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POLYADIC METHODS IN ELASTODYNAMICS

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Abstract–To see the similarities and differences with eletromagnetics, basic concepts and equations of elastodynamics are formulated in coordinate-free form applying concepts from Polyadic Algebra. Plane-wave propagation is studied for time-harmonic equations in isotropic and simple uniaxially anisotropic media and the Green dyadic is derived for the isotropic medium. As an extension, the Green dyadic for the anisotropic elastic medium is derived in perturbational approximation. In the Appendix A, some basic properties for tetradics, useful for practical analysis both in elastodynamics as well as in electromagnetics, are derived.

1. INTRODUCTION

Being coordinate free, the dyadic algebra, as originally introduced by Gibbs [1], has more recently been applied to electromagnetic analysis in its full power [2]. While studies in elastodynamics have also applied the dyadic algebra ([3], see Appendix A therein), they have generally been limited to shorthand notation without direct analytical advantage and often coordinate-dependent matrix formalism has been invoked whenever inconvenience in notation has been encountered. This may be due to the fact that elastodynamic analysis involves tensors of the fourth rank whose coordinate-free counterparts, known as tetradics, have not been much studied in the literature. It is the purpose of this paper to introduce basic properties of tetradics and, alongside with the knowledge obtained while working with dyadics in electromagnetics, apply these concepts to basic problems in elastodynamics. It was originally hoped that, because dyadic algebra has recently revealed new interesting analytic solutions to Green dyadics corresponding to various anisotropic and bi-anisotropic electromagnetic media, that similar progress could be made with the elastodynamic problems. However, in this study it is seen that even the simplest anisotropic generalization of the well-known isotropic medium leads to a great difficulty in solving the resulting equations. Instead, a solution corresponding to an arbitrary perturbational anisotropy could be found. The main purpose of this study is, however, to extend the dyadic analysis to tetradics which requires suitable identities to be found as tools for the analysis.

2. BASIC QUANTITIES

This section reviews basic elastodynamic quantities and equations using dyadics and tetradics. A short introduction to their basic definitions and properties is given in the Appendix A.

2.1 Strain Dyadic

In an elastic medium a part of the body is deformed by an exterior force so that it may have average translation by a constant vector $\mathbf{T}$, average rotation of the body by a vector $\mathbf{P}(\mathbf{r})$ and an additional displacement $\mathbf{s}(\mathbf{r})$ due to the deformation of the body [4]:

$$\mathbf{D}(\mathbf{r}) = \mathbf{T} + \mathbf{P}(\mathbf{r}) + \mathbf{s}(\mathbf{r}). \tag{1}$$

If the body is totally rigid, $\mathbf{s} = 0$ and the body is just translated and rotated. In general $\mathbf{s}$ is much smaller than the other displacement vectors, which from now on will be omitted.

$\mathbf{s}$ can be understood as the total relative displacement with respect to a point of gravity which is assumed invariant. It is an integral of a more interesting quantity, the strain at the point $\mathbf{r}$. This is obtained from the change in the differential vector $d\mathbf{r}$ connecting two points $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ when the body is distorted and it does not depend on the distance of the center of gravity. In fact, if we write $d\mathbf{r}'$ for the differential vector after the displacement, it has the form

$$d\mathbf{r}' = d\mathbf{r} \cdot \left(\overline{\overline{I}} + \nabla\mathbf{s} \right). \tag{2}$$

The dyadic $\nabla\mathbf{s}(\mathbf{r})$ measures the strain at the point $\mathbf{r}$. It can be split in its symmetric and antisymmetric parts as

$$\nabla\mathbf{s} = \frac{1}{2}\left(\nabla\mathbf{s} + (\nabla\mathbf{s})^T\right) + \frac{1}{2}(\nabla\mathbf{s} - \left(\nabla\mathbf{s})^T\right) = \overline{\overline{\epsilon}} + \overline{\overline{R}}, \tag{3}$$

$$\overline{\overline{\epsilon}} = \frac{1}{2}\left(\nabla\mathbf{s} + (\nabla\mathbf{s})^T\right), \qquad \overline{\overline{R}} = -\frac{1}{2}(\nabla \times \mathbf{s}) \times \overline{\overline{I}}. \tag{4}$$

The dyadic $\overline{\overline{R}}(\mathbf{r})$ gives the rotation of the element of volume around $\mathbf{r}$ due to the distortion. Its axis of rotation is in the direction of $\nabla \times \mathbf{s}$ and its angle in radians is $|\nabla \times \mathbf{s}|$. This has nothing to do with the total rotation $\mathbf{T}$ of the body. The dyadic $\overline{\overline{\epsilon}}$ gives the pure strain at $\mathbf{r}$. The strain dyadic can be written in terms of three orthogonal unit eigenvectors $\mathbf{e}_i$ (principal directions of strain) as

$$\overline{\overline{\epsilon}} = \sum_{i=1}^{3} \epsilon_i \mathbf{e}_i \mathbf{e}_i. \tag{5}$$

A differential parallelepiped with sides of length dL_i parallel to the respective three eigenvectors will change to $L_i' = L_i(1+\epsilon_i)$ after deformation, whence the ϵ_i are the relative principal extensions of the medium. We now assume that the strain in the medium is small, $\epsilon_i \ll 1$.

The change of the volume of the differential parallelepiped is

$$dV' = dL_1' dL_2' dL_3' = (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3) dL_1 dL_2 dL_3$$
$$\approx \left(1 + \overline{\overline{\epsilon}} : \overline{\overline{I}}\right) dV, \tag{6}$$

$$\overline{\overline{\epsilon}} : \overline{\overline{I}} = \epsilon_1 + \epsilon_2 + \epsilon_2. \tag{7}$$

Because the trace of the strain dyadic $\mathrm{tr}\,\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}} : \overline{\overline{I}}$ is the relative change of volume for small deformations, it also gives the relative change of density of the medium (with opposite sign). The symmetric stress dyadic $\overline{\overline{\epsilon}}$ can be split in two parts as

$$\overline{\overline{\epsilon}} = \epsilon \overline{\overline{I}} + \overline{\overline{\epsilon}}^{\circ}, \qquad \epsilon = \frac{1}{3} \overline{\overline{\epsilon}} : \overline{\overline{I}}, \tag{8}$$

where $\overline{\overline{\epsilon}}^{\circ}$ is the trace-free part: $\overline{\overline{\epsilon}}^{\circ} : \overline{\overline{I}} = 0$, and defines the shear in the medium.

Examples

Assuming the displacement is of the radial form $\mathbf{s} = \epsilon \mathbf{r}$, we have

$$\nabla \mathbf{s} = \overline{\overline{\epsilon}} = \epsilon \overline{\overline{I}}. \tag{9}$$

In this case the strain is simple isotropic expansion with no rotation. An example of static strain with no change of differential volume can be obtained from $\mathbf{s} = \epsilon(x\mathbf{u}_x + y\mathbf{u}_y - 2z\mathbf{u}_z)$ as

$$\nabla \mathbf{s} = \overline{\overline{\epsilon}} = \epsilon \left(\overline{\overline{I}} - 3\mathbf{u}_z \mathbf{u}_z \right), \tag{10}$$

which extends the volume in the plane transverse to z-axis and contracts along the z-axis. Here $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$ denote the orthogonal unit vectors.

For $\mathbf{s} = \epsilon(x\mathbf{u}_x - y\mathbf{u}_y)$ we have

$$\nabla \mathbf{s} = \overline{\overline{\epsilon}} = \epsilon(\mathbf{u}_x \mathbf{u}_x - \mathbf{u}_y \mathbf{u}_y), \tag{11}$$

which is called simple shear. Again, no rotation is involved. The medium is expanded in the x direction and contracted in the same ratio in the y direction. The differential volume is not changed: $\mathrm{tr}\,\overline{\overline{\epsilon}} = 0$. Another type of shear called pure shear is obtained from $\mathbf{s} = \epsilon(x\mathbf{u}_y + y\mathbf{u}_x)$:

$$\nabla\mathbf{s} = \overline{\overline{\epsilon}} = \epsilon(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x), \tag{12}$$

which also has $\mathrm{tr}\,\overline{\overline{\epsilon}} = 0$. Because we can write

$$\epsilon(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x) = \epsilon(\mathbf{u}_x\mathbf{u}_y - \mathbf{u}_y\mathbf{u}_x) + 2\epsilon\mathbf{u}_y\mathbf{u}_x, \tag{13}$$

it can be understood as a rotation of the angle 2ϵ and a shear in the y direction, $\epsilon\mathbf{u}_y\mathbf{u}_x$. This latter displacement is in the y direction similar to that of a pack of sliding cards.

As a more general example consider $\mathbf{s} = \epsilon x(y\mathbf{u}_z - z\mathbf{u}_y)$ with

$$\nabla\mathbf{s} = \epsilon\mathbf{u}_x(y\mathbf{u}_z - z\mathbf{u}_y) + \epsilon x(\mathbf{u}_y\mathbf{u}_x - \mathbf{u}_z\mathbf{u}_y) = \overline{\overline{R}} + \overline{\overline{\epsilon}}, \tag{14}$$

$$\overline{\overline{R}} = -\frac{\epsilon}{2}(2x\mathbf{u}_x - y\mathbf{u}_y - z\mathbf{u}_z) \times \overline{\overline{I}}, \tag{15}$$

$$\overline{\overline{\epsilon}} = \epsilon y(\mathbf{u}_x\mathbf{u}_z + \mathbf{u}_z\mathbf{u}_x) - \epsilon z(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x). \tag{16}$$

2.2 Stress Dyadic

Forces producing strains in a medium are called stresses. Consider the differential force $d\mathbf{F}$ on a differential directed area $d\mathbf{A} = \mathbf{n}dA$, where the unit vector $\mathbf{n}$ is normal to the surface dA. $d\mathbf{F}$ is in general in a direction different from $\mathbf{n}$. Because it is a linear vector function of $d\mathbf{A}$, we can express the relation in the form

$$d\mathbf{F} = \overline{\overline{\sigma}} \cdot d\mathbf{A}, \tag{17}$$

in terms of a dyadic $\overline{\overline{\sigma}}$, the stress dyadic. $\overline{\overline{\sigma}}$ is a symmetric dyadic, which can be seen from the following consideration [3, pp. 4–5]. Assume any finite volume V in the medium bounded by a closed surface A. The forces consist of the impressed body force with density $\mathbf{F}(\mathbf{r})$ and of the surface force whose density is given by the stress dyadic $\overline{\overline{\sigma}}$. In static equilibrium the total force must be zero:

$$\oint_A \overline{\overline{\sigma}} \cdot d\mathbf{A} + \int_V \mathbf{F}dV = \int_V \left(\nabla \cdot \overline{\overline{\sigma}}^T + \mathbf{F}\right) dV = 0. \tag{18}$$

Since this is valid for any volume V, from Gauss' theorem we obtain

$$\nabla\cdot\overline{\overline{\sigma}}^T+\mathbf{F}=0. \tag{19}$$

Also, in a stationary state the torque with respect to any point must vanish:

$$\oint_A \mathbf{r}\times\overline{\overline{\sigma}}\cdot d\mathbf{A}+\int_V \mathbf{r}\times\mathbf{F}dV=\int_V \left(\nabla\cdot\left(\mathbf{r}\times\overline{\overline{\sigma}}\right)^T+\mathbf{r}\times\mathbf{F}\right)dV=0. \tag{20}$$

Applying again Gauss' theorem and the equation above we obtain

$$\nabla\cdot\left(\mathbf{r}\times\overline{\overline{\sigma}}\right)^T+\mathbf{r}\times\mathbf{F}=-\left(\nabla\cdot\overline{\overline{\sigma}}^T\right)\times\mathbf{r}-(\nabla\mathbf{r})\dot{{}_\times}\,\overline{\overline{\sigma}}^T-\mathbf{r}\times\left(\nabla\cdot\overline{\overline{\sigma}}^T\right)=0. \tag{21}$$

This leaves us with

$$(\nabla\mathbf{r})\dot{{}_\times}\,\overline{\overline{\sigma}}^T=\overline{\overline{I}}\dot{{}_\times}\,\overline{\overline{\sigma}}^T=0, \tag{22}$$

which is equivalent to

$$\overline{\overline{\sigma}}^T=\overline{\overline{\sigma}}, \tag{23}$$

or that the stress dyadic must be symmetric. The stress in any plane with normal unit vector $\mathbf{n}$ can be written as the sum

$$\overline{\overline{\sigma}}\cdot\mathbf{n}=\mathbf{n}(\mathbf{n}\cdot\overline{\overline{\sigma}}\cdot\mathbf{n})+(\overline{\overline{I}}-\mathbf{nn})\cdot\overline{\overline{\sigma}}\cdot\mathbf{n}, \tag{24}$$

of which the first term is called the normal stress and the second term the shear stress on the plane.

The symmetric stress dyadic can be expressed in terms of its orthogonal unit eigenvectors $\mathbf{s}_i$ as

$$\overline{\overline{\sigma}}=\sum_{n=1}^{3}\sigma_i\mathbf{s}_i\mathbf{s}_i. \tag{25}$$

The eigenvalues σ_i are principal stresses along the eigendirections (principal directions) $\mathbf{s}_i$. When $\sigma_2=\sigma_3=0$, stress is called tension in the $\mathbf{s}_1$ direction. When $\sigma_1=-\sigma_2,\ \sigma_3=0$, it is called a shearing stress. Any stress dyadic can be split in two terms as

$$\overline{\overline{\sigma}}=\sigma\overline{\overline{I}}+\overline{\overline{\sigma}}^\circ,\qquad \sigma=\frac{1}{3}\overline{\overline{\sigma}}:\overline{\overline{I}}=\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3), \tag{26}$$

where $\overline{\overline{\sigma}}^{\circ}$ is a trace-free dyadic.

2.3 The Elastic Tetradic

The medium is called elastic when it can remain in equilibrium under a shearing stress. In a linear medium there is a linear relation between the stress and strain dyadics:

$$\overline{\overline{\sigma}} = \mathcal{C} : \overline{\overline{\epsilon}}. \tag{27}$$

$\mathcal{C}$ is a tetradic with 81 components (in general), which in an orthonormal basis $\{\mathbf{u}_i\}$ can be written as

$$\mathcal{C} = \sum_{i,j,k,\ell=1}^{3} C_{ijk\ell}\mathbf{u}_i\mathbf{u}_i\mathbf{u}_k\mathbf{u}_\ell. \tag{28}$$

$\mathcal{C}$ is called the elastic tetradic.

Deformation of a material body requires energy which is stored in the form of strain. The differential energy function corresponding to $dW = \mathbf{F} \cdot d\mathbf{s}$ in particle mechanics can be represented as

$$dW = \overline{\overline{\sigma}} : d\overline{\overline{\epsilon}} = \left(d\overline{\overline{\epsilon}}\right) : \mathcal{C} : \overline{\overline{\epsilon}}. \tag{29}$$

From this one can obtain laws of symmetry for the elastic tetradic $\mathcal{C}$. Another starting point is to consider the energy function as a function of the strain dyadic, $W\left(\overline{\overline{\epsilon}}\right)$. For small deformations it can be expressed as

$$W\left(\overline{\overline{\epsilon}}\right) = W_o + \overline{\overline{A}} : \overline{\overline{\epsilon}} + \frac{1}{2}\overline{\overline{\epsilon}} : \mathcal{C} : \overline{\overline{\epsilon}} + \cdots \tag{30}$$

If the energy in the unstrained state is denoted by zero, we have $W_o = 0$. Also, we must always have $W\left(\overline{\overline{\epsilon}}\right) \geq 0$, because otherwise the body would give out energy. This requires $\overline{\overline{A}} = 0$, because otherwise changing the sign of $\overline{\overline{\epsilon}}$ would change the sign of W. Thus, the first term must be quadratic in $\overline{\overline{\epsilon}}$. Neglecting the higher-order terms as being small, we have

$$W\left(\overline{\overline{\epsilon}}\right) = \frac{1}{2}\overline{\overline{\epsilon}} : \mathcal{C} : \overline{\overline{\epsilon}} = \frac{1}{2}\overline{\overline{\sigma}} : \overline{\overline{\epsilon}}. \tag{31}$$

From this it is seen that, because of the symmetry of $\overline{\overline{\epsilon}}$, the tetradic $\mathcal{C}$ can be defined to satisfy the following symmetry conditions:

$$\mathcal{C} = \mathcal{C}^T = {}^T\mathcal{C} \quad \Rightarrow \quad C_{ijk\ell} = C_{ij\ell k} = C_{jik\ell}. \tag{32}$$

This reduces the number of parameters from 81 to 36. Also, because the same dyadic $\overline{\overline{\epsilon}}$ is multiplying $\mathcal{C}$ from both sides, we have the following symmetry condition:

$$\mathcal{C} = \mathcal{C}^{\tau}, \quad \Rightarrow \quad C_{ijk\ell} = C_{k\ell ij}, \tag{33}$$

which further reduces the number of free parameters to 21 [5, 6].

2.4 Field and Energy Equations

Euler's equation of motion for the displacement $\mathbf{s}$ has the form

$$\nabla \cdot \overline{\overline{\sigma}} - \rho \partial_t^2 \mathbf{s} = -\mathbf{F}, \tag{34}$$

where ρ is the density of the medium. To have an equation for $\mathbf{s}$, one has to substitute the stress $\overline{\overline{\sigma}}$ in terms of the strain $\overline{\overline{\epsilon}}$ which is a function of $\mathbf{s}$. Multiplying (34) by $\cdot(-\partial_t \mathbf{s})$ and denoting

$$\mathbf{S} = -\overline{\overline{\sigma}} \cdot \partial_t \mathbf{s} = -\overline{\overline{\epsilon}} : \mathcal{C} \cdot \partial_t \mathbf{s}, \tag{35}$$

we obtain

$$\nabla \cdot \mathbf{S} + \overline{\overline{\sigma}} : \partial_t (\nabla \mathbf{s}) + \rho \left(\partial_t^2 \mathbf{s}\right) \cdot (\partial_t \mathbf{s}) = \nabla \cdot \mathbf{S} + \frac{1}{2} \partial_t \overline{\overline{\sigma}} : \overline{\overline{\epsilon}} + \frac{1}{2} \rho \left(\partial_t \mathbf{s}\right)^2 = \mathbf{F} \cdot \partial_t \mathbf{s}. \tag{36}$$

This is an energy balance equation of the form

$$\nabla \cdot \mathbf{S} + \partial_t E = \mathbf{F} \cdot \partial_t \mathbf{s}, \tag{37}$$

where E is the total energy, the sum of the respective kinetic and potential energy functions

$$E = K + W, \qquad K = \frac{1}{2} \rho \left(\partial_t \mathbf{s}\right)^2, \qquad W = \frac{1}{2} \overline{\overline{\epsilon}} : \overline{\overline{\sigma}}. \tag{38}$$

Thus, the vector $\mathbf{S}$ (the Umov vector, introduced before the corresponding Poynting vector of electromagnetism) represents the power flow in the elastic medium.

2.5 Isotropic Medium

For an isotropic elastic medium the elastic tetradic $\mathcal{C}$ must be of the form of an isotropic tetradic $\mathcal{E}$ which has the general form with three scalar parameters (see Appendix A)

$$\mathcal{C} = \mu \mathcal{I} + \eta \mathcal{T} + \lambda \overline{\overline{I}}\,\overline{\overline{I}}. \tag{39}$$

Applying the first symmetry condition

$$\mathcal{C}^T = \mu\mathcal{T} + \eta\mathcal{I} + \lambda\overline{\overline{I}}\,\overline{\overline{I}} = \mathcal{C} = \mu\mathcal{I} + \eta\mathcal{T} + \lambda\overline{\overline{I}}\,\overline{\overline{I}}, \tag{40}$$

gives $\mu = \eta$. The same is obtained from the condition ${}^T\mathcal{C} = \mathcal{C}$. The last condition $\mathcal{C}^\mathcal{T} = \mathcal{C}$ is satisfied by any isotropic dyadic. Thus, the isotropic elastic tetradic has the special form

$$\mathcal{C} = \mu(\mathcal{I} + \mathcal{T}) + \lambda\overline{\overline{I}}\,\overline{\overline{I}}, \tag{41}$$

where λ and μ are the Lamé constants. Writing the strain dyadic as $\overline{\overline{\epsilon}} = \epsilon\overline{\overline{I}} + \overline{\overline{\epsilon}}^\circ$, where $\overline{\overline{\epsilon}}^\circ$ is trace-free, we have

$$\overline{\overline{\sigma}} = \mathcal{C} : \overline{\overline{\epsilon}} = 2\mu\overline{\overline{\epsilon}}^\circ + 3\lambda\epsilon\overline{\overline{I}} = 2\mu\overline{\overline{\epsilon}}^\circ + (2\mu + 3\lambda)\epsilon\overline{\overline{I}}. \tag{42}$$

Writing, correspondingly, the stress dyadic as $\overline{\overline{\sigma}} = \sigma\overline{\overline{I}} + \overline{\overline{\sigma}}^\circ$, we have for the isotropic medium the following relation between the stress and strain components:

$$\sigma = (2\mu + 3\lambda)\epsilon, \qquad \overline{\overline{\sigma}}^\circ = 2\mu\overline{\overline{\epsilon}}^\circ. \tag{43}$$

From this property we can form the inverse relation

$$\overline{\overline{\epsilon}} = \mathcal{C}^{-1} : \overline{\overline{\sigma}} = \frac{1}{2\mu}\overline{\overline{\sigma}}^\circ + \frac{1}{2\mu + 3\lambda}\epsilon\overline{\overline{I}} = \frac{1}{2\mu}\left(\overline{\overline{\sigma}} - \frac{\lambda}{2\mu + 3\lambda}\left(\overline{\overline{\sigma}} : \overline{\overline{I}}\right)\overline{\overline{I}}\right). \tag{44}$$

This can also be obtained through the formula for the inverse tetradic given in the Appendix A.

The elastic coefficients μ, λ obey certain inequalities due to the positive definiteness of the potential energy function $W\left(\overline{\overline{\epsilon}}\right) > 0$ for any strain dyadic $\overline{\overline{\epsilon}} \neq 0$. For the isotropic medium we can write this condition as

$$W\left(\overline{\overline{\epsilon}}\right) = \frac{1}{2}\left[2\mu\overline{\overline{\epsilon}} : \overline{\overline{\epsilon}} + \lambda\left(\overline{\overline{\epsilon}} : \overline{\overline{I}}\right)^2\right] = \frac{1}{2}\left[3(2\mu + 3\lambda)\epsilon^2 + 2\mu\overline{\overline{\epsilon}}^\circ : \overline{\overline{\epsilon}}^\circ\right] > 0. \tag{45}$$

Because ϵ and $\overline{\overline{\epsilon}}^\circ$ can be chosen independently, we must require

$$\mu > 0, \qquad 2\mu + 3\lambda > 0. \tag{46}$$

Examples

As an example, if a long and thin wire parallel to the z-axis is stretched, we have

$$\overline{\overline{\sigma}} = \sigma \mathbf{u}_z \mathbf{u}_z, \tag{47}$$

and the strain dyadic becomes

$$\overline{\overline{\epsilon}} = \frac{\sigma}{2\mu(2\mu + 3\lambda)} \left[2(\mu + \lambda)\mathbf{u}_z\mathbf{u}_z - \lambda \left(\overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z \right) \right]. \tag{48}$$

Thus, there is extension along the axis and contraction in the transverse plane. The quantity called Young's modulus $Y = \mu(2\mu + 3\lambda)/(\mu + \lambda)$ gives the ratio of the axial stress and strain components in the elastic medium: $\overline{\overline{\sigma}} : \mathbf{u}_z\mathbf{u}_z = Y\overline{\overline{\epsilon}} : \mathbf{u}_z\mathbf{u}_z$.

As another example, consider pure shear of a bar:

$$\overline{\overline{\sigma}} = \overline{\overline{\sigma}}^\circ = \sigma(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x). \tag{49}$$

In this case the strain dyadic becomes

$$\overline{\overline{\epsilon}} = \frac{\sigma}{2\mu}(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x). \tag{50}$$

Finally, hydrostatic pressure p on a body immersed in water produces the stress $\overline{\overline{\sigma}} = -p\overline{\overline{I}}$. The corresponding strain in the body is

$$\overline{\overline{\epsilon}} = -\frac{p}{2\mu + 3\lambda}\overline{\overline{I}}. \tag{51}$$

The increase in volume is, thus,

$$\nabla \cdot \mathbf{s} = \overline{\overline{\epsilon}} : \overline{\overline{I}} = -\frac{3p}{2\mu + 3\lambda}. \tag{52}$$

2.6 Uniaxial Medium

Let us consider the simplest possible generalization of the isotropic medium. Because of the symmetry constraints, tetradics of only special form can be added to the isotropic elastic tetradic. In fact, when adding a term $\mathcal{C}_1$ to the isotropic tetradic $\mathcal{C}_i$ as

$$\mathcal{C} = \mathcal{C}_i + \mathcal{C}_1, \tag{53}$$

$\mathcal{C}_1$ must satisfy the same rules of symmetry:

$$\mathcal{C}_1 = \mathcal{C}_1^T = {}^T\mathcal{C}_1 = \mathcal{C}_1^\tau. \tag{54}$$

Thus, a single tetrad

$$\mathcal{C}_1 = \mathbf{abcd} \tag{55}$$

must actually be of the form

$$\mathcal{C}_1 = C_1\mathbf{vvvv}, \tag{56}$$

where $\mathbf{v}$ is a unit vector. Obviously, a medium in this case can be labeled as uniaxially anisotropic because there exists a special axial direction defined by the vector $\mathbf{v}$. Because of its simple form, we call the medium defined by the tetradic $\mathcal{C}_i+\mathcal{C}_1$ the simple uniaxial medium. For the definition of the general uniaxial medium, see Appendix A.

Similarly, adding a tetradic $\mathcal{C}_2$ consisting of a sum of two tetrads, we can show that the symmetry requirement limits its form to either of the two possibilities

$$\mathcal{C}_2 = C_2(\mathbf{vvww} + \mathbf{wwvv}), \quad \text{or} \quad \mathcal{C}_2 = C_2(\mathbf{vvvv} + \mathbf{wwww}), \tag{57}$$

where $\mathbf{v}$ and $\mathbf{w}$ are two unit vectors.

In this study we concentrate only on isotropic media or simple uniaxial media with the single additive term (56). Let us for convenience denote the coefficient by $C_1 = \alpha\mu$ where α is a dimensionless number. Thus, the elastic tetradic becomes

$$\mathcal{C} = \mu\left(\mathcal{I} + \mathcal{T}\right) + \lambda\overline{\overline{I}}\,\overline{\overline{I}} + \alpha\mu\mathbf{vvvv}. \tag{58}$$

The energy function has the form

$$W\left(\overline{\overline{\epsilon}}\right) = \frac{1}{2}\overline{\overline{\epsilon}} : \mathcal{C} : \overline{\overline{\epsilon}} = \frac{1}{2}\left[2\mu\overline{\overline{\epsilon}} : \overline{\overline{\epsilon}} + \lambda\left(\overline{\overline{\epsilon}} : \overline{\overline{I}}\right)^2 + \alpha\mu\left(\overline{\overline{\epsilon}} : \mathbf{vv}\right)^2\right]. \tag{59}$$

Now we can again substitute $\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^\circ + \epsilon\overline{\overline{I}}$, where $\overline{\overline{\epsilon}}^\circ$ is trace-free. In this case we have

$$W\left(\overline{\overline{\epsilon}}\right) = \frac{1}{2}\left[\mu\overline{\overline{\epsilon}}^\circ : \overline{\overline{\epsilon}}^\circ + 3\mu\epsilon^2 + \lambda 9\epsilon^2 + \alpha\mu\left(\overline{\overline{\epsilon}}^\circ : \mathbf{vv}\right)^2 + \alpha\mu\epsilon^2\right], \tag{60}$$

which must be positive for all possible strain dyadics. Because the two terms $\overline{\overline{\epsilon}}^{\circ}$, $\epsilon\overline{\overline{I}}$ are independent, the condition splits in two parts as

$$2\mu + 3\lambda + \frac{1}{3}\alpha\mu > 0, \tag{61}$$

$$\mu\left[2\overline{\overline{\epsilon}}^{\circ} : \overline{\overline{\epsilon}}^{\circ} + \alpha\left(\overline{\overline{\epsilon}}^{\circ} : \mathbf{vv}\right)^2\right] > 0. \tag{62}$$

Taking $\overline{\overline{\epsilon}}^{\circ} = A(\mathbf{ww} - \mathbf{ww}{}^{\times}_{\times}\mathbf{vv})$ with $\mathbf{v}\cdot\mathbf{w} = 0$, it satisfies $\overline{\overline{\epsilon}}^{\circ} : \overline{\overline{I}} = \overline{\overline{\epsilon}}^{\circ} : \mathbf{vv} = 0$ and (62) leads to

$$\mu > 0. \tag{63}$$

The rest of (62) gives one more condition for α. Assuming $\overline{\overline{\epsilon}}^{\circ} : \mathbf{vv} \neq 0$ it can be written as

$$\alpha > -2\frac{\overline{\overline{\epsilon}}^{\circ} : \overline{\overline{\epsilon}}^{\circ}}{\left(\overline{\overline{\epsilon}}^{\circ} : \mathbf{vv}\right)^2}. \tag{64}$$

The most stringent condition is obtained for the axially symmetric dyadic $\overline{\overline{\epsilon}}^{\circ} = A\left(\mathbf{vv} - \frac{1}{3}\overline{\overline{I}}\right)$, in terms of which we have

$$\alpha > -3. \tag{65}$$

In summary, the three conditions for the positive-definite uniaxial medium are

$$\mu > 0, \qquad 2\mu + 3\lambda > \alpha\mu/3, \qquad \alpha > -3. \tag{66}$$

3. TIME-HARMONIC PROBLEMS

In the time-harmonic case Euler's equation of motion becomes

$$\nabla\cdot\overline{\overline{\sigma}} + \omega^2\rho\mathbf{s} = -\mathbf{F} \tag{67}$$

and the power balance equation in complex form can be derived in the form

$$\nabla\cdot\left(\overline{\overline{\sigma}}\cdot\mathbf{s}^*\right) = \overline{\overline{\sigma}} : \overline{\overline{\epsilon}}^* - \omega^2\rho\mathbf{s}\cdot\mathbf{s}^* - \mathbf{F}\cdot\mathbf{s}^*, \tag{68}$$

from which we can identify the complex Umov vector and the average energy quantities as

$$\mathbf{S} = \frac{1}{4}\overline{\overline{\sigma}}\cdot\mathbf{s}^*, \qquad K = \frac{1}{4}\omega^2\mathbf{s}\cdot\mathbf{s}^*, \qquad W = \frac{1}{4}\overline{\overline{\sigma}} : \overline{\overline{\epsilon}}^*. \tag{69}$$

The elastodynamic problem can be expressed in terms of two coupled equations for two dyadic and one vector unknown as

$$\nabla \cdot \overline{\overline{\sigma}} = -\,\omega^2 \rho \mathbf{s} - \mathbf{F}, \tag{70}$$

$$\frac{1}{2}\left[(\nabla \mathbf{s}) + (\nabla \mathbf{s})^T\right] = \overline{\overline{\epsilon}}, \tag{71}$$

with the constitutive equation $\overline{\overline{\sigma}} = \mathcal{C} : \overline{\overline{\epsilon}}$. The solution is essentially dependent on the medium tetradic $\mathcal{C}$.

Eliminating the dyadic unknowns, the problem can be reduced to a second-order equation for the displacement $\mathbf{s}$ as

$$\nabla \cdot \mathcal{C} : \overline{\overline{\epsilon}} + \omega^2 \rho \mathbf{s} = \nabla \cdot \mathcal{C} : \nabla \mathbf{s} + \omega^2 \rho \mathbf{s} = -\mathbf{F}. \tag{72}$$

Because of symmetry, we can also write

$$\overline{\overline{L}}(\nabla) \cdot \mathbf{s} = \mathcal{C} \;③\; \nabla\nabla \mathbf{s} + \omega^2 \rho \mathbf{s} = -\mathbf{F}. \tag{73}$$

The second-order dyadic operator is of the form

$$\overline{\overline{L}}(\nabla) = \nabla \cdot \mathcal{C} \cdot \nabla + \omega^2 \rho \overline{\overline{I}}. \tag{74}$$

3.1 Plane Waves

A set of plane-wave solutions to the equation (73) is obtained by studying the class of exponential functions

$$\mathbf{s}(\mathbf{r}) = \mathbf{s}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{75}$$

with no sources $\mathbf{F}$ in the finite region. In this case we have the algebraic vector equation

$$\overline{\overline{L}}(-j\mathbf{k}) \cdot \mathbf{s}_o = 0. \tag{76}$$

Assuming $\mathbf{s}_o \neq 0$ gives us the sixth-order dispersion equation for the vector $\mathbf{k}$:

$$\det \overline{\overline{L}}(-j\mathbf{k}) = -\det\left(\mathbf{k} \cdot \mathcal{C} \cdot \mathbf{k} - \omega^2 \rho \overline{\overline{I}}\right) = 0. \tag{77}$$

The determinant can be expanded through dyadic operations as [2]

$$\det \overline{\overline{L}}(-j\mathbf{k}) = \frac{1}{6} \overline{\overline{L}}(-j\mathbf{k})^{\times}_{\times} \overline{\overline{L}}(-j\mathbf{k}) : \overline{\overline{L}}(-j\mathbf{k}) = 0. \tag{78}$$

Isotropic Medium

In the isotropic case the dyadic becomes

$$\overline{\overline{L}}_i(-j\mathbf{k}) = -\mathbf{k}\cdot\left[\mu(\mathcal{I}+\mathcal{T})+\lambda\overline{\overline{I}}\,\overline{\overline{I}}\right]\cdot\mathbf{k}+\omega^2\rho\overline{\overline{I}}$$
$$= -(\mu+\lambda)\mathbf{k}\mathbf{k}-\left[\mu(\mathbf{k}\cdot\mathbf{k})-\omega^2\rho\right]\overline{\overline{I}}, \qquad (79)$$

and the determinant can be further expanded as

$$\det\overline{\overline{L}}_i(-j\mathbf{k}) = -\left[\mu(\mathbf{k}\cdot\mathbf{k})-\omega^2\rho\right]^3-\left[\mu(\mathbf{k}\cdot\mathbf{k})-\omega^2\rho\right]^2(\mu+\lambda)(\mathbf{k}\cdot\mathbf{k})$$
$$= -\left[\mu(\mathbf{k}\cdot\mathbf{k})-\omega^2\rho\right]^2\left[(2\mu+\lambda)(\mathbf{k}\cdot\mathbf{k})-\omega^2\rho\right]. \qquad (80)$$

Thus, the equation which is of third order for $\mathbf{k}\cdot\mathbf{k}$ can be split in two parts. Denoting the direction of wave propagation by the unit vector $\mathbf{u}$, the solutions $\mathbf{k}=\mathbf{u}k(\mathbf{u})$ for the two waves are

$$\mu(\mathbf{k}\cdot\mathbf{k})-\omega^2\rho=0, \qquad k_1(\mathbf{r})=\omega\sqrt{\frac{\rho}{\mu}}, \qquad (81)$$

$$(2\mu+\lambda)(\mathbf{k}\cdot\mathbf{k})-\omega^2\rho=0, \qquad k_2(\mathbf{u})=\omega\sqrt{\frac{\rho}{2\mu+\lambda}}. \qquad (82)$$

When these are substituted in the dyadic $\overline{\overline{L}}(-j\mathbf{k})$, the equation (76) becomes

$$\overline{\overline{L}}_i(-j\mathbf{k}_1)\cdot\mathbf{s}_{o1} = -\omega^2\rho\frac{\mu+\lambda}{\mu}\,\mathbf{u}\mathbf{u}\cdot\mathbf{s}_{o1}=0, \qquad (83)$$

$$\overline{\overline{L}}_i(-j\mathbf{k}_2)\cdot\mathbf{s}_{o2} = \omega^2\rho\frac{\mu+\lambda}{2\mu+\lambda}\left[\overline{\overline{I}}-\mathbf{u}\mathbf{u}\right]\cdot\mathbf{s}_{o2}=0. \qquad (84)$$

Thus, we can see that the polarization of the wave 2 is longitudinal (P-wave, compressional wave), $\mathbf{s}_{o2}=\mathbf{u}$, while that of the wave 1 is transversal (S-wave, shear wave), because any $\mathbf{s}_{o1}$ satisfying $\mathbf{u}\cdot\mathbf{s}_{o1}=0$ will do. There are two possible transversely polarized S-waves with orthogonal polarizations.

The strain dyadics corresponding to these two waves are

$$\overline{\overline{\epsilon}}_1 = \frac{1}{2}\left[\nabla\mathbf{s}_1(\mathbf{r})+(\nabla\mathbf{s}_1(\mathbf{r}))^T\right] = -\frac{j}{2}\left[\mathbf{k}_1\mathbf{s}_{o1}+\mathbf{s}_{o1}\mathbf{k}_1\right]e^{-j\mathbf{k}_1\cdot\mathbf{r}}$$
$$= -\frac{j\omega}{2}\sqrt{\frac{\rho}{\mu}}\left[\mathbf{u}\mathbf{s}_{o1}+\mathbf{s}_{o1}\mathbf{u}\right]e^{-j\mathbf{k}_1\cdot\mathbf{r}}, \qquad (85)$$

$$\overline{\overline{\epsilon}}_2 = \frac{1}{2}\left[\nabla\mathbf{s}_2(\mathbf{r})+(\nabla\mathbf{s}_2(\mathbf{r}))^T\right] = -\frac{j}{2}\left[\mathbf{k}_2\mathbf{s}_{o2}+\mathbf{s}_{o2}\mathbf{k}_2\right]e^{-j\mathbf{k}_2\cdot\mathbf{r}}$$
$$= -j\omega\sqrt{\frac{\rho}{2\mu+\lambda}}\,\mathbf{u}\mathbf{s}_{o2}e^{-j\mathbf{k}_2\cdot\mathbf{r}}. \qquad (86)$$

The shear wave satisfies $\overline{\overline{\epsilon}}_1 : \overline{\overline{I}} = \overline{\overline{\epsilon}}_1 : \mathbf{uu} = 0$. The stress dyadics corresponding to these waves are

$$\overline{\overline{\sigma}}_1 = \mathcal{C} : \overline{\overline{\epsilon}}_1 = 2\mu\overline{\overline{\epsilon}}_1, \qquad \overline{\overline{\sigma}}_2 = \mathcal{C} : \overline{\overline{\epsilon}}_2 = \epsilon_2 \left(2\mu\mathbf{uu} + \lambda\overline{\overline{I}}\right), \tag{87}$$

when we denote $\overline{\overline{\epsilon}}_2 = \epsilon_2\mathbf{uu}$. Because the strain dyadics are orthogonal in the sense that $\overline{\overline{\epsilon}}_1 : \mathcal{C} : \overline{\overline{\epsilon}}_2 = \overline{\overline{\epsilon}}_1 : \overline{\overline{\sigma}}_2 = 0$, the energy of the sum wave is the sum of energies of the two waves:

$$W\left(\overline{\overline{\epsilon}}_1 + \overline{\overline{\epsilon}}_2\right) = W\left(\overline{\overline{\epsilon}}_1\right) + W\left(\overline{\overline{\epsilon}}_2\right). \tag{88}$$

Thus, these partial waves propagate without coupling energy to each other.

Uniaxial Medium

Let us now consider wave propagation in the simple uniaxial medium defined by (58). In this case the dispersion equation becomes

$$\begin{aligned}
&\det\left[\overline{\overline{L}}_i(-j\mathbf{k}) + \alpha\mu(\mathbf{v}\cdot\mathbf{k})^2\mathbf{vv}\right] \\
&\quad = \det\overline{\overline{L}}_i(-j\mathbf{k}) + \alpha\mu(\mathbf{v}\cdot\mathbf{k})^2\mathbf{vv} : \overline{\overline{L}}_i^{(2)}(-j\mathbf{k}), \qquad (89)\\
\overline{\overline{L}}_i^{(2)}(-j\mathbf{k}) &= \frac{1}{2}\left[(\mu+\lambda)\mathbf{kk} + \left[\mu(\mathbf{k}\cdot\mathbf{k}) - \omega^2\rho\right]\overline{\overline{I}}\right] \\
&\quad {}^{\times}_{\times}\left[(\mu+\lambda)\mathbf{kk} + \left[\mu(\mathbf{k}\cdot\mathbf{k}) - \omega^2\rho\right]\overline{\overline{I}}\right] \\
&= \left[\mu(\mathbf{k}\cdot\mathbf{k}) - \omega^2\rho\right]\left[\mu(\mathbf{k}\cdot\mathbf{k}) - \omega^2\rho + (\mu+\lambda)\mathbf{kk}{}^{\times}_{\times}\overline{\overline{I}}\right] \\
&= \mu(2\mu+\lambda)\left(\mathbf{k}\cdot\mathbf{k} - k_1^2\right)\left[\left(\mathbf{k}\cdot\mathbf{k} - k_2^2\right)\overline{\overline{I}} - \frac{\mu+\lambda}{2\mu+\lambda}\mathbf{kk}\right]. \qquad (90)
\end{aligned}$$

Combining these and (80), the dispersion equation becomes

$$\begin{aligned}
&\det\left[\overline{\overline{L}}_i(-j\mathbf{k}) + \alpha\mu\left(\mathbf{v}\cdot\mathbf{k}\right)^2\mathbf{vv}\right] \\
&= -\mu^2(2\mu+\lambda)\left(\mathbf{k}\cdot\mathbf{k} - k_1^2\right)^2\left(\mathbf{k}\cdot\mathbf{k} - k_2^2\right) \\
&\quad + \mu^2(2\mu+\lambda)\alpha(\mathbf{v}\cdot\mathbf{k})^2\left(\mathbf{k}\cdot\mathbf{k} - k_1^2\right)\left[\mathbf{k}\cdot\mathbf{k} - k_2^2 - \frac{\mu+\lambda}{2\mu+\lambda}(\mathbf{v}\cdot\mathbf{k})^2\right] \\
&= 0, \qquad (91)
\end{aligned}$$

or

$$\left(\mathbf{k}\cdot\mathbf{k}-k_1^2\right)\left(\left(\mathbf{k}\cdot\mathbf{k}-k_1^2\right)\left(\mathbf{k}\cdot\mathbf{k}-\frac{\alpha\mu}{2\mu+\lambda}(\mathbf{v}\cdot\mathbf{k})^2-k_2^2\right)\right.$$
$$\left.-\alpha\frac{\mu+\lambda}{2\mu+\lambda}(\mathbf{v}\cdot\mathbf{k})^2(\mathbf{v}\times\mathbf{k})^2\right)=0. \quad (92)$$

It is seen that the $\mathbf{k}$-vector surfaces consist of a sphere $\mathbf{k}=k_1\mathbf{u}$ and a fourth-order surface with two wave vectors for the general direction $\mathbf{u}$. For $\alpha\to 0$ this latter surface is split into two spheres, one coinciding with the S-wave sphere $\mathbf{k}=k_1\mathbf{u}$ and the other one corresponding to the P-wave with $\mathbf{k}=k_2\mathbf{u}$. As is seen from Figure 1, when α grows from zero, these two wave-vector surfaces start to depart from spherical form, eventually touching each other for a certain value of α. After that there will be complex k values for certain directions of propagation corresponding to attenuating waves.

For waves propagating transverse to the axis $\mathbf{v}$ of the medium, $\mathbf{k}\cdot\mathbf{v}=0$, the dispersion equation (92) reduces to

$$\left(\mathbf{k}\cdot\mathbf{k}-k_1^2\right)^2\left(\mathbf{k}\cdot\mathbf{k}-k_2^2\right)=0, \quad (93)$$

which is the same as that of the isotropic medium. Thus, the $\mathbf{k}$-vector surfaces cut the plane transverse to $\mathbf{v}$ in two circles of radii k_1 and k_2. The two waves propagating along the axis, $\mathbf{k}=\mathbf{v}k_v$, can be shown to satisfy

$$\left(k_v^2-k_1^2\right)^2\left(k_v^2-k_\alpha^2\right)=0, \qquad k_\alpha^2=\frac{\omega^2\rho}{2\mu+\lambda-\alpha\mu}, \quad (94)$$

or, again, there are only two possible wave vectors. For $\alpha\to 0$ we have $k_\alpha\to k_2$.

Let us now study the fourth-order surface for the general direction of propagation. Writing

$$c=\cos\theta=\mathbf{v}\cdot\mathbf{u}, \qquad s^2=\sin^2\theta=(\mathbf{v}\times\mathbf{u})\cdot(\mathbf{v}\times\mathbf{u})=1-c^2, \quad (95)$$

after some algebraic work the fourth-order part of (92) can be written for the wavenumber k as

$$k^{-4}-k^{-2}\left[k_2^{-2}+\left(1-\alpha c^2\right)k_1^{-2}\right]$$
$$+k_1^{-2}\left[\left(1+\alpha c^2 s^2\right)k_2^{-2}-\alpha\left(1-s^4\right)k_1^{-2}\right]=0. \quad (96)$$

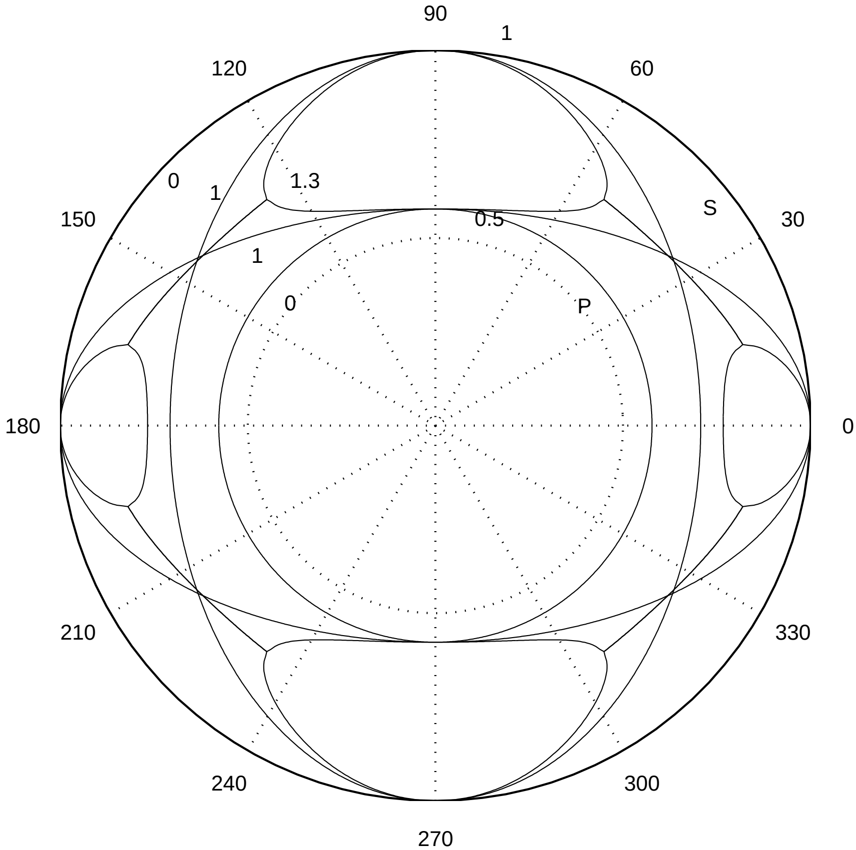

Figure 1. Normalized wave-vector surfaces $k(\theta)/k_1$ for uniaxial elastic medium for parameter values $(\mu+\lambda)/\mu = 2$ and $\alpha = 0, 1, 1.3$. The surfaces are rotationally symmetric around the horizontal axis parallel to the unit vector $\mathbf{v}$. The outer (two S-waves) and inner (one P-wave) spheres correspond to the isotropic case $\alpha = 0$. For the critical value $\alpha = 1$ the exterior quasiS-wave surface touches the interior quasiP-wave surface along two circles (four points in the figure). For $\alpha = 1.3$ the real surfaces form two caps at the poles and a belt around the equator, connected by regions of complex wave-vectors. In the figure these regions are depicted by the real part of the wave-vectors which are the same for both waves and seen as a single surface connecting the real-valued regions.

The solution can then be expressed as

$$2k^{-2} = \left(1 - \alpha c^2\right) k_1^{-2} + k_2^{-2} \pm \sqrt{\left[(1-\alpha c^2)k_1^{-2} - k_2^{-2}\right]^2 + 4\alpha(1-s^4)k_1^{-2}\left(k_1^{-2} - k_2^{-2}\right)}, \quad (97)$$

or

$$(k_1/k)^2 = \frac{1}{2}\left[2 + \frac{\mu+\lambda}{\mu} - \alpha c^2 \pm \sqrt{\left(\frac{\mu+\lambda}{\mu} - \alpha c^2\right)^2 - 4\alpha c^2 s^2 \frac{\mu+\lambda}{\mu}}\,\right]. \quad (98)$$

In addition to these two wave-number solutions there is the third one with the simple condition $k = k_1$.

At this point we can check the final expression (98) for the known special cases. Both $\alpha = 0$ and $c = 0$ are readily seen to give the two solutions $k = k_2, k_1$ corresponding to the upper and lower signs of the square root. For $c = 1$ we respectively obtain $k = k_\alpha, k_1$. It is interesting to find whether the two solutions of (98) coincide for any real value of θ. In fact, requiring the square-root term to vanish gives us an equation for $c = \cos\theta$:

$$\left(x - c^2\right)^2 = 4c^2\left(1 - c^2\right)x, \qquad x = \frac{\mu+\lambda}{\alpha\mu}, \quad (99)$$

whose solution is

$$c^2 = \cos^2\theta = \frac{3x}{1+4x}\left(1 \pm \sqrt{1 - \frac{1+4x}{9}}\right). \quad (100)$$

There is only one solution for c^2 when the square-root term vanishes, i.e., for

$$x = 2, \quad \Rightarrow \quad \alpha = \frac{\mu+\lambda}{2\mu}. \quad (101)$$

This corresponds to the two values $c = \cos\theta = \pm\sqrt{2/3}$ or $\theta = 35.3^\circ, 144.7^\circ$ whose sum is 180°. Beyond this value of α the square-root term becomes imaginary and $k(\theta)$ has complex values for some values of θ. To check the angular ranges with complex $k(\theta)$ values of Figure 1, we set $x = 2/1.3 = 1.54$ which gives us the ranges $\theta = 14.5^\circ - 53.5^\circ$ and $\theta = 126.5^\circ - 165.5^\circ$. For $x = 1$ the caps with real values for $k(\theta)$ vanish in Figure 1.

To find the polarization of the plane wave in the uniaxial medium we can again study the original equation (76), which can be written as

$$\overline{\overline{L}}(-j\mathbf{k})\cdot\mathbf{s}_o = -\left[(\mu+\lambda)k^2\mathbf{uu} + \mu\left(k^2-k_1^2\right)\overline{\overline{I}} - \alpha\mu k^2 c^2\mathbf{vv}\right]\cdot\mathbf{s}_o = 0. \tag{102}$$

When studying this close enough we find that $\mathbf{s}_o$ can satisfy $\mathbf{s}_o\cdot\mathbf{u}=0$ or $\mathbf{s}_o\times\mathbf{u}=0$ only for axial and transverse propagation, i.e., for $\mathbf{u}=\mathbf{v}$ or $\mathbf{u}\cdot\mathbf{v}=0$. For transverse propagation $c=0$ the anisotropy does not actually have any effect, whence the waves are like in isotropic medium: two S-waves with $k=k_1$ and one P-wave with $k=k_2$. For axial propagation, again, there are two S-waves with $k=k_1$, but the P-wave satisfies $k=k_\alpha$. For all other directions there is one S-wave with $k=k_1$ and two waves which can be labeled as quasiS- and quasiP-waves which reduce to pure S- and P-waves for $\alpha\to 0$.

3.2 Green Functions

To solve elastic fields for a given source function $\mathbf{F}(\mathbf{r})$ in integral form we need the Green dyadic, field from a point source, which is the solution of the equation

$$\overline{\overline{L}}(\nabla)\cdot\overline{\overline{G}}(\mathbf{r}) = -\delta(\mathbf{r})\overline{\overline{I}}, \qquad \overline{\overline{G}}(\mathbf{r}) = -\overline{\overline{L}}^{-1}(\nabla)\delta(\mathbf{r}). \tag{103}$$

For uniqueness, outgoing-wave conditions in the infinity are needed. The solution depends essentially on the elastic tetradic $\mathcal{C}$ and analytic solutions can be found only in some special cases. In fact, expressing the inverse of the dyadic operator in the analytic form [2]

$$\overline{\overline{L}}^{-1}(\nabla) = \frac{1}{\det\overline{\overline{L}}(\nabla)}\overline{\overline{L}}^{(2)T}(\nabla), \tag{104}$$

$$\overline{\overline{L}}^{(2)}(\nabla) = \frac{1}{2}\overline{\overline{L}}(\nabla)\,{}^{\times}_{\times}\,\overline{\overline{L}}(\nabla), \qquad \det\overline{\overline{L}}(\nabla) = \frac{1}{6}\overline{\overline{L}}(\nabla)\,{}^{\times}_{\times}\,\overline{\overline{L}}(\nabla):\overline{\overline{L}}(\nabla), \tag{105}$$

one can reduce the dyadic problem to a scalar Green function problem

$$\det\overline{\overline{L}}(\nabla)\; G(\mathbf{r}) = -\delta(\mathbf{r}), \tag{106}$$

whose solution will give the Green dyadic as

$$\overline{\overline{G}}(\mathbf{r}) = \overline{\overline{L}}^{(2)T}(\nabla)G(\mathbf{r}). \tag{107}$$

Now the problem to find the scalar Green function $G(\mathbf{r})$ leads to solving a partial differential equation of the sixth order in general. Hopes for its solution can be concentrated to those cases in which the equation reduces to one of lower order or when the determinant operator can be expressed in factorized form. The order is lower in the case when there is the same scalar operator as a factor of both $\overline{\overline{L}}^{(2)T}(\nabla)$ and $\det\overline{\overline{L}}(\nabla)$ operators, whence it can be canceled out. In cases when the inverse operator can be formulated directly there is no need to form the operators $\overline{\overline{L}}^{(2)T}(\nabla)$ and $\det\overline{\overline{L}}(\nabla)$ explicitly.

Isotropic Medium

For the isotropic tetradic (41) the dyadic operator is of the form

$$\begin{aligned}\overline{\overline{L}}_i(\nabla) &= \nabla\cdot\left[\mu(\mathcal{I}+\mathcal{T})+\lambda\overline{\overline{I}}\,\overline{\overline{I}}\right]\cdot\nabla+\omega^2\rho\overline{\overline{I}}\\ &= (\mu+\lambda)\nabla\nabla+\mu\left(\nabla^2+k_1^2\right)\overline{\overline{I}}.\end{aligned} \tag{108}$$

The adjoint operator $\overline{\overline{L}}_i^{(2)}(\nabla)$ and the determinant operator can be expressed as [2]

$$\overline{\overline{L}}_i^{(2)}(\nabla) = \mu(2\mu+\lambda)\left(\nabla^2+k_1^2\right)\left[\left(\nabla^2+k_2^2\right)\overline{\overline{I}}-\frac{\mu+\lambda}{2\mu+\lambda}\nabla\nabla\right], \tag{109}$$

$$\det\overline{\overline{L}}_i(\nabla) = \mu^2(2\mu+\lambda)\left(\nabla^2+k_1^2\right)^2\left(\nabla^2+k_2^2\right). \tag{110}$$

Thus, in this case one operator $\nabla^2+k_1^2$ can be canceled out in forming the inverse operator:

$$\begin{aligned}\overline{\overline{L}}_i^{-1}(\nabla) &= \frac{\left(\nabla^2+k_2^2\right)\overline{\overline{I}}-\dfrac{\mu+\lambda}{2\mu+\lambda}\nabla\nabla}{\mu\left(\nabla^2+k_1^2\right)\left(\nabla^2+k_2^2\right)}\\ &= \frac{\overline{\overline{I}}}{\mu\left(\nabla^2+k_1^2\right)}-\frac{\mu+\lambda}{2\mu+\lambda}\frac{\nabla\nabla}{\mu\left(\nabla^2+k_1^2\right)\left(\nabla^2+k_2^2\right)}.\end{aligned} \tag{111}$$

The factorized denominator in the last term can be dissolved through partial fractions as

$$\frac{1}{\left(\nabla^2+k_1^2\right)\left(\nabla^2+k_2^2\right)} = \frac{1}{k_2^2-k_1^2}\left(\frac{1}{\nabla^2+k_1^2}-\frac{1}{\nabla^2+k_2^2}\right). \tag{112}$$

Inserting this, the inverse operator can finally be expressed quite simply as

$$\overline{\overline{L}}_i^{-1}(\nabla) = \frac{\overline{\overline{I}}}{\mu\left(\nabla^2 + k_1^2\right)} + \frac{1}{\omega^2\rho}\left(\frac{\nabla\nabla}{\nabla^2 + k_1^2} - \frac{\nabla\nabla}{\nabla^2 + k_2^2}\right)$$
$$= \frac{1}{\omega^2\rho}\left(\frac{\nabla\nabla + k_1^2\overline{\overline{I}}}{\nabla^2 + k_1^2} - \frac{\nabla\nabla}{\nabla^2 + k_2^2}\right). \quad (113)$$

Because the scalar Green function corresponding to the general scalar Helmholtz operator $\nabla^2 + k^2$ is of the known form (satisfying the radiation condition in infinity)

$$G(k, \mathbf{r}) = -\frac{1}{\nabla^2 + k^2}\delta(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r}, \quad (114)$$

we can finally write the Green dyadic as a sum of three terms

$$\overline{\overline{G}}(\mathbf{r}) = \frac{1}{\omega^2\rho}\left(\nabla \times \overline{\overline{I}}\right)^2 G(k_1, \mathbf{r}) - \frac{1}{\omega^2\rho}\nabla\nabla G(k_2, \mathbf{r}) - \frac{1}{\omega^2\rho}\overline{\overline{I}}\delta(\mathbf{r}). \quad (115)$$

The last term represents a singularity at the source point which does not affect fields outside the source and can be simply omitted unless the field within the source region is under interest. The other two terms give solenoidal and irrotational components which propagate with different velocities corresponding to the two wavenumbers k_1 and k_2.

Given a source $\mathbf{F}(\mathbf{r})$, we can find the solution $\mathbf{s}(\mathbf{r})$ outside the source region V as the sum of two vector functions

$$\mathbf{s}(\mathbf{r}) = -\int_V \overline{\overline{G}}\left(\mathbf{r} - \mathbf{r}'\right) \cdot \mathbf{F}\left(\mathbf{r}'\right) dV' = \mathbf{s}_1(\mathbf{r}) + \mathbf{s}_2(\mathbf{r}), \quad (116)$$

with

$$\mathbf{s}_1(\mathbf{r}) = -\frac{1}{\omega^2\rho}\nabla \times \nabla \times \int_V G\left(k_1, \mathbf{r} - \mathbf{r}'\right)\mathbf{F}\left(\mathbf{r}'\right), \quad (117)$$

$$\mathbf{s}_2(\mathbf{r}) = \frac{1}{\omega^2\rho}\nabla\nabla \cdot \int_V G\left(k_2, \mathbf{r} - \mathbf{r}'\right)\mathbf{F}\left(\mathbf{r}'\right) dV'. \quad (118)$$

It is seen that the former is a solenoidal vector function and, the latter, an irrotational vector. By inserting the differentiations inside the well-behaved integrals (the source and field points do not coincide) and making partial integrations we can show that if the source is irrotational: $\nabla \times \mathbf{F}(\mathbf{r}) = 0$, we have $\mathbf{s}_1(\mathbf{r}) = 0$ and, if the source is solenoidal: $\nabla \cdot \mathbf{F} = 0$, we have $\mathbf{s}_2(\mathbf{r}) = 0$. This is, however, valid only for fields outside the source region.

Uniaxial Medium

Let us finally consider the simple uniaxial medium defined by (58). In this case, the dyadic operator becomes

$$\overline{\overline{L}}(\nabla) = \overline{\overline{L}}_i(\nabla) + \alpha\mu(\mathbf{v}\cdot\nabla)^2\mathbf{vv}, \tag{119}$$

where $\overline{\overline{L}}_i(\nabla)$ denotes the operator of the isotropic medium (108). Let us again find the inverse operator through the formula (104) by expanding the operators as [2]

$$\begin{aligned}\overline{\overline{L}}^{-1}(\nabla) &= \frac{\left[\overline{\overline{L}}_i(\nabla) + \alpha\mu(\mathbf{v}\cdot\nabla)^2\mathbf{vv}\right]^{(2)}}{\det\left[\overline{\overline{L}}_i(\nabla) + \alpha\mu(\mathbf{v}\cdot\nabla)^2\mathbf{vv}\right]}\\ &= \frac{\overline{\overline{L}}_i^{(2)}(\nabla) + \alpha\mu(\mathbf{v}\cdot\nabla)^2\mathbf{vv}\,{}^{\times}_{\times}\overline{\overline{L}}_i(\nabla)}{\det\overline{\overline{L}}_i(\nabla) + \alpha\mu(\mathbf{v}\cdot\nabla)^2\mathbf{vv} : \overline{\overline{L}}_i^{(2)}(\nabla)}.\end{aligned} \tag{120}$$

This time progress in general analytical form appears hopeless, because the fourth-order operator in $\det\overline{\overline{L}}(\nabla)$ doesn't seem to be factorizable for general parameter values. This can be understood from the dispersion equation of the plane wave, because the wave-vector surfaces include one of fourth order and, apparently, it cannot be reduced to two second-order surfaces unless the parameter α vanishes. However, for small values of $|\alpha|$ we can find a perturbational solution for the Green dyadic. In fact, the Green dyadic can be found for perturbational anisotropy of general form, as is seen below.

General Perturbational Anisotropy

Let us assume that the elastic tetradic is of the form

$$\mathcal{C} = \mathcal{C}_i + \mathcal{C}_\alpha, \tag{121}$$

where $\mathcal{C}_i$ is the isotropic tetradic

$$\mathcal{C}_i = \mu(\mathcal{I} + \mathcal{T}) + \lambda\overline{\overline{I}}\,\overline{\overline{I}} \tag{122}$$

and $\mathcal{C}_\alpha = \alpha\mathcal{C}_1$ is a small tetradic with a finite tetradic $\mathcal{C}_1$ and vanishing coefficient $\alpha \to 0$.

The operator dyadic can be written as

$$\overline{\overline{L}}(\nabla) = \nabla\cdot\mathcal{C}\cdot\nabla + \omega^2\rho\overline{\overline{I}} = \overline{\overline{L}}_i(\nabla) + \alpha\overline{\overline{L}}_1(\nabla), \quad \overline{\overline{L}}_1(\nabla) = \nabla\cdot\mathcal{C}_1\cdot\nabla. \tag{123}$$

The operator $\overline{\overline{L}}_i(\nabla)$ corresponding to the isotropic part of the elastic tetradic has a known inverse $\overline{\overline{L}}_i^{-1}(\nabla)$ as given in (113). To find the Green dyadic corresponding to the perturbationally anisotropic medium,

$$\overline{\overline{G}}(\mathbf{r}) = -\overline{\overline{L}}^{-1}(\nabla)\delta(\mathbf{r}), \tag{124}$$

we have to form the inverse of the operator $\overline{\overline{L}}(\nabla)$. This can be expanded to first order in α as

$$\begin{aligned}\overline{\overline{L}}^{-1}(\nabla) &= \left[\overline{\overline{I}} + \alpha\overline{\overline{L}}_i^{-1}(\nabla)\cdot\overline{\overline{L}}_1(\nabla)\right]^{-1}\cdot\overline{\overline{L}}_i^{-1}(\nabla)\\ &\approx \left[\overline{\overline{I}} - \alpha\overline{\overline{L}}_i^{-1}(\nabla)\cdot\overline{\overline{L}}_1(\nabla)\right]\cdot\overline{\overline{L}}_i^{-1}(\nabla) = \overline{\overline{L}}_i^{-1}(\nabla) + \alpha\overline{\overline{M}}(\nabla),\\ \overline{\overline{M}}(\nabla) &= -\overline{\overline{L}}_i^{-1}(\nabla)\cdot\overline{\overline{L}}_1(\nabla)\cdot\overline{\overline{L}}_i^{-1}(\nabla).\end{aligned} \tag{125}$$

The last term represents the perturbational correction to the inverse operator. In its expression we can now substitute from (113) as

$$\begin{aligned}&\overline{\overline{M}}(\nabla)\\ &= -\frac{1}{\omega^4\rho^2}\left(\frac{\nabla\nabla + k_1^2\overline{\overline{I}}}{\nabla^2 + k_1^2} - \frac{\nabla\nabla}{\nabla^2 + k_2^2}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\frac{\nabla\nabla + k_1^2\overline{\overline{I}}}{\nabla^2 + k_1^2} - \frac{\nabla\nabla}{\nabla^2 + k_2^2}\right)\\ &= -\frac{1}{\omega^4\rho^2}\left[\frac{\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)}{\left(\nabla^2 + k_1^2\right)^2} + \frac{\nabla\nabla\cdot\overline{\overline{L}}_1(\nabla)\cdot\nabla\nabla}{\left(\nabla^2 + k_2^2\right)^2}\right.\\ &\quad\left. - \frac{\nabla\nabla\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right) + \left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\nabla\nabla}{\left(\nabla^2 + k_1^2\right)\left(\nabla^2 + k_2^2\right)}\right].\end{aligned} \tag{126}$$

The last inverse operator can be expressed as

$$\frac{1}{\left(\nabla^2 + k_1^2\right)\left(\nabla^2 + k_2^2\right)} = \frac{1}{k_2^2 - k_1^2}\left(\frac{1}{\nabla^2 + k_1^2} - \frac{1}{\nabla^2 + k_2^2}\right), \tag{127}$$

which gives

$$\overline{\overline{M}}(\nabla) = -\frac{1}{\omega^4\rho^2}\left[\frac{\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)}{\left(\nabla^2 + k_1^2\right)^2}\right.$$
$$+\frac{\nabla\nabla\cdot\overline{\overline{L}}_1(\nabla)\cdot\nabla\nabla}{\left(D^2 + k_2^2\right)^2} - \frac{1}{k_2^2 - k_1^2}\left[\nabla\nabla\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\right.$$
$$\left.\left. + \left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\nabla\nabla\right]\left(\frac{1}{\nabla^2 + k_1^2} - \frac{1}{\nabla^2 + k_2^2}\right)\right]. \tag{128}$$

The scalar Green function corresponding to the square of the scalar Helmholtz operator $(\nabla^2 + k^2)^2$ can be easily derived from the Green function (114) of the Helmholtz operator through differentiation with respect to the quantity k^2 (whence $k = \sqrt{k^2}$). Choosing the function satisfying the radiation condition in infinity we have

$$G'(k, \mathbf{r}) = -\frac{1}{\left(\nabla^2 + k^2\right)^2}\delta(\mathbf{r}) = \frac{\partial}{\partial k^2}\frac{1}{\nabla^2 + k^2}\delta(\mathbf{r})$$
$$= -\frac{\partial}{\partial k^2}\frac{e^{-jkr}}{4\pi r} = -\frac{e^{-jkr}}{j8\pi k}. \tag{129}$$

Thus, the approximate Green dyadic for the perturbational uniaxial anisotropic elastic medium has the form

$$\overline{\overline{G}}(\mathbf{r}) = -\overline{\overline{L}}^{-1}(\nabla)\delta(\mathbf{r}) \approx -\overline{\overline{L}}_i^{-1}(\nabla)\delta(\mathbf{r}) - \alpha\overline{\overline{M}}(\nabla)\delta(\mathbf{r}) = \overline{\overline{G}}_i(\mathbf{r}) + \alpha\overline{\overline{G}}'(\mathbf{r}), \tag{130}$$

where the perturbational Green dyadic can be expressed in closed form as

$$\overline{\overline{G}}'(\mathbf{r}) = -\overline{\overline{M}}(\nabla)\delta(\mathbf{r})$$
$$= \frac{1}{\omega^4\rho^2}\left[\frac{\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)}{\left(\nabla^2 + k_1^2\right)^2} + \frac{\nabla\nabla\cdot\overline{\overline{L}}_1(\nabla)\cdot\nabla\nabla}{\left(\nabla^2 + k_2^2\right)^2}\right.$$
$$- \frac{1}{k_2^2 - k_1^2}\left[\nabla\nabla\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right) + \left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\nabla\nabla\right]$$
$$\left.\cdot\left(\frac{1}{\nabla^2 + k_1^2} - \frac{1}{\nabla^2 + k_2^2}\right)\right]\delta(\mathbf{r}). \tag{131}$$

Finally, we arrive at the expression

$$\overline{\overline{G}}'(\mathbf{r}) = \frac{1}{\omega^4\rho^2}\Big[-\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)G'(k_1, r)$$
$$-\nabla\nabla\cdot\overline{\overline{L}}_1(\nabla)\cdot\nabla\nabla G'(k_2,\mathbf{r}) + \frac{1}{k_2^2-k_1^2}\Big[\nabla\nabla\cdot\overline{\overline{L}}_1(\nabla)\cdot\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)$$
$$+\left(\nabla\nabla + k_1^2\overline{\overline{I}}\right)\cdot\overline{\overline{L}}_1(\nabla)\cdot\nabla\nabla\Big]\left[G(k_1,\mathbf{r}) - G(k_2,\mathbf{r})\right]\Big], \tag{132}$$

where we have yet to insert the operator $\overline{\overline{L}}_1(\nabla)$ in question. The perturbational addition (132) for the Green dyadic is believed to be new, [8, 9].

In some cases the expression is simplified. For example, if the anisotropic elastic tetradic has the special form

$$\mathcal{C}_1 = C_1\overline{\overline{I}}\times\overline{\overline{I}}\times\overline{\overline{I}}, \tag{133}$$

we have

$$\overline{\overline{L}}_1(\nabla) = C_1\nabla\times\overline{\overline{I}}\times\nabla = C_1\left(\nabla\nabla - \nabla^2\overline{\overline{I}}\right). \tag{134}$$

Because of $\nabla\cdot\overline{\overline{L}}_1(\nabla) = \overline{\overline{L}}_1(\nabla)\cdot\nabla = 0$, the expression (132) is simplified to

$$\overline{\overline{G}}'(\mathbf{r}) = -\frac{k_1^4}{\omega^4\rho^2}\overline{\overline{L}}_1(\nabla)G'(k_1,r) = \frac{k_1^3 C_1}{j8\pi\omega^4\rho^2}\left(\nabla\nabla - \nabla^2\overline{\overline{I}}\right)e^{-jk_1 r}. \tag{135}$$

APPENDIX A. TETRADICS

Here we give some definitions and properties concerning working with coordinate-free vectors, dyadics and tetradics. It is assumed that basic dyadic algebra is known, e.g., from [2], where an Appendix of useful identities are given. Since tetradic identities could not be found from any source, a number of them are derived in the present Appendix.

A.1 Basic Expansions and Operations

Dyadic Expansions

Let us study some basic properties of tetradics. A tetradic can be expanded in general in terms of a basis of orthogonal unit vectors

$\{\mathbf{u}_i\}$ as

$$\mathcal{A} = \sum_{ijk\ell=1}^{3} A_{ijk\ell}\mathbf{u}_i\mathbf{u}_j\mathbf{u}_k\mathbf{u}_\ell. \tag{136}$$

There are $3^4 = 81$ components $A_{ijk\ell}$. Another basic expansion is in terms of two sets of nine dyadics as

$$\mathcal{A} = \sum_{i=1}^{9} \overline{\overline{A}}_i \overline{\overline{B}}_i, \tag{137}$$

which is not unique. Actually one set of nine dyadics can be chosen at will. For example, we can write

$$\mathcal{A} = \sum_{i=1}^{9} \overline{\overline{U}}_i \overline{\overline{B}}_i, \tag{138}$$

where $\left\{\overline{\overline{U}}_i\right\}$ is some given set of basis dyadics (a linearly independent set of dyadics). Thus, a tetradic can be defined through a set of nine dyadics $\left\{\overline{\overline{B}}_i\right\}$.

Let us apply the products of dyadic algebra for tetradics operated from both sides separately. The double-dot product of a tetradic with a dyadic from the right and from the left gives a dyadic:

$$\mathcal{A} : \overline{\overline{X}} = \sum_{i=1}^{9} \overline{\overline{A}}_i \left(\overline{\overline{B}}_i : \overline{\overline{X}}\right), \qquad \overline{\overline{X}} : \mathcal{A} = \sum_{i=1}^{9} \left(\overline{\overline{X}} : \overline{\overline{A}}_i\right) \overline{\overline{B}}_i. \tag{139}$$

Dot product with a vector gives a triadic:

$$\mathcal{A} \cdot \mathbf{a} = \sum_{i=1}^{9} \overline{\overline{A}}_i \left(\overline{\overline{B}}_i \cdot \mathbf{a}\right), \qquad \mathbf{a} \cdot \mathcal{A} = \sum_{i=1}^{9} \left(\mathbf{a} \cdot \overline{\overline{A}}_i\right) \overline{\overline{B}}_i. \tag{140}$$

Dot product to other entries of the tetradic can be made through the transpose operators defined below. Occasionally we also use the triple-dot product between a tetradic and a triadic which is defined as

$$\begin{aligned} \mathcal{A} \circledS \mathbf{abc} &= \sum_{i=1}^{9} \left(\overline{\overline{A}}_i \cdot \mathbf{a}\right) \left(\overline{\overline{B}}_i : \mathbf{bc}\right), \\ \mathbf{abc} \circledS \mathcal{A} &= \sum_{i=1}^{9} \left(\mathbf{ab} : \overline{\overline{A}}_i\right) \left(\mathbf{c} \cdot \overline{\overline{B}}_i\right), \end{aligned} \tag{141}$$

and the result is a vector. The quadruple-dot product between two tetradics gives a scalar:

$$\mathcal{A} \circled{4} \mathbf{abcd} = \mathbf{abcd} \circled{4} \mathcal{A} = \sum_{i=1}^{9} \left(\mathbf{ab} : \overline{\overline{A}}_i\right) \left(\mathbf{cd} : \overline{\overline{B}}_i\right). \tag{142}$$

The unit tetradic is defined by

$$\mathcal{I} : \overline{\overline{X}} = \overline{\overline{X}}, \qquad \text{for all } \overline{\overline{X}}. \tag{143}$$

Its expansion is

$$\mathcal{I} = \sum_{1=1}^{9} \overline{\overline{U}}_i \overline{\overline{U}}'_i, \tag{144}$$

where $\left\{\overline{\overline{U}}'_i\right\}$ is a basis of dyadics reciprocal to $\left\{\overline{\overline{U}}_i\right\}$. This means the property

$$\overline{\overline{U}}_i : \overline{\overline{U}}'_j = \delta_{ij}. \tag{145}$$

In terms of an orthonormal vector basis $\{\mathbf{u}_i\}$ we have

$$\mathcal{I} = \sum_{ij=1}^{3} \mathbf{u}_i \mathbf{u}_j \mathbf{u}_i \mathbf{u}_j. \tag{146}$$

Because we can write

$$\begin{aligned} \mathcal{I} : \mathbf{ab} &= \sum_{ij=1}^{3} \mathbf{u}_i \mathbf{u}_j (\mathbf{u}_i \cdot \mathbf{a})(\mathbf{u}_j \cdot \mathbf{b}) = \sum_{i=1}^{3} \mathbf{u}_i (\mathbf{u}_i \cdot \mathbf{a}) \sum_{j=1}^{3} \mathbf{u}_j (\mathbf{u}_j \cdot \mathbf{b}) \\ &= \left(\overline{\overline{I}} \cdot \mathbf{a}\right) \left(\overline{\overline{I}} \cdot \mathbf{b}\right) = \mathbf{ab}, \end{aligned} \tag{147}$$

and we know that the unit dyadic $\overline{\overline{I}} = \sum \mathbf{u}_i \mathbf{u}_i$ is independent of the basis, the unit tetradic $\mathcal{I}$ is independent of the vector basis. In contrast, for example, the tetradic

$$\mathcal{Q} = \sum_{i=1}^{3} \mathbf{u}_i \mathbf{u}_i \mathbf{u}_i \mathbf{u}_i \tag{148}$$

is not independent of the vector basis.

Corresponding to the trace of a dyadic we can define the trace of a tetradic as

$$\mathrm{tr}\,\mathcal{A} = \mathrm{tr}\sum \mathbf{a}_i\mathbf{b}_i\mathbf{c}_i\mathbf{d}_i = \mathcal{A} \,④\, \mathcal{I} = \mathcal{I} \,④\, \mathcal{A} = \sum(\mathbf{a}_i\cdot\mathbf{b}_i)(\mathbf{c}_i\cdot\mathbf{d}_i). \quad (149)$$

The transpose tetradic is defined by

$$\mathcal{T} : \mathbf{ab} = \mathbf{ba}, \quad \Rightarrow \quad \mathcal{T} : \overline{\overline{X}} = \overline{\overline{X}} : \mathcal{T} = \overline{\overline{X}}^T, \quad \text{for all } \overline{\overline{X}}, \quad (150)$$

and it is also independent of the vector basis. Its expansions are

$$\mathcal{T} = \sum_{1=1}^{9} \overline{\overline{U}}_i^T \overline{\overline{U}}'_i, \quad \text{or} \quad \mathcal{T} = \sum_{ij=1}^{3} \mathbf{u}_i\mathbf{u}_j\mathbf{u}_j\mathbf{u}_i = \sum_{i=1}^{3} \mathbf{u}_i\overline{\overline{I}}\mathbf{u}_i. \quad (151)$$

The difference of $\mathcal{I}$ and $\mathcal{T}$ can be expanded as

$$\begin{aligned}\mathcal{I} - \mathcal{T} &= \sum_{i,j} \mathbf{u}_i\mathbf{u}_j(\mathbf{u}_i\mathbf{u}_j - \mathbf{u}_j\mathbf{u}_i) = \frac{1}{2}\sum_{i,j}(\mathbf{u}_i\mathbf{u}_j - \mathbf{u}_j\mathbf{u}_i)(\mathbf{u}_i\mathbf{u}_j - \mathbf{u}_j\mathbf{u}_i)\\ &= \frac{1}{2}\sum_{i,j}\overline{\overline{I}}\times(\mathbf{u}_i\times\mathbf{u}_j)(\mathbf{u}_i\times\mathbf{u}_j)\times\overline{\overline{I}} = \sum_{i,j}\overline{\overline{I}}\times\frac{1}{2}\left(\mathbf{u}_i\mathbf{u}_i \,{}^{\times}_{\times}\, \mathbf{u}_j\mathbf{u}_j\right)\times\overline{\overline{I}}\\ &= \overline{\overline{I}}\times\overline{\overline{I}}\times\overline{\overline{I}}, \end{aligned} \quad (152)$$

whence we obtain the simple rule

$$\overline{\overline{I}}\times\overline{\overline{I}}\times\overline{\overline{I}} = \mathcal{I} - \mathcal{T}. \quad (153)$$

To check this we can operate on a dyadic product of two vectors:

$$\begin{aligned}\left(\overline{\overline{I}}\times\overline{\overline{I}}\times\overline{\overline{I}}\right) : \mathbf{ab} &= \overline{\overline{I}}\times\overline{\overline{I}}\cdot\left(\overline{\overline{I}} \,{}^{\times}_{\cdot}\, \mathbf{ab}\right) = -\overline{\overline{I}}\times(\mathbf{a}\times\mathbf{b})\\ &= -\mathbf{ba} + \mathbf{ab} = (\mathcal{I} - \mathcal{T}) : \mathbf{ab}, \end{aligned} \quad (154)$$

which remains valid when $\mathbf{ab}$ is replaced by the sum $\sum \mathbf{a}_i\mathbf{b}_i = \overline{\overline{A}}$.

Basis of Dyads

As an example of a dyadic basis let us consider one formed from a basis of orthogonal unit vectors $\{\mathbf{u}_i\}$, $i = 1, 2, 3$. Let us denote the unit dyadics by two indices as

$$\overline{\overline{U}}_{ij} = \mathbf{u}_i\mathbf{u}_j. \quad (155)$$

Obviously, the reciprocal basis coincides with the original one:

$$\overline{\overline{U}}'_{ij} = \overline{\overline{U}}_{ij}, \tag{156}$$

because they satisfy

$$\overline{\overline{U}}_{ij} : \overline{\overline{U}}_{k\ell} = \delta_{ij}^{k\ell}. \tag{157}$$

Any tetradic can be expressed in terms of the unit dyadics in different forms as

$$\mathcal{A} = \sum_{k,\ell=1}^{3} \overline{\overline{A}}_{k\ell}\overline{\overline{U}}_{k\ell} = \sum_{i,j=1}^{3} \overline{\overline{U}}_{ij}\overline{\overline{B}}_{ij} = \sum_{ijk\ell=1}^{3} A_{ijk\ell}\overline{\overline{U}}_{ij}\overline{\overline{U}}_{k\ell}, \tag{158}$$

with connections between the coefficients as

$$\overline{\overline{A}}_{k\ell} = \sum_{ij=1}^{3} A_{ijk\ell}\overline{\overline{U}}_{ij}, \qquad \overline{\overline{B}}_{ij} = \sum_{k\ell=1}^{3} A_{ijk\ell}\overline{\overline{U}}_{k\ell}. \tag{159}$$

Transpose Operations

We can define the following transpose operators for the tetradics:

$$\left(\overline{\overline{A}}\,\overline{\overline{B}}\right)^T = \overline{\overline{A}}\left(\overline{\overline{B}}^T\right), \quad {}^T\!\left(\overline{\overline{A}}\,\overline{\overline{B}}\right) = \left(\overline{\overline{A}}^T\right)\overline{\overline{B}}, \quad {}^T\!\left(\overline{\overline{A}}\,\overline{\overline{B}}\right)^T = \left(\overline{\overline{A}}^T\right)\left(\overline{\overline{B}}^T\right). \tag{160}$$

For the product of vectors they read

$$(\mathbf{abcd})^T = \mathbf{abdc}, \quad {}^T(\mathbf{abcd}) = \mathbf{bacd}, \quad {}^T(\mathbf{abcd})^T = \mathbf{badc}. \tag{161}$$

Another transpose operator is

$$\left(\overline{\overline{A}}\,\overline{\overline{B}}\right)^\tau = {}^\tau\!\left(\overline{\overline{A}}\,\overline{\overline{B}}\right) = \overline{\overline{B}}\,\overline{\overline{A}}. \tag{162}$$

Note that these operators do not commute:

$$\left(\overline{\overline{A}}\,\overline{\overline{B}}\right)^{T\tau} = \overