actually an asymmetric tensor with certain properties. In a short and condensed course it was found expedient to keep tensor analysis and the formal details to the absolutely necessary minimum. Thus we already know that two juxtaposed vectors $\mathbf{AB}$ constitute a dyadic, or a matrix with components $A_i B_j$. It is easy to see that a construct $A_i B_j - A_j B_i$ is an antisymmetric matrix. This defines the Curl operation in general, where we now have $A_i = \partial_{x_i}$. For $i, j = 1, 2, 3$, there are only three independent entries in the matrix, therefore the Curl operation in three dimensional space could be disguised as a vector operation, on the other hand in four dimensional space $i, j = 1, 2, 3, 4$, there are six independent entries, therefore there is no way that such an entity could be represented as a four-vector. This discussion is considered sufficient for a first course in applied Relativistic Electrodynamics.

There are many cases where the six equations $A_i B_j - A_j B_i = 0$, $i, j = 1, 2, 3, 4$ must be satisfied. There is no harm in symbolically writing $\mathbf{A} \times \mathbf{B} = 0$, or $\partial_{\mathbf{X}} \times \mathbf{A} = 0$, as long as we know what we are

doing. This facilitates a mental association to already known concepts, such as $\partial_{\mathbf{x}} \times \partial_{\mathbf{x}} a = 0$ where a is a scalar field. Similarly $\partial_{\mathbf{X}} \times \partial_{\mathbf{X}} a = 0$ will be understood as

$$\frac{\partial}{\partial X_i} \frac{\partial a}{\partial X_j} - \frac{\partial}{\partial X_j} \frac{\partial a}{\partial X_i} = 0, \qquad i, j = 1, 2, 3, 4 \tag{57}$$

and it is seen that for a smooth function a, such that the order of differentiation is commutative, (57) is identically satisfied. The analogy cannot be taken too far, for example the analog $\partial_{\mathbf{X}} \cdot \partial_{\mathbf{X}} \times \mathbf{A} = 0$, when $\mathbf{A}$ is a four-vector, does not exist. Simple examples $\mathbf{X} \times \mathbf{X} = 0$, $\partial_{\mathbf{X}} \times \mathbf{X} = 0$, are easily verified. We can also apply the $\partial_{\mathbf{X}} \times$ operation to Φ_e, Φ_m, to show that this yields the Maxwell's equations fields appearing in (54). The details will be left for the reader to be worked out. The operation $\partial_{\mathbf{X}} \times$ applied to a four-vector yields six independent equations. This is sometimes referred to as a six-vector [7]. For more detail and the relation to the electromagnetic tensor, see for example [5, 7]. The formal elegance of Relativistic Electrodynamics is an aspect which should be sacrificed here in order to be able to focus on some applications.

9. PROPER TIME AND RELATED CONCEPTS

In a subsequent section ray equations are considered. The concept of a ray is intimately associated with wave packets and their motion in space. For that and other purposes we have to include a short section on the concept of proper time and related concepts of velocity and acceleration. Actually it is also warranted on ground of intrinsically involving an ingenious idea due to Minkowski: The creation of new four-vectors by associating four-vectors with invariants, e.g., differentiating $\mathbf{X}$ with respect to the proper time to derive the four-velocity, as done below.

In analogy with a three-dimensional space we define the Minkowski space four-dimensional arc length dS in terms of four-vectors

$$dS = \sqrt{d\mathbf{X} \cdot d\mathbf{X}} = \sqrt{d\mathbf{x} \cdot d\mathbf{x} - c^2 (dt)^2} \tag{58}$$

and this is an invariant. Using (58) we further define an invariant having the dimensionality of time

$$d\tau = \frac{dS}{ic} = dt \sqrt{1 - \frac{d\mathbf{x} \cdot d\mathbf{x}}{c^2 (dt)^2}} = dt' \sqrt{1 - \frac{d\mathbf{x}' \cdot d\mathbf{x}'}{c^2 (dt')^2}} \tag{59}$$

We now introduce the proper time τ. In (59) we attribute $d\tau$ to the time increment of an observer co-moving with (i.e., at rest in) the primed frame of reference Γ'.

Consequently in (59) $d\mathbf{x}' = 0$, i.e., $d\tau = dt'$. The inertial frame Γ' is observed from Γ to be moving at the velocity $\mathbf{u} = d\mathbf{x}/dt$, hence

$$d\tau = dt\sqrt{1 - \frac{d\mathbf{x} \cdot d\mathbf{x}}{c^2(dt)^2}} = dt\sqrt{1 - \frac{\mathbf{u} \cdot \mathbf{u}}{c^2}} = dt/\gamma \qquad (60)$$

This is the celebrated relativistic time dilatation phenomenon, already mentioned above in connection with the twin paradox. We used $\mathbf{u}$ for the relative velocity, because $\mathbf{v}$ is used below for the velocity as a general concept.

The four-velocity is now defined as

$$\mathbf{V} = \frac{d\mathbf{X}}{d\tau} = \left(\frac{d\mathbf{x}}{d\tau}, ic\frac{dt}{d\tau}\right) = \frac{dt}{d\tau}\left(\frac{d\mathbf{x}}{dt}, ic\right) = \frac{dt}{d\tau}(\mathbf{v}, ic) = \gamma(\mathbf{v}, ic) \quad (61)$$

If $\mathbf{v}$ is a constant velocity and we consider the proper frame in which $\mathbf{v} = 0$, which also implies $\mathbf{u} = 0$, then (61) reduces to the temporal component ic, and the four-velocity is now an imaginary constant. The length of the four-velocity vector is therefore given by

$$\mathbf{V} \cdot \mathbf{V} = \mathbf{V}' \cdot \mathbf{V}' = -c^2 \qquad (62)$$

for all observers in inertial systems. By taking differentials of the Lorentz transformation (6) (using $\mathbf{u}$ for the relative velocity between frames of reference) and taking the ratio of the two equations, it is established that the relativistic transformation formula for $\mathbf{v}$ is given by

$$\mathbf{v}' = \frac{\widetilde{\mathbf{U}} \cdot (\mathbf{v} - \mathbf{u})}{\gamma(1 - \mathbf{v} \cdot \mathbf{u}/c^2)} \qquad (63)$$

We could also consider the similarity of the four-vectors $\mathbf{V} = \left(\frac{d\mathbf{x}}{d\tau}, ic\frac{dt}{d\tau}\right)$ and (2), and by inspection of (6) derive

$$\begin{aligned} \frac{d\mathbf{x}'}{d\tau} &= \mathbf{U} \cdot \left(\frac{d\mathbf{x}}{d\tau} - \mathbf{u}\frac{dt}{d\tau}\right) \\ \frac{dt'}{d\tau} &= \gamma\left(\frac{dt}{d\tau} - \frac{\mathbf{u} \cdot d\mathbf{x}}{c^2 d\tau}\right) \end{aligned} \qquad (64)$$

the same way the transformations (8), (11), (37), (39), (55) were derived by inspection. Obviously by dividing the two equations (64), (63) is obtained once more.

The process of creating such new four-vectors can be continued. We define the four-acceleration as

$$\mathbf{W} = \frac{d\mathbf{V}}{d\tau} = \left(\frac{d^2\mathbf{x}}{d\tau^2}, ic\frac{d^2t}{d\tau^2} \right) \tag{65}$$

and similarly to (64), the pertinent transformation formulas can be derived. It is an interesting result that subject to (62) the following invariant vanishes

$$\mathbf{V} \cdot \mathbf{W} = \mathbf{V} \cdot \frac{d\mathbf{V}}{d\tau} = \frac{1}{2}\frac{d|\mathbf{V}|^2}{d\tau} = -\frac{1}{2}\frac{dc^2}{d\tau} = 0 \tag{66}$$

i.e., the two four-vectors are always perpendicular, in a formal sense. In this context the phrase "relativistic acceleration is always centrifugal" is sometimes found.

It is not our intention to discuss in detail Relativistic Mechanics, because this will once again divert us from the main theme. It is however straightforward to associate with the four-velocity the momentum-energy four-vector

$$\mathbf{P} = m\mathbf{V} \tag{67}$$

where the proportionality factor m is the rest mass of a particle, measured by an observer at rest with respect to the object. The associated invariant is probably the popularly best known result of Special Relativity (e.g., see [30])

$$-c^2\mathbf{P} \cdot \mathbf{P} = \gamma^2 m^2 c^4 - \gamma^2 m^2 \nu^2 c^2 = -c^2\mathbf{P}' \cdot \mathbf{P}' = m^2 c^4 \tag{68}$$

where in (68) the primed frame of reference Γ' has been chosen as the co-moving (i.e., rest frame). The term mc^2 is now interpreted as the rest mass energy, and consistently γmc^2 is the total energy, including the effect of the motion. The difference term in (68) $\gamma^2 m^2 \nu^2 c^2 = p^2 c^2$ is associated with the momentum, i.e., the kinetic energy, where the three-dimensional momentum is given as $\mathbf{p} = \gamma m\mathbf{v}$.

Newton's law in four-vector form follows as

$$\mathbf{F} = \frac{d\mathbf{P}}{d\tau} = m\frac{d\mathbf{V}}{d\tau} = m\mathbf{W} \tag{69}$$

Now is a good time to pick up the subject of the Coulomb and Lorentz force formulas stated in (3), (4). We shall state the Lorentz force formula in four-vector form and check our stipulation:

$$\mathbf{F} = \gamma(\mathbf{f}, iq_e\mathbf{v} \cdot \mathbf{E}/c) \tag{70}$$

where $\mathbf{F}$ is the force four-vector, and $\mathbf{f}$ is given in (3). For a point charge at rest in Γ' substitute $\mathbf{v} = 0$, and apply primes, thus (70) becomes $\mathbf{F}' = (q_e\mathbf{E}', 0)$. Therefore if (70) defines a four-vector we must have

$$\mathbf{F} \cdot \mathbf{F} = \mathbf{F}' \cdot \mathbf{F}' = q_e{}^2\mathbf{E}' \cdot \mathbf{E}' \tag{71}$$

Note that the right hand side of (71) expresses the Coulomb force formula (squared), hence dimensionally we already deal with an expression describing force. Using the definition of $\mathbf{F}$ in (70), the definition of the constant for the scalar product (71) and the transformation formula for $\mathbf{E}'$ given in (10) it can be shown (a good exercise!) that (70) indeed defines a four-vector. Finally it is easy to verify that (70) satisfies $\mathbf{F} \cdot \mathbf{V} = 0$, hence it is a properly defined four-force. The relation of (3) and (4) to the respective Coulomb force formulas for $\mathbf{v} = 0$ is now clear.

10. THE BREAKDOWN OF SPECIAL RELATIVITY

The dramatic caption above is intended to draw attention. The subject of proper time in the presence of acceleration and its implications needs to be emphasized. For a discussion on this subject see Bohm [30]. The proper time has been introduced above (58)–(60), based on the assumption that $d\mathbf{X} \cdot d\mathbf{X}$ is a relativistic invariant. This holds as long as the velocity is constant, i.e., in the absence of acceleration. If this condition is not met, then we have

$$\begin{aligned}
d\mathbf{x}' &= \widetilde{\mathbf{U}} \cdot (d\mathbf{x} - \mathbf{v}dt - (d\mathbf{v})t) + d\widetilde{\mathbf{U}} \cdot (\mathbf{x} - \mathbf{v}t) \\
dt' &= \gamma\left(dt - \mathbf{v} \cdot d\mathbf{x}/c^2 - (d\mathbf{v}) \cdot \mathbf{x}/c^2\right) + (d\gamma)(t - \mathbf{v} \cdot \mathbf{x}/c^2)
\end{aligned} \tag{72}$$

i.e., the differentials of the terms involving the velocity must be taken into account too. Consequently $d\mathbf{X} \cdot d\mathbf{X}$ ceases to be a Lorentz or relativistic invariant, strictly speaking. If we insist on (60) to still be valid, there is involved a drastic *heuristic* assumption that we are allowed to replace an accelerated frame of reference with a sequence of

instantaneous inertial systems. It is the similar situation encountered in a movie or computer animation, whereby real motion is replaced by a sequence of "frozen" frames, each slightly different from the other. Here the acceleration is simulated by a transition from one inertial system to another, with a gradually changing relative velocity. Obviously, this was not included in the fundamental model of Special Relativity. The usual *verbal* argument justifying the instantaneous inertial frame concept is that during a short time interval dt the incremental $d\mathbf{v}$ is small, i.e., the acceleration is small, and therefore the effect of the acceleration on the proper clock carried along by the accelerated frame of reference will be negligible. In other words, the behavior of the clock will be according to Newtonian physics, whereby time measurement is absolute and not affected by motion.

The problem has immediate repercussions regarding the four-velocity (61) and its consequences. If the velocity $\mathbf{v}$ is not a constant, then the differentiation in (65) must take this into account. I.e., if one accepts the form $\mathbf{V} = \gamma(\mathbf{v}, ic)$ then in deriving the acceleration (65) it must be understood that γ is not a constant.

