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CONSTITUTIVE RELATIONS IN INHOMOGENEOUS SYSTEMS AND THE PARTICLE-FIELD CONUNDRUM

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Abstract–Recently a general framework has been proposed for constitutive relations. This theoretical approach attempted to represent constitutive relations as spatiotemporal differential operators acting on the physically observable fields. The general statement is sufficiently broad to embrace linear and nonlinear systems, and dispersive as well as inhomogeneous systems. The present study investigates specific examples related to polarizable and chiral media. It was immediately realized that prior to working out the examples, we have to better understand the relation of the kinematics of particles to field concepts. Throughout, the Minkowski space notation and related relativistic ideas are exploited for simpler notation and deeper understanding.

1. INTRODUCTION: A GENERALIZED CONCEPTUALIZATION OF CONSTITUTIVE RELATIONS

Maxwell's Electromagnetism theory (Electrodynamics), e.g., see [1, 2], is a branch of physics which reached maturity and can serve as prototypical for a general conceptualization of physical models in general. This then reflects back on other theories like Continuum Mechanics (e.g., Fluid Mechanics, Acoustics, Elastodynamics) [3–8]). The common denominator is that the mathematical formalism is indeterminate and requires supplemental relations, dubbed Constitutive Relations. Thus for example in linear acoustics the equations are based on Newton's Mechanics, but in order to become determinate, the supplemental compressibility relation is required, mediating between volume and pressure. Similarly in elastodynamics, in addition to the Newtonian mechanical laws, the Hooke relation mediating stress and strain must be supplemented. The distinction between the "laws" proper, i.e., Newton's Mechanics, and the supplemental constitutive relations is often fuzzy, indicated by calling the latter "laws" as well; this fuzziness characterizes electromagnetism as well, especially when a phenomenological-empirical methodology is used to introduce the subject to engineers and applied scientists. In view of Einstein's Special Relativity theory, and it's edifice of the covariance of Maxwell's laws, i.e., the equations governing Electromagnetism, the picture becomes more focused.

Conversely, constitutive relations can be exploited to characterize certain media and propose experiments to establish certain properties of materials. However the general observations above sometimes pass unnoticed, namely that the aim of acquiring constitutive relations, is first and foremost to render the system of equations determinate. Elevating constitutive relations to this level, there is nothing intrinsic in them. Whether it is called "dielectric parameter", "conductivity", etc., is a matter of convenience for categorizing certain material properties. Historically this general conceptualization arose in Relativistic Electrodynamics [9] in an effort to derive the constitutive relations for moving media. Researchers spent a lot of effort trying to figure out the proper transformations for the constitutive parameters themselves (see also [10] for some references). It was Minkowski [11] who noticed that the objective is to provide relations amongst the fields, which are the measurable quantities. Only *a-posteriori* statements about the new relations are of interest. Thus for example, moving simple media, possessing constant scalar ε, μ parameters in their rest frame of reference, display bi-anisotropic properties when observed in moving inertial systems, but their intrinsic properties are the parameters ε, μ in their rest frame of reference, and only the relations between fields show the effect of the motion. To later recast the results as displaying bi-anisotropicity is optional. In some cases, e.g., for nonlinear media, the rearrangement of the results might be too complicated, and not worth the effort, but nonetheless, the supplemental equations are available.

Recently [12] an attempt was made to implement this general methodology and present general integral and differential expressions for constitutive relations. The class of media discussed included nonlinear media initially described by Volterra's functional series, which were represented in equivalent differential operator form. One important outcome of this investigation was that dispersion and inhomogeneity must be put on the same footing: We are familiar with the phenomenon of dispersion, i.e., the selective response of media to various frequencies. But spatiotemporal inhomogeneities also effect spectral responses. New frequencies appear when the inhomogeneity is temporal, introducing Doppler like phenomena, and new wave vectors enter when a spatial inhomogeneity is present. These effects are sometimes referred to as "geometrical dispersion". At the other end of the gamut, "material dispersion" can often be attributed to the local inhomogeneity, i.e., the structure of the constituent elemental particles comprising the medium.

The quest for a systematic approach to the question of constitutive relations was mainly theoretical [12]. Consequently the need arose for some detailed analyses of examples. This is done here.

In the course of formulating the general background for working out the examples, another interesting question posed itself, namely the particle-field conundrum mentioned in the title. Only after clarifying this facet, we will be ready for the examples themselves.

2. PARTICLES AND FIELDS

The particle-field conundrum is not new. The problem is quite basic to our conception of physics and physical laws. In a strict sense the conundrum can only be solved by introducing nonlinear relations into the initially linear field and particle problems, and this makes it even more intriguing. It seems that historically the problem arose in continuum mechanics, where it is referred to as the Lagrangian versus Eulerian descriptions of material motion [3–5] (note in passing that Sommerfeld attributes both descriptions to Euler), but the same features exist in Electromagnetic theory, as well as any other field model. To begin with, the question is geometrical and kinematical, although in the relevant continuum mechanics treatment the dynamics is introduced in an early stage, thus somewhat losing the basic general aspects. Also, it is noted that a full four-dimensional treatment as given below is lacking. The question revolves around the role of the spatiotemporal coordinates and how dependent and independent variables are defined.

Mechanics in its elementary form, whether Newtonian or relativistic, is the study of material point masses (henceforth "particles"). Let us recall how the motion of particles is dealt with in mechanics. Except for statics, mechanics deals with moving particles. To keep track of the changing position of a particle in space we need to define its trajectory. From geometry we know that a curve in space (except in one and two dimensions) needs to be defined parametrically. We thus have a trajectory defined as

$$\mathbf{x} = \mathbf{x}(\tau) \tag{1}$$

where $\mathbf{x} = (x, y, z)$ is a triplet of Cartesian spatial coordinates, and τ is a parameter along the trajectory, usually τ is identified with time (more precisely, in the context of Special Relativity, this should be the particle's proper time, as discussed below), but all we really need is a scalar, real variable τ. Thus far (1) describes the kinematics of the single particle. It also facilitates the definition of the particle's velocity and acceleration as the ordinary derivatives of (1), e.g.,

$$\mathbf{v}(\tau) = \frac{d\mathbf{x}(\tau)}{d\tau} \tag{2}$$

provided it is understood that $\tau = t$. See Fig. 1. Similarly, for defining the dynamics, Newton's inertia law starts with,

$$\mathbf{a}(\tau) = \frac{d^2\mathbf{x}(\tau)}{d\tau^2} = \frac{d\mathbf{v}}{d\tau} \tag{3}$$

whereby acceleration, the ordinary derivative of (2) is involved. In (1)–(3) the independent variable is τ. In contradistinction to the above particle description, we have field descriptions. As a representative

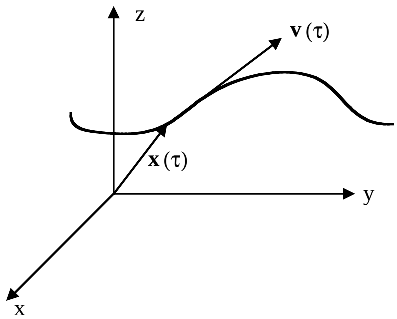

Figure 1. Depicts a trajectory $\mathbf{x}(\tau)$ and its associated $\mathbf{v}(\tau)$ embedded in a Cartesian coordinate system.

of a field model, Maxwell's equations of the electromagnetic field are chosen. In contemporary form and notation, in MKS units, we write the set of vectorial, coupled partial differential equations as

$$\begin{aligned} \partial_{\mathbf{x}} \times \mathbf{E} &= -\partial_t \mathbf{B} - \mathbf{j}_m \\ \partial_{\mathbf{x}} \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{j}_e \\ \partial_{\mathbf{x}} \cdot \mathbf{D} &= \rho_e \\ \partial_{\mathbf{x}} \cdot \mathbf{B} &= \rho_m \end{aligned} \tag{4}$$

where $\partial_{\mathbf{x}}$ and ∂_t denote the space derivative (usually referred to as "Del" or "Nabla"), and the partial time derivative, operators, respectively. All the fields are space and time dependent, e.g., $\mathbf{E} = \mathbf{E}(\mathbf{X})$. Here

$$\mathbf{X} = (\mathbf{x}; ict) \tag{5}$$

symbolizes the space-time dependence. Actually we exploit the Minkowski-space location vector notation, where c is the universal constant of the speed of light, and i is the unit imaginary complex number $i^2 = -1$ but at this point there is no attempt to include any relativistic considerations, and the same notation could also apply to continuum mechanics models. For symmetry and completeness, in the present representation, the Maxwell equations include the conventional electric (index e), as well as the fictitious magnetic (index m), current and charge density sources.

