Progress In Electromagnetics Research, *PIER 30, 59–84, 2001*

POLYNOMIAL OPERATORS AND GREEN FUNCTIONS

I. V. Lindell

Electromagnetics Laboratory
Helsinki University of Technology
Otakaari 5A, Espoo FIN-02015HUT, Finland

F. Olyslager

Department of Information Technology
University of Ghent
Sint-Pietersnieuwstraat 41, B-9000, Ghent, Belgium

Abstract–Green functions corresponding to various polynomial partial differential operators of second, fourth and higher order are derived and the results are collected in tabular form for quick reference. The results and the methods suggested for their derivation are of importance in solving electromagnetic field problems associated with various linear (bi-anisotropic) media.

1. INTRODUCTION

Green functions represent basic solutions to differential equations subject to some additional (boundary) conditions. Here we consider problems without finite boundaries. Green functions of this kind can be interpreted as fields arising from point, line or plane sources (corresponding to three, two and one dimensional Green functions, respectively) in a medium of infinite extent. Knowledge of the Green function of a certain medium can essentially simplify solving electromagnetic boundary-value problems in the medium in question. In fact, instead of solving differential equations with boundary conditions, one can derive integral equations for unknown boundary sources or obtain solutions for source problems through simple integrations.

For the moment, knowledge of Green functions of different media appears quite limited, due to insufficient knowledge of solutions associated with corresponding differential operators. It is well known that problems in general linear (bi-anisotropic) media [1] lead to polynomial partial-differential operators of the fourth order [2]. Their solutions have not, however, been widely discussed in scientific literature. On the other hand, dyadic Green functions corresponding to different boundary conditions in simple isotropic media have been thoroughly treated in the famous monograph by C. T. Tai [3].

The main purpose of the present paper is to derive Green function solutions to some polynomial operators of the fourth order which have importance in building solutions for Green dyadics in different media. Only problems of static or time-harmonic origin leading to equations of the elliptic type are discussed here. As an introduction, solutions for some second-order operators are also given. The results are displayed in tabular form in an Appendix for convenience. Also methods of solution, grown from the experience of these authors in solving Green dyadics for various media, are shown in concise form and they may have application to other similar problems.

2. GREEN FUNCTIONS

Green function $G(\mathbf{r})$ is a solution to a differential equation defined by the partial differential operator[1] $L(\nabla)$:

[1] The minus sign in front of the delta function is a convenience not obeyed by all authors. Sometimes it is absorbed in the definition of the operator.

$$L(\nabla)G(\mathbf{r}) = -\delta(\mathbf{r}). \tag{1}$$

Symbolically, we can write the solution in the form

$$G(\mathbf{r}) = -\frac{1}{L(\nabla)}\delta(\mathbf{r}). \tag{2}$$

To solve time-harmonic electromagnetic fields in linear bi-anisotropic media the basic problem is to find the dyadic Green function depending on the dyadic parameters of the medium. The basic problem involves a dyadic operator of the second order operating on the dyadic unknown which consists of nine scalar functions to be solved. This can always be reduced to the problem of a single scalar Green function involving a scalar fourth-order operator, the Helmholtz determinant operator. For the electric field this operator can be written as [2]

$$L(\nabla) = \det \overline{\overline{H}}_e(\nabla) = \det\left[-\left(\nabla\times\overline{\overline{I}} - jk_o\overline{\overline{\xi}}_r\right)\cdot\overline{\overline{\mu}}_r^{-1}\cdot\left(\nabla\times\overline{\overline{I}} + jk_o\overline{\overline{\zeta}}_r\right) + k_o^2\overline{\overline{\epsilon}}_r\right], \tag{3}$$

$\overline{\overline{\epsilon}}_r$, $\overline{\overline{\mu}}_r$, $\overline{\overline{\xi}}_r$, and $\overline{\overline{\zeta}}_r$ denoting the relative dyadic parameters of the medium. In the general case, when expanded, the Helmholtz determinant operator has nonvanishing differential operators of all orders from 0 to 4. For some special media the expression can be simplified. For example, for the class of so-called decomposable media, the fourth-order operator can be factorized, written as the product of two second-order operators [8].

In addition to the differential equation, suitable radiation conditions are needed at infinity for the uniqueness of the solution. Normally it is required that power is transported by the electromagnetic fields towards infinity, away from the source, and not the other direction. What this means for a solution to the fourth-order operator, which is a kind of potential function, is not so obvious. When the solutions are applied to making Green dyadics representing electromagnetic fields, the solutions of the fourth-order operator equations should match the radiation conditions of the Green dyadics. This problem is not addressed here. The intention is just to find analytic Green-function solutions $G(\mathbf{r})$ corresponding to operators $L(\nabla)$. For more information on radiation conditions in bi-anisotropic media, see [9, 10].

A few words on the notation applied in the analysis: operations with dyadics are defined in [2]. Arbitrary vectors are denoted by $\mathbf{a}$, $\mathbf{b}$ and unit vectors by $\mathbf{u}$, $\mathbf{v}$. Three and two-dimensional radial position vectors are respectively denoted by $\mathbf{r}$ and $\boldsymbol{\rho} = \mathbf{u}_x x + \mathbf{u}_y y$ and their lengths by r and ρ. The two-dimensional differential operator is denoted by ∇_t. $\overline{\overline{S}}$ denotes an arbitrary symmetric dyadic and $\overline{\overline{S}}_t$ a two-dimensional symmetric dyadic. The operator $\overline{\overline{A}} : \nabla\nabla = \nabla \cdot \overline{\overline{A}} \cdot \nabla$ can always be replaced by $\overline{\overline{S}} : \nabla\nabla$ where $\overline{\overline{S}}$ is the symmetric part of the dyadic $\overline{\overline{A}}$. Affine transformation through a symmetric dyadic is understood so that the identity $\nabla\mathbf{r} = \overline{\overline{I}}$ is invariant. Thus, if ∇ is transformed to $\overline{\overline{S}} \cdot \nabla$, we must transform $\mathbf{r}$ to $\overline{\overline{S}}^{-1} \cdot \mathbf{r}$ and similarly in two dimensions. Uniaxial dyadics are denoted by $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ with their axes along the unit vector $\mathbf{u}_z$. For example, we have

$$\overline{\overline{\alpha}} = \alpha_t \overline{\overline{I}}_t + \alpha_z \mathbf{u}_z \mathbf{u}_z. \tag{4}$$

3. EVALUATION OF SOME GREEN FUNCTIONS

Green function expressions corresponding to some typical operators are derived here in concise form to demonstrate methods used for obtaining the solutions. The results are summarized in the Appendix as a table in the form $L(\nabla) \Rightarrow G(\mathbf{r})$.

3.1 Second-Order Operators

The basic Green functions in three, two and one dimensions are well known:

$$\left(\nabla^2 + k^2\right) G(k;\mathbf{r}) = -\delta(\mathbf{r}), \qquad G(k;\mathbf{r}) = \frac{e^{-jkr}}{4\pi r}, \tag{5}$$

$$\left(\nabla_t^2 + k^2\right) G_t(k;\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho}), \qquad G_t(k;\mathbf{r}) = \frac{1}{4j} H_0^{(2)}(k\rho), \tag{6}$$

$$\left(\partial_z^2 + k^2\right) G_z(k;z) = -\delta(z), \qquad G_z(k;z) = \frac{1}{2jk} e^{-jk|z|}. \tag{7}$$

$\underline{L(\nabla) = \overline{\overline{S}} : \nabla\nabla + k^2}$

The basic Green function problem and solution (5) can be transformed affinely through a symmetric dyadic $\overline{\overline{S}}^{1/2}$ [2] as

$$\mathbf{r} \rightarrow \overline{\overline{S}}^{-1/2} \cdot \mathbf{r}, \qquad \nabla \rightarrow \overline{\overline{S}}^{1/2} \cdot \nabla, \tag{8}$$

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} \rightarrow D_s = \sqrt{\overline{\overline{S}}^{-1} : \mathbf{rr}}, \qquad \nabla^2 \rightarrow \overline{\overline{S}} : \nabla\nabla \tag{9}$$

to the form

$$\left(\overline{\overline{S}}:\nabla\nabla+k^2\right)G\left(k;\overline{\overline{S}}^{-1/2}\cdot\mathbf{r}\right)=-\delta\left(\overline{\overline{S}}^{-1/2}\cdot\mathbf{r}\right)=-\sqrt{\det\overline{\overline{S}}}\,\delta(\mathbf{r}). \quad (10)$$

Thus, the solution of

$$\left(\overline{\overline{S}}:\nabla\nabla+k^2\right)G_s(k;\mathbf{r})=-\delta(\mathbf{r}) \quad (11)$$

can be expressed as

$$G_s(k;\mathbf{r})=\frac{1}{\sqrt{\det\overline{\overline{S}}}}G\left(k;\overline{\overline{S}}^{-1/2}\cdot\mathbf{r}\right)=\frac{e^{-jkD_s}}{4\pi\sqrt{\det\overline{\overline{S}}}\,D_s}. \quad (12)$$

Here we have assumed that the symmetric dyadic $\overline{\overline{S}}$ is complete, i.e., that it has the inverse $\overline{\overline{S}}^{-1}$. In the converse case this leads to a two-dimensional problem considered next.

One has yet to define the branch of the distance function D_s so that the result satisfies radiation conditions in the infinity, which definition obviously depends on the dyadic $\overline{\overline{S}}$. For example, if $\overline{\overline{S}}$ is real or Hermitian and positive definite, we have $\overline{\overline{S}}:\mathbf{rr}>0$ for $\mathbf{r}\neq 0$, and the positive square root must be chosen to ensure outward propagation of the wave.

$\underline{L(\nabla_t)=\overline{\overline{S}}_t:\nabla_t\nabla_t+k^2}$

To study the affine transformation of the two-dimensional Green function problem (6) we introduce a two-dimensional symmetric dyadic $\overline{\overline{S}}_t$ as a part of the three-dimensional dyadic

$$\overline{\overline{S}}=\overline{\overline{S}}_t+S\mathbf{u}_z\mathbf{u}_z,\qquad \mathbf{u}_z\cdot\overline{\overline{S}}_t=\overline{\overline{S}}_t\cdot\mathbf{u}_z=0. \quad (13)$$

From the inverse [2]

$$\overline{\overline{S}}^{-1}=\frac{1}{S}\mathbf{u}_z\mathbf{u}_z+\frac{\mathbf{u}_z\mathbf{u}_z\,{}^{\times}_{\times}\,\overline{\overline{S}}_t}{\mathrm{spm}\overline{\overline{S}}_t},\qquad \mathrm{spm}\overline{\overline{S}}_t=\frac{1}{2}\overline{\overline{S}}_t\,{}^{\times}_{\times}\,\overline{\overline{S}}_t:\overline{\overline{I}} \quad (14)$$

we can identify its two-dimensional part and call it the two-dimensional inverse of $\overline{\overline{S}}_t$:

$$\overline{\overline{S}}_t^{-1}=\frac{\mathbf{u}_z\mathbf{u}_z\,{}^{\times}_{\times}\,\overline{\overline{S}}_t}{\mathrm{spm}\overline{\overline{S}}_t}. \quad (15)$$

Thus, we can define the two-dimensional affine transformation as

$$\boldsymbol{\rho} \to \overline{\overline{S}}_t^{-1/2} \cdot \boldsymbol{\rho}, \quad \rho = \sqrt{\boldsymbol{\rho}\cdot\boldsymbol{\rho}} \to \sqrt{\overline{\overline{S}}_t^{-1} : \boldsymbol{\rho}\boldsymbol{\rho}} = \sqrt{\frac{\overline{\overline{S}}_t : (\mathbf{u}_z \times \boldsymbol{\rho})(\mathbf{u}_z \times \boldsymbol{\rho})}{\mathrm{spm}\overline{\overline{S}}_t}}, \tag{16}$$

$$\nabla_t \to \overline{\overline{S}}_t^{1/2} \cdot \nabla_t, \qquad \nabla_t^2 \to \overline{\overline{S}}_t : \nabla_t\nabla_t, \tag{17}$$

and transform the problem to

$$\left(\overline{\overline{S}}_t : \nabla_t\nabla_t + k^2\right) G_t\left(k; \overline{\overline{S}}_t^{-1/2} \cdot \rho\right) = -\delta\left(\overline{\overline{S}}_t^{-1/2} \cdot \boldsymbol{\rho}\right) = -\sqrt{\mathrm{spm}\overline{\overline{S}}_t}\,\delta(\boldsymbol{\rho}). \tag{18}$$

The solution to

$$\left(\overline{\overline{S}}_t : \nabla_t\nabla_t + k^2\right) G_{st}(k;\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho}) \tag{19}$$

can thus be written as

$$\begin{aligned} G_{st}(k;\boldsymbol{\rho}) &= \frac{1}{\sqrt{\mathrm{spm}\overline{\overline{S}}_t}} G_t\left(k; \overline{\overline{S}}_t^{-1/2} \cdot \rho\right) = \frac{1}{4j\sqrt{\mathrm{spm}\overline{\overline{S}}_t}} H_0^{(2)}\left(k\sqrt{\overline{\overline{S}}_t^{-1} : \boldsymbol{\rho}\boldsymbol{\rho}}\right) \\ &= \frac{1}{4j\sqrt{\mathrm{spm}\overline{\overline{S}}_t}} H_0^{(2)}\left(k\frac{\sqrt{\overline{\overline{S}}_t : (\mathbf{u}_z \times \boldsymbol{\rho})(\mathbf{u}_z \times \boldsymbol{\rho})}}{\sqrt{\mathrm{spm}\overline{\overline{S}}_t}}\right). \end{aligned} \tag{20}$$

Here we assume that $\mathrm{spm}\overline{\overline{S}}_t \neq 0$. In the converse case the problem becomes basically one dimensional.

The previous two-dimensional problem (19) can also be written in three-dimensional form by adding the delta function $\delta(z)$:

$$\left(\overline{\overline{S}}_t : \nabla\nabla + k^2\right) [G_{st}(k;\mathbf{r})\delta(z)] = -\delta(\mathbf{r}), \tag{21}$$

whence the corresponding solution is $G_{st}(k;\boldsymbol{\rho})\delta(z)$, where $G_{st}(k;\boldsymbol{\rho})$ equals (20).

$\underline{L(\nabla) = \mathbf{ab} : \nabla\nabla + k^2}$

This problem is related to the two-dimensional one because we can assume without restricting generality that $\mathbf{a}$ and $\mathbf{b}$ are vectors orthogonal to $\mathbf{u}_z$. The case of parallel vectors $(\mathbf{a} \times \mathbf{b} = 0)$ leads basically to the operator of the one-dimensional type $\partial_x^2 + k^2$ and is excluded. The Green function satisfies

$$L(\nabla)G(\mathbf{r}) = \left(\overline{\overline{S}}_t : \nabla\nabla + k^2\right) G(\mathbf{r}) = -\delta(\mathbf{r}), \tag{22}$$

where $\overline{\overline{S}}_t$ is the symmetric part of the dyad $\mathbf{ab}$,

$$\overline{\overline{S}}_t = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}). \tag{23}$$

Assuming

$$\mathrm{spm}\overline{\overline{S}}_t = -\frac{1}{4}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) \neq 0, \tag{24}$$

we can apply (20) to give the solution

$$\begin{aligned} G(\mathbf{r}) &= G_{st}(k; \boldsymbol{\rho})\delta(z) \\ &= \frac{1}{2j\sqrt{-(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})}} H_0^{(2)}\left(2k\frac{\sqrt{(\mathbf{u}_z \cdot \mathbf{a} \times \boldsymbol{\rho})(\mathbf{u}_z \cdot \mathbf{b} \times \boldsymbol{\rho})}}{\sqrt{-(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})}}\right)\delta(z). \end{aligned} \tag{25}$$

The question about the branches of the square roots has been discussed in [6].

$\underline{L(\nabla) = \overline{\overline{S}} : \nabla\nabla + 2\mathbf{a} \cdot \nabla + k^2}$

Assuming $\overline{\overline{S}}$ complete and symmetric, this operator can be expressed as

$$\begin{aligned} L(\nabla) &= \overline{\overline{S}} : \nabla\nabla + 2\mathbf{a} \cdot \nabla + k^2 \\ &= \overline{\overline{S}} : \left(\nabla + \overline{\overline{S}}^{-1} \cdot \mathbf{a}\right)\left(\nabla + \overline{\overline{S}}^{-1} \cdot \mathbf{a}\right) + k_a^2, \quad k_a = \sqrt{k^2 - \overline{\overline{S}}^{-1} : \mathbf{aa}}. \end{aligned} \tag{26}$$

Because for any function $f(\mathbf{r})$ we can write

$$\left(\nabla + \overline{\overline{S}}^{-1} \cdot \mathbf{a}\right) f(\mathbf{r}) = \exp\left(-\mathbf{a} \cdot \overline{\overline{S}}^{-1} \cdot \mathbf{r}\right) \nabla \left[\exp\left(\mathbf{a} \cdot \overline{\overline{S}}^{-1} \cdot \mathbf{r}\right) f(\mathbf{r})\right], \tag{27}$$

the equation for the Green function can be written as

$$\left(\overline{\overline{S}} : \nabla\nabla + k_a^2\right)\left[\exp\left(\mathbf{a} \cdot \overline{\overline{S}}^{-1} \cdot \mathbf{r}\right) G(\mathbf{r})\right] = -\exp\left(\mathbf{a} \cdot \overline{\overline{S}}^{-1} \cdot \mathbf{r}\right) \delta(\mathbf{r}) = -\delta(\mathbf{r}). \tag{28}$$

This has the solution

$$G(\mathbf{r}) = \exp\left(-\mathbf{a}\cdot\overline{\overline{S}}^{-1}\cdot\mathbf{r}\right) G_s(k_a,\mathbf{r}) = \exp\left(\mathbf{a}\cdot\overline{\overline{S}}^{-1}\cdot\mathbf{r}\right)\frac{e^{-jk_aD_s}}{4\pi D_s}. \quad (29)$$

The special case $\mathbf{a}=0$ gives the earlier result (12).