11. THE MINKOWSKI CONSTITUTIVE RELATIONS

Sommerfeld [7] discusses the Minkowski constitutive relations for moving media. The question is an old one, and can be asked in various ways. If you ask "how does a moving medium behave, for example, does it appear to be a different medium with different constitutive parameters?", then the answer to the question is given in terms of the transformation formulas for the constitutive parameters. This has been amply discussed in the literature, e.g., see Post [31], see also Hebenstreit [32, 33], and Hebenstreit and Suchy [34], but we adopt here the Minkowski methodology. Accordingly, the above manner of asking the question does not contribute to any problem of application-oriented Relativistic Electrodynamics. The question should be put in the way Minkowski asked it: What are the relations between the fields in a moving medium, given the properties of the medium in the co-moving (rest) frame of reference. This methodology is also adopted by Kong [18], presenting a general discussion of various bianisotropic media, and also cites previous studies. The Minkowskian methodology is carried one step further by realizing that it is not even necessary to derive explicit expressions for the constitutive relations — it suffices to derive a determinate system of equations and unknowns [2, 35]. Sommerfeld

[7] considers the simple case of a medium which is linear, isotropic, nondispersive, and homogeneous, in its rest frame, i.e., the co-moving frame of reference. The treatment is not much more complicated when anisotropic dispersive media are assumed. A bonus of this approach is that we can now mention, through the subject of dispersive systems, the problem of generally non-local and non-instantaneous processes, and its relation to the light cone and causality. More on that will be given below in the section on differential operator constitutive relations.

In the co-moving frame of reference Γ', in the Fourier transform representation space the constitutive relations

$$
\begin{aligned}
\mathbf{D}'(\mathbf{K}') &= \widetilde{\varepsilon}(\mathbf{K}') \cdot \mathbf{E}'(\mathbf{K}') \\
\mathbf{B}'(\mathbf{K}') &= \widetilde{\mu}(\mathbf{K}') \cdot \mathbf{H}'(\mathbf{K}')
\end{aligned}
\tag{73}
$$

are assumed to hold, where the constitutive parameters here are dyadics (or call them matrices, or second rank tensors). The frequency dependent dispersive medium is very common and familiar, e.g., $\mathbf{D}'(\omega') = \widetilde{\varepsilon}(\omega') \cdot \mathbf{E}'(\omega')$, pertinent to the dielectric medium at rest within a capacitor, say. See for example Jackson [11]. It follows that in the time domain the constitutive relation becomes the convolution integral

$$
\mathbf{D}'(t') = \int_{-\infty}^{t'} d\tau' \widetilde{\varepsilon}(\tau') \cdot \mathbf{E}'(t' - \tau')
\tag{74}
$$

where the upper limit is taken as t' in order to have effects at time t' only from retarded (previous) causes occurring before t'. In view of (74), the ω' dependent case is termed temporal dispersion. It provides an example for processes observed at time t', caused by effects initiated previously, i.e., not simultaneously. This is a simple but important case, it has nothing intrinsic to do with relativistic considerations. However, the introduction of a general dependence on $\mathbf{K}$, (34), ties the problem of causality to Special Relativity. Thus in $\mathbf{X}$ space the first line (73) becomes a four-dimensional integral

$$
\mathbf{D}'(\mathbf{X}') = \int_{\Xi_1}^{\Xi_2} (d^4 \Xi') \widetilde{\varepsilon}(\Xi') \cdot (\mathbf{E}' - \Xi')
\tag{75}
$$

0 where $\Xi' = (\boldsymbol{\xi}', ic\tau')$ denotes the integration variables. The choice of the integration limits in (75) is subject to (14), but in addition we must

ensure that the effect on the present is due to past only, i.e., only the part of the light cone satisfying $\tau' \leq t'$ is scanned in the integration (75).

The **K** space field transformation formulas, i.e., (10) with the argument changed according to (44) are now substituted into (73), and both sides are premultiplied by $\widetilde{\mathbf{V}}^{-1} = \widetilde{\mathbf{U}}/\gamma$ yielding

$$
\begin{aligned}
\mathbf{D} + \mathbf{v} \times \mathbf{H}/c^2 &= \widetilde{\boldsymbol{\varepsilon}}_{\mathbf{v}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
\mathbf{B} - \mathbf{v} \times \mathbf{H}/c^2 &= \widetilde{\boldsymbol{\mu}}_{\mathbf{v}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \\
\widetilde{\boldsymbol{\varepsilon}}_{\mathbf{v}} = \widetilde{\mathbf{V}}^{-1} \cdot \widetilde{\boldsymbol{\varepsilon}} \cdot \widetilde{\mathbf{V}}, &\quad \widetilde{\boldsymbol{\mu}}_{\mathbf{v}} = \widetilde{\mathbf{V}}^{-1} \cdot \widetilde{\boldsymbol{\mu}} \cdot \widetilde{\mathbf{V}}
\end{aligned} \tag{76}
$$

where $\mathbf{D} = \mathbf{D}(\mathbf{K})$ etc. One could stop here and call (76) the appropriate Minkowski constitutive relations. Together with (1) they now constitute a determinate system of equations in the Γ frame of reference. However, in the present case it is possible to get explicit expressions for $\mathbf{D}$ and $\mathbf{B}$. Multiply the second line of (76) by $\mathbf{v} \times$ and substitute $\mathbf{v} \times \mathbf{B}$ into the first line. After some additional manipulation, we obtain the Minkowski constitutive relations for the present case

$$
\begin{aligned}
\mathbf{D} &= \left[\widetilde{\mathbf{I}} + \widetilde{\boldsymbol{\varepsilon}}_{\mathbf{v}} \cdot \mathbf{v} \times \widetilde{\boldsymbol{\mu}}_{\mathbf{v}} \cdot \mathbf{v} \times \widetilde{\mathbf{I}} \right]^{-1} \\
&\quad \cdot \left[\widetilde{\boldsymbol{\varepsilon}}_{\mathbf{v}} \cdot \left(\widetilde{\mathbf{I}} + \frac{\mathbf{v} \times \mathbf{v} \times \widetilde{\mathbf{I}}}{c^2} \right) \cdot \mathbf{E} + \left(\widetilde{\boldsymbol{\varepsilon}}_{\mathbf{v}} \cdot \mathbf{v} \times \widetilde{\boldsymbol{\mu}}_{\mathbf{v}} - \frac{\mathbf{v} \times \widetilde{\mathbf{I}}}{c^2} \right) \cdot \mathbf{H} \right] \\
\mathbf{B} &= \left[\widetilde{\mathbf{I}} + \widetilde{\boldsymbol{\mu}}_{\mathbf{v}} \cdot \mathbf{v} \times \widetilde{\boldsymbol{\varepsilon}}_{\mathbf{v}} \cdot \mathbf{v} \times \widetilde{\mathbf{I}} \right]^{-1} \\
&\quad \cdot \left[\widetilde{\boldsymbol{\mu}}_{\mathbf{v}} \cdot \left(\widetilde{\mathbf{I}} + \frac{\mathbf{v} \times \mathbf{v} \times \widetilde{\mathbf{I}}}{c^2} \right) \cdot \mathbf{H} + \left(\widetilde{\boldsymbol{\mu}}_{\mathbf{v}} \cdot \mathbf{v} \times \widetilde{\boldsymbol{\varepsilon}}_{\mathbf{v}} - \frac{\mathbf{v} \times \widetilde{\mathbf{I}}}{c^2} \right) \cdot \mathbf{E} \right]
\end{aligned} \tag{77}
$$

The result (77) reduces to the simple form given by Sommerfeld [7], for example. For special cases of bianisotropic media in motion see Kong [18]. In conclusion it is noted that the present discussion is based on the existence of (73) and the validity of (10) only. This remark is important for the case where one attempts to incorporate losses into the definition of the constitutive parameters, in that case (73) will have to be augmented by constitutive relations relating the current densities and the fields *via* conductivity parameters.

12. DISPERSION EQUATIONS IN MOVING MEDIA

The dispersion equation concept is central to wave propagation in general, and especially in connection with ray propagation in dispersive media, discussed subsequently. It is therefore essential for engineers and applied physicists to cover this subject. In the present context of Relativistic Electrodynamics the question of the dispersion equation in various inertial frames of reference is discussed [29]. An interesting aspect is added below, in tackling the question of tracing rays in a moving medium.

Consider Maxwell's equations in the co-moving frame Γ', given by (41), and for the case of vanishing charge densities, i.e., $\rho'_e = 0$, $\rho'_m = 0$ within the region of interest, and substitute the constitutive relations from (73). Furthermore, "Ohm's law" is assumed, i.e., the currents are not source currents prescribed as constraints, but depend on the fields in the form

$$
\begin{aligned}
\mathbf{j}'_e &= \widetilde{\boldsymbol{\sigma}}_e \cdot \mathbf{E}' \\
\mathbf{j}'_m &= \widetilde{\boldsymbol{\sigma}}_m \cdot \mathbf{H}'
\end{aligned}
\tag{78}
$$

and are also substituted into (41). Consequently it is possible to define new parameters and rewrite (41) in the form

$$
\begin{aligned}
\mathbf{k}' \times \mathbf{E}' - \omega' \widetilde{\boldsymbol{\mu}}^{\dagger} \cdot \mathbf{H}' &= 0 \\
\mathbf{k}' \times \mathbf{H}' + \omega' \widetilde{\boldsymbol{\mu}}^{\dagger} \cdot \mathbf{E}' &= 0 \\
\mathbf{k}' \cdot \mathbf{D}' &= 0 \\
\mathbf{k}' \cdot \mathbf{B}' &= 0
\end{aligned}
\tag{79}
$$

The last two equations merely state that $\mathbf{D}'$ and $\mathbf{B}'$ are perpendicular to $\mathbf{k}'$. The first two equations in (79) and their solution provides wave modes which are of interest. Mathematically they provide a system of six scalar homogeneous equations, for which the condition for nontrivial solutions is that the determinant of the system must vanish. This condition prescribes a scalar relation between ω' and $\mathbf{k}'$, the so called dispersion equation, which can be written in the form

$$
F'(\mathbf{K}') = \mathbf{0}
\tag{80}
$$

Inasmuch as (80) is a scalar, it is very suggestive to assume that the mere substitution of (37) to obtain

$$
F'(\mathbf{K}'[\mathbf{K}]) = \bar{F}'(\mathbf{K}) = 0
\tag{81}
$$

provides the dispersion equation for the unprimed frame of reference Γ. What we have done in the transition from (80) to (81)is merely to express F' in terms of the Γ frame coordinates $\mathbf{K}$. This does not imply that $\bar{F}' = 0$ is the dispersion equation measurable by an observer in Γ. The confusion is compounded by the fact that indeed

$$F(\mathbf{K}) = \bar{F}'(\mathbf{K}) = 0 \tag{82}$$

is Lorentz invariant and is the dispersion equation in Γ, but this must be shown!

One must start with the first two vector equations of (79). The first can be rewritten as

$$\mathbf{H}' = \frac{1}{\omega'}\widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \mathbf{E}' \tag{83}$$

and substituted into the second, yields

$$\left(\mathbf{k}' \times \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\varepsilon}}^{\dagger}\right) \cdot \mathbf{E}' = 0 \tag{84}$$

Or, isolating $\mathbf{E}'$ first, we obtain

$$\mathbf{E}' = -\frac{1}{\omega'}\widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot \mathbf{k}' \times H' \tag{85}$$

and substituting into the first equation in (79), yields

$$\left(\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\mu}}^{\dagger}\right) \cdot \mathbf{H}' = 0 \tag{86}$$

In the primed reference frame Γ' the dispersion equations are therefore

$$\det\left[\mathbf{k}' \times \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\varepsilon}}^{\dagger}\right] = 0$$
$$\det\left[\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\mu}}^{\dagger}\right] = 0 \tag{87}$$

It is easy to show that the two conditions are identical (as they should, because for a given wave mode there exists only one dispersion equation governing both the $\mathbf{E}'$ and $\mathbf{H}'$ fields). Multiplying (84) from the left by $\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot$ and using the rule that in a product of matrices, the product of determinants is equal to the determinant of the product yields,

$$\det\left[\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}}\right] \det\left[\mathbf{k}' \times \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\varepsilon}}^{\dagger}\right]$$
$$= \det\left[\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} + \omega'^2\widetilde{\mathbf{I}}\right] \det\left[\mathbf{k}' \times \widetilde{\mathbf{I}}\right] = 0 \quad (88)$$

and because in (88) $\det\left[\mathbf{k}' \times \widetilde{\mathbf{I}}\right] \neq 0$, we have

$$\det\left[\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} + \omega'^2\widetilde{\mathbf{I}}\right] = 0 \qquad (89)$$

This is manipulated to yield

$$\det\left[\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} + \omega'^2\widetilde{\boldsymbol{\mu}}^{\dagger} \cdot \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}}\right]$$
$$= \det\left[\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\mu}}^{\dagger}\right]\det\left[\widetilde{\boldsymbol{\mu}}^{\dagger^{-1}}\right] = 0 \qquad (90)$$

and since it is assumed that $\det\left[\widetilde{\boldsymbol{\mu}}^{\dagger^{-1}}\right] \neq 0$, we obtain the second representation (87).

We are now ready to explore the question of the corresponding dispersion equations for an observer attached to the unprimed frame of reference Γ. Consider first the case where there are no magnetic currents, $\mathbf{j}_m = 0$. For this case we substitute from (10) into (84) and use the fact that in Γ we have $\mathbf{k} \times \mathbf{E} = \omega\mathbf{B}$, obtaining

$$\left[\left(\mathbf{k}' \times \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\varepsilon}}^{\dagger}\right) \cdot \widetilde{\mathbf{V}} \cdot \left(\widetilde{\mathbf{I}} + \mathbf{v} \times \mathbf{k} \times \widetilde{\mathbf{I}}/\omega\right)\right] \cdot \mathbf{E} = 0 \quad (91)$$

where the vanishing of the determinant of the dyadic in (91) constitutes the dispersion equation in Γ. However, $\det\left[\widetilde{\mathbf{V}} \cdot (\widetilde{\mathbf{I}} + \mathbf{v} \times \mathbf{k}\times \widetilde{\mathbf{I}}/\omega)\right] \neq 0$, hence the dispersion equation is again given by (87). Thus (82) is established. The process can be retraced for the analog case $\mathbf{j}_e = 0$, or when both e and m type current densities vanish. But if neither $\mathbf{k} \times \mathbf{E} = \omega\mathbf{B}$ or $\mathbf{k} \times \mathbf{H} = -\omega\mathbf{D}$ can be assumed, then the best we can say is that (84), (85) and (10) prescribe

$$\left(\mathbf{k}' \times \widetilde{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\varepsilon}}^{\dagger}\right) \cdot \left[\widetilde{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B})\right] = 0$$
$$\left(\mathbf{k}' \times \widetilde{\boldsymbol{\varepsilon}}^{\dagger^{-1}} \cdot \mathbf{k}' \times \widetilde{\mathbf{I}} + \omega'^2\widetilde{\boldsymbol{\mu}}^{\dagger}\right) \cdot \left[\widetilde{\mathbf{V}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D})\right] = 0 \qquad (92)$$

therefore (87) is satisfied if the determinants of the matrices in brackets in (92) are nonvanishing.

The question of modes and velocity dependent modes have been discussed in [29]. For ω' in Γ' taken as a constant, the roots of the

dispersion equation define wave modes. These are the modes observed in the frame of reference Γ'. According to (81), (82), the dispersion equation is an invariant. By substitution of the Doppler effect (37) into the dispersion equation, the dispersion equation in terms of **K** space variables is obtained. Choosing a constant ω in the unprimed frame Γ, new roots are obtained. Inasmuch as (37) transforms **k** components into ω and vice-versa, in general the value of the roots and their number differ from those encountered in Γ'. This means that new velocity induced wave modes are created. The discussion of the various pertinent modes is a complicated matter which will not be covered here (and is not recommended for the syllabus of a course based on the present article). See for example Chawla and Unz [36].

13. APPLICATION TO HAMILTONIAN RAY PROPAGATION

The use of Minkowski's four-vectors, whether we are discussing a relativistic problem or a problem posed in a single frame of reference, facilitates compact notation, and will be used extensively below. The subject of ray propagation in dispersive media is important for applied physicists and electromagnetic radiation engineers. It serves to compute field problems in dispersive inhomogeneous time-varying media, e.g., problems involving magnetized plasma appearing in connection with ionospheric radio wave propagation. Usually the computation of rays is presented in the literature as a consequence of the celebrated Fermat principle, which is mathematically stated in terms of a variational principle. Usually the problem is considered in space, but not in time. See for example Kelso [37], Van Bladel [38], Ghatak [39], Sommerfeld [40]. Here the full spatiotemporal theory is presented, allowing temporal variations as well. Initially the subject is presented here in a simplified, although concise manner, which obviates the necessity of introducing the Fermat principle as a variational principle. This was found as a pedagogically preferable approach for the author's students. The Fermat principle (discussed here in the following section), is then presented when the student is already familiar with the Hamilton ray equations and possesses a basis for comparison. Ray propagation also serves here as an example for using four-vectors, for extending the **K** space beyond the Fourier transform in the sense of the eikonal approximation, and it clarifies the role of the group velocity in ray theory.

In order to introduce the subject and relevant concepts, we start with the transition from general wave functions to wave packets in homogeneous media. This development is an extension of Stratton's [5] one-dimensional argument. Consider an arbitrary function as in (33). In order for this function to be a solution of a wave system (e.g., Maxwell's equations rendered determinate by supplementing them by constitutive relations), it must satisfy the pertinent dispersion equation $F(\mathbf{K}) = 0$. This can be built into (33) by rewriting it in the form

$$f(\mathbf{X}) = q \int (d^4\mathbf{K})\delta(F)f(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}} = q \int (d^3\mathbf{k})g(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x} - i\Omega(\mathbf{k})t},$$
$$g(\mathbf{k}) = f(\mathbf{f}, \mathbf{\Omega}(\mathbf{k})) \tag{93}$$

where δ denotes the Dirac impulse function which is zero for all values of the argument except $\delta(0)$, where it becomes singular, and $F(\mathbf{K}) = \omega - \mathbf{\Omega}(\mathbf{k}) = \mathbf{0}$ is the dispersion equation which, provided we can solve for ω, can be written as $\omega = \Omega(\mathbf{k})$. Thus the four-dimensional integral collapses into a three-dimensional integral, and of course we lose the identity of $f(\mathbf{X})$ as a four-dimensional Fourier transform integral. The closest we can approach a Fourier inverse transformation is to perceive t as a parameter and write

$$g(\mathbf{k})e^{-i\Omega(\mathbf{k})t} = \int (d^3\mathbf{x})g(\mathbf{x}, t)e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{94}$$

Inasmuch as t is a parameter, (94) is valid for any value of t. Usually we will find little use for (94), but the mathematical result is interesting. See also [41].

Equation (93) is a general wave function for the wave system in question. The transition to a wave packet is facilitated by considering a narrow-band spectrum in $\mathbf{k}$, such that only the leading terms in the following Taylor expansion need to be retained:

$$\Omega(\mathbf{k}) = \Omega(\mathbf{k}_0) + \partial_\mathbf{k}\Omega(\mathbf{K})|_{\mathbf{k}=\mathbf{k}_0} \cdot (\mathbf{k} - \mathbf{k}_0) = \omega_0 + \mathbf{v}_g \cdot (\mathbf{k} - \mathbf{k}_0) \tag{95}$$

where $\mathbf{k}_0$ is the narrow-band's central value of the spectrum in $\mathbf{k}$, the vector derivative symbolizes the gradient operation in the representation space $\mathbf{k}$, and $\mathbf{v}_g$ will be identified below as the group velocity. Substituting (95) into (93) yields after some manipulation

$$f(\mathbf{X}) = e^{i\mathbf{K}_0\cdot\mathbf{X}}q \int (d^3\mathbf{k})g(\mathbf{k})e^{i(\mathbf{k}-\mathbf{k}_0)\cdot(\mathbf{x}-\mathbf{v}_g t)} \tag{96}$$

which is interpreted as a wave packet consisting of a carrier wave times an envelope (or modulation), the latter is a constant on the trajectory $\mathbf{x} - \mathbf{v}_g t = \text{constant}$, defining the group velocity $\mathbf{v}_g = d\mathbf{x}/dt$. Apparently (93)–(96) are easier to handle in terms of the three-velocity $\mathbf{v}_g$. However, just as an exercise, let us see that the same can be handled in four-vector notation too. Thus instead of (95) we write

$$F(\mathbf{K}) = F(\mathbf{K}_0) + \frac{\partial F}{\partial \mathbf{K}_0} \cdot d\mathbf{K} = F(\mathbf{K}_0) + \frac{\partial F}{\partial \mathbf{K}_0} \cdot (\mathbf{K} - \mathbf{K}_0) = 0 \quad (97)$$

where the differentiation with respect to $\mathbf{K}_0$ means that this value is substituted into the derivative after differentiation. Inasmuch as $F(\mathbf{K}_0) = 0$ too, we conclude that within the approximation where (97) holds the term involving the derivative vanishes too. Adding this vanishing term in the exponent in (93) yields

$$\begin{aligned} f(\mathbf{X}) &= q \int (d^4\mathbf{K})\delta(F)f(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}} \\ &= e^{i\mathbf{K}_0\cdot\mathbf{X}}q \int (d^4\mathbf{K})\delta(F)f(\mathbf{K})e^{i(\mathbf{K}-\mathbf{K}_0)\cdot\left(\mathbf{X}+\alpha\frac{\partial F}{\partial \mathbf{K}_0}\right)} \end{aligned} \quad (98)$$

where α is an arbitrary Lagrange multiplier constant. Once again (98) displays the wave packet structure of a carrier wave multiplying the envelope function, and the envelope is constant on a trajectory defined by

$$\mathbf{X} + \alpha \partial_{\mathbf{K}_0} F = \mathbf{\Xi} \quad (99)$$

where $\mathbf{\Xi}$ is a constant four-vector, and therefore the origin of the coordinate system can be redefined, and $\mathbf{\Xi}$ absorbed into a new $\mathbf{X}$. This amounts to taking $\mathbf{\Xi} = 0$ in (99). Expressing (99) in three-dimensional components yields

$$\frac{\mathbf{x}}{t} = -\frac{\partial_{\mathbf{k}_0} F}{\partial_{\omega_0} F} \quad (100)$$

and from (97) we have

$$\begin{aligned} dF &= \frac{\partial F}{\partial \mathbf{K}} \cdot d\mathbf{k} = \left(\frac{\partial F}{\partial \mathbf{k}} + \frac{\partial F}{\partial \omega}\frac{\partial \Omega}{\partial \mathbf{k}}\right) \cdot d\mathbf{k} = 0 \\ \mathbf{v}_g &= \partial_{\mathbf{k}}\Omega = -\frac{\partial_{\mathbf{k}} F}{\partial_\omega F} \end{aligned} \quad (101)$$

therefore in (100) the group velocity appears once again. In view of the homogeneous medium, (100) applies to $\mathbf{X}$ components as well as incremental $d\mathbf{X}$ components, therefore $\mathbf{v}_g = d\mathbf{x}/dt$ is obtained again. Note the minus sign in (101), which would be missing if one (erroneously) treats a partial derivative as a ratio of differentials. It appears that in this case the four-vector treatment is somewhat more cumbersome, although still feasible.

The definition of wave packets in inhomogeneous, time dependent media is impossible within the context of the Fourier transformation. However, for "slow variation" such that the variation of the properties of the medium over distances and time intervals commensurate with the wavelength and the period of the signal, respectively, an approximate procedure can be defined. This is usually referred to as working in the high frequency limit. Clearly spatial and temporal changes in the constitutive parameters do not fit into our formalism for homogeneous media. Such changes cannot be included in the constitutive relations stipulated for the co-moving frame, e.g., see (73) or in the corresponding equations for the laboratory frame of reference Γ, (76), (77), because they are not consistent with a Fourier transform representation. Consequently the dispersion equations (87) are invalid too. Nor is a representation of a wave function in terms of a superposition of plane waves, as in (93) a legitimate solution. In order to overcome this difficulty we introduce the so called eikonal approximation (this is usually called in the mathematicians jargon "the WKB approximation", or "method of characteristics"). For further explanation and previous literature citations see for example Censor [42], and Molcho and Censor [43]. In time-invariant, homogeneous media the basic solution is a plane wave $Ae^{i\theta(\mathbf{X})}$, $\theta(\mathbf{X}) = \mathbf{K} \cdot \mathbf{X}$ where the amplitude A is a constant. In slowly varying spatiotemporally varying media the fundamental solution is chosen as

$$A(\mathbf{X})e^{i\theta(\mathbf{X})}, \partial_{\mathbf{x}}\theta(\mathbf{X}) = \mathbf{K} \qquad (102)$$

Therefore $\mathbf{K}$ is obtained as the four-gradient of the phase, as in the simple case, but not through the Fourier transform. This is the eikonal approximation. The existence and the representation of the new function θ is at this point an open question and will be discussed shortly. The idea of slow variation is mathematically stated by assuming that derivatives of the amplitude in (102) are negligibly small compared to

the derivatives of the exponential, e.g.,

$$\partial_t \left(A(\mathbf{X}) e^{i\theta(\mathbf{X})} \right) = (\partial_t A(\mathbf{X})) \, e^{i\theta(\mathbf{X})} - i\omega A(\mathbf{X}) e^{i\theta(\mathbf{X})} \approx -i\omega A(\mathbf{X}) e^{i\theta(\mathbf{X})} \tag{103}$$

i.e., $|\left(\partial_t A(\mathbf{X})\right)/A(\mathbf{X})| \ll |i\omega|$, and similarly for the spatial components $\mathbf{x}$. Therefore the eikonal approximation has the same property as the Fourier transformation in (36), namely that the differential operation $\partial_{\mathbf{X}}$ is equivalent to algebraization, by producing a factor $i\mathbf{K}$.

The simplest way to introduce the representation of θ, which is also very appealing to students familiar with electrostatics, is the following: The four-gradient operation in (102) is reminiscent of the way the electrostatic potential $\mathbf{E} = -\partial_{\mathbf{x}} \phi(\mathbf{x})$ was derived. Writing

$$\phi(\mathbf{x}) = \int_{\phi(\mathbf{x}_0)}^{\phi(\mathbf{x})} d\phi, \qquad \phi(\mathbf{x}_0) = 0 \tag{104}$$

we chose the lower limit, the so called reference potential as zero, and the integral depends on the limits only, hence in the mathematician's language $d\phi$ is a total or exact differential. Using the chain rule of calculus we write $d\phi = \partial_{\mathbf{x}} \phi(\mathbf{x}) = -\mathbf{E} \cdot d\mathbf{x}$ and (104) becomes

$$\phi(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\boldsymbol{\xi}) \cdot d\boldsymbol{\xi} \tag{105}$$

where $\boldsymbol{\xi}$, denotes the integration (dummy) variable, but henceforth we shall write $\mathbf{x}$ also under the integration symbol, except in cases where confusion might arise. Recall that $\mathbf{E}$ was dubbed as a conservative field which satisfies $\nabla \times \mathbf{E}(\mathbf{x}) = \partial_{\mathbf{x}} \times \mathbf{E}(\mathbf{x}) = 0$. The last condition amounts to $\frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_i}$ i.e., It is a statement on the smoothness of the function ϕ, permitting to exchanging the order of differentiation. Applying all this to ray theory, we now use the four-dimensional analogs and write

$$\theta(\mathbf{X}) = \int_{\theta(\mathbf{X}_0)}^{\theta(\mathbf{X})} d\theta = \int_{\mathbf{x}_0}^{\mathbf{x}} \partial_{\mathbf{x}} \theta(\mathbf{X}) \cdot d\mathbf{X} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{K}(\mathbf{X}) \cdot d\mathbf{X} \tag{106}$$

where the reference phase is chosen as zero. The last two expressions in (106) are line integrals in four-space. By inspection of (57), it is clear that

$$\partial_{\mathbf{X}} \times \partial_{\mathbf{X}} \theta(\mathbf{X}) = \partial_{\mathbf{X}} \times \mathbf{K} = 0 \tag{107}$$

which can also be written in terms of three-vectors as

$$\partial_{\mathbf{x}} \times \mathbf{k}(\mathbf{X}) = 0$$
$$\partial_t \mathbf{k}(\mathbf{X}) + \partial_{\mathbf{x}} \omega(\mathbf{X}) = 0 \tag{108}$$

see Poeverlein [44]. The first line (108) is Snell's law in disguise and is referred to as the Sommerfeld-Runge law of refraction. Recall that in electrostatics $\nabla \times \mathbf{E}(\mathbf{x}) = \partial_{\mathbf{x}} \times \mathbf{E}(\mathbf{x}) = 0$ implied the continuity of the tangential component of $\mathbf{E}$ at the interface between media with different constitutive parameters. In analogy, the first line (108) prescribes the continuity of the tangential component of $\mathbf{k}$ at the interface between media with different constitutive parameters. But exactly this is what Snell's law states! Consequently we now know that in general Snell's law holds in time dependent systems as well.

Using the eikonal approximation in the $\mathbf{X}$ space Maxwell equations, (1), and including slowly varying constitutive relations

$$\mathbf{D}(\mathbf{K}, \mathbf{X}) = \widetilde{\boldsymbol{\varepsilon}}(\mathbf{K}, \mathbf{X}) \cdot \mathbf{E}(\mathbf{K}, \mathbf{X})$$
$$\mathbf{B}(\mathbf{K}, \mathbf{X}) = \widetilde{\boldsymbol{\mu}}(\mathbf{K}, \mathbf{X}) \cdot \mathbf{H}(\mathbf{K}, \mathbf{X}) \tag{109}$$

where (109) assumes that $\mathbf{X}$ space is the co-moving frame, i.e., the frame where the medium is at rest, otherwise instead of (109) we could use the corresponding Minkowski constitutive relations (77), in which it is assumed that $\mathbf{X}'$ space is the co-moving frame. We are led to a space and time dispersion equation

$$F(\mathbf{K}, \mathbf{X}) = 0 \tag{110}$$

which can also be written as

$$F(\partial_{\mathbf{X}}\theta(\mathbf{X}), \mathbf{X}) = 0 \tag{111}$$

The last form is a differential equation on θ, referred to as the eikonal differential equation. It is usually nonlinear and difficult to solve. The idea of deriving ray equations is to replace (110), hence also (111) by a set of coupled first order ordinary differential equations (this is usually called in the mathematician's jargon "the method of characteristics" and the electrical engineers sometimes refer to "state space equations"). In the next section it is shown how to derive the ray equations using the generalized Fermat principle due to Synge [45].

However, it must be realized that electrical engineering and applied physics students, even if they have been exposed to variational analysis, say if they had a course in analytical mechanics, can hardly cope with the subject. It was found advantageous to obviate this approach by using the following methodology. To satisfy (110) it suffices to satisfy $dF = 0$, which implies $F = \text{constant}$, and provided this constant is set to zero at least for one set of values of $\mathbf{K}$, $\mathbf{X}$, we have $F = 0$ everywhere and always. The last condition is taken care of by the initial and boundary conditions, so all we have to worry about is the solution of $dF = 0$. Choosing a real monotonous parameter r (not necessarily the proper time), we now write

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \mathbf{K}} \cdot \frac{d\mathbf{K}}{d\tau} + \frac{\partial F}{\partial \mathbf{X}} \cdot \frac{d\mathbf{X}}{d\tau} = 0 \qquad (112)$$

for which we "guess" a solution

$$\begin{aligned}
\frac{d\mathbf{X}}{d\tau} &= \lambda(\tau)\frac{\partial F}{\partial \mathbf{K}} \\
\frac{d\mathbf{K}}{d\tau} &= -\lambda(\tau)\frac{\partial F}{\partial \mathbf{X}}
\end{aligned} \qquad (113)$$

which is easily verified by substitution into (112). In (113) $\lambda(\tau)$ is an arbitrary Lagrange multiplier function. The role of the various dependent and independent variables in (113) must be amplified. A solution of (113) (if we know how to solve it, e.g., using the Runge-Kutta numerical method) yields a spatiotemporal trajectory $\mathbf{X}(\tau)$. The field $\mathbf{K}(\mathbf{X})$ is found as $\mathbf{K}(\mathbf{X}(r))$ on this trajectory. Note that we have defined $\mathbf{X}(\tau)$, i.e., $\mathbf{X}$ as a function of τ, but not τ as a function of $\mathbf{X}$, i.e., τ is the independent variable here. If a sufficiently dense pattern of rays is computed in a certain region, then in principle we have, at our disposal the field $\mathbf{K}(\mathbf{X})$ everywhere in this region. Inasmuch as the integration (106) is independent of the specific path of integration, the phase θ can be computed according to the definition (106), whether we integrate along a specific ray path or use an arbitrary integration path. Note that ray theory in its simplest form enables us to compute the phase, or wave fronts, but is mute as to the amplitude and the polarization of the wave. The intensity (absolute value of the amplitude) can sometimes be heuristically determined by applying energy flux considerations to ray tubes. Information regarding polarization is almost always lost in a ray computation procedure.

Obviously (113) satisfies (112), hence subject to initial condition also (110), (111) are satisfied. However, since (113) was based on a guess, its uniqueness is not guaranteed. As an example for a different choice see the subsequent discussion on ray propagation in lossy media.

What makes the choice (113) special is the fact that it also satisfies the uniqueness conditions expressed as (107), (108). Thus applying the $\partial_{\mathbf{X}}\times$ operation to the second equation of (113) leads to

$$d_\tau \partial_{\mathbf{X}} \times \mathbf{K} = -\lambda(\tau)\partial_{\mathbf{X}} \times \partial_{\mathbf{X}}F = 0 \tag{114}$$

and consequently $\partial_{\mathbf{X}} \times \mathbf{K} = \text{constant}$ along the path. Implementing the argumentation already used above, we say that if $\partial_{\mathbf{X}} \times \mathbf{K} = 0$ is satisfied at the initial point of the ray, say, then it is everywhere and always satisfied. In performing the operations indicated in (114) it is assumed that we have at our disposal a ray and also neighboring rays in its vicinity, otherwise the $\partial_{\mathbf{X}}$ operations cannot be performed. Also it is noted that operations in (114) do not involve τ. We conclude that the set of equations (113) uniquely determines the phase, and therefore can be considered as equivalent to a direct solution of the eikonal equation (111). In performing the partial differentiations indicated in (113) it is assumed that (110) and therefore also derivatives of it are available as algebraic expressions in terms of $\mathbf{K}$, $\mathbf{X}$. In this context $\mathbf{K}$, $\mathbf{X}$ are considered as independent variables. The field $\mathbf{K}(\mathbf{X})$ is only available as a solution of the ray equations (113): After a certain region is sufficiently densely covered with rays, yielding $\mathbf{K}(\tau)$, $\mathbf{X}(\tau)$, the mapping out of $\mathbf{K}(\mathbf{X})$ can be performed. Consequently expressions for $\partial_{\mathbf{X}} \times \mathbf{K}$ can only be derived after solving (113) and mapping $\mathbf{K}(\mathbf{X})$ sufficiently densely in a certain region.

At this point it is advantageous to identify τ as the proper time, which is a relativistic invariant and therefore serves to preserve the four-vector nature of $d\mathbf{X}/d\tau$, $d\mathbf{K}/d\tau$ in (113). Moreover, this defines $d\mathbf{X}/d\tau$ as a four-velocity as in (61) and the associated $d\mathbf{x}/dt$ as a conventional three-dimensional velocity which transforms from one reference frame to another according to the special relativistic formula for the transformation of velocity (63). Special Relativity now enters upon realizing that we have at our disposal dispersion equations in both co-moving and other inertial frames of reference. This means that we know how to trace rays in moving media. Geometrical notions as to how will ray trajectories appear to an observer in another frame of reference are not a valid method, because rays are the loci of

lines whose tangent indicates the direction of the group velocity. It is the latter that must be computed in every instance. Identifying τ in (113) as the proper time implies that $d\mathbf{X}/d\tau$ in (113) is now a velocity four-vector, for which (62) applies, hence

$$\lambda(\tau) = ic / \sqrt{\frac{\partial F}{\partial \mathbf{K}} \cdot \frac{\partial F}{\partial \mathbf{K}}} \tag{115}$$

The result (115) appears rather puzzling at a first glance: On one hand we announced that $\lambda(\tau)$ is a function of τ, while on the other hand (115) declares λ as a function of the derivatives of F. What (115) means is that after performing the differentiations, the arguments $\mathbf{K}(\tau)$, $\mathbf{X}(\tau)$ are expressed as a function of τ along the ray.

Dividing all the equations (113) by $dt/d\tau$, a set of equations is obtained in which t is the parameter along the ray. This has the advantage of eliminating $\lambda(\tau)$:

$$\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \mathbf{v}_g = -\frac{\partial_{\mathbf{k}} F}{\partial_\omega F} \\
\frac{d\mathbf{k}}{dt} &= \frac{\partial_{\mathbf{x}} F}{\partial_\omega F} \\
\frac{d\omega}{dt} &= -\frac{\partial_t F}{\partial_\omega F}
\end{aligned} \tag{116}$$

Note that the relativistic nature of the variables is thus obscured, hence a transformation of trajectories and associated group velocities becomes complicated. Furthermore, in the present form (116), the application of (114) is invalid. It is easy to see that in (116) we actually deal with the same group velocity as defined in (101), this is a direct result of holding $\mathbf{X} = \text{constant}$ during this operation:

$$dF\,|_{\mathbf{X}=const.} = \left.\frac{\partial F}{\partial \mathbf{K}}\right|_{\mathbf{X}=const.} \cdot d\mathbf{K} = \left(\frac{\partial F}{\partial \mathbf{k}} + \frac{\partial F}{\partial \omega}\frac{\partial \Omega}{\partial \mathbf{k}}\right) \cdot d\mathbf{k} = 0 \tag{117}$$

As an aside, it is observed that (117) suggests an interesting analog

$$\begin{aligned}
dF|_{\mathbf{K}=const.} &= \left.\frac{\partial F}{\partial \mathbf{X}}\right|_{\mathbf{K}=const.} \cdot d\mathbf{X} = \left(\frac{\partial F}{\partial \mathbf{x}} + \frac{\partial F}{\partial \mathbf{t}}\frac{\partial T}{\partial \mathbf{x}}\right) \cdot d\mathbf{x} = 0 \\
\mathbf{s}_g &= \partial_{\mathbf{x}} T(\mathbf{x}) = -\frac{\partial_{\mathbf{x}} F}{\partial_t F}
\end{aligned} \tag{118}$$

Quite unexpectedly (118) suggests a spatiotemporal surface function $t = T(\mathbf{x})$ whose gradient is a slowness vector. See also [41, 46].