As a field model, the quadruplet (5) constitutes the four independent spatiotemporal coordinates, in contradistinction to the particle description (1)–(3). I.e., (4) is from the outset written in the Eulerian method, whereby a geometrical space-time frame of reference is assumed, and (4) relates what happens in this space-time, in contradistinction to the Lagrangian method, where a certain material particle, or parcel, is followed along a trajectory. In order to understand why the two descriptions cannot coexist independently side by side, we have to deeper peruse Maxwell's equations (4): To begin with, (4) does not involve any of the familiar quantities encountered in mechanics or the rest of physics: Force, momentum, energy, etc. Consequently additional mediating relations are required in order to connect electromagnetism to the rest of physics. The Coulomb or Lorentz force formulas (which are part of the same concept under Special Relativity), or the alternative Poynting power flux formula try to fill this gap. The Lorentz force formulas are given by

$$\begin{aligned} \mathbf{f}_e &= q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ \mathbf{f}_m &= q_m(\mathbf{H} - \mathbf{v} \times \mathbf{D}) \end{aligned} \tag{6}$$

where q_e, q_m are point charges moving on known trajectories according to (2), hence (6) is already part of the conundrum, in that (6) involves a mixture of particle and field descriptions. Indeed, the conundrum keeps popping up any time we try to relate (4) to mechanics. Thus the Poynting vector formula in its raw form is given by

$$\partial_{\mathbf{x}} \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \partial_{\mathbf{x}} \times \mathbf{E} - \mathbf{E} \cdot \partial_{\mathbf{x}} \times \mathbf{H} = -(\mathbf{H} \cdot \partial_t \mathbf{B} + \mathbf{E} \cdot \partial_t \mathbf{D} + \mathbf{H} \cdot \mathbf{j}_m + \mathbf{E} \cdot \mathbf{j}_e) \tag{7}$$

which is then volume-integrated and the left side of (7) represented as a surface integral using the Gauss theorem. Evidently the system (4) is indeterminate, i.e., there are more unknowns than equations. Therefore in order to derive any solution of (4) we need additional relations involving the dependent field variables, namely the constitutive relations. These have to be determined by the nature of the media involved and their properties, and are not an integral part of the electromagnetic hypothesis, or "law", (4). As shown below, the constitutive relations once again involve particles moving on trajectories, which brings us back to the particle-field conundrum. Without constitutive relations (7) is practically useless, alas, even when constitutive relations are available, the usefulness of (7) as an expression for energy conservation is limited to special cases. A complete fundamental relation connecting electromagnetism to mechanics is still lacking.

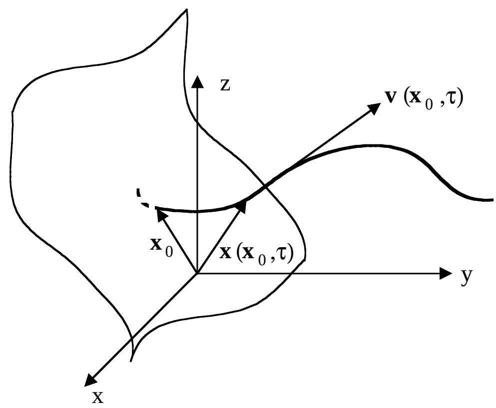

Figure 2. Depicts a member $\mathbf{x}(\mathbf{x}_0, \tau)$ of a family of trajectories, individually defined by the position vector $\mathbf{x}_0$ locating the point of intersection with a reference surface.

3. PARTICLES AND SPATIOTEMPORAL TRAJECTORIES

In (1) a single particle and its trajectory are described, i.e., the particle in its motion traces out this trajectory. In order to describe a stream of particles moving on the same line in space, an additional parameter is required, e.g., the particle's initial time at some reference point in space. To that end we now define in (1)–(3)

$$\tau = t - t_0 \tag{8}$$

and accordingly all particles traverse the same trajectory in space, each according to its initial time. Another extension of (1) provides a family of trajectories: A surface is defined, providing a reference position for all trajectories. The trajectories are given now by

$$\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \tau) \tag{9}$$

see Fig. 2. Upon combining (8) and (9), we get streams of particles on these trajectories. This is still a three-dimensional depiction of the trajectories. Consider now the generalization of (9) to the form

$$\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t_0, \tau) \tag{10}$$

In (10) for each initial time a different trajectory is defined, thus if (8) is understood, the family of trajectories is constantly changing, with each particle tracing out its own trajectory. At this point it is realized that a much broader statement is feasible, namely a four-dimensional Minkowski-space formulation: Let the reference surface be given by

$$f(\mathbf{X}_0) = 0 \tag{11}$$

which adds another degree of freedom, allowing the surface to change as a function of the initial time t_0, but of course we do not have to use this most general feature, leaving (11) as a three-dimensional surface nonvarying in time $f(\mathbf{x}_0) = 0$. The trajectories are then given by

$$\mathbf{X} = \mathbf{X}(\mathbf{X}_0, \tau) \tag{12}$$

This is a very interesting result, more so if we consider the spatial and temporal components

$$\mathbf{x} = \mathbf{x}(\mathbf{X}_0, \tau) \tag{13}$$

$$t = t(\mathbf{X}_0, \tau) \tag{14}$$

While (13) is already expressed by (10), the temporal part (14) is a general relation of t to the parameter τ along the trajectories. Obviously (8), in a narrow sense, is one choice of this relation. The way (12)–(14) work is the following: A location $\mathbf{X}_0$ is chosen that satisfies (11). This defines time and space reference location for a trajectory on which $\mathbf{X} = \mathbf{X}(\mathbf{X}_0, \tau)$, i.e., (12), or (13), (14) are computed for a specific particle. Obviously in (12)–(14), $\mathbf{X}_0$ is independent of τ, therefore

$$\mathbf{V}_\tau(\mathbf{X}_0, \tau) = \left.\frac{d\mathbf{X}}{d\tau}\right|_{\mathbf{X}_0} = \frac{\partial \mathbf{X}}{\partial \tau} = \left(\frac{\partial \mathbf{x}}{\partial \tau}; ic\frac{\partial t}{\partial \tau}\right) \tag{15}$$

defines the four-vector velocity $\mathbf{V}_\tau$, where $\mathbf{X}_0$ is held constant. The components of (15) yield the three-dimensional velocity $\mathbf{v}$ as

$$\mathbf{v}_\tau(\mathbf{X}_0, \tau) = \frac{\partial_\tau \mathbf{x}}{\partial_\tau t} \tag{16}$$

which is everywhere tangential to the trajectory. If the constancy of $\mathbf{X}_0$ is understood, then we can write (16) in a form conforming with the chain rule of calculus as

$$\mathbf{v}_\tau = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{d\tau}\frac{d\tau}{dt} \tag{17}$$

The corresponding four-acceleration is given by

$$\mathbf{W}_\tau(\mathbf{X}_0,\tau) = \left.\frac{d^2\mathbf{X}}{d\tau^2}\right|_{\mathbf{X}_0} = \frac{\partial^2\mathbf{X}}{\partial\tau^2} = \frac{\partial\mathbf{V}_\tau}{\partial\tau} = \left(\frac{\partial^2\mathbf{x}}{\partial\tau^2};\frac{\partial^2 t}{\partial\tau^2}\right) \tag{18}$$

with the three-dimensional acceleration given similarly to (16) by

$$\mathbf{a}_\tau(\mathbf{X}_0,\tau) = \frac{\partial_\tau\mathbf{v}_\tau}{\partial_\tau t} \tag{19}$$

We do not have to, but it is very suggestive to consider τ as the proper time in the Special Relativistic sense. As a consequence we now have (15), (18) denoting the four-velocity and four-acceleration, respectively, in the Minkowskian sense.