3.2 Fourth-Order Operators

The fourth-order operators considered here are all products of second-order operators which means that the most general cases are not covered. Some of the factorized fourth-order operator problems can be solved in terms of solutions to the second-order problems. As a simple example of such a problem we consider the following:

$$\nabla_t^2\left(\partial_z^2+k^2\right)G(\mathbf{r}) = -\delta(\mathbf{r}). \quad (30)$$

This can be solved by splitting it in two factors as

$$G(\mathbf{r}) = \frac{-1}{\nabla_t^2\left(\partial_z^2+k^2\right)}\delta(\mathbf{r}) = -\frac{-1}{\nabla_t^2}\delta(\boldsymbol{\rho})\frac{-1}{\partial_z^2+k^2}\delta(z) = \frac{1}{4\pi jk}\ln(\gamma\rho)e^{-jk|z|}. \quad (31)$$

$$\underline{L(\nabla) = \left(D(\nabla)+k_1^2\right)\left(D(\nabla)+k_2^2\right)}$$

Here $D(\nabla)$ is a second-order operator. The Green function for the factorized operator $L(\nabla)$ can be solved in terms of the Green functions of the factor operators. This can be done by making the partial fraction expansion

$$\begin{aligned} G(\mathbf{r}) &= \frac{-1}{\left(D(\nabla)+k_1^2\right)\left(D(\nabla)+k_2^2\right)}\delta(\mathbf{r}) \\ &= \frac{1}{k_2^2-k_1^2}\frac{-1}{D(\nabla)+k_1^2}\delta(\mathbf{r}) + \frac{1}{k_1^2-k_2^2}\frac{-1}{D(\nabla)+k_2^2}\delta(\mathbf{r}) \\ &= \frac{1}{k_2^2-k_1^2}G_D(k_1;\mathbf{r}) + \frac{1}{k_1^2-k_2^2}G_D(k_2;\mathbf{r}). \end{aligned} \quad (32)$$

The Green function $G_D(k;\mathbf{r})$ is the solution to

$$\left(D\left(\nabla\right)+k^2\right)G_D(k;\mathbf{r}) = -\delta(\mathbf{r}). \quad (33)$$

This method can be applied to various operators. For example, when $D(\nabla) = \partial_z^2$, we have from (7)

$$G_D(k; \mathbf{r}) = \frac{e^{-jk|z|}}{2jk}\delta(\boldsymbol{\rho}), \tag{34}$$

$$G(\mathbf{r}) = \left(\frac{1}{k_2^2 - k_1^2} \frac{e^{-jk_1|z|}}{2jk_1} + \frac{1}{k_1^2 - k_2^2} \frac{e^{-jk_2|z|}}{2jk_2} \right) \delta(\boldsymbol{\rho}). \tag{35}$$

The latter can be rewritten by introducing new parameters k, K as

$$k_2 = k + K, \qquad k_1 = k - K, \tag{36}$$

whence we have

$$G(\mathbf{r}) = \frac{e^{-jk|z|}}{4jkk_1k_2} \left[\cos(K|z|) + jk|z| \frac{\sin(K|z|)}{K|z|} \right] \delta(\boldsymbol{\rho}). \tag{37}$$

$\underline{L(\nabla) = \left(D(\nabla) + k^2\right)^2}$

The Green function corresponding to an operator of the square type is obtained as the limit $k_1 \rightarrow k_2 \rightarrow k$ from the previous case. Denoting $k_1 = k$ and $k_2 = k + \Delta$, we have

$$\begin{aligned} G(\mathbf{r}) &= \frac{-1}{\left(D(\nabla) + k^2\right)^2}\delta(\mathbf{r}) = -\lim_{\Delta \to 0} \frac{G_D(k_2; \mathbf{r}) - G_D(k_1; \mathbf{r})}{k_2^2 - k_1^2} \\ &= -\lim_{\Delta \to 0} \frac{G(k + \Delta, \mathbf{r}) - G(k; \mathbf{r})}{\Delta(2k + \Delta)} = -\frac{1}{2k}\frac{\partial}{\partial k}G(k; \mathbf{r}). \end{aligned} \tag{38}$$

Applying this to the basic operators in one, two and three dimensions gives us

$$\frac{-1}{(\partial_z^2 + k^2)^2}\delta(\mathbf{r}) = (1 + jk|z|)\frac{e^{-jk|z|}}{4jk^3}\delta(\boldsymbol{\rho}), \tag{39}$$

$$\frac{-1}{\left(\nabla_t^2 + k^2\right)^2}\delta(\mathbf{r}) = \frac{\rho}{8jk}H_1^{(2)}(k\rho)\delta(z), \tag{40}$$

$$\frac{-1}{\left(\nabla^2 + k^2\right)^2}\delta(\mathbf{r}) = \frac{je^{-jkr}}{8\pi k}. \tag{41}$$

These expressions can be checked by operating once by $D(\nabla)+k^2$. For example,

$$\left(D(\nabla)+k^2\right)\frac{-1}{\left(D(\nabla)+k^2\right)^2}\delta(\mathbf{r})=\frac{-1}{D(\nabla)+k^2}\delta(\mathbf{r})=G_D(k;\mathbf{r}) \quad (42)$$

gives the Green dyadic corresponding to the operator in the previous section. For example, operating (39) by $\partial_z^2+k^2$ is seen to give $(e^{-jk|z|}/2jk)\delta(\boldsymbol{\rho})$.

$\underline{L(\nabla)=\nabla_t^2\left(\nabla^2+k^2\right)}$

The Green function corresponding to this operator can be expressed as

$$G(\boldsymbol{\rho},z)=\frac{-1}{\nabla_t^2\left(\nabla^2+k^2\right)}\delta(\mathbf{r})=\frac{1}{\nabla_t^2}\frac{e^{-jkr}}{4\pi r}. \quad (43)$$

This can be solved through the following trick [4]. Writing the spherically symmetric Green function in the form

$$\frac{e^{-jkr}}{4\pi r}=\frac{1}{\rho}\partial_\rho\left(\frac{je^{-jkr}}{4\pi k}\right), \quad (44)$$

the axially symmetric Green function $G(\boldsymbol{\rho},z)$ satisfies

$$\nabla_t^2 G(\boldsymbol{\rho},z)=\frac{1}{\rho}\partial_\rho\left[\rho\partial_\rho G(\rho,z)\right]=\frac{1}{\rho}\partial_\rho\left[\frac{je^{-jkr}}{4\pi k}\right]. \quad (45)$$

The expressions in square brackets must be the same except for an arbitrary function of z denoted by $f(z)$:

$$\partial_\rho G(\boldsymbol{\rho},z)=\frac{je^{-jkr}}{4\pi k\rho}+\frac{1}{\rho}f(z). \quad (46)$$

The Green function $G(\boldsymbol{\rho},z)$ can be obtained through integration [7] as

$$G(\boldsymbol{\rho},z)=-\int_\rho^\infty\frac{je^{-jk\sqrt{\rho'^2+z^2}}}{4\pi k\rho'}d\rho'+f(z)\ln\gamma\rho$$
$$=-\frac{je^{-jkr}}{8\pi k}\int_0^\infty\left(\frac{e^{-jkt}}{t+r-z}+\frac{e^{-jkt}}{t+r+z}\right)dt+f(z)\ln(\gamma\rho). \quad (47)$$

Here, γ is an arbitrary scalar making $\gamma\rho$ a dimensionless number. The integrals can be expanded as

$$\int_0^\infty \frac{e^{-jy}}{y+x} dy = E_1(jx)e^{jx}, \tag{48}$$

where $E_1(x)$ is the exponential integral [5]

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt, \qquad \partial_x E_1(x) = -\frac{e^{-x}}{x}. \tag{49}$$

The Green function has thus the form

$$G(\boldsymbol{\rho}, z) = \frac{1}{8j\pi k} \left[E_1(jk(r-z))e^{-jkz} + E_1(jk(r+z))e^{jkz} + f(z)\ln(\gamma\rho) \right], \tag{50}$$

where a numerical factor has been absorbed in the arbitrary function $f(z)$. Let us postpone its choice to the next example. The final form is given in (55).

As a simple check of (50), let us differentiate it by ∂_ρ and apply (49) together with the property $\partial_\rho = (\rho/r)\partial_r$ when operating on a function of r. Doing this, we arrive at

$$\partial_\rho G(\boldsymbol{\rho}, z) = -\frac{\rho}{8\pi r} \left(\frac{e^{-jk(r-z)}}{r-z} e^{-jkz} + \frac{e^{-jk(r+z)}}{r+z} e^{jkz} \right) = \frac{je^{-jkr}}{4\pi k\rho}, \tag{51}$$

which coincides with (46), the starting point. Here one should note that $\partial_\rho G(\boldsymbol{\rho}, z)$ is sufficient when forming the Green dyadic and the Green function $G(\boldsymbol{\rho}, z)$ itself is not needed [4].

$\underline{L(\nabla) = \left(\partial_z^2 + k^2\right)\left(\nabla^2 + k^2\right)}$

This operator can be reduced to the previous one through a partial fraction expansion:

$$G(z, \boldsymbol{\rho}) = \frac{-1}{(\partial_z^2 + k^2)(\nabla^2 + k^2)} \delta(\mathbf{r}) = \frac{-1}{\nabla_t^2 (\partial_z^2 + k^2)} \delta(\mathbf{r}) - \frac{-1}{\nabla_t^2 (\nabla^2 + k^2)} \delta(\mathbf{r}). \tag{52}$$

Applying now the results (31) and (50), the Green function can be expressed as

$$G(z,\boldsymbol{\rho}) = \frac{j}{8\pi k}\Big[2e^{-jk|z|}\ln(\gamma\rho) + E_1(jk(r-z))e^{-jkz} + E_1(jk(r+z))e^{jkz} - f(z)\ln(\gamma\rho)\Big]. \qquad (53)$$

At this point we may consider the term containing the arbitrary function $f(z)$. Since in the present case the operator contains ∂_z^2 and ∇^2, it is natural to have a solution in terms of r and z variables only. The variable ρ can be eliminated by choosing $f(z) = 2e^{-jk|z|}$. Thus the present Green function is simplified in form to

$$G(z,\boldsymbol{\rho}) = \frac{j}{8\pi k}\Big[E_1(jk(r-z))e^{-jkz} + E_1(jk(r+z))e^{jkz}\Big]. \qquad (54)$$

and that of the previous operator, (50), to the form

$$G(\boldsymbol{\rho},z) = \frac{1}{8j\pi k}\Big[E_1(jk(r-z))e^{-jkz} + E_1(jk(r+z))e^{jkz} - 2e^{-jk|z|}\ln(k\rho)\Big]. \qquad (55)$$

Here the arbitrary factor γ has been chosen as $\gamma = k$. (54) can also be derived using Fourier transform techniques showing that the terms with $\ln(\gamma\rho)$ must vanish. This is shown explicitly for the more general case below.

<u>$L(\nabla) = \left(\partial_z^2 + k_2^2\right)\left(\nabla^2 + k_1^2\right)$</u>

The Green function of this operator can be written in the form of a series expansion. Let us Fourier transform the equation

$$L(\nabla)G(\mathbf{r}) = \left(\partial_z^2 + k_2^2\right)\left(\nabla^2 + k_1^2\right)G(\mathbf{r}) = -\delta(\mathbf{r}) \qquad (56)$$

with respect to z and denote the corresponding spectral variable by β. The spectral Green function $G(\boldsymbol{\rho},\beta)$ then satisfies

$$\left(\nabla_t^2 + k_1^2 - \beta^2\right)G(\boldsymbol{\rho},\beta) = -\frac{\delta(\boldsymbol{\rho})}{k_2^2 - \beta^2}. \qquad (57)$$

The solution of this equation is given by

$$G(\boldsymbol{\rho}, \beta) = -\frac{j}{4} \frac{H_0^{(2)}\left(\sqrt{k_1^2 - \beta^2}\rho\right)}{k_2^2 - \beta^2}. \tag{58}$$

Now we can apply the Neumann series expansion [13, p. 130]

$$H_0^{(2)}\left(\sqrt{k_1^2 - \beta^2}\rho\right) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left[\frac{\rho(k_2^2 - k_1^2)}{2\sqrt{k_2^2 - \beta^2}}\right]^n H_n^{(2)}\left(\rho\sqrt{k_2^2 - \beta^2}\right). \tag{59}$$

After some manipulations this allows us to write $G(\boldsymbol{\rho}, \beta)$ as

$$G(\boldsymbol{\rho}, \beta) = -\frac{j}{4} \sum_{n=0}^{+\infty} \frac{1}{n!} \left[\frac{\rho(k_2^2 - k_1^2)}{2}\right]^n \int_{\rho}^{+\infty} \frac{1}{(\rho')^n} \frac{H_{n+1}^{(2)}\left(\rho'\sqrt{k_2^2 - \beta^2}\right)}{\left(\sqrt{k_2^2 - \beta^2}\right)^{n+1}} d\rho'. \tag{60}$$

In [14, p. 56, Eq. (41)] the following integral can be found

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{j}{4} \frac{H_{n+1}^{(2)}\left(\rho\sqrt{k_2^2 - \beta^2}\right)}{\left(\sqrt{k_2^2 - \beta^2}\right)^{n+1}} e^{-j\beta z} d\beta = -\frac{r}{4\pi\rho} \left(\frac{r}{k_2\rho}\right)^n h_n^{(2)}(k_2 r), \tag{61}$$

with $h_n^{(2)}(k_2 r)$ denoting the spherical Hankel function of order n and second kind given by

$$h_n^{(2)}(k_2 r) = \frac{j^{n+1}}{k_2 r} e^{-jk_2 r} \sum_{m=0}^{n} \left(-\frac{j}{2k_2 r}\right)^m \frac{(n+m)!}{m!(n-m)!}. \tag{62}$$

This means that the inverse Fourier transformation $G(\mathbf{r})$ of $G(\boldsymbol{\rho}, \beta)$ can be written as

$$G(\mathbf{r}) = \frac{1}{4\pi} \sum_{n=0}^{+\infty} \frac{1}{n!} \left[\frac{\rho\left(k_2^2 - k_1^2\right)}{2k_2}\right]^n \int_{\rho}^{+\infty} \frac{r'}{\rho'} \left(\frac{r'}{(\rho')^2}\right)^n h_n^{(2)}\left(k_2 r'\right) d\rho', \tag{63}$$

with $r' = \sqrt{(\rho')^2 + z^2}$.

Now let us consider the following integral

$$I = \int_{\rho}^{+\infty} \frac{(r')^{n-m}}{(\rho')^{2n+1}} e^{-jk_2 r'} d\rho', \tag{64}$$

with $m = 0, 1, \ldots, n$. We can also write this as

$$I = j^{n-m} \frac{\partial^{n-m}}{\partial k_2^{n-m}} \int_r^{+\infty} \frac{e^{-jk_2\tau}}{(\tau^2 - z^2)^{n+1}} \tau d\tau. \tag{65}$$

If we now expand the rational factor in the integrand in partial fractions this becomes

$$I = j^{n-m} \frac{\partial^{n-m}}{\partial k_2^{n-m}} \int_r^{+\infty} e^{-jk_2\tau} \sum_{i=1}^{n+1} \alpha_i^{(n)} \left[\frac{1}{(\tau - z)^i} - \frac{(-1)^i}{(\tau + z)^i} \right] d\tau, \tag{66}$$

where the coefficients $\alpha_i^{(n)}$ only depend on z and follow from identifying (66) with (65). It can be checked that $\alpha_1^{(n)} = 0$ for $n > 0$. A general expression for $\alpha_i^{(n)}$ is not too easy, although not impossible, to obtain [15]. Some functions $\alpha_i^{(n)}(z)$ can be written simply as

$$\alpha_1^{(0)} = 1/2,\ \alpha_2^{(1)} = 1/(4z),\ \alpha_2^{(2)} = -1/\left(16z^3\right),\ \alpha_3^{(2)} = 1/\left(8z^2\right). \tag{67}$$

Now we can evaluate (66) using exponential integrals ([16] p.228)

$$\begin{aligned} I = & j^{n-m} \frac{\partial^{n-m}}{\partial k_2^{n-m}} \sum_{i=1}^{n+1} \alpha_i^{(n)} \left\{ \frac{e^{-jk_2 z} E_i \left[jk_2(r - z) \right]}{(r - z)^{i-1}} \right. \\ & \left. - \frac{(-1)^i e^{jk_2 z} E_i \left[jk_2(r + z) \right]}{(r + z)^{i-1}} \right\}, \end{aligned} \tag{68}$$

where E_i is the exponential integral of order i. With this result the expression for $G(\mathbf{r})$ finally becomes

$$\begin{aligned} G(\mathbf{r}) = & \frac{j}{4\pi k_2} \sum_{n=0}^{+\infty} \left\{ \frac{1}{n!} \left[\frac{\rho\left(k_1^2 - k_2^2\right)}{2k_2} \right]^n \left[\sum_{m=0}^{n} \left(-\frac{1}{2k_2} \right)^m \frac{(n+m)!}{m!(n-m)!} \frac{\partial^{n-m}}{\partial k_2^{n-m}} \right] \right. \\ & \left. \sum_{i=1}^{n+1} \alpha_i^{(n)} \left\{ \frac{e^{-jk_2 z} E_i \left[jk_2(r - z) \right]}{(r - z)^{i-1}} - \frac{(-1)^i e^{jk_2 z} E_i \left[jk_2(r+z) \right]}{(r + z)^{i-1}} \right\} \right\}. \end{aligned} \tag{69}$$

Note that when $k_2 = k_1$ only the term with $n = 0$ survives and the result of (54) is recovered.