For the special case of a medium not varying in time the third equation in (116) reduces to $\omega = $ constant. If we furthermore represent F as $F = \omega - \Omega(\mathbf{k}, \mathbf{x}) = 0$, then we obtain

$$
\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \mathbf{v}_g = -\frac{\partial F}{\partial \mathbf{k}} = \frac{\partial \Omega}{\partial \mathbf{k}} \\
\frac{d\mathbf{k}}{dt} &= = \frac{\partial F}{\partial \mathbf{x}} = -\frac{\partial \Omega}{\partial \mathbf{x}}
\end{aligned}
\tag{119}
$$

14. THE FERMAT PRINCIPLE AND ITS RELATIVISTIC CONNOTATIONS

The Fermat principle is usually stated as saying that the ray will traverse the distance between two points in extremal (usually minimal) time. For media not varying in time, after integration with respect to time, the phase becomes a line integral in $\mathbf{x}$-space and has the form

$$
\theta(\mathbf{X}) = \left[\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{k}(\mathbf{x}) \cdot d\mathbf{x} \right] - \omega t
\tag{120}
$$

where the brackets emphasize that ωt is not included in the integral. Taking θ at $t = 0$, say, taking the upper limit in (120) as a fixed value $\mathbf{x}_1$ and dividing (120) by the constant ω yields a function $T(\mathbf{x}_0, \mathbf{x}_1)$ whose dimension is time, and it depends on the fixed endpoints $\mathbf{x}_0, \mathbf{x}_1$

$$
T(\mathbf{x}_0, \mathbf{x}_1) = \frac{1}{\omega} \int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{k}(\mathbf{x}) \cdot d\mathbf{x}
\tag{121}
$$

The statement of the Fermat principle is that $\delta T(\mathbf{x}_0, \mathbf{x}_1) = 0$, where δ denotes the variation operation. Obviously, dealing with a definite integral we cannot find the extremum by differentiating T and equating the result to zero as done for functions in calculus. The variation operator, which for all other purposes acts as the conventional differentiation operator, operates on the functional, i.e., operates on the functions within the integrand, (121). Exchanging order of integration and variation, this yields

$$
\begin{aligned}
0 = \delta T(\mathbf{x}_0, \mathbf{x}_1) &= \frac{1}{\omega} \int_{\mathbf{x}_0}^{\mathbf{x}_1} \{ \delta[\mathbf{k}(\mathbf{x})] \cdot d\mathbf{x} + \mathbf{k}(\mathbf{x}) \cdot d\delta\mathbf{x} \} \\
&= \frac{1}{\omega} \int_{\mathbf{x}_0}^{\mathbf{x}_1} \{ \delta\mathbf{k} \cdot d\mathbf{x} - d\mathbf{k} \cdot \delta\mathbf{x} \} + \frac{1}{\omega} \int_{\mathbf{x}_0}^{\mathbf{x}_1} d\left[\mathbf{k} \cdot \delta\mathbf{x} \right]
\end{aligned}
\tag{122}
$$

The last integral in (122) is directly integrable, and since at the fixed endpoints the variation vanishes (that is what is meant by fixed endpoints), this integral vanishes. For arbitrary $\delta\mathbf{k}$, $\delta\mathbf{x}$ the integrand $\delta\mathbf{k}\cdot d\mathbf{x} - d\mathbf{k}\cdot\delta\mathbf{x}$ in (122) must vanish. Another constraint that must be met is the dispersion equation, i.e., its variation δF must vanish too. This yields a second equation. After slightly modifying $\delta\mathbf{k}\cdot d\mathbf{x} - d\mathbf{k}\cdot\delta\mathbf{x}$ by introducing an arbitrary parameter w and exchanging derivatives for the differentials, we have

$$
\begin{aligned}
\frac{d\mathbf{x}}{dw}\cdot\delta\mathbf{k} - \frac{d\mathbf{k}}{dw}\cdot\delta\mathbf{x} &= 0 \\
\frac{\partial F}{\partial\mathbf{k}}\cdot\delta\mathbf{k} + \frac{\partial F}{\partial\mathbf{x}}\cdot\delta\mathbf{x} &= 0
\end{aligned}
\tag{123}
$$

consistent with (119) when the arbitrary w is identified with t. We could also include F in the integrand in (122), because $F = 0$ and thus does not change the value of the integral. This will be implemented as an illustration in the derivation of the generalized Fermat principle below. The equations (119) resulting from the variational principle and therefore from (123) are called the Euler-Lagrange equations (of the pertinent variational principle).

The generalization of the Fermat principle to include time-varying media is given by Synge [45]. Here the notation is simplified by the use of four-vectors. The Fermat principle is represented in the form (again one must keep in mind that $\mathbf{X}$ in the integrand is the dummy variable)

$$
\delta\theta(\mathbf{X}) = 0 = \delta\int_{\mathbf{x}_0}^{\mathbf{x}_1}\mathbf{K}(\mathbf{X})\cdot d\mathbf{X}
\tag{124}
$$

where we have a line integral in four-space between two fixed so called world points $\mathbf{X}_0$, $\mathbf{X}_1$. Equation (124) expresses the idea that the integral path has to be chosen in such a way that the sum of the increments $d\theta$ along the path will be minimal (or extremal, in general). Inasmuch as the points $\mathbf{X}_0$, $\mathbf{X}_1$ already define a fixed time interval $t_1 - t_0$, the question arises as to what can be minimized (or in general extremized) in this process. The answer is fascinating, and can only be given in the context of Special Relativity theory: The quantity to be minimized is $d\theta = \mathbf{K}\cdot d\mathbf{K} = \mathbf{K}\cdot(d\mathbf{X}/d\tau)d\tau$ where τ is the proper time. The components of the $\mathbf{K}$ vector, as well as the components of the four-velocity $\frac{d\mathbf{X}}{d\tau}$ are slowly varying functions and may be considered as constant for an incremental $d\theta$, i.e., when $\mathbf{X}_0$, $\mathbf{X}_1$ are close world

points. Therefore the integral (124) amounts to finding the trajectory which minimizes (in general—extremizes) the proper time. Another way of looking at it is to exploit the invariance of $d\theta = \mathbf{K}\cdot d\mathbf{X} = \mathbf{K}'\cdot d\mathbf{X}'$ which in the proper frame where $\mathbf{v}_g = 0$ and $\gamma = 1$ becomes $d\theta = -\omega' d\tau$. If ω', which is a slowly v arying function, is considered to be constant over the distance and time of $d\theta$, then the same conclusion is reached, i.e., that $d\theta \propto d\tau$, i.e., minimizing θ means that the proper time along the ray is minimized. This interpretation has been previously proposed, see [42, 43]. The variational integral (124) is now rewritten as

$$\delta\theta(\mathbf{X}) = 0 = \delta \int_{\mathbf{X}_0}^{\mathbf{X}_1} \left\{ \mathbf{K}(\mathbf{X}) \cdot \frac{d\mathbf{X}}{d\tau} - \lambda F(\mathbf{K}, \mathbf{X}) \right\} d\tau \qquad (125)$$

where in the integrand $\lambda = \lambda(\tau)$, $\mathbf{X} = \mathbf{X}(\tau)$. The variation operations are performed in (125). To illustrate the technique, this time the constraint $F = 0$ is included in the integral by adding a term $-\lambda F$, where $\lambda(\tau)$ is an arbitrary Lagrange multiplier function. Using the same technique as in (122) and noting that $\delta(\lambda F) = F\delta\lambda + \lambda\delta F = \lambda\delta F$ we now obtain

$$\delta\theta(\mathbf{X}) = 0 = \int_{\mathbf{X}_0}^{\mathbf{X}_1} \left\{ \frac{d\mathbf{X}}{d\tau} \cdot \delta\mathbf{K} - \frac{d\mathbf{K}}{d\tau} \cdot \delta\mathbf{X} \right.$$
$$\left. - \lambda \left[\frac{\partial F}{\partial \mathbf{K}} \cdot \delta\mathbf{K} + \frac{\partial F}{\partial \mathbf{X}} \cdot \delta\mathbf{X} \right] \right\} d\tau \qquad (126)$$

and for arbitrary $\delta\mathbf{K}$, $\delta\mathbf{X}$ the expression in braces in (126) yields once again the ray equations (113). Note that in the present development the step of proving that the ray equations define $\mathbf{K}$ which satisfies (114) is not necessary. The two methods (i.e., "guessing" the result and verifying its validity using (114), and finding the ray equations by deriving the Euler-Lagrange equations of the Fermat-Synge variational principle (which by the way must also be viewed as an ingenious "guess" because it is stated axiomatically!) lead to the same ray equations.

Both methods start by assuming the dispersion equation (110) and the phase integral function (106). The first method stipulated that the value of the phase integral must depend on the end points of the path, and be independent of the particular integration path chosen. This was necessary for establishing the uniqueness of the "guess" (113). The second method, stipulated that the first variation of the phase

integral vanishes, as in (126). The second method leads to the same ray equations (113), which are now the Euler-Lagrange equations of the variational principle. As strange as it sounds, the logical conclusion is that the Fermat principle, an edifice of physics, is equivalent to the "guess" (113), plus (107), a modest statement on the smoothness of the function $\theta(\mathbf{X})$. It would be a good thing for our students to know this and to disperse some of the mystique involved in the attempts to explain the Fermat principle.

15. APPLICATION TO RAY PROPAGATION IN LOSSY MEDIA

Another application which invokes questions of Lorentz invariance and relativistic transformations, coupled with analyticity of functions, is the question of ray propagation in lossy media. At a first glance this appears as a very unlikely candidate for this role, but there are some fundamental questions involved, tied in with relativistic problems which posses important engineering implications. In lossy media as discussed above, when the losses are introduced through currents, say, as in (78), (79), the ensuing dispersion equations as in (110) are complex. Consequently the group velocity according to the original definition (101) will become complex too, in general, in turn implying complex space and time. A previous study, [47], cites earlier work in this area. Recent theoretical and numerical investigations are available too, [48, 49]. The main problem is that numerous models are feasible for extending the group velocity to the present case of complex dispersion equations. All the models define group velocities which reduce to the conventional definition in lossless media, all models are mathematically valid (albeit not always clarifying how a pertinent variational principle might be stated), but the physical consequences vary from one definition to another. There are essentially two main schools of thought: Some researchers do not perceive any difficulty in continuing the concepts of a real group velocity, and real space and time, into the complex domain. The trouble is that complex group velocities are mathematically possible also in dispersion equations for lossless media. A ray which starts in a lossless region in real space-time is propagated into a lossy region. According to the complex ray tracing method the group velocity and the trajectory will become complex. Upon reentering a lossless region, the group velocity will not automatically revert to real values. Thus, in addition to the conceptual difficulties of deal-

ing with complex space and time, and having to come to terms with a complex group velocity for which no physical explanation, such as packets of energy propagating through space, can be found, advocates of this approach are also confronted with complex group velocities in lossless media, completely losing the physical appeal of the group velocity concept. The other group of researchers advocates the use of real group velocities even in the presence of lossy media. The present model belongs to this class. The difficulty with many of these models is that they do not maintain analyticity, consequently differential operators as appearing in (113), if the functions do not satisfy the Cauchy-Riemann conditions for analyticity, become meaningless. Furthermore, transformation formulas such as (63) apply to complex functions only if they are analytic: If the group velocity is given by (116) as $\mathbf{v}_g = -\partial_{\mathbf{k}} F / \partial_\omega F$ and is part of a velocity four-vector, then in another inertial system we will have a corresponding $\mathbf{v}'_g = -\partial_{\mathbf{k}'} F / \partial_{\omega'} F$ (involving the same invariant F), and the two complex quantities are related through the transformation formula (63). We cannot arbitrarily define say $\mathbf{v}_g = -\Re\{\partial_{\mathbf{k}} F / \partial_\omega F\}$, by taking the real part only, because with the corresponding $\mathbf{v}'_g = -\Re\{\partial_{\mathbf{k}'} F / \partial_{\omega'} F\}$ (63) will not be satisfied by this pair $\mathbf{v}_g$, $\mathbf{v}'_g$. We also mention in passing that analyticity has a lot to do with causality, e.g., via the Kramers-Kronig relations, see for example Jackson [11], and Kong [18], and also due to the fact that the zeroes of the dispersion equation are the poles determining the free space Green function for the medium at hand, see for example Felsen and Marcuvitz [50]. The following model offers a definition for the ray equations and group velocity which keeps the group velocity simultaneously real (i.e., confined to the real axes of the relevant $\mathbf{X}$ space complex variables complex planes) and analytic, therefore commensurate with the pertinent relativistic transformation formulas. This goal is achieved, [51], see also [52, 53], by modifying the ray equations (113), (116). A new degree of freedom is introduced and an additional constraint is added, as explained below.

It is assumed that in addition to the ray equations (113), now extended to the complex domains $\mathbf{X}$, $\mathbf{K}$, there exists also the constraint that everywhere along the ray path, $\Im\mathbf{V}(\tau) = 0$, i.e., the imaginary part of the four velocity vanishes. In carrying out this operation, the factor i in the temporal part of the four-vector is treated as a real constant, not as the imaginary unit. In view of (113) and noting that

$\lambda(\tau)$ is real, this constraint is introduced in the form

$$\Im\frac{d}{d\tau}\frac{\partial F}{\partial\mathbf{K}}=0 \tag{127}$$

along the ray path, which guarantees $\Im\mathbf{V}(\tau)=0$ if it is satisfied at least at one point. Inasmuch as (127) is a four-vector constraint, it amounts to four scalar constraints. Similarly to (112), we expand (127) as

$$\Im\left(\frac{\partial^2 F}{\partial\mathbf{K}\partial\mathbf{K}}\cdot\frac{d\mathbf{K}}{d\tau}+\frac{\partial^2 F}{\partial\mathbf{X}\partial\mathbf{K}}\cdot\frac{d\mathbf{X}}{d\tau}\right)=0 \tag{128}$$

We cannot solve simultaneously (112) and (128). We have to add to (112) an additional (four-vector) degree of freedom, hence (113) are replaced now by

$$\begin{aligned}\frac{d\mathbf{X}}{d\tau}&=\lambda(\tau)\frac{\partial F}{\partial\mathbf{K}}\\\frac{d\mathbf{K}}{d\tau}&=-\lambda(\tau)\frac{\partial F}{\partial\mathbf{X}}-\lambda(\tau)\mathbf{\Lambda}(\tau)\end{aligned} \tag{129}$$

By substitution of (129) into (128) and some manipulation

$$\Im\mathbf{\Lambda}=\left(\Re\frac{\partial^2 F}{\partial\mathbf{K}\partial\mathbf{K}}\right)^{-1}\cdot\Im\left(\frac{\partial^2 F}{\partial\mathbf{X}\partial\mathbf{K}}\cdot\frac{\partial F}{\partial\mathbf{K}}-\frac{\partial^2 F}{\partial\mathbf{K}\partial\mathbf{K}}\cdot\frac{\partial F}{\partial\mathbf{X}}\right) \tag{130}$$

is derived, subject to a choice $\mathbf{\Lambda}=\Re\mathbf{\Lambda}+i\Im\mathbf{\Lambda}$, $\Re\mathbf{\Lambda}=0$. When (129) is solved by means of some numerical method, the initial values are chosen in real space $\mathbf{X}$. at each step in the solution the new increment $d\mathbf{K}$ is computed such that the next computed increment $d\mathbf{X}$ will be in real space once again. Therefore the value of $\mathbf{\Lambda}$ and its change from step to step will be small compared to the change of other parameters. One can think of iterating the same step with the new $d\mathbf{X}$, $d\mathbf{K}$, to finally render $\mathbf{\Lambda}=0$ along a ray, and propagate in real $\mathbf{X}$ space. Therefore we can say that (129) satisfies the uniqueness condition (114).

In terms of the three-dimensional representation (116), and using the same technique, the analogs of (129), (130), are derived. Instead of (116) we now have

$$\begin{aligned}\frac{d\mathbf{x}}{dt}&=\mathbf{v}_g=-\frac{\partial_{\mathbf{k}}F}{\partial_\omega F}\\\frac{d\mathbf{k}}{dt}&=\frac{\partial_{\mathbf{x}}F}{\partial_\omega F}+i\boldsymbol{\beta}\\\frac{d\omega}{dt}&=-\frac{\partial_t F}{\partial_\omega F}+i\alpha\end{aligned} \tag{131}$$

These ray equations have to satisfy the dispersion equation. By substitution of (131) into the three-dimensional version of (112), i.e.,

$$\frac{dF}{dt} = \frac{\partial F}{\partial \mathbf{k}} \cdot \frac{d\mathbf{k}}{dt} + \frac{\partial F}{\partial \omega} \frac{d\omega}{dt} + \frac{\partial F}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial F}{\partial t} = 0 \qquad (132)$$

yielding

$$\alpha = \mathbf{v}_g \cdot \boldsymbol{\beta} \qquad (133)$$

We find the conditions for the additional constraint, similarly to (127), from $\Im d_t \mathbf{v}_g = 0$, yielding

$$\Im \left(\frac{\partial \mathbf{v}_g}{\partial \mathbf{k}} \cdot \frac{d\mathbf{k}}{dt} + \frac{\partial \mathbf{v}_g}{\partial \omega} \frac{d\omega}{dt} + \frac{\partial \mathbf{v}_g}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial \mathbf{v}_g}{\partial t} \right)$$
$$= \Im \left(\frac{\partial \mathbf{v}_g}{\partial \mathbf{k}} \cdot \frac{\partial_{\mathbf{x}} F}{\partial_\omega F} + \frac{\partial \mathbf{v}_g}{\partial \mathbf{k}} \cdot i\boldsymbol{\beta} - \frac{\partial \mathbf{v}_g}{\partial \omega} \frac{\partial_t F}{\partial_\omega F} \right.$$
$$\left. + \frac{\partial \mathbf{v}_g}{\partial \omega} i\boldsymbol{\beta} \cdot \mathbf{v}_g + \frac{\partial \mathbf{v}_g}{\partial \mathbf{x}} \cdot \mathbf{v}_g + \frac{\partial \mathbf{v}_g}{\partial t} \right) = 0 \qquad (134)$$

finally finding

$$\boldsymbol{\beta} = - \left[\Re \left(\frac{d\mathbf{v}_g}{d\mathbf{k}} + \frac{d\mathbf{v}_g}{d\omega} \mathbf{v}_g \right) \right]^{-1}$$
$$\cdot \Im \left(\frac{d\mathbf{v}_g}{d\mathbf{k}} \cdot \frac{\partial_{\mathbf{x}} F}{\partial_\omega F} - \frac{d\mathbf{v}_g}{d\omega} \frac{\partial_t F}{\partial_\omega F} + \frac{d\mathbf{v}_g}{d\mathbf{x}} \cdot \mathbf{v}_g + \frac{d\mathbf{v}_g}{dt} \right) \qquad (135)$$

The result $\alpha = \mathbf{v}_g \cdot \boldsymbol{\beta}$, coupled with the four-vector product yields a scalar product of two perpendicular four-vectors

$$(\boldsymbol{\beta}, ic\alpha) \cdot (\partial_{\mathbf{k}} F, -i\partial_\omega F/c) = \Im \boldsymbol{\Lambda} \cdot \partial_{\mathbf{K}} F = 0 \qquad (136)$$

i.e., $\Im \boldsymbol{\Lambda} \cdot \mathbf{X} = 0$. Therefore the appropriate Fermat variational principle whose Euler-Lagrange equations are (129) can be written as

$$\delta \theta(\mathbf{X}) = 0 = \delta \int_{\mathbf{X_0}}^{\mathbf{X_1}} \left\{ \mathbf{K}(\mathbf{X}) \cdot \frac{d\mathbf{X}}{d\tau} - \lambda(\tau) F(\mathbf{K}, \mathbf{X}) - \lambda(\tau) \boldsymbol{\Lambda}(\tau) \cdot \mathbf{X} \right\} d\tau$$
$$(137)$$

The algorithm above, in the form (131) has been implemented for ray propagation in an absorptive ionosphere, see Sonnenschein et al., [54, 55].

The above ray tracing model for lossy media guarantees that the group velocity remains real, also that the ray paths are confined to real space and time and the appropriate dispersion equation is satisfied. This is achieved by choosing both $\mathbf{X}$ and $\mathbf{K}$ complex quantities, and adding a constraint that keeps the rays to real space. Inasmuch as the group velocity is a multivariate analytic function, it obeys the conventional relativistic transformation for velocities (63). It should be noticed that if the dispersion equation $F(\mathbf{K}, \mathbf{X}) = 0$ and $\mathbf{v}_g(\mathbf{K}, \mathbf{X})$ are analytic in all the components of $\mathbf{K}$, $\mathbf{X}$, then all the derivatives in this section involve only analytic functions, although by its definition, the conditions (127) itself is nonanalytic (a real or imaginary part of an analytic function is not analytic, as we know). The ray equations (129), (131) involve nonanalytic functions, namely those derived by the constraint (127). It is noted, however, that in the phase integral (106) they do not feature: As $\mathbf{K}$ changes, new increments $d\mathbf{K}$ appear in the integral, and multiply $d\mathbf{X}$, i.e., the velocity. But according to (136) and $\Im\Gamma \cdot \mathbf{X} = 0$ they get cancelled. Hence the phase integral is once again analytic, both if used as a definition, or if derived by integration (summation) of computed values.

16. NONLINEAR MEDIA AND VOLTERRA SERIES

There are a few reasons for including this subject in a course on application-oriented electrodynamics. The first reason is that the subject is important and timely, with fascinating physical effects. Loosely speaking, nonlinear media are characterized by constitutive parameters depending on fields. This gives rise to a plethora of new phenomena, both academically interesting *per se,* and of interest for applications. Paramount are the phenomena of harmonic generation, which one finds also in nonlinear lumped elements (e.g., magnetic materials which become saturated when flux increases, or electronic devices possessing nonlinear voltage-current characteristic curves), and new wave-specific phenomena such as self-focusing. In the latter, due to the field dependent constitutive parameters, the wave, depending on the intensity profile, "creates for itself" a "lens", thus a self-focusing phenomenon appears, e.g., see [56]. Once the tools of the Minkowski four-vectors and the associated Fourier transforms are at our disposal, it can be introduced in a compact and consistent manner. The second reason is that the new nonlinear constitutive relations demonstrate the feasibility of applying the Minkowski methodology, discussed above, to

complicated media.

Firstly, homogeneous media will be discussed and the role of dispersion in nonlinear media considered. In homogeneous linear media, in the $\mathbf{X}$ domain, convolution integrals like (74), (75), follow from the $\mathbf{K}$ space representation of the constitutive relations, e.g., (73), (76), (77). How can one extend these concepts to nonlinear systems? What are the characteristics of a representation that will make it valid? Note that the Maxwell equations (1), considered as axiomatically stipulated "law of nature", are indeterminate, i.e., the number of unknowns exceeds the number of available equations. The additional constitutive relations supplement Maxwell's equations, but are not uniquely determined: They depend on the material properties of the media at hand. We are not entering the extensive field of characterizing various materials here, rather looking for general characteristics.

It is noted that (74), (75), our prototype linear model for constitutive relations, is given by means of functionals, i.e., the dependence between the fields involves an integral. Volterra's functional series [2, 57, 58, 35], are the "natural" extension of the simple convolution integrals of the kind (74), (75). These "super convolutions" also satisfy a few basic requirements for a nonlinear media model: They satisfy a "correspondence principle" by which they reduce to the linear case in the limit of vanishing nonlinearity. Moreover, the model also indicates the various modes of nonlinear interaction, displaying products of fields. We start this discussion in the co-moving frame where the medium is at rest, and for the time being, for sake of simpler notation, this will be taken as the unprimed frame of reference. Also, for simplicity of the discussion only dielectric media are considered. Accordingly, instead of the convolution integral (75) we now have

$$\mathbf{D}(\mathbf{X}) = \sum_{n=1}^{\infty} \mathbf{D}^n(\mathbf{X}) = \sum_{n=1}^{\infty} \mathbf{P}^n\{\mathbf{X}, \mathbf{E}\} \qquad (138)$$

where the series suggests a hierarchy of increasingly complex nonlinear interactions, such that the most significant ones are the leading terms, and $\mathbf{P}^n\{\mathbf{X}, \mathbf{E}\}$ are adequate functionals depending on the coordinates $\mathbf{X}$ and the fields $\mathbf{E}$. The Volterra series of functionals, which is the *functional* counterpart and generalization of the Taylor series for *functions*, provides an adequate model for a hierarchical system that in

practice can be truncated after a certain number of terms:

$$\mathbf{D}^{(n)}(\mathbf{X}) = \int (d^4\mathbf{X}_1) \cdots \int (d^4\mathbf{X}_n)\widetilde{\varepsilon}^{(n)}(\mathbf{X}_1, \cdots, \mathbf{X}_n)$$
$$\because \mathbf{E}(\mathbf{X} - \mathbf{X}_1) \cdots \mathbf{E}(\mathbf{X} - \mathbf{X}_n) \qquad (139)$$

In (139) we have n four-fold integrations which for $n = 1$ reduce to the linear case (75), the n-the order constitutive parameter $\widetilde{\varepsilon}^{(n)}$ is now a dyadic (in the generic sense, some would refer to it as a tensor) acting on the indicated fields. The cluster symbol $\because$ denotes all the inner multiplications of the constitutive dyadic and the fields. Relativistic causality as discussed above must be incorporated into the model, i.e., $|\mathbf{X} - \mathbf{X}_n| \leq 0$ and only the past part of the cones is permitted. The Volterra functionals have an inverse Fourier transformation, the relation in $\mathbf{K}$ space is given by

$$\mathbf{D}^{(n)}(\mathbf{K}) = q^{n-1} \int (d^4\mathbf{K}_1) \cdots \int (d^4\mathbf{K}_{n-1})\widetilde{\varepsilon}^{(n)}(\mathbf{K}_1 \cdots \mathbf{K}_n)$$
$$\because \mathbf{E}(\mathbf{K}_1) \cdots \mathbf{E}(\mathbf{K}_n) \qquad (140)$$

This now is an $n - 1$ four-fold integration expression which for $n = 1$ reduces to the algebraic linear case (73). A scrutiny of (140) reveals that $\mathbf{K}_n$ is undefined, indeed, (140) must be supplemented by the constraint

$$\mathbf{K} = \mathbf{K}_1 + \cdots + \mathbf{K}_n \qquad (141)$$

i.e., $\mathbf{k} = \mathbf{k}_1 + \cdots + \mathbf{k}_n$, $\omega = \omega_1 + \cdots + \omega_n$. This is a remarkable relation. In the quantum-mechanical context it is an expression of conservation of energy (for frequencies) and momenta (for wave vectors). Moreover, as far as nonlinear processes are concerned, (141) is a statement of the production of harmonics and mixing of frequencies, and the production of new propagation vectors for these waves. The Volterra model is therefore very plausible for the purposes of modeling nonlinear constitutive relations.

An outline of the proof for the Fourier transform leading from (139) to (140) in conjunction with (141) is given now. In order to avoid cumbersome notation, consider (139) for $n = 2$, i.e.,

$$\mathbf{D}^{(2)}(\mathbf{X}) = \int (d^4\mathbf{X}_1) \int (d^4\mathbf{X}_2)\widetilde{\varepsilon}^{(2)}(\mathbf{X}_1, \mathbf{X}_2)$$
$$\because \mathbf{E}(\mathbf{X} - \mathbf{X}_1)\mathbf{E}(\mathbf{X} - \mathbf{X}_2) \qquad (142)$$

and show that according to (140) we obtain

$$\mathbf{D}^{(2)}(\mathbf{K}) = q \int (d^4\mathbf{K}_1)\widetilde{\varepsilon}^{(2)}(\mathbf{K}_1, \mathbf{K}_2) \because \mathbf{E}(\mathbf{K}_1)\mathbf{E}(\mathbf{K}_2) \tag{143}$$

and according to (141) $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$. The general proof for $n > 2$ follows the same pattern: Incorporating (33) into (142) for the fields and interchanging the integration order, (142) is now recast in the form

$$\mathbf{D}^{(2)}(\mathbf{X}) = q^2 \int (d^4\mathbf{K}_1) \int (d^4\mathbf{K}_2) e^{i(\mathbf{K}_1+\mathbf{K}_2)\cdot\mathbf{X}}\widetilde{\varepsilon}^{(2)}(\mathbf{K}_1, \mathbf{K}_2)$$
$$\because \mathbf{E}(\mathbf{K}_1)\mathbf{E}(\mathbf{K}_2) \tag{144}$$

Now apply (35) to the two sides of (144), this yields

$$\mathbf{D}^{(2)}(\mathbf{K}) = q \int (d^4\mathbf{K}_1) \int (d^4\mathbf{K}_2)\widetilde{\varepsilon}^{(2)}(\mathbf{K}_1, \mathbf{K}_2)$$
$$\because \mathbf{E}(\mathbf{K}_1)\mathbf{E}(\mathbf{K}_2) \left[q \int (d^4\mathbf{X}) e^{i(\mathbf{K}_1+\mathbf{K}_2-\mathbf{K})\cdot\mathbf{X}} \right] \tag{145}$$

The last expression in brackets in (145) is recognized as the four-dimensional δ or Dirac's impulse function

$$\delta(\mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}) \tag{146}$$

and (143) and $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$ follow.

Constitutive relations, characterizing a medium, should define an empirically measurable system, otherwise all this constitutes empty mathematical manipulations. In order to investigate this question, we start with a simple signal. We cannot simply assume a single frequency harmonic signal, because the nonlinearity creates harmonics. Therefore the starting point is a periodic solution

$$\mathbf{E}(\mathbf{X}) = \sum_{m=-\infty}^{\infty} \mathbf{E}_m(\mathbf{K}) e^{im\mathbf{K}\cdot\mathbf{X}} = \sum_m \mathbf{E}_m e^{im\mathbf{K}\cdot\mathbf{X}} \tag{147}$$

This already includes the assumption of phase coherence or phase synchronism, meaning that all harmonics have the same phase velocity. Substituting (147) into (139) yields

$$\mathbf{D}^{(n)}(\mathbf{X}) = \sum_{m_1,\cdots,m_n} \widetilde{\mathbf{I}} \because \mathbf{E}_{m_1} \cdots \mathbf{E}_{m_n} e^{i(m_1+\cdots+m_n)\mathbf{K}\cdot\mathbf{X}}$$
$$\widetilde{\mathbf{I}} = \int (d^4\mathbf{X}_1) \cdots \int (d^4\mathbf{X}_n)\widetilde{\varepsilon}^{(n)}(\mathbf{X}_1, \cdots, \mathbf{X}_n) \tag{148}$$
$$\cdot e^{-im_1\mathbf{K}\cdot\mathbf{X}_1} \cdots e^{-im_n\mathbf{K}\cdot\mathbf{X}_n} = \widetilde{\varepsilon}^{(n)}(m_1\mathbf{K}, \cdots, m_n\mathbf{K})$$

where the integration turns out to be a Fourier transform. Equation (148) implies that

$$\mathbf{D}^{(n)}(\mathbf{X}) = \sum_p \mathbf{D}_p^{(n)} e^{ip\mathbf{K}\cdot\mathbf{X}} \tag{149}$$

be periodic too, implying in turn

$$\mathbf{D}_p^{(n)}(\mathbf{K}) = {\sum_p}' \widetilde{\varepsilon}^{(n)}(m_1\mathbf{K}, \cdots, m_n\mathbf{K}) \because \mathbf{E}_{m_1} \cdots \mathbf{E}_{m_n} \tag{150}$$

where the prime indicates that in (150) terms are regrouped according to $p = m_1 + \cdots + m_n$ as prescribed by the orthogonality properties of the exponentials.

It is obvious that in general (150) is too complicated for experimentally determining the material parameters $\widetilde{\varepsilon}^{(n)}$. One way of simplifying the model is to define

$$\begin{aligned} \mathbf{D}_p^{(n)}(\mathbf{K}) &= \overline{\overline{\widetilde{\varepsilon}}}^{(n)}(p\mathbf{K}) \because \mathbf{E}_p \cdots \mathbf{E}_p \\ &= {\sum_p}' \widetilde{\varepsilon}^{(n)}(m_1\mathbf{K}, \cdots, m_n\mathbf{K}) \because \mathbf{E}_{m_1} \cdots \mathbf{E}_{m_n} \end{aligned} \tag{151}$$

It appears that we have accomplished the task, as now in (151) $\overline{\overline{\widetilde{\varepsilon}}}^{(n)}$ depend on one specific harmonic p only. Of course, we cannot filter out one frequency, because the harmonics are interdependent through the nonlinear interactions. But there is more to it: A little scrutiny reveals that the new parameters $\overline{\overline{\widetilde{\varepsilon}}}^{(n)}$ are field-dependent, i.e., they are not true medium parameters, but depend on the fields. Therefore, strictly speaking, the effort of characterizing the system failed. But there is a redeeming feature: From (151) it is clear that $\overline{\overline{\widetilde{\varepsilon}}}^{(n)}$ involves ratios of field components, and as the amplitude changes, the effect on all field components is similar, hence the new equivalent constitutive parameters provide a good approximation.

From the point of view of Relativistic Electrodynamics, forms like (139), when given in the co-moving frame Γ' with primed fields and coordinates, raise the question of applying the Minkowski methodology in order to derive the corresponding constitutive relations in the unprimed frame of reference Γ. One obvious step to take is to substitute for the Γ' fields from (10), and in the spectral domain (44) etc.

However, there remains the question of the integration. Exploiting the properties of the Jacobian as expressed in (19)–(21), the transformation of the integration volume is then given by (43), for (140), and (45) for (139). The difficulty of defining the limits in (139) within the past region of the corresponding light cones, and the recasting of the constraint (141) still linger. The Volterra differential operators introduced next offer a straightforward alternative. Moreover, this method is also applicable to inhomogeneous media.

17. DIFFERENTIAL OPERATOR CONSTITUTIVE RELATIONS

The integral representation (139) accounts for dispersion effects in non-linear media, but does not include inhomogeneity effects. This aspect will be dealt with in the present section [2, 35].

We start with homogeneous media with a constitutive relation written in the form

$$\mathbf{D}(\mathbf{K}) = \widetilde{\varepsilon}(i\mathbf{K}) \cdot \mathbf{E}(\mathbf{K}) \tag{152}$$

Now apply the Fourier transform (33) to (152) and note, like in (36), that $\partial_{\mathbf{X}} \Leftrightarrow i\mathbf{K}$ are interchangeable. Consequently in the spatiotemporal domain $\mathbf{X}$ the constitutive relation (152) is given by

$$\mathbf{D}(\mathbf{X}) = \widetilde{\varepsilon}(\partial_{\mathbf{X}}) \cdot \mathbf{E}(\mathbf{X}) \tag{153}$$

where we have simply replaced in the constitutive parameter $\partial_{\mathbf{X}} \Leftrightarrow i\mathbf{K}$. On the other hand, a formal transformation of (152) leads to a convolution integral like in (75)

$$\mathbf{D}(\mathbf{X}) = \int (d^4\mathbf{X}_1)\overset{*}{\widetilde{\varepsilon}}(\mathbf{X}_1) \cdot \mathbf{E}(\mathbf{X} - \mathbf{X}_1) \tag{154}$$

where $\widetilde{\varepsilon}(i\mathbf{K})$ is the Fourier transform of $\overset{*}{\widetilde{\varepsilon}}(\mathbf{X})$, see [35]. From (153), (154) it follows that we have found a differential operator equivalent to the convolution integral. This differential operator is dubbed as Volterra differential operator, because it works for the general case (139) too.

Rewriting (153) in the form

$$\mathbf{D}(\mathbf{X}) = \widetilde{\varepsilon}(\partial_{\mathbf{X}_1})|_{\mathbf{X}_1=\mathbf{X}} \cdot \mathbf{E}(\mathbf{X}_1) \tag{155}$$

indicates that we first apply the operator, then replace the operation variable $\mathbf{X}_1$ by $\mathbf{X}$. It is now easy to show that (139) will be replaced by

$$\mathbf{D}^{(n)}(\mathbf{X}) = \widetilde{\varepsilon}^{(n)}(\partial_{\mathbf{X}_1}|_{\mathbf{X}_1=\mathbf{X}}, \cdots, \partial_{\mathbf{X}_n}|_{\mathbf{X}_n=\mathbf{X}}) \because \mathbf{E}(\mathbf{X}_1) \cdots \mathbf{E}(\mathbf{X}_n) \quad (156)$$

For example

$$(a|_{\mathbf{X}_1=\mathbf{X}}, \partial_{\mathbf{X}_2}|_{\mathbf{X}_2=\mathbf{X}}, \partial_{\mathbf{X}_3}|_{\mathbf{X}_3=\mathbf{X}}) A(\mathbf{X}_1)A(\mathbf{X}_2)A(\mathbf{X}_3) = aA(\mathbf{X})(\partial_{\mathbf{X}}A(\mathbf{X}))^2 \tag{157}$$

indicates that the nonlinear term involves a constant factor a, and a product of the function A and the square of its four-gradient.

Thus far only homogeneous media have been considered. The generalization to inhomogeneous media, characterized by varying coefficients, is introduced in the form

$$\mathbf{D}^{(n)}(\mathbf{X}) = \widetilde{\varepsilon}^{(n)}(\mathbf{X}, \partial_{\mathbf{X}_1}|_{\mathbf{X}_1=\mathbf{X}}, \cdots, \partial_{\mathbf{X}_n}|_{\mathbf{X}_n=\mathbf{X}}) \because \mathbf{E}(\mathbf{X}_1) \cdots \mathbf{E}(\mathbf{X}_n) \tag{158}$$

The question of the Minkowski methodology for deriving the constitutive relations in different inertial systems is now much simpler, even for nonlinear media. Given (158) in the primed frame of reference Γ'

$$\mathbf{D}'^{(n)}(\mathbf{X}') = \widetilde{\varepsilon}^{(n)}(\mathbf{X}', \partial_{\mathbf{X}'_1}|_{\mathbf{X}'_1=\mathbf{X}'}, \cdots, \partial_{\mathbf{X}'_n}|_{\mathbf{X}'_n=\mathbf{X}'})$$
$$\because \mathbf{E}'(\mathbf{X}'_1) \cdots \mathbf{E}'(\mathbf{X}'_n) \tag{159}$$

we substitute from (6) for coordinates, from (8) for spatiotemporal derivatives, and from (10) for fields, and obtain the corresponding Minkowski constitutive relation in the unprimed frame of reference Γ.

This short overview should suffice for an introduction to the subject.

18. SCATTERING BY A MOVING CYLINDER

Finally, the theoretical results will be applied to the question of scattering by moving objects. The problem of scattering by a moving mirror has been discussed by Einstein [3]. The general scattering problem in two and three dimensions has been considered too [59]. The formalism was also applied to the fascinating problem of the inverse Doppler effect, an effect that still waits for more investigation and experimentation [60]. As a simple example of the formalism, we consider here

scattering of a normally incident harmonic plane wave, polarized along the cylindrical axis, by a cylinder moving perpendicularly to the axis [59], [41]. We start in the unprimed frame of reference Γ with a plane wave, whose field $\mathbf{f}$ (which can be either the electric $\mathbf{E}$-field or the magnetic $\mathbf{H}$-field) is polarized along the cylindrical z-axis, and propagating in the x-axis direction

$$\mathbf{f} = \hat{\mathbf{z}} f_0 e^{ikx - i\omega t} \tag{160}$$

A cylinder with a finite cross section in the $x - y$ plane is given, moving in the x-axis direction according to $\mathbf{v} = \hat{\mathbf{x}}\nu$. The choice of the specific direction of propagation (160), which is also the direction of the velocity, does not impair the generality of the model.

In the cylinder's co-moving frame of reference Γ' the incident wave (160), undergoing the appropriate relativistic transformations, is observed as

$$\mathbf{f}' = \hat{\mathbf{z}} f_0' e^{ik'x' - i\omega' t'}, \qquad \frac{\omega'}{\omega} = \frac{k'}{k} = \frac{f_0'}{f_0} = \left(\frac{1 - \beta}{1 + \beta} \right)^{1/2} \tag{161}$$

where the wave parameters are related as indicated by (161). The incident wave in the co-moving frame is now recast in (cylindrical) space, and time coordinates $(\mathbf{r}', t')$, in terms of a Bessel-Fourier series, yielding

$$\hat{\mathbf{z}} f_0' \sum_{m=-\infty}^{\infty} i^m J_m(k'r') e^{im\theta' - i\omega' t'} \tag{162}$$

in terms of the nonsingular Bessel functions J_m and $\mathbf{r}' = (r', \theta')$. The scattered wave is obtained by solving the wave equation in cylindrical coordinates in terms of the location vector in the cross-sectional plane $\mathbf{r}' = (r', \theta')$, and as a Hankel-Fourier series. The solution is given by

$$\mathbf{u}'(\mathbf{r}', t') = \hat{\mathbf{z}} f_0' \sum_{m=-\infty}^{\infty} i^m a_m H_m^{(1)}(k'r') e^{im\theta' - i\omega' t'} \tag{163}$$

where $H_m^{(1)}$ denotes Hankel functions of the first kind. The choice of $H_m^{(1)}$ together with the time exponent in (153) guarantees outgoing waves. Using the Sommerfeld integral representation for the cylindrical functions, (163) is recast as a sum of plane waves,

$$\mathbf{u}'(\mathbf{r}', t') = \hat{\mathbf{z}} \frac{f_0'}{\pi} \int_{\theta' - (\pi/2) + i\infty}^{\theta' + (\pi/2) - i\infty} e^{ik'r'\cos(\theta' - \tau') - i\omega' t'} g(\tau') d\tau' \tag{164}$$

propagating in complex directions indicated by the complex angles τ', where the scattering amplitude, i.e., the weight function for each such plane wave, is given by the Fourier series,

$$g(\theta') = \sum_{m=-\infty}^{\infty} a_m e^{im\theta'} \tag{165}$$

Upon applying the appropriate relativistic transformation (10) for the field assumed in (164), noting that the field is polarized perpendicularly to the velocity, and using the perpendicularity of the fields and the direction of propagation for a plane wave in free space, (164) transforms into the corresponding field in Γ in the form

$$\mathbf{u} = \mathbf{u}(\mathbf{r}(\mathbf{r}',t'),t(\mathbf{r}',t')) \tag{166}$$

which is conveniently expressed in terms of the Γ' coordinates. The last step of substituting the Lorentz transformation (6) into the result is cumbersome and straightforward, hence it is not shown here explicitly. Accordingly, (166) is represented as

$$\mathbf{u} = \hat{\mathbf{z}}\frac{\gamma f_0'}{\pi} \int_{\theta'-(\pi/2)+i\infty}^{\theta'+(\pi/2)-i\infty} e^{ik'r'\cos(\theta'-\tau')-i\omega't'}(1+\boldsymbol{\beta}\cos\tau')g(\tau')d\tau' \tag{167}$$

Some additional manipulation yields (167) once again in the form of a Hankel- Fourier series,

$$\mathbf{u} = \hat{\mathbf{z}}\gamma f_0' \sum_{m=-\infty}^{\infty} i^m b_m H_m^{(1)}(k'r')e^{im\theta'-i\omega't'} \tag{168}$$

with the new coefficients now given in terms of the original ones as

$$b_m = a_m + (a_{m-1} + a_{m+1})\beta/2 \tag{169}$$

clearly showing the velocity effects producing interaction between various multipoles. For the corresponding three-dimensional formulas, as well as graphical simulations, see [59].

19. CONCLUDING REMARKS

Relativistic Electrodynamics is now a tangibly needed subject in the education of electromagnetic radiation engineers, as well as physics

graduates who discover that they are more application-oriented and therefore drift towards modern electromagnetic theory and applications. The experience of the present author is dictating a pedagogical approach which is very unorthodox from the point of view of physicists, whose way of presenting the subject also percolated into the electrical engineering electromagnetic theory textbooks. It is suggested that the rudiments of Relativistic Electrodynamics be stipulated axiomatically, according to the "topsy-turvy" scheme given here, and the practical implications and conclusions be introduced by keeping the mathematical machinery to the absolutely necessary minimum. It has been found that four-vectors and dyadics (i.e., matrices) is practically all the mathematical equipment needed (of course, previous courses in electromagnetic field theory are assumed). The various applications and examples given here are of course optional. It is expected that educators will be biased by their own interest in relevant subjects.

ACKNOWLEDGMENT

REFERENCES

1. Censor, D., "Application-oriented relativistic electrodynamics," *PIER, Progress In Electromagnetics Research,* Editor J. A. Kong, Vol. 4, 119–158, Elsevier, 1991.
2. Censor, D., "Electrodynamics, topsy-turvy Special Relativity, and generalized Minkowski constitutive relations for linear and nonlinear systems," *PIER — Progress in Electromagnetics Research,* Editor J. A. Kong, Vol. 18, 261–284, Elsevier, 1998.
3. Einstein, A., "Zur Elektrodynamik bewegter Körper," *Ann. Phys. (Lpz.),* 17, 891–921, 1905; English translation: "On the electrodynamics of moving bodies," *The Principle of Relativity,* Dover.

4. Becker, R., *Electromagnetic Fields and Interactions,* Dover, 1982.

5. Stratton, J. A., *Electromagnetic Theory,* McGraw-Hill, 1941.

6. Fano, R. M., L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces,* Wiley, 1960.

7. Sommerfeld, A., *Electrodynamics,* Dover, 1964.

8. Jordan, E. C. and K. G. Balmain, *Electromagnetic Waves and Radiating Systems,* Prentice-Hall, 1968.

9. Panofsky, W. K. H. and M. Phillips, *Classical Electricity and Magnetism,* Addison-Wesley, 1971.

10. Shadowitz, A., *The Electromagnetic Field,* McGraw-Hill, 1975.

11. Jackson, J. D., *Classical Electrodynamics,* Wiley, 1975.

12. Portis, A. M., *Electromagnetic Fields, Sources and Media,* Wiley, 1978.

13. Lorrain, P. and D. R. Corson, *Electromagnetism, Principles and Applications,* W. H. Freeman, 1970.

14. Wangness, R. K., *Electromagnetic Fields,* Wiley, 1979.

15. Griffiths, D. J., *Introduction to Electrodynamics,* Prentice-Hall, 1981.

16. Frankl, D. R., *Electromagnetic Theory,* Prentice-Hall, 1986.

17. Chen, H. C., *Theory of Electromagnetic Waves,* McGraw-Hill, 1985.

18. Kong, J. A., *Electromagnetic Wave Theory,* Wiley, 1986.

19. Plonus, M. A., *Applied Electromagnetics,* McGraw-Hill, 1988.

20. Eringen, A. C. and G. A. Maugin, *Electrodynamics of Continua,* Vol. 2, Springer, 1990.

21. Schwartz, M., *Principles of Electrodynamics,* Dover, 1987.

22. Van Bladel, J., *Relativity and Engineering,* Springer, 1984.

23. Lindell, I. V., *Methods for Electromagnetic Field Analysis,* Oxford, 1992.

24. Minkowski, H., "Die Grundgleichungen für die elektromagnetische Vorgänge in bewegten Körpern," *Göttinger Nachrichten,* 53–116, 1908.

25. Zangari, M. and D. Censor, "Spectral representations: An alternative to the spatiotemporal world view," *Synthese,* Vol. 112, 97–123, 1997.

26. Censor, D., "Simultaneity, causality, and spectral representations," submitted for publication.

27. Abraham, M., "Zur theorie der strahlung und des strahlungdruckes," *Ann. Phys. (Lpz.),* Vol. 14, 236–287, 1904.

28. Pauli, W., *Theory of Relativity,* Pergamon, 1958, also Dover Publications.

29. Censor, D., "Dispersion equations in moving media," *Proceedings IEEE,* Vol. 68, 528–529, 1980.

30. Bohm, D., *The Special Theory of Relativity,* Benjamin, 1965.
31. Post, E. J., *Formal Structure of Electromagnetics,* North-Holand, 1962.
32. Hebenstreit, H., "Constitutive relations for moving plasmas," *Z. Naturforsch. A,* Vol. 34a, 147–154, 1979.
33. Hebenstreit, H., "Calculation of covariant dispersion equations for moving plasmas," *Z. Naturforsch. A,* Vol. 34a, 155–162, 1979.
34. Hebenstreit, H. and K. Suchy, "Polarization relations and dispersion equations for anisotropic moving media," *Z. Naturforsch. A,* Vol. 34a, 1147–1157, 1979.
35. Censor, D., "A quest for systematic constitutive formulations for general field and wave systems based on the Volterra differential operators," *PIER — Progress In Electromagnetics Research,* Editor J. A. Kong, Vol. 25, 261–284, Elsevier, 2000.
36. Chawla, B. R. and H. Unz, *Electromagnetic Waves in Moving Magnetoplasmas,* Univ. Press, Lawrence, Kansas, 1971.
37. Kelso, J. M., *Radio Ray Propagation in the Ionosphere,* McGraw-Hill, 1964.
38. Van Bladel, J., *Electromagnetic Fields,* Hemisphere, 1985.
39. Ghatak, A., *Optics,* McGraw-Hill, 1977.
40. Sommerfeld, A., *Optics,* Academic Press, 1964.
41. Censor, D., "Quasi doppler effects associated with spatiotemporal translatory, moving, and active boundaries," *JEMWA — Journal of Electromagnetic Waves and Applications,* Vol. 13, 145–174, 1999.
42. Censor, D., "Application-oriented ray theory," *International. J. Electrical. Eng. Education,* Vol. 15, 215–223, 1978.
43. Molcho, J. and D. Censor, "A simple derivation and an example of Hamiltonian ray propagation," *Am. J. Phys,* Vol. 54, 351–353, 1986.
44. Poeverlein, H., "Sommerfeld-Runge law in three and four dimensions," *The Physical Review,* Vol. 128, 956–964, 1962.
45. Synge, J. L., *Geometrical Mechanics and De Broglie Waves,* Cambridge University Press, 1954.
46. Censor, D., ""Waves", "objects" and special relativity," *JEMWA — Journal of Electromagnetic Waves and Applications,* Vol. 5, 1365–1391, 1991.
47. Censor, D. and J. J. Gavan, "Wave packets, group velocities and rays in lossy media revisited," *IEEE Trans. on Electromagnetic Compatibility,* Vol. 31, 262–272, 1989.
48. Sonnenschein, E., I. Rutkevich, and D. Censor, "Wave packets, rays and the role of real group velocity in absorbing media," *Physical Review E,* Vol. 57, 1005–1016, 1998.

49. Sonnenschein, E., I. Rutkevich, and D. Censor, "Wave packet and group velocity in absorbing media: Solutions of the telegrapher's equation," to appear in *JEMWA — Journal of Electromagnetic Waves and Applications,* 2000.

50. Felsen, L. B. and N. Marcuvitz, *Radiation and Scattering of Waves,* Prentice-Hall, 1973.

51. Censor, D. and K. Suchy, "Wave packets and ray tracing in lossy media," *Proc. URSI Natl. Comm.,* Fed. Rep. of Germany, Kleinheubacher, 617–623, 1975.

52. Censor, D., "Fermat's principle and real space-time rays in absorbing media," *J. Phys. A: Math. Gen.,* Vol. 10, 1781–1790, 1977.

53. Censor, D., "Fermat's principle, Hamiltonian ray equations, group velocity and wave packets in absorptive media," *Advances in Thermodynamics,* Vol. 3: Theory and Extremum Principles, Editors, S. Sieniuycz and P. Salamon, 448–481, Taylor and Francis, 1990.

54. Sonnenschein, E., D. Censor, I. Rutkevich, and J. A. Bennett, "Ray trajectories in an absorptive ionosphere," *JASTRP — Journal of Atmospheric and Solar-Terrestrial Physics,* Vol. 59, 2101–2110, 1997.

55. Sonnenschein, E., N. Blaunstein, and D. Censor, "HF ray propagation in the presence of resonance heated ionospheric plasma," *JASTRP — Journal of Atmospheric and Solar-Terrestrial Physics,* Vol. 60, 1605–1623, 1998.

56. Sonnenschein, M. and D. Censor, "Simulation of Hamiltonian light beam propagation in nonlinear media," *JOSA — Journal of the Optical Society of America, B,* Vol. 15, 1335–1345, 1998.

57. Volterra, V., *Theory of Functionals and of Integral and Integro-Differential Equations,* Dover, 1959.

58. Censor, D., "Ray tracing in weakly nonlinear moving media," *Journal of Plasma Physics,* Vol. 16, 415–426, 1976.

59. Censor, D., "Scattering in velocity dependent systems," *Radio Science,* Vol. 7, 331–337, 1972.

60. Ben-Shimol, Y. and D. Censor, "Contribution to the problem of near zone inverse Doppler effect," *Radio Science,* Vol. 33, 463–474, 1998.