At this point we are ready to discuss the transition from the trajectory representation (12)–(19) to a field representation. As we cover a certain region with trajectories, each such trajectory defines points $\mathbf{X}$ in space-time according to (12)–(14), and at each point the velocity (16) and acceleration (19) (as well as higher derivatives, if needed) are computed according to (15)–(19). In principle the data enables us to map out the fields

$$\begin{aligned}\mathbf{V}_\tau &= \mathbf{V}_\tau(\mathbf{X})\\ \mathbf{W}_\tau &= \mathbf{W}_\tau(\mathbf{X})\end{aligned} \tag{20}$$

including the three-dimensional functions of interest

$$\mathbf{v}_\tau = \mathbf{v}_\tau(\mathbf{X}) \tag{21}$$

$$\mathbf{a}_\tau = \mathbf{a}_\tau(\mathbf{X}) \tag{22}$$

Hence in principle the transition from (15)–(19) to (20)–(22) is feasible, but the index tag τ must be kept, reminding us how the new fields were derived.

Obviously the relation of (21) to (22) is not simple. In view of (16), (19), one cannot simply differentiate (21) in order to obtain (22). How is this question related to the straightforward application of the chain rule of calculus in the form

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial\mathbf{x}}\cdot\frac{d\mathbf{x}}{dt} + \frac{\partial\mathbf{v}}{\partial t} = \frac{\partial\mathbf{v}}{\partial\mathbf{x}}\cdot\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t} \tag{23}$$

which appears in practically every textbook on continuum mechanics? Expression (23) is usually referred to as the “material derivative”, or "moving derivative", and supposedly describes the acceleration of a medium parcel. Let us consider the expression

$$\frac{d\mathbf{v}_\tau}{d\tau} = \frac{\partial \mathbf{v}_\tau}{\partial \mathbf{X}} \cdot \frac{d\mathbf{X}}{d\tau} = \frac{\partial \mathbf{v}_\tau}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{d\tau} + \frac{\partial \mathbf{v}_\tau}{\partial t}\frac{dt}{d\tau} = \frac{\partial \mathbf{v}_\tau}{\partial \mathbf{x}} \cdot \mathbf{v}_\tau + \frac{\partial \mathbf{v}_\tau}{\partial t}\frac{dt}{d\tau} \quad (24)$$

where the derivatives with respect to $\mathbf{X}$ and $\mathbf{x}$ (i.e., the four-gradient and three-dimensional gradient, respectively) are obtained by differentiating (20), (21), respectively, the derivatives with respect to τ are obtained from (12)–(14). If it is assumed that τ is the special-relativistic proper time in the moving particle's frame of reference, then for slow velocities relative to c, we have $\tau = t$. Consequently (24) becomes

$$\mathbf{a}_\tau = \frac{d\mathbf{v}_\tau}{dt} = \frac{\partial \mathbf{v}_\tau}{\partial \mathbf{x}} \cdot \mathbf{v}_\tau + \frac{\partial \mathbf{v}_\tau}{\partial t} \quad (25)$$

which for $\tau = t$ tallies with (19) on one hand, and on the other hand is very similar to (23). However the index tag τ in (25) must be kept, and means that all the differentiations were made along the trajectory.

In order to discuss constitutive relations, electrically charged material point particles (i.e., electrons, atoms or molecules) must be considered. This again can only be done in the context of the mechanical particle concept, discussed above.

The classical derivation of constitutive relations based on polarizable media assumes simple conditions [13] i.e., homogeneous media. Moreover, either explicitly or implicitly (23) or (25) is assumed in a simplistic manner, and only the linear term is retained

$$\mathbf{a}_\tau = \frac{d\mathbf{v}_\tau}{dt} = \frac{\partial \mathbf{v}_\tau}{\partial t} \quad (26)$$

This is essentially the classical linear oscillator model, leading to the associated frequency domain permittivity. It assumes that if a spatiotemporal driving field exists, the particle is affected by the temporal part only, and the spatial position $\mathbf{x}$ is held fixed in spite of the motion of the particle. This stratagem circumvents the need for a trajectory description of the particle's motion, simply by assuming a particle at rest. The aim of the present section is to perform the calculations and derive a spatiotemporal domain differential operator representation for the constitutive relation, which will also take into account the possibility of inhomogeneity. This will serve as a model for more complicated cases, discussed subsequently.

We start by defining a number density

$$N = N(\mathbf{X}_0, \tau) \tag{27}$$

along trajectories and their vicinities. this takes into account the possibility that a bunch of particles moves along a single trajectory, i.e., trajectories can coalesce in certain regions, or adjacent trajectories might be considered a single one. This is the key to describing inhomogeneous media. After applying to (27) the mapping out process, as explained above, we derive a particle density field in the form

$$N = N(\mathbf{X}) \tag{28}$$

similar to the method resulting in (20)–(22). The conventional definition of current density as the product of charge density and velocity, now yields

$$\mathbf{j}(\mathbf{X}_0, \tau) = qN(\mathbf{X}_0, \tau)\mathbf{v}_\tau(\mathbf{X}_0, \tau) \tag{29}$$

and mapping out the data over a region of space-time will yield the field representation

$$\mathbf{j}(\mathbf{X}) = qN(\mathbf{X})\mathbf{v}(\mathbf{X}) = \rho(\mathbf{X})\mathbf{v}(\mathbf{X}) \tag{30}$$

where the charge and current densities, whether indexed e or m, appear also in Maxwell's equations (4). Note that (29), (30), i.e., the definition of current density as moving charge density, already amounts to a constitutive relation, because it relates two fields given in the original Maxwell equations! Evidently, we are assuming in (29), (30) a single species of charges. There is no need to unnecessarily complicate the present discussion by assuming a mixture of species [13]. We will also assume that for the time being only e type entities are involved, and suppress the index.

Current is sometimes defined as the time rate of change of polarization. We could start with this idea too: The traditional definition of polarization as $\mathbf{p} = q\mathbf{x}$ cannot be used here except for the trivial case of straight line trajectories. There the conventional definition starts with the mute assumption of having a homogeneous medium. The definition is usually extended heuristically to slowly varying media, which are locally approximated as homogeneous. This locality property has to be carefully examined in each case, i.e., the electromagnetic field space and time scales must be carefully compared to the corresponding scales of the medium parameters. For example, this appears in the context of ray theory, i.e., as a high frequency approximation. Inasmuch as we strive here to present a general case, a differential definition is adopted

$$d\mathbf{p} = q d\mathbf{x} \tag{31}$$

where according to (13) the displacement is along the trajectory. Thus we obtain

$$\frac{d\mathbf{p}}{dt} = q\frac{\partial_\tau \mathbf{x}}{\partial_\tau t} = q\mathbf{v}_\tau \tag{32}$$

and for a number density given by (27), the previous form (29) follows.

For a single particle of mass m the equation of motion is governed by

$$m\mathbf{a} + \alpha\mathbf{v} + m\omega_0^2\mathbf{x} = \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \tag{33}$$

where the total force is the sum of the inertial force, the velocity dependent (so called wet-) friction force, and the displacement dependent ("spring action") force. The inertial force is associated with the mass m, the friction force is characterized by the coefficient α, and the spring constant is the eigenvalue of the frictionless free oscillation motion expressed by the equation $\mathbf{a} + \omega_0^2\mathbf{x} = 0$, reduced from (33). In the electromagnetic case it is assumed that the total force is balanced by the Coulomb force

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = q\mathbf{E} \tag{34}$$

where (34) involves a single particle, and $\mathbf{E}$ is the locally effective field acting on this particle. It is noted that the right hand side of (34) expressing the Coulomb force is only adequate for charges at rest, otherwise it must be replaced by the Lorentz force formula, first line in (6). This means that low velocities (relative to the speed of light c) are assumed. At this time there is no external static magnetic field assumed, as in the case of a magnetized plasma (e.g., in the terrestrial ionosphere). Obviously the difficulties associated with the particle-field conundrum are already indicated in (33), (34): As a field entity we have $\mathbf{E} = \mathbf{E}(\mathbf{X})$ with $\mathbf{X} = \mathbf{X}(\mathbf{x}, ict)$, i.e., four independent variables as in (4), while the mechanical forces are expressed in particle form, i.e., by using ordinary derivatives and trajectories as in (1) etc.