$\underline{L(\nabla) = \nabla_t^2 \left(\overline{\overline{\alpha}} : \nabla\nabla + k^2\right)}$

The dyadic $\overline{\overline{\alpha}}$ in the operator is assumed uniaxial:

$$\overline{\overline{\alpha}} = \alpha_t \overline{\overline{I}}_t + \alpha_z \mathbf{u}_z \mathbf{u}_z. \tag{70}$$

The Green function is obtained through affine transformation and the previous result. Starting with

$$\nabla \to \overline{\overline{\alpha}}^{1/2} \cdot \nabla = \sqrt{\alpha_t} \nabla_t + \sqrt{\alpha_z} \mathbf{u}_z \partial_z, \quad \mathbf{r} \to \overline{\overline{\alpha}}^{-1/2} \cdot \mathbf{r} = \frac{\boldsymbol{\rho}}{\sqrt{\alpha_t}} + \mathbf{u}_z \frac{z}{\sqrt{\alpha_z}}, \tag{71}$$

we can transform

$$\nabla_t^2 \left(\nabla^2 + k^2\right) G(\boldsymbol{\rho}, z) = -\delta(\mathbf{r}) \tag{72}$$

to

$$\begin{aligned} &\alpha_t \nabla_t^2 \left(\overline{\overline{\alpha}} : \nabla\nabla + k^2\right) G\left(\frac{\boldsymbol{\rho}}{\sqrt{\alpha_t}}, \frac{z}{\sqrt{\alpha_z}}\right) \\ &\quad = -\delta\left(\overline{\overline{\alpha}}^{-1/2} \cdot \mathbf{r}\right) = -\sqrt{\det \overline{\overline{\alpha}}}\, \delta(\mathbf{r}) - \alpha_t \sqrt{\alpha_z}\, \delta(\mathbf{r}). \end{aligned} \tag{73}$$

Thus, the solution of

$$L(\nabla) G_\alpha(\mathbf{r}) = \nabla_t^2 \left(\overline{\overline{\alpha}} : \nabla\nabla + k^2\right) G_\alpha(\mathbf{r}) = -\delta(\mathbf{r}) \tag{74}$$

becomes

$$G_\alpha(\mathbf{r}) = \frac{1}{\sqrt{\alpha_z}} G\left(\frac{\boldsymbol{\rho}}{\sqrt{\alpha_t}}, \frac{z}{\sqrt{\alpha_z}}\right). \tag{75}$$

When we substitute the expression (55), the resulting Green function is

$$\begin{aligned} G_\alpha(\mathbf{r}) = \frac{1}{8j\pi k \sqrt{\alpha_z}} &\left[E_1\left(jk\left(D_\alpha - \frac{z}{\sqrt{\alpha_z}}\right)\right) e^{-jkz/\sqrt{\alpha_z}} \right. \\ &\left. + E_1\left(jk\left(D_\alpha + \frac{z}{\sqrt{\alpha_z}}\right)\right) e^{jkz/\sqrt{\alpha_z}} - 2e^{-jk|z|/\sqrt{\alpha_z}} \ln\left(k\rho/\sqrt{\alpha_t}\right) \right], \end{aligned} \tag{76}$$

where the distance function $D_\alpha(\mathbf{r})$ is defined by

$$D_\alpha = \sqrt{\overline{\overline{\alpha}}^{-1} : \mathbf{rr}} = \sqrt{\frac{\rho^2}{\alpha_t} + \frac{z^2}{\alpha_z}}. \tag{77}$$

$\underline{L(\nabla) = \left(\partial_z^2 + k^2\right)\left(\alpha\nabla_t^2 + \partial_z^2 + k^2\right)}$

We can also generalize the Green function of the operator $(\partial_z^2+k^2)(\nabla^2+k^2)$ affinely through the uniaxial dyadic $\overline{\overline{\alpha}}$ by assuming $\alpha_z = 1$ and denoting $\alpha_t = \alpha$. The Green function corresponding to the operator

$$L(\nabla) = \left(\partial_z^2 + k^2\right)\left(\alpha\nabla_t^2 + \partial_z^2 + k^2\right) \tag{78}$$

is obtained from (54) by setting $\rho\to\rho/\sqrt{\alpha}$ and multiplying by α :

$$G(z,\rho) = \frac{j\alpha}{8\pi k}\left[E_1\left(jk(D_\alpha - z)\right)e^{-jkz} + E_1\left(jk(D_\alpha + z)\right)e^{jkz}\right]. \tag{79}$$

The distance function is now

$$D_\alpha = \sqrt{\frac{\rho^2}{\alpha} + z^2}. \tag{80}$$

$\underline{L(\nabla) = \left(\overline{\overline{S}} : \nabla\nabla + k_1^2\right)\left(\overline{\overline{S}} : \nabla\nabla + k_2^2\right)}$

From the knowledge of the Green function

$$G_s(k;\mathbf{r}) = \frac{-1}{\overline{\overline{S}} : \nabla\nabla + k^2}\delta(\mathbf{r}) = \frac{e^{-jkD_s}}{4\pi D_s\sqrt{\det\overline{\overline{S}}}}, \quad D_s = \sqrt{\overline{\overline{S}}^{-1} : \mathbf{rr}}, \tag{81}$$

and applying the partial fraction expansion method, we can form the Green function corresponding to the fourth-order operator equation

$$L(\nabla)G_{12}(\mathbf{r}) = \left(\overline{\overline{S}} : \nabla\nabla + k_1^2\right)\left(\overline{\overline{S}} : \nabla\nabla + k_2^2\right)G_{12}(\mathbf{r}) = -\delta(\mathbf{r}) \tag{82}$$

as

$$G_{12}(\mathbf{r}) = \frac{G_s(k_1;\mathbf{r})}{k_2^2 - k_1^2} + \frac{G_s(k_2;\mathbf{r})}{k_1^2 - k_2^2}. \tag{83}$$

To find the Green function for the fourth-order operator $\left(\overline{\overline{S}}:\nabla\nabla+k^2\right)^2$ through the limit $k_1\to k_2\to k$ is somewhat tedious. The result can however be simply obtained through differentiation as

$$\begin{aligned} G_2(\mathbf{r}) &= \frac{-1}{\left(\overline{\overline{S}} : \nabla\nabla + k^2\right)^2}\delta(\mathbf{r}) = -\frac{1}{2k}\partial_k\frac{-1}{\overline{\overline{S}} : \nabla\nabla + k^2}\delta(\mathbf{r}) \\ &= -\frac{1}{2k}\partial_k\frac{e^{-jkD_s}}{4\pi D_s\sqrt{\det\overline{\overline{S}}}} = \frac{je^{-jkD_s}}{8\pi k\sqrt{\det\overline{\overline{S}}}}. \end{aligned} \tag{84}$$

3.3 Higher-Order Operators

$L(\nabla) = \left(D(\nabla) + k^2\right)^n$

The method given previously for the squares of operators can be generalized to any positive powers through differentiation as follows:

$$\partial_{k^2} \frac{-1}{D(\nabla) + k^2} \delta(\mathbf{r}) = -\frac{-1}{(D(\nabla) + k^2)^2} \delta(\mathbf{r}), \tag{85}$$

where we denote

$$\partial_{k^2} = \frac{1}{2k} \frac{\partial}{\partial k}. \tag{86}$$

It must be assumed that the operator $D(\nabla)$ does not depend on the parameter k.

The differentiation principle can be applied many times:

$$\partial^2_{k^2} \frac{-1}{D(\nabla) + k^2} \delta(\mathbf{r}) = 2 \frac{-1}{(D(\nabla) + k^2)^3} \delta(\mathbf{r}), \tag{87}$$

$$\partial^{n-1}_{k^2} \frac{-1}{D(\nabla) + k^2} \delta(\mathbf{r}) = (-1)^{n-1} (n-1)! \frac{-1}{(D(\nabla) + k^2)^n} \delta(\mathbf{r}). \tag{88}$$

This makes it possible to derive expressions for Green functions corresponding to any powers of operators of this kind. Assuming

$$G_1(\mathbf{r}) = G(k; \mathbf{r}) = \frac{-1}{D(\nabla) + k^2} \delta(\mathbf{r}), \tag{89}$$

we have

$$G_n(\mathbf{r}) = \frac{-1}{(D(\nabla) + k^2)^n} \delta(\mathbf{r}) = \frac{(-1)^{n-1}}{(n-1)!} \partial^{n-1}_{k^2} G(k; \mathbf{r}). \tag{90}$$

For example, we have quite straightforwardly

$$\frac{-1}{(\nabla^2 + k^2)^3} \delta(\mathbf{r}) = \frac{1}{2} \partial^2_{k^2} \frac{e^{-jkr}}{4\pi r} = \frac{1}{4k} \partial_k \left(\frac{1}{2k} \partial_k \frac{e^{-jkr}}{4\pi r} \right)$$
$$= \frac{j(1 + jkr)}{32\pi k^3} e^{-jkr}. \tag{91}$$

This satisfies

$$\left(\nabla^2 + k^2\right)^2 \frac{j(1 + jkr) e^{-jkr}}{32\pi k^3} = \left(\nabla^2 + k^2\right) \frac{j e^{-jkr}}{8\pi k} = -\delta(\mathbf{r}), \tag{92}$$

as can be easily checked.

$L(\nabla) = \left(\overline{\overline{S}} : \nabla\nabla + k^2\right)^n$

The previous method can now be applied to find higher-order Green functions defined by the operator

$$G_n(\mathbf{r}) = \frac{-1}{\left(\overline{\overline{S}} : \nabla\nabla + k^2\right)^n}\delta(\mathbf{r}). \tag{93}$$

In fact, identifying $G_1(\mathbf{r}) = G_s(\mathbf{r})$ of (12), we can write

$$\begin{aligned} G_n(\mathbf{r}) &= \frac{1}{n-1}\left(-\frac{1}{2k}\partial_k\right)G_{n-1}(k;\mathbf{r}) = \frac{1}{(n-1)(n-2)}\left(-\frac{1}{2k}\partial_k\right)^2 G_{n-2}(k;\mathbf{r}) \\ &= \frac{1}{(n-1)!}\left(-\frac{1}{2k}\partial_k\right)^{n-1} G_s(\mathbf{r}). \end{aligned} \tag{94}$$

It is easy to find the sequence of Green functions as

$$G_3(\mathbf{r}) = \frac{1}{2}\left(-\frac{1}{2k}\partial_k\right)^2 G_s(k;\mathbf{r}) = \frac{je^{-jkD_s}}{32\pi k^3\sqrt{\det\overline{\overline{S}}}}(1 + jkD_s), \tag{95}$$

$$G_4(\mathbf{r}) = \frac{1}{3}\left(-\frac{1}{2k}\partial_k\right)G_3(k;\mathbf{r}) = \frac{je^{-jkD_s}}{64\pi k^5\sqrt{\det\overline{\overline{S}}}}\left(1 + jkD_s + \frac{1}{3}(jkD_s)^2\right). \tag{96}$$

More generally, the Green function defined by an operator of the order $2n$:

$$G_{12\cdots n}(\mathbf{r}) = \frac{-1}{\prod_{i=1}^{n}\left(\overline{\overline{S}} : \nabla\nabla + k_i^2\right)}\delta(\mathbf{r}) \tag{97}$$

can be solved through the partial fraction expansion. For example, the Green function corresponding to $n = 3$, is

$$\begin{aligned} G_{123}(\mathbf{r}) &= \frac{-1}{\left(\overline{\overline{S}} : \nabla\nabla + k_1^2\right)\left(\overline{\overline{S}} : \nabla\nabla + k_2^2\right)\left(\overline{\overline{S}} : \nabla\nabla + k_3^2\right)}\delta(\mathbf{r}) \\ &= \frac{-1}{\left(k_2^2 - k_1^2\right)\left(k_3^2 - k_1^2\right)\left(\overline{\overline{S}} : \nabla\nabla + k_1^2\right)}\delta(\mathbf{r}) \end{aligned}$$

$$+\frac{-1}{\left(k_1^2-k_2^2\right)\left(k_3^2-k_2^2\right)\left(\overline{\overline{S}}:\nabla\nabla+k_2^2\right)}\delta(\mathbf{r})$$
$$+\frac{-1}{\left(k_1^2-k_3^2\right)\left(k_2^2-k_3^2\right)\left(\overline{\overline{S}}:\nabla\nabla+k_3^2\right)}\delta(\mathbf{r})$$
$$=\frac{G_s(k_1;\mathbf{r})}{\left(k_2^2-k_1^2\right)\left(k_3^2-k_1^2\right)}+\frac{G_s(k_2;\mathbf{r})}{\left(k_1^2-k_2^2\right)\left(k_3^2-k_2^2\right)}$$
$$+\frac{G_s(k_3;\mathbf{r})}{\left(k_1^2-k_3^2\right)\left(k_2^2-k_3^2\right)}. \tag{98}$$

The obvious generalization is

$$G_{12\cdots n}(\mathbf{r})=\sum_{i=1}^{n}\frac{G_s(k_i;\mathbf{r})}{\prod\limits_{\substack{j=1\\(j\neq i)}}^{n}\left(k_j^2-k_i^2\right)}. \tag{99}$$

3.4 Factorized Fourth-Order Operator

For a certain class of bi-anisotropic media, so-called decomposable media [8], the fourth-order Helmholtz determinant operator has been shown to be factorizable as a product of two second-order operators,

$$L(\nabla)=L_1(\nabla)L_2(\nabla). \tag{100}$$

In this case, by expressing

$$L_2(\nabla)=L_1(\nabla)+\Delta(\nabla), \tag{101}$$

the corresponding Green function

$$G(\mathbf{r})=-\frac{1}{L_1(\nabla)L_2(\nabla)}\delta(\mathbf{r}) \tag{102}$$

can be expressed as a formal operator series by applying the previous Green functions. In fact, assuming that the operator $\Delta(\nabla)$ is 'small', we can formally write the series expansion

$$G(\mathbf{r})=-\frac{1}{L_1(\nabla)\left[L_1(\nabla)+\Delta(\nabla)\right]}\delta(\mathbf{r})=-\frac{1}{L_1^2(\nabla)}\sum_{n=0}^{\infty}\frac{\left[-\Delta(\nabla)\right]^n}{L_1^n(\nabla)}\delta(\mathbf{r}). \tag{103}$$

If we can express the Green function of the inverse power operators, the resulting Green function is given as as a series of operators operating on known functions.

For example, consider the operator of the previous example,

$$L_1(\nabla) = \overline{\overline{S}} : \nabla\nabla + k^2. \tag{104}$$

Thus, the Green function satisfying

$$\left(\overline{\overline{S}} : \nabla\nabla + k^2\right)\left(\overline{\overline{S}} : \nabla\nabla + k^2 + \Delta(\nabla)\right) G(\mathbf{r}) = -\delta(\mathbf{r}) \tag{105}$$

is obtained in the form of an expansion involving operators,

$$G(\mathbf{r}) = -\sum_{n=0}^{\infty} \frac{[-\Delta(\nabla)]^n}{\left(\overline{\overline{S}} : \nabla\nabla + k^2\right)^{n+2}}\delta(\mathbf{r}) = -\sum_{n=0}^{\infty} \frac{[\Delta(\nabla)]^n\, \partial_{k^2}^{n+1}}{(n+1)!} G_s(k;\mathbf{r}). \tag{106}$$

This series has use in finding corrections to Green functions of square operators when the factorizable operator only slightly deviates from a square operator. A perturbative series method of this kind was first discussed by K. K. Mei [11] to find solution for the Green dyadic in an anisotropic medium. The same idea was used to find a series expression for the Green dyadic in a class of bi-anisotropic media, [12].

As a simple check of (106) let us take $\Delta = k_2^2 - k^2$, a constant. In this case, (106) reduces to

$$\begin{aligned} G(\mathbf{r}) &= -\frac{1}{\Delta}\sum_{n=0}^{\infty} \frac{[\Delta\partial_{k^2}]^{n+1}}{(n+1)!} G_s(k;\mathbf{r}) = -\frac{1}{\Delta}\left[e^{\Delta\partial_{k^2}} - 1\right] G_s(k;\mathbf{r}) \\ &= \frac{1}{k_2^2 - k^2}\left[G_s(k;\mathbf{r}) - G_s(k_2;\mathbf{r})\right], \end{aligned} \tag{107}$$

which coincides with (83) with $k_1 = k$. Here we have used the shifting operator as

$$e^{\Delta\partial_{k^2}} f(k) = e^{\Delta\partial_{k^2}} f\left(\sqrt{k^2}\right) = f\left(\sqrt{k^2 + \Delta}\right) = f(k_2). \tag{108}$$

3.5 Factorized Higher-Order Operators

For general higher-order operators that can be factorized in a product of second order operators it is possible to express the Green function as a double integral over finite intervals. Consider the operator of the form

$$L(\nabla) = \prod_{i=1}^{N} L_i(\nabla), \tag{109}$$

with

$$L_i(\nabla) = \overline{\overline{S}}_i : \nabla\nabla + 2\mathbf{a}_i \cdot \nabla + k_i^2. \tag{110}$$

Let us perform a spatial Fourier transformation in three dimensions where $\mathbf{k}$ becomes the Fourier variable. In spectral domain the expression (1) then becomes

$$L(-j\mathbf{k})G(\mathbf{k}) = -1. \tag{111}$$

To obtain the solution in physical space, the inverse Fourier transformation is formed as

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \iiint G(\mathbf{k}) e^{-j\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}, \tag{112}$$

where the integration runs over the whole three-dimensional $\mathbf{k}$-space,

Using spherical coordinates (k, θ, ϕ) in the $\mathbf{k}$-space we can write the inverse transformation as

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} dk \int_{0}^{\pi/2} d\theta \int_{0}^{2\pi} d\phi G(\mathbf{k}) e^{-j\mathbf{k}\cdot\mathbf{r}} k^2 \sin\theta. \tag{113}$$

Note that the k-integration is running from $-\infty$ to $+\infty$ and the θ-integration from 0 to $\pi/2$, which is somewhat unconventional but essential to the further evaluation.

Expressing the position vector $\mathbf{k}$ in the Fourier space in spherical coordinates as

$$\mathbf{k} = k\mathbf{u}, \qquad \mathbf{u} = \sin\theta\cos\phi\mathbf{u}_x + \sin\theta\sin\phi\mathbf{u}_y + \cos\theta\mathbf{u}_z, \tag{114}$$

we can write $G(\mathbf{k})$ as

$$G(\mathbf{k}) = \frac{-1}{\prod_{i=1}^{N} \left(-\overline{\overline{S}}_i : \mathbf{uu}\right)\left(k - k_i^+\right)\left(k - k_i^-\right)}, \tag{115}$$

with

$$k_i^{\pm} = \frac{\mathbf{a}_i \cdot \mathbf{u}}{\overline{\overline{S}}_i : \mathbf{uu}} \pm \sqrt{\left(\frac{\mathbf{a}_i \cdot \mathbf{u}}{\overline{\overline{S}}_i : \mathbf{uu}} \right)^2 + \frac{k_i^2}{\overline{\overline{S}}_i : \mathbf{uu}}}. \tag{116}$$

Now we assume that the branch cuts of the square roots in (116) are such that k_i^+ have a negative imaginary part and that k_i^- have a positive imaginary part. For real $k_i^{\pm}$ we assume that k_i^+ is positive and k_i^- is negative. In general this is not always possible however from physical grounds this will often be the case. This choice implements some kind of radiation condition.

Using the previous assumptions we can evaluate the k-integration with the Cauchy residue theorem in closed form for $\mathbf{u} \cdot \mathbf{r} > 0$ as

$$\begin{aligned} G(\mathbf{r}, \theta, \phi) &= \int_{-\infty}^{+\infty} G(\mathbf{k}) e^{-jk\mathbf{u}\cdot\mathbf{r}} k^2 dk \\ &= \frac{2\pi j}{\prod_{i=1}^{N} \left(-\overline{\overline{S}}_i : \mathbf{uu} \right)} \sum_{j=1}^{N} \frac{e^{-jk_j^+ \mathbf{ur}} \left(k_j^+ \right)^2}{\prod_{i=1, i \neq j}^{N} \left(k_j^+ - k_i^+ \right) \prod_{i=1}^{N} \left(k_j^+ - k_i^- \right)}. \end{aligned} \tag{117}$$

Here we have assumed that all the k_i^+ are different. When some of the k_i^+ are equal then the Cauchy residue theorem can be used for a higher-order pole. When $\mathbf{u} \cdot \mathbf{r} < 0$ the residues in the poles k_i^- have to be considered. The space domain Green function

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi G(\mathbf{r}, \theta, \phi) \sin\theta, \tag{118}$$

has two remaining integrals over ϕ and θ that cannot be evaluated in closed form. Since they run over finite intervals a numerical integration using for example Gaussian quadrature formulas is straight forward.

APPENDIX. TABLE OF GREEN FUNCTIONS

A number of Green functions $G(\mathbf{r})$ corresponding to a number of operators $L(\nabla)$ and satisfying (1) are presented in the following table in the form

$$L(\nabla) \Rightarrow G(\mathbf{r}). \tag{119}$$

Green functions corresponding to two- and one-dimensional operators are given in three-dimensional form, i.e., with explicit extra delta functions.

A.1 Second-Order Operators $L(\nabla)$

$$\partial_z^2 \Rightarrow -\frac{|z|}{2}\delta(\boldsymbol{\rho}) \tag{120}$$

$$\partial_z^2 + k^2 \Rightarrow \frac{e^{-jk|z|}}{2jk}\delta(\boldsymbol{\rho}) \tag{121}$$

$$\nabla_t^2 \Rightarrow -\frac{1}{2\pi}\ln\gamma\rho\ \delta(z) \tag{122}$$

$$\nabla_t^2 + k^2 \Rightarrow \frac{1}{4j}H_0^{(2)}(k\rho)\delta(z) \tag{123}$$

$$\overline{\overline{S}}_t : \nabla\nabla \Rightarrow -\frac{1}{2\pi\sqrt{\mathrm{spm}\overline{\overline{S}}_t}}\ln\left(\gamma\sqrt{\overline{\overline{S}}_t^{-1} : \boldsymbol{\rho\rho}}\right)\delta(z) \tag{124}$$

$$\overline{\overline{S}}_t : \nabla\nabla + k^2 \Rightarrow \frac{1}{4j\sqrt{\mathrm{spm}\overline{\overline{S}}_t}}H_0^{(2)}\left(k\sqrt{\overline{\overline{S}}_t^{-1} : \boldsymbol{\rho\rho}}\right) \tag{125}$$

$$\nabla^2 + k^2 \Rightarrow \frac{e^{-jkr}}{4\pi r} \tag{126}$$

$$\overline{\overline{S}} : \nabla\nabla \Rightarrow \frac{1}{4\pi\sqrt{\det\overline{\overline{S}}}D_s}, \quad D_s = \sqrt{\overline{\overline{S}}^{-1} : \mathbf{rr}} \tag{127}$$

$$\overline{\overline{S}} : \nabla\nabla + k^2 \Rightarrow G_s(k;\mathbf{r}) = \frac{e^{-jkD_s}}{4\pi\sqrt{\det\overline{\overline{S}}}\ D_s} \tag{128}$$

$$\overline{\overline{S}}:\nabla\nabla + 2\mathbf{a}\cdot\nabla + k^2 \Rightarrow \exp\left(-\mathbf{a}\cdot\overline{\overline{S}}^{-1}\cdot\mathbf{r}\right)G_s\left(\sqrt{k^2 - \overline{\overline{S}}^{-1}:\mathbf{aa}};\mathbf{r}\right) \tag{129}$$

$$\mathbf{ab}:\nabla\nabla + k^2 = \overline{\overline{S}}_t:\nabla\nabla + k^2 \Rightarrow \frac{1}{4j\sqrt{\mathrm{spm}\overline{\overline{S}}_t}}H_0^{(2)}\left(k\sqrt{\overline{\overline{S}}_t^{-1}:\boldsymbol{\rho\rho}}\right)\delta(z) \tag{130}$$

$$\overline{\overline{S}}_t = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}), \qquad \mathrm{spm}\overline{\overline{S}}_t = -(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})/4$$

A.2 Fourth-Order Operators $L(\nabla)$

$$\left(\partial_z^2 + k_1^2\right)\left(\partial_z^2 + k_2^2\right) \Rightarrow \frac{e^{-jk|z|}}{4jkk_1k_2}\left[\cos(K|z|) + jk|z|\frac{\sin(K|z|)}{K|z|}\right]\delta(\boldsymbol{\rho}) \tag{131}$$

$$k = \frac{k_2 + k_1}{2}, \quad K = \frac{k_2 - k_1}{2}$$

$$\left(\partial_z^2 + k^2\right)^2 \Rightarrow \frac{1}{4jk^3}\left(1 + jk|z|\right)e^{-jk|z|}\delta(\boldsymbol{\rho}) \tag{132}$$

$$\left(\partial_z^2 + k^2\right)\nabla_t^2 \Rightarrow \frac{e^{-jk|z|}}{4j\pi k}\ln\gamma\rho \tag{133}$$

$$\left(\nabla_t^2 + k_1^2\right)\left(\partial_z^2 + k_2^2\right) \Rightarrow \frac{e^{-jk_2|z|}}{8k_2}H_o^{(2)}(k_1\rho) \tag{134}$$

$$\left(\nabla_t^2 + k_1^2\right)\left(\nabla_t^2 + k_2^2\right) \Rightarrow \frac{H_0^{(2)}(k_1\rho)}{4j\left(k_2^2 - k_1^2\right)} + \frac{H_0^{(2)}(k_2\rho)}{4j\left(k_1^2 - k_2^2\right)} \tag{135}$$

$$\left(\nabla_t^2 + k^2\right)^2 \Rightarrow \frac{\rho}{8jk}H_1^{(2)}(k\rho)\delta(z) \tag{136}$$

$$\nabla_t^4 \Rightarrow \frac{1}{8\pi}\rho^2(\ln k\rho + 1)\delta(z) \tag{137}$$

$$\nabla^4 \Rightarrow \frac{r}{8\pi} \tag{138}$$

$$\left(\nabla^2 + k_1^2\right)\left(\nabla^2 + k_2^2\right) \Rightarrow -\frac{e^{-jk_1r} - e^{-jk_2r}}{4\pi\left(k_1^2 - k_2^2\right)r} = \frac{je^{-jkr}}{8\pi k}\frac{\sin Kr}{Kr} \tag{139}$$

$$\nabla^2\left(\nabla^2 + k^2\right) \Rightarrow \frac{1 - e^{-jkr}}{4\pi k^2 r} \tag{140}$$

$$\left(\nabla^2 + k^2\right)^2 \Rightarrow j\frac{e^{-jkr}}{8\pi k} \tag{141}$$

$$\left(\partial_z^2 + k^2\right)\left(\nabla^2 + k^2\right) \Rightarrow \frac{j}{8\pi k}\left[E_1(jk(r-z))e^{-jkz} + E_1(jk(r+z))e^{jkz}\right] \tag{142}$$

$$\nabla_t^2\left(\nabla^2 + k^2\right) \Rightarrow \frac{1}{8j\pi k}\Big[E_1\left(jk(r-z)\right)e^{-jkz} + E_1\left(jk(r+z)\right)e^{jkz} - 2e^{-jk|z|}\ln(k\rho)\Big] \tag{143}$$

$$\nabla_t^2\left(\overline{\overline{\alpha}} : \nabla\nabla + k^2\right) \Rightarrow \frac{1}{8j\pi k\sqrt{\alpha_z}}\left[E_1\left(jk\left(D_\alpha - \frac{z}{\sqrt{\alpha_z}}\right)\right)e^{-jkz/\sqrt{\alpha_z}} + E_1\left(jk\left(D_\alpha + \frac{z}{\sqrt{\alpha_z}}\right)\right)e^{jkz/\sqrt{\alpha_z}} - 2e^{-jk|z/\sqrt{\alpha_z}|}\ln\left(k\rho/\sqrt{\alpha_t}\right)\right], \quad D_\alpha = \sqrt{\overline{\overline{\alpha}}^{-1} : \mathbf{rr}} \tag{144}$$

$$\left(\overline{\overline{\alpha}} : \nabla\nabla + k^2\right)\left(\overline{\overline{\beta}} : \nabla\nabla + k^2\right) \Rightarrow \frac{1}{8j\pi(\beta_t - \alpha_t)k}\Bigg[\left[E_1\left(jk(D_\alpha - z)\right)\right.$$
$$\left.-E_1\left(jk(D_\beta - z)\right)\right] e^{-jkz} + \left[E_1\left(jk(D_\alpha + z)\right)\right.$$
$$\left.-E_1\left(jk(D_\beta + z)\right)\right] e^{jkz}\Bigg], \quad (\text{with } \alpha_z = \beta_z = 1) \tag{145}$$

$$\left(\overline{\overline{S}} : \nabla\nabla + k_1^2\right)\left(\overline{\overline{S}} : \nabla\nabla + k_2^2\right) \Rightarrow \frac{G_s(k_1;\mathbf{r})}{k_2^2 - k_1^2} + \frac{G_s(k_2;\mathbf{r})}{k_1^2 - k_2^2} \tag{146}$$

$$\left(\overline{\overline{S}} : \nabla\nabla + k^2\right)^2 \Rightarrow \frac{jD_s}{2k} G_s(k;\mathbf{r}) = \frac{je^{-jkD_s}}{8\pi k\sqrt{\det \overline{\overline{S}}}} \tag{147}$$

A.3 Higher-Order Operators $L(\nabla)$

$$\left(\overline{\overline{S}}:\nabla\nabla+k_1^2\right)\left(\overline{\overline{S}}:\nabla\nabla+k_2^2\right)\left(\overline{\overline{S}}:\nabla\nabla+k_3^2\right) \Rightarrow \frac{G_s(k_1;\mathbf{r})}{\left(k_2^2 - k_1^2\right)\left(k_3^2 - k_1^2\right)} \tag{148}$$

$$+ \frac{G_s(k_2;\mathbf{r})}{\left(k_1^2 - k_2^2\right)\left(k_3^2 - k_2^2\right)} + \frac{G_s(k_3;\mathbf{r})}{\left(k_1^2 - k_3^2\right)\left(k_2^2 - k_3^2\right)} \tag{149}$$

$$\left(\overline{\overline{S}} : \nabla\nabla + k^2\right)^3 \Rightarrow \frac{j\left(1 + jkD_s\right)e^{-jkD_s}}{32\pi k^3\sqrt{\det \overline{\overline{S}}}} \tag{150}$$

$$\left(\overline{\overline{S}} : \nabla\nabla + k^2\right)^4 \Rightarrow \frac{j\left(1 + jkD_s + \frac{1}{3}\left(jkD_s\right)^2\right)e^{-jkD_s}}{64\pi k^5\sqrt{\det \overline{\overline{S}}}} \tag{151}$$

$$\prod_{i=1}^{n}\left(\overline{\overline{S}} : \nabla\nabla + k_i^2\right) \Rightarrow \sum_{i=1}^{n} \frac{G_s(k_i;\mathbf{r})}{\prod\limits_{\substack{j=1\\(j\neq i)}}^{n}\left(k_j^2 - k_i^2\right)} \tag{152}$$

REFERENCES

1. Kong, J. A., *Theory of Electromagnetic Waves,* 15–16, Wiley, New York, 1975.
2. Lindell, I. V., *Methods for Electromagnetic Field Analysis,* 2nd ed., University Press, Oxford, 1995.
3. Tai, C. T., *Dyadic Green Functions in Electromagnetic Theory,* 2nd ed., IEEE Press, New York, 1994.

4. Lindell, I. V., and W. S. Weiglhofer, "Green dyadic for a uniaxial bianisotropic medium," *IEEE Trans. Antennas Propagat.,* Vol. 42, No. 7, 1013–1016, July 1994.
5. Arfken, G., *Mathematical Methods for Physicists,* 3rd ed., 342, Academic Press, Boston, 1985.
6. Lindell, I. V., and F. Olyslager, "TE/TM decomposition of electromagnetic sources in a nonreciprocal anisotropic medium," *IEE Proc., Microwaves, Ant., Propag.,* Vol. 145, No. 1, 109–115, Feb. 1998.
7. Lindell, I. V., and F. Olyslager, "TE/TM decomposition of the Green dyadic in uniaxial anisotropic media," *Electromagnetics,* Vol. 18, 383–394, 1998.
8. Lindell, I. V., and F. Olyslager, "Factorization of Helmholtz determinant operator for decomposable bi-anisotropic media," *J. Electro. Waves Appl.,* Vol. 13, No. 4, 429–444, 1999.
9. Jacoby, B., and F. Olyslager, "Asymptotic expansions for Green's dyadics in bianisotropic media," *Prog. Electro. Res.,* 12, J. A. Kong (Ed.), EMW Publishing, Cambridge, 1996.
10. Olyslager, F., *Electromagnetic Waveguides and Transmission Lines,* 128–130, Clarendon Press, Oxford, 1999.
11. Mei, K. K., "On the perturbational solution to the dyadic Green's function of Maxwell's equations in anisotropic media," *IEEE Trans. Antennas Propagat.,* Vol. 19, 665–669, 1971.
12. Lindell, I. V., and F. Olyslager, "Green dyadic for a class of nonreciprocal bi-anisotropic media," *Micro. Opt. Tech. Lett.,* Vol. 19, No. 3, 216–221, 1998.
13. Magnus, W., F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics,* Springer-Verlag, Berlin, 1966.
14. Erdélyi, A., W. Magnus, F. Oberhettinger, and F. G. Tricomi, *The Bateman Project: Tables of Integral Transforms,* Vol. 1, McGraw-Hill, New York, 1954.
15. Prudnikov, A. P., Y. A. Brychkov, and O. I. Marichev, *Integrals and Series,* Vol. 1, Elementary Functions, Gordon and Breach Science Publishers, New York, 1988.
16 Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions,* Dover Publications, New York, 1965.