Consider the spring force first. Similarly to the argument leading to (17), we have a differential form

$$\frac{d\mathbf{F}_3(\tau)}{d\tau} = m\omega_0^2 \frac{d\mathbf{x}(\tau)}{d\tau} = m\omega_0^2 \mathbf{v}_\tau(\tau) \tag{35}$$

for a single particle, and for many, noninteracting, single species particles characterized by the same parameter ω_0^2, we adopt the methodology leading to (29), yielding

$$\frac{d\mathbf{F}_3(\mathbf{X}_0,\tau)}{d\tau} = \mathbf{F}_3' = m\omega_0^2 N(\mathbf{X}_0,\tau)\mathbf{v}_\tau(\mathbf{X}_0,\tau) \tag{36}$$

where for brevity the apostrophe denotes differentiation with respect to τ. In view of (29), in terms of current density, we get

$$\mathbf{F}_3' = \frac{m\omega_0^2}{q}\mathbf{j}(\mathbf{X}_0,\tau) \tag{37}$$

Similarly, for the friction term, including the density effect, we have

$$\mathbf{F}_2(\mathbf{X}_0,\tau) = \alpha N(\mathbf{X}_0,\tau)\mathbf{v}_\tau(\mathbf{X}_0,\tau) \tag{38}$$

hence

$$\frac{d\mathbf{F}_2}{d\tau} = \mathbf{F}_2' = \alpha\mathbf{v}_\tau\frac{dN}{d\tau} + \alpha N\frac{d\mathbf{v}_\tau}{d\tau} = \alpha(\mathbf{v}_\tau N' + N\mathbf{a}_\tau) \tag{39}$$

It is noted that in (39) N', the rate of change of the particle density, is not neglected. This is the key to our systematic approach of treating dispersion and inhomogeneity on the same footing. Differentiating (29) yields

$$\mathbf{j}' = q(\mathbf{v}_\tau N' + N\mathbf{a}_\tau) \tag{40}$$

hence from (39), (40) we obtain

$$\mathbf{F}_2' = \frac{\alpha}{q}\mathbf{j}' \tag{41}$$

The inertia force for single species particles of density N is given by

$$\mathbf{F}_1 = Nm\mathbf{a}_\tau \tag{42}$$

hence we derive

$$\frac{d\mathbf{F}_1}{d\tau} = \mathbf{F}_1' = m\left(\mathbf{a}_\tau N' + N\mathbf{a}_\tau'\right) \tag{43}$$

By taking derivatives of (29) and substituting into (43), it can be recast as

$$\mathbf{F}_1' = \frac{m}{q}\left(\mathbf{j}'' - \mathbf{j}'\frac{N'}{N} - \mathbf{j}\left(\frac{N'}{N}\right)'\right) \tag{44}$$

Adding up the various contributions, we now have an expression for the total force derivative

$$\mathbf{F}' = \mathbf{F}_1' + \mathbf{F}_2' + \mathbf{F}_3' \tag{45}$$

differentiated with respect to τ, in terms of the current density (29) and the particle density of charge carriers. It is assumed that N and its derivatives are known coefficients. On the other hand, $\mathbf{j}$ is one of the dependent variables appearing also in the electromagnetic system of equations (4).

Our mission now is to relate the force derivative (45) to the electromagnetic field (4), and derive the appropriate constitutive relations. This invokes the particle-field conundrum: Once again we consider the process of mapping out functions included in (45) on a region of the spatiotemporal domain, as in (20)–(22), (28), (30). From (28) defining a local instantaneous density, and (34) describing the force on a single particle, the Coulomb force density (per unit volume) is given by

$$\mathbf{F}(\mathbf{X}) = qN(\mathbf{X})\mathbf{E}(\mathbf{X}) \tag{46}$$

therefore

$$\mathbf{F}' = q(N\mathbf{E}' + N'\mathbf{E}) \tag{47}$$

In all above formulas, culminating in (45), (47), the apostrophe denotes differentiation with respect to τ along the trajectory, i.e.

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau} + \mathbf{v}_\tau \cdot \frac{\partial}{\partial\mathbf{x}} = \frac{\partial}{\partial\tau} + \frac{\mathbf{j}}{qN} \cdot \frac{\partial}{\partial\mathbf{x}} \tag{48}$$

Sometimes nonlinear effects in material media are attributed to nonlinear mechanisms on the level of the single material particle. While this may be true, the nonlinear nature is already displayed in (24), (48) as a general feature of the particle-wave conundrum. In view of the second derivatives in (44), the explicit representation of (45), (47), leads to a very complicated nonlinear expression. The decision whether to "linearize" the equations, i.e., neglect the gradient in (48), or to deal with the full complexity of (48), is an arbitrary heuristic step which must be decided on the merit of the individual problem at hand. For the time being, we decide to reduce the problem to its linear part. Moreover, as was done above, in the transition from (24) to (25), it is assumed that τ is the special-relativistic proper time in the moving particle's frame of reference, and for slow velocities relative to c, we have $\tau = t$. Therefore in (45) and all the previous constituent formulas, and (47), the apostrophe *de facto* denotes the partial time derivative. Consequently (45) becomes

$$\begin{aligned}\mathbf{F}' &= \left[\frac{m}{q}\left(\left(\omega_0^2 - \left(\frac{N'}{N}\right)'\right) + \left(\frac{\alpha}{m} - \frac{N'}{N}\right)\partial_t + \partial_t^2\right)\right]\mathbf{j} \\ &= \mathcal{A}(\mathbf{X}, \partial_{\mathbf{X}})\mathbf{j}(\mathbf{X})\end{aligned} \tag{49}$$

symbolized as an operator involving the spatiotemporal coordinates and their associated differential operators (although for the present case the left side (49) involves time derivatives only), operating on the current density field. Similarly (47) is rewritten as an operator acting on the field $\mathbf{E}$

$$\mathbf{F}' = q(N' + N\partial_t)\mathbf{E} \tag{50}$$

Applying $\mathcal{A}$ in (49) to the second equation (4), and substituting from (50), we obtain

$$\begin{aligned}\mathcal{A}\partial_{\mathbf{x}} \times \mathbf{H} &= \varepsilon_0\mathcal{A}\partial_t\mathbf{E} + \mathcal{A}\,\mathbf{j} = \mathcal{B}\mathbf{E} \\ \mathcal{B}(\mathbf{X}, \partial_{\mathbf{X}}) &= \varepsilon_0\mathcal{A}\partial_t + q(N' + N\partial_t)\end{aligned} \tag{51}$$

Evidently in (51) we have accomplished our goal stated in Section 1 above: All the extraneous constitutive factors, namely the mechanical variables, have been eliminated, hence in the general conceptual sense the job is finished. The rest from here on is optional. In the general case it has only a notational and *symbolical* value: We recast (51) in the symbolic form

$$\partial_{\mathbf{x}} \times \mathbf{H} = \frac{\mathcal{B}}{\mathcal{A}}\mathbf{E} = \sigma\mathbf{E} \tag{52}$$

and *symbolically* the ratio of operator multiplying $\mathbf{E}$ can be referred to as the conductivity differential operator. In order to define a corresponding dielectric differential operator, one must recast (52) as

$$\partial_{\mathbf{x}} \times \mathbf{H} = \sigma \mathbf{E} = \partial_t(\varepsilon \mathbf{E}) \tag{53}$$

Obviously a general way of relating σ, ε, is not feasible. Therefore it should be realized that our ability to reduce the general form (51) to (52) and thus derive something that can be called "conductivity" is just incidental. But according to the general Minkowskian methodology that the constitutive relations are a means to render Maxwell's equations (4) determinate, the identification of dielectric, conductivity, etc., entities is not essential for deriving solutions, indeed, it is not even necessary for characterizing a medium. It is also realized that Heaviside's operational calculus can only be applied in a system with constant coefficients, i.e., an homogeneous medium, hence the derivation of an inverse operator that will bring $\mathcal{A}$ in (52) into the numerator is not feasible in general. A similar way of looking at the same situation is to realize that a Fourier or Laplace transform of (51) will not simply replace the derivatives by algebraic factors, it will also transform the spatiotemporally dependent coefficients, and since products of spatiotemporal terms are present in (51), this will lead to convolution integrals in the transform space. The related case of nonlinear constitutive relations is even more complicated [12]. If the coefficients are treated as constants, i.e., their derivatives are neglected, then we are in the realm of the ray, or WKB approximation, which applies only to spectrally narrow-band signals. Therefore the ratio of operators in (52) must be understood in the original sense of (51), i.e., that in (52) the operator in the denominator acts on the left side of the equation.

With the above analysis we have provided a concrete example for the abstract discussion given before [12].

4. ADDITION OF AN EXTERNAL MAGNETIC FIELD

In gyrotropic materials an external magnetic field $\mathbf{B}_0$ is present, which affects the force balance equation (45), adding a force term, including a spatiotemporal density N

$$\mathbf{F}_4 = qN\mathbf{v} \times \mathbf{B}_0 = -\mathbf{B}_0 \times \mathbf{j} \tag{54}$$

The expression (54) follows from the Lorentz force (6) assuming vanishing $\mathbf{E}$ field and subject to (29). Assuming space-time dependent fields for $\mathbf{B}_0$ and $\mathbf{j}$, (54) yields on differentiation

$$\mathbf{F}_4' = -\mathbf{B}_0' \times \mathbf{j} - \mathbf{B}_0 \times \mathbf{j}' = -(\mathbf{B}_0' + \mathbf{B}_0 \partial_t) \times \mathbf{j} \tag{55}$$

where in the last term (55) we already made the concession that the problem is linearized, and $\tau = t$. Consequently (55) must be added to (45), and the operator in brackets in (49) is accordingly modified to include the right hand side of (55). Note that the operator now involves the vectorial cross operation, and becomes therefore a dyadic $\widetilde{\boldsymbol{\mathcal{A}}}(\mathbf{X}, \partial_\mathbf{X})$. However, this does not introduce any conceptual difficulties into our problem. As in (51), the new operator is now applied to the second equation (4), and a new dyadic operator $\widetilde{\boldsymbol{\mathcal{B}}}(\mathbf{X}, \partial_\mathbf{X})$ is defined. Finally similarly to (51), we obtain

$$\begin{aligned} \widetilde{\boldsymbol{\mathcal{A}}} \cdot \partial_\mathbf{x} \times \mathbf{H} &= \widetilde{\boldsymbol{\mathcal{B}}} \cdot \mathbf{E} \\ \widetilde{\boldsymbol{\mathcal{B}}}(\mathbf{X}, \partial_\mathbf{X}) &= \varepsilon_0 \widetilde{\boldsymbol{\mathcal{A}}} \partial_t + \widetilde{\boldsymbol{\mathcal{I}}} q (N' + N \partial_t) \end{aligned} \tag{56}$$

where $\widetilde{\boldsymbol{\mathcal{I}}}$ is the idemfactor (unit) dyadic. Finally the symbolical form (52) is derived once again, with the appropriately modified operator

$$\partial_\mathbf{x} \times \mathbf{H} = \frac{\widetilde{\boldsymbol{\mathcal{B}}} \cdot}{\widetilde{\boldsymbol{\mathcal{A}}} \cdot} \mathbf{E} \tag{57}$$

provided we keep in mind that it represents (56), otherwise a division by a dyadic that does not have an inverse is meaningless. We can still call the new construct in (57) "conductivity", especially because of its dimensions, but its use as a separate entity is dubious. Nevertheless, it accomplishes the goal of rendering the system (4) determinate.

5. A MODEL FOR A CHIRAL MEDIUM

The discussion above suggests that we use the same methodology for more complicated media as well, e.g., chiral media. Chiral media in the presence of time harmonic fields have been extensively investigated for many years now. See [14], a comprehensive study including many references. For time harmonic waves such a model of a chiral medium accounts for the characteristic elliptical wave polarization effects found in such media, i.e., constitutive relations for chiral media in the presence of time harmonic fields involve complex numbers, and i factor (i.e., the imaginary unit number) parameters indicates that two effects are present, with a $\pi/2$ phase shift between them.

This phenomenon must also be explained by the model proposed here, however in the present analysis the objective is to transcend the steady state harmonic time dependence and deal with general spatiotemporal behavior. As a concrete example, a model of a "chiral particle" is proposed and analyzed. This facilitates the derivation of the pertinent constitutive operators, which will also take into account the particular elements of the linear oscillator model used for the non-chiral media above. I.e., we take here into account chirality, inertia, friction, and "spring action". In addition, the effect of gyrotropicity, caused by an external magnetic field, can also be included.

Unlike the Tellegen particle [14], involving permanent electric and magnetic dipoles, our model involves induced polarization only, consistent with the linear oscillator model above, which has no permanent remnant polarization. This is mandatory for combining the properties of the linear oscillator and the present chiral model. Also it is shown that such a model displays the expected temporal phase shift between electric and magnetic fields, which is important for properly describing the induced elliptical polarization effects for time harmonic waves.

It must be kept in mind that the thrust of the present paper is mainly to discuss constitutive relations conceptually, and provide some examples, not to classify or analyze in detail chiral media. Additional information on such specifics can be found in [14–17], which also provide starting points to the extensive literature on the subjects. Our model patently belongs to the "wire and loop" class (but not the special uniaxial variety discussed in some of the papers cited above and their references). Also unlike [16] for example, the present analysis assumes material scales small compared to wavelength. The direct consequence is that resonance effects on the present structures do not occur, or in other words, structural dispersion related to the chiral particles is excluded from our discussion. This vastly simplifies the analysis and obviates a lot of detail, while leaving enough space to bring out the essentials of the present problem. The two right and left handed chiral particles are depicted in Figs. 3, 4, respectively. Geometrically we have here right and left handed structures, respectively. Physically, it is assumed that the structures are made of conducting wires. Hence approximately, in the circuit sense, we deal here with a capacitor-inductor (L-C) circuit, where the two arms of the lineal wire

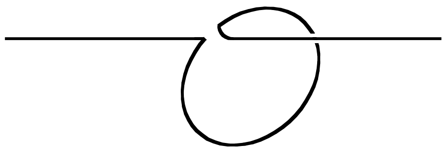

Figure 3. The chiral particle model consists of a lineal conducting segment connected to a perpendicular loop. The present particle is defined as right handed.

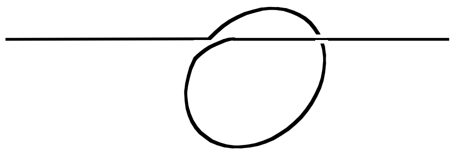

Figure 4. Like in Fig. 3, but being a mirror image, the present particle is left handed.

segment provide (most of) the capacitance, while the loop provides (most of) the inductive element. Of course, this is only an idealized lumped elements approximation, because the electric and magnetic fields are distributed.

A little reflection immediately reveals that the combination of elements depicted in Figs. 3, 4, is actually a bare-bones rendition of the celebrated helical chiral particles, cited in many books and articles [14], which possesses the combined lineal and the loop properties in its structure. If you stretch the helix, the linear dimension becomes emphasized; if you squeeze the helix, the lineal dimension diminishes and the loop characteristics is emphasized.

The circuit analogy immediately explains the phase shifts encountered in the system: Consider the external electric field as the driving force. It charges and discharges the capacitor. A charged capacitor, in the sense of Thevenin's equivalence theorems, can be replaced by a voltage source in series with the (uncharged) capacitor. Alternatively, if the driving force is the external magnetic field, then we consider the current induced in the loop by the external magnetic field to be equivalent to a current source connected in parallel to the inductor. Thus our basic circuit is depicted in Fig. 5. When the capacitor is charged by the external field we get the equivalent circuit Fig. 6a, which can be redrawn in the form shown in Fig. 6b. Similarly, Fig. 7a depicts

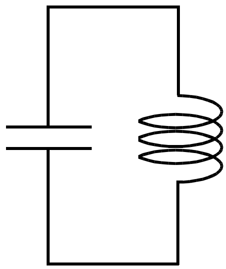

Figure 5. The basic L-C circuit model for the chiral particle.

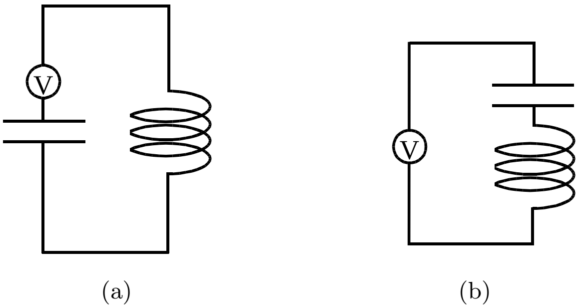

Figure 6. (a) Adding a Thevenin representation of the charged capacitor; (b) recasting (a) as a voltage generator connected to a series L-C circuit.

the equivalent current source parallel to the inductor, resulting from the magnetic field, and Fig. 7b is just another rendition of the same circuit.

For both voltage and current excitations there exists a $\pi/2$ phase shift between voltage and current at the reactive elements, this is indicated by the i factor in the capacitive and inductive impedances. Hence the same phase shift exists between the electric and magnetic fields, respectively, created by the chiral particles. In the time domain this will show up in the time derivatives in the pertaining differential equation.

Let us consider now the behavior of our chiral particle in the presence of a plane, linearly polarized, time harmonic excitation wave. Let the wave propagate in the z-direction, its **E**-field polarized in the x-direction and its **H**-field polarized in the y-direction, pointing out of the page, as depicted in Fig. 8. We present three prototypical orienta-

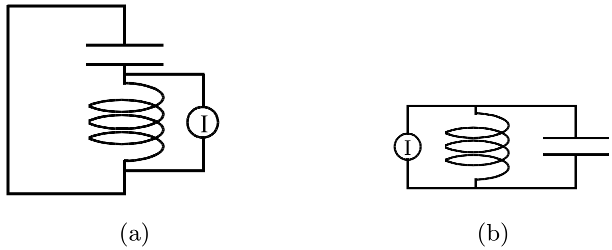

Figure 7. (a) Adding a Thevenin representation of the current carrying inductor; (b) recasting (a) as a current generator connected to a parallel L-C circuit.

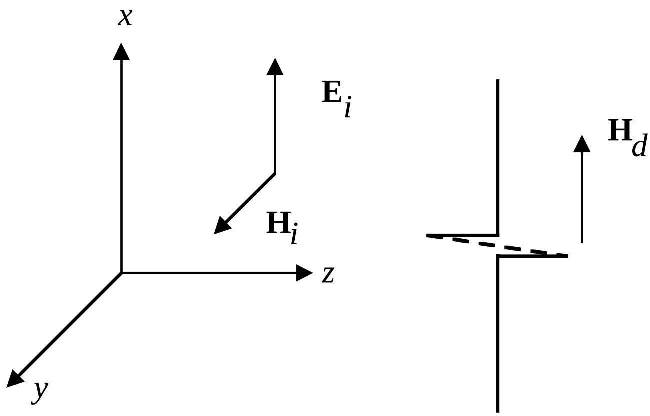

Figure 8. Positioning the chiral element of Fig. 3 for **E**-field excitation. The dashed segment is the part of the loop below the page.

tions of the chiral particles. If the orientation of the chiral particle in space is a random variable, the ensemble average will yield an equivalent bi-isotropic medium.

Imagine the lineal part of the chiral particle in Fig. 3 to be aligned in the x-direction, Fig. 8, where the dashed line indicates the part of the loop below the page. Here the incident wave $\mathbf{E}_i$ excites the lineal part of the chiral particle, acting like a short lineal wire antenna. Also note that $\mathbf{E}_i$, being perpendicular to the loop, cannot induce current in it (recall the dimensions of the loop are small compared to wavelength!). The charges that are created and moving along the lineal segment, are hurled through the loop part, thus creating the induced field $\mathbf{H}_d$. The direction of the field is dictated by the sense of rotation of the

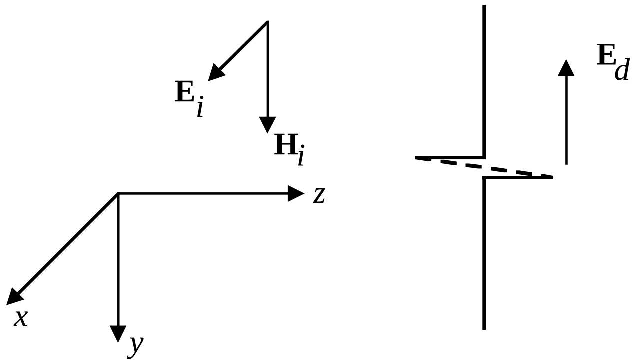

Figure 9. Positioning the chiral element of Fig. 3 for **H**-field excitation. The dashed segment is the part of the loop below the page.

loop, which in Fig. 8 is a right handed helix. As long as the loop, be it right or left handed, is perpendicular to the lineal segment, we get an induced field $\mathbf{H}_d$ which is perpendicular to the initial magnetic field $\mathbf{H}_i$. It is shown below that $\mathbf{H}_i$ and $\mathbf{H}_d$ are shifted by $\pi/2$, thus giving rise to elliptically polarized waves in the chiral medium in the time harmonic regime. Also note that the magnetic excitation $\mathbf{H}_i$, being parallel to the loop, cannot induce voltage in it. In other words, no E.M.F. voltage, in the sense of Faraday's "law" (referred to as "law" because it is already included in (4)), can be present. Also note that $\mathbf{H}_i$ cannot induce current in the lineal segment, because they are perpendicular. Of course, the new field $\mathbf{H}_d$ is associated with a corresponding $\mathbf{E}_d$ prescribed by (4), and in the far field these two fields behave like the components of a plane wave.

The analogous configuration is depicted in Fig. 9, where we retain the chirality of the particle, but change its orientation in space. Here the incident field $\mathbf{E}_i$ affects neither the lineal segment of the chiral particle, which is perpendicular, nor the loop segment which is parallel. On the other hand, $\mathbf{H}_i$ induces an E.M.F. in the loop, producing current that is forced to flow through the lineal segment and thus causing polarization and the appearance of an induced field $\mathbf{E}_d$. The incident $\mathbf{E}_i$ and induced $\mathbf{E}_d$ fields are perpendicular and possess a $\pi/2$ phase shift, thus again conducive to elliptically polarized waves.

In the configuration depicted in Fig. 10 no induced fields are created. This is an interesting observation, telling us that when the present

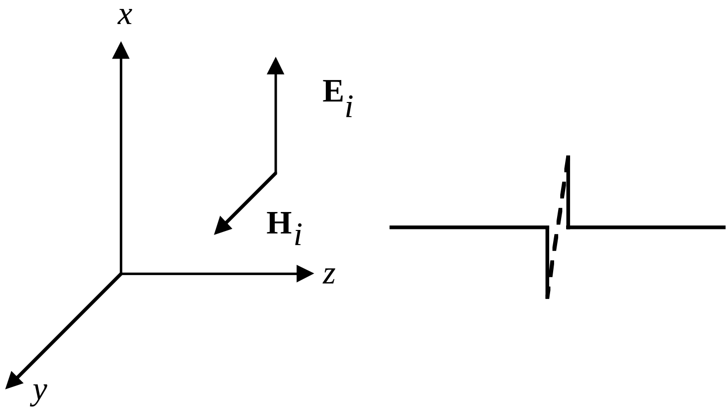

Figure 10. Positioning the element perpendicularly with respect to the excitation fields. The lineal and loop elements can be excited neither by the excitation **E** nor **H** fields.

chiral elements are envisioned as small spirals, as usually presented in many texts, they have no effect on the total field when aligned with their axes along the direction of propagation of the incident wave. Any orientation of a chiral element in space can be considered as the superposition of three elements depicted in Figs. 8–10.

We will now analyze the circuit parameters and the corresponding fields in more detail: Consider the chiral particle depicted in Fig. 3, and positioned as in Fig. 8. Obviously this corresponds to the equivalent circuit in Fig. 6b. Assuming small particles and avoiding the more intricate details of antenna theory, it is seen that the incident field induces an in-phase voltage V, which drives a current I through the reactive elements, and therefore the current's phase is shifted by $\pi/2$. Whether the current is leading or lagging depends on the combined capacitive and inductive reactance. Usually the latter will be dominant. In any case, the current is then proportional and in phase with respect to the associated magnetic field, and thus $\mathbf{H}_d$ is created. The incident wave's magnetic field $\mathbf{H}_i$ is in phase with $\mathbf{E}_i$, hence we get two fields, $\mathbf{H}_i$ and $\mathbf{H}_d$ which are normal in space and in quadrature in time, associated with their mate fields $\mathbf{E}_i$ and $\mathbf{E}_d$ according to (4), thus giving rise to a combined elliptically polarized wave once again. Essentially the same analysis and the same conclusions are derived from Figs. 7b, 8.

6. CHIRAL MEDIUM: ELECTRO-MECHANICAL PROPERTIES AND CONSTITUTIVE RELATIONS

We will now consider a mechanical model for a chiral medium, consistent with the model given above for polarizable media. Of course, the chiral particles themselves are not moving under the influence of the fields, they only provide the constrains for the motion of the e and m type charges, as described above for the single wire and loop chiral particle. Thus for our case electric (e type) charges move on the metallic conductors of the lineal segment and the loop comprising the chiral particle, but the magnetic fields originating on the loop part can be attributed to associated magnetic charges. The transition from a lumped element to a continuous medium concept is now needed. As for simple polarizable materials, our goal here is to express the mechanical properties in terms of the current density $\mathbf{j}$, and substitute (in the operator sense) into (4) to get rid of $\mathbf{j}$ and thus have a "pure" electromagnetic representation in terms of constitutive differential operators acting on the fields $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, $\mathbf{D}$.

Let us recall what we did before: In polarizable materials, e.g., an ionized plasma, we interpret the situation as having a matrix which keeps the environment electrically neutral, and within this space mobile electrons can move. The inertia is provided by their mass, the friction can be explained in terms of inelastic interactions with the embedding matrix, and the "spring action" as due to the tendency of the medium to remain electrically neutral, i.e., when charges move, they are disturbing the neutrality and thus Coulomb forces appear which pull them back. In the case of a medium consisting of polarizable atoms or molecules (as opposed to an ionized plasma with a free moving "cloud" of electrons), the same situation is understood, in spite of the fact that the charges are attached to the polarizable particles and not completely free to move around in the embedding medium. This is the backdrop for the classical linear oscillator model [13]. This approach must now be adapted to describe a continuous chiral medium. Thus according to our lumped chiral element depicted in Figs. 3, 4, we envisage a medium which causes charges to move along trajectories under the influence of fields, and simultaneously perform helical motion. Conceptually this is somewhat similar to the transition from a lumped element to a distributed element transmission line. Consider the case of homogeneous media first. By inspection of Fig. 8, we see that electrical polarization $\mathbf{p}_e$ is induced in the lineal segment by the source electrical field $\mathbf{E}_i$. In Fig. 9, due to the source magnetic field $\mathbf{H}_i$, and by virtue of Faraday's "law", an E.M.F. voltage is created. The voltage created along the loop wire appears between the terminals of the loop, driving a current through the lineal segment and thus producing an electrical polarization proportional to the time-derivative of the magnetic field. The Faraday "law" invoked here is already embodied in Maxwell's equations, in the first equation (4). Hence we combine the two effects writing

$$\mathbf{p}_e = \widetilde{\mathbf{a}} \cdot \mathbf{E} - \widetilde{\mathbf{b}} \cdot \partial_t \mathbf{H} \tag{58}$$

where $\widetilde{\mathbf{a}}$, $\widetilde{\mathbf{b}}$ are the pertinent dyadics which take into account the geometry and orientation of the (bi-isotropically average) chiral particle. The time derivative $-\partial_t \mathbf{H}$ including the sign takes into account the Faraday law formula.

Analogously, we have magnetic polarization too. The analog of the first term in (58) is derived from Fig. 9. Here the source magnetic field $\mathbf{H}_i$ excites magnetic polarization by inducing a current in the loop, acting like the equivalent current source in Fig. 7b. The field $\mathbf{H}_d$ in Fig. 8, due to the current produced in the lineal segment and driven into the loop, creates magnetic polarization. The current is related to the time derivative of the electrical field. Therefore $\mathbf{H}_d$ and the associated magnetic polarization are proportional to $\partial_t \mathbf{E}$. This relation of the magnetic field to the derivative of the electric field is based on the second equation (4), and therefore a change of sign has been incorporated. The magnetic polarization is therefore written as

$$\mathbf{p}_m = \widetilde{\mathbf{c}} \cdot \mathbf{H} + \widetilde{\mathbf{d}} \cdot \partial_t \mathbf{E} \tag{59}$$

where again we have dyadics $\widetilde{\mathbf{c}}$, $\widetilde{\mathbf{d}}$, depending on the geometry of the chiral particle in some average sense. No attempt is done here to decide common characteristics and symmetries in the dyadics above.

Assuming spatiotemporally constant dyadics in (58), (59), and recognizing in the linear regime current densities as the partial time derivatives of polarization densities, the time derivatives of (58), (59), yield

$$\mathbf{j}_e = \widetilde{\mathbf{a}} \cdot \partial_t \mathbf{E} - \widetilde{\mathbf{b}} \cdot \partial_t^2 \mathbf{H} \tag{60}$$

$$\mathbf{j}_m = \widetilde{\mathbf{c}} \cdot \partial_t \mathbf{H} + \widetilde{\mathbf{d}} \cdot \partial_t^2 \mathbf{E} \tag{61}$$

The combined set of (4) and (60), (61) provides a determinate system of equations that can now be solved for specific problems. Moreover, upon substitution of (60), (61) into (4), the current densities are eliminated, and new so-called chiral constitutive relations can be defined. In the present case everything is assumed to take place in the spatiotemporal domain, and the chiral constitutive relations are stated as differential operators.

Our goal is to derive the pertinent relations for inhomogeneous chiral media, and include all the elements that came into play in the linear oscillator analysis above, including the external magnetic field introduced in Section 4. Here however, we encounter magnetic $\mathbf{q}_m$ as well as electric $\mathbf{q}_e$ charges, and their corresponding current densities $\mathbf{j}_m$, $\mathbf{j}_e$. It follows that we have to include both e and m type forces, as given in (6). To be consistent with our above analysis for the electrically polarizable media, we assume a magnetic Coulomb force obtained from (6) by taking $\mathbf{v} = 0$.

The analogue of the Lorentz force for the external magnetic field (54) follows from (6) on taking $\mathbf{E} = 0$ and subject to the current definition as in (29), where now m-type current is involved. Consequently, similarly to (40), we now add to our system of forces for a single particle also

$$\mathbf{F}_{m4} = -q_m \mathbf{v} \times \mathbf{D}_m \tag{62}$$

Consequently we can have gyrotropic effects of e and m types also in the presence of chiral media, in principle.

The rest is now a systematic implementation of this outline. Corresponding to (46) we now have

$$\mathbf{F}_e(\mathbf{X}) = q_e N(\mathbf{X}) \left(\widetilde{\mathbf{a}} \cdot \mathbf{E}(\mathbf{X}) - \widetilde{\mathbf{b}} \cdot \partial_t \mathbf{H}(\mathbf{X}) \right) \tag{63}$$

where now the varying density as in (46) has been included. For concrete applications it is of course a question of the actual medium at hand. Here we have chosen the chiral properties to be constant in space and time. In the case of a bi-anisotropic medium this prescribes that all particles are identically aligned in space. For bi-isotropic media constant dyadics imply some local averaging. One can also assume spatiotemporally dependent dyadics, and in addition include dispersion in the form $\widetilde{\mathbf{a}}(\mathbf{X}, \partial_{\mathbf{X}})$ etc. In any case, the dyadic fields must be given *a-priori* as known functions and cannot depend on the electromagnetic or mechanical fields themselves, because this would already constitute a nonlinear effect. Corresponding to (59) and similarly to (63), we now have

$$\mathbf{F}_m(\mathbf{X}) = q_m N(\mathbf{X}) \left(\widetilde{\mathbf{c}} \cdot \mathbf{H}(\mathbf{X}) + \widetilde{\mathbf{d}} \cdot \partial_t \mathbf{E}(\mathbf{X}) \right) \tag{64}$$

with the same provisos applying to the chiral dyadics.

On the other side of the equation stand the mechanical forces. The argument (27)–(30), (35)–(37) for the spring action is retraced, and it is again noted that we do not differentiate the parameters of the particles as they move along their trajectories. Hence we reproduce (37) with the appropriate modifications, deriving

$$\mathbf{F}'_{e3} = \frac{m_e \omega_{0e}^2}{q_e} \mathbf{j}_e(\mathbf{X}) \tag{65}$$

and for the corresponding magnetic force

$$\mathbf{F}'_{m3} = \frac{m_m \omega_{0m}^2}{q_m} \mathbf{j}_m(\mathbf{X}) \tag{66}$$

Note that we have assumed different free resonance for the electric and magnetic parameters, but nevertheless, we have a single species system of particles. However this implies that we attribute some mass m_m to the magnetic charge q_m. This perplexing problem is actually a different way of expressing the mechanical properties, actually the mass m_m already enters the free oscillations eigenvalue ω_{0m}^2 appearing in (66) and should not be more surprising than the concept of magnetic current density $\mathbf{j}_m$. Ultimately it is the mass of the electrons hurled around the loop that enters the expression for the mass of the fictitious magnetic charges.

Following the argument (38)–(41), the friction term yields

$$\mathbf{F}'_{e2} = \frac{\alpha_e}{q_e} \mathbf{j}'_e \tag{67}$$

and

$$\mathbf{F}'_{m2} = \frac{\alpha_m}{q_m} \mathbf{j}'_m \tag{68}$$

for the electric, magnetic particles, respectively. The inertia force term (44) is now rewritten with the appropriate indices $e,\ m$. Therefore (44) now becomes

$$\mathbf{F}'_{e1} = \frac{m_e}{q_e} \left(\mathbf{j}''_e - \mathbf{j}'_e \frac{N'}{N} - \mathbf{j}_e \left(\frac{N'}{N} \right)' \right) \tag{69}$$

and

$$\mathbf{F}'_{m1} = \frac{m_m}{q_m}\left(\mathbf{j}''_m - \mathbf{j}'_m\frac{N'}{N} - \mathbf{j}_m\left(\frac{N'}{N}\right)'\right) \tag{70}$$

for the electric and magnetic forces, respectively.

The gyrotropic forces follow from (54)–(55). We thus have for the time derivatives of the forces

$$\mathbf{F}'_{e4} = -(\mathbf{B}'_0 + \mathbf{B}_0\partial_t) \times \mathbf{j}_e \tag{71}$$

and

$$\mathbf{F}'_{m4} = (\mathbf{D}'_0 + \mathbf{D}_0\partial_t) \times \mathbf{j}_m \tag{72}$$

where the fields $\mathbf{B}_0$, $\mathbf{D}_0$ are considered to be known, as before. Moreover, the chiral and gyrotropic aspects are treated here as additive, which means that there is no interaction between the two aspects. If the fields $\mathbf{B}_0$, $\mathbf{D}_0$ cannot be treated as *quasi*-static fields, obviously the model (71), (72) is inadequate, and the fields must be considered as part of the source fields. If the time derivatives of $\mathbf{B}_0$, $\mathbf{D}_0$ are negligent, only the second term in parenthesis in (71), (72) applies.

We now repeat the process (49)–(52), this time including the gyrotropic effects (71), (72). Thus we combine all the force derivative terms to get an operator as in (49). Note that including terms as in (71), (72), will lead to dyadic operators, thus we have for the analog of (49)

$$\mathbf{F}'_e = \widetilde{\boldsymbol{\mathcal{A}}}_e(\mathbf{X}, \mathbf{B}_0, \partial_{\mathbf{X}}) \cdot \mathbf{j}_e(\mathbf{X}) \tag{73}$$

and

$$\mathbf{F}'_m = \widetilde{\boldsymbol{\mathcal{A}}}_m(\mathbf{X}, \mathbf{D}_0, \partial_{\mathbf{X}}) \cdot \mathbf{j}_m(\mathbf{X}) \tag{74}$$

for the electric, magnetic forces, respectively. The force $\mathbf{F}'_e$, (73) is equated to the electromagnetic force derivative obtained by differentiating (63)

$$\begin{aligned}\mathbf{F}'_e &= \left(q_e N\left(\widetilde{\mathbf{a}}\cdot\mathbf{E} - \widetilde{\mathbf{b}}\cdot\partial_t\mathbf{H}\right)\right)' \\ &= q_e N'\left(\widetilde{\mathbf{a}}\cdot\mathbf{E} - \widetilde{\mathbf{b}}\cdot\partial_t\mathbf{H}\right) + q_e N\left(\widetilde{\mathbf{a}}\cdot\partial_t\mathbf{E} - \widetilde{\mathbf{b}}\cdot\partial_t^2\mathbf{H}\right)\end{aligned} \tag{75}$$

Similarly $\mathbf{F}'_m$, (74) is equated to the electromagnetic force derivative obtained by differentiating (64)

$$\mathbf{F}'_m = q_m N'\left(\widetilde{\mathbf{c}}\cdot\mathbf{H} + \widetilde{\mathbf{d}}\cdot\partial_t\mathbf{E}\right) + q_m N\left(\widetilde{\mathbf{c}}\cdot\partial_t\mathbf{H} + \widetilde{\mathbf{d}}\cdot\partial_t^2\mathbf{E}\right) \tag{76}$$

The same strategy used in (51) is now implemented. The second Maxwell equation in (4) is multiplied by $\widetilde{\boldsymbol{\mathcal{A}}}_e(\mathbf{X}, \mathbf{B}_0, \partial_{\mathbf{X}})\cdot$ given in (73), and the first equation (4) is multiplied by $\widetilde{\boldsymbol{\mathcal{A}}}_m(\mathbf{X}, \mathbf{D}_0, \partial_{\mathbf{X}})\cdot$ in (74). Now substitute from (75), (76), respectively, to eliminate in (4) the current densities $\mathbf{j}_e$, $\mathbf{j}_m$. The final result is a "pure" electromagnetic representation in terms of **E**, **H**, **B**, **D**.

7. CONCLUDING REMARKS

Maxwell's Electromagnetism model (4) is indeterminate. In spite of its seemingly perfection, we need to add constitutive relations in order to solve any problem. Recently it was attempted to state constitutive relations for dispersive as well as inhomogeneous media, linear as well as nonlinear, in a general and consistent manner [12]. The present study shows that at least for a certain class of media, as described above, the approach is valid.

In the course of examining the constitutive relations, which stem from mechanical considerations relevant to the medium at hand, it became clear that a much deeper question is involved, what we called the particle-field conundrum. While Maxwell's equations are field equations, with the dependent variables depending on the spatiotemporal coordinates, the mechanics of particles are stated for ordinary derivatives and trajectories, and the spatiotemporal coordinates become dependent on the parameters defining the trajectories. In continuum mechanics this puzzling situation has been recognized long ago, and appears under the titles of Eulerian and Lagrangian descriptions. But the difficulty is not resolved by recognizing the problem itself, and we end up with using approximations, or confronting the question of nonlinear systems, although Maxwell's Electromagnetism is essentially a linear theory.

It remains to show how intrinsically nonlinear systems can be dealt with, using the methods used in this article.

A cardinal aspect of electromagnetic research in general is the Minkowski methodology, which appears in all its acuteness in the present study too. It started in connection with finding the constitutive relations for moving media. The Minkowski paradigm prescribes that we do not try to derive the "equivalent dielectric constant" etc. for the new system, but rather express the new relations in terms of the original parameters and the fields involved. This gave rise to the celebrated Minkowski constitutive relations for moving media, which contain neither a definite dielectric parameter, nor a definite magnetic parameter. Later, it prompted the research in general bi-anisotropic media in general. However, Minkowski's original idea got lost in the process, but should be always guiding us: *The name of the game is not finding specific constitutive relations, but rather providing sufficient equations for rendering Maxwell's equation determinate.* Sometimes we succeed in deriving new parameters that can be dubbed in terms of old ones, like the conductivity differential operator in (52), (57), which is already quite contrived. We must be ready for constitutive relations that cannot be categorized in terms of the already known classes of materials, in fact, *the class of constitutive relations is infinite, and the only criterion is the solvability of Maxwell's equation.*

The above argumentations are not electromagnetic-specific: The particle-field conundrum and the Minkowski methodology are general to all branches of continuum physics.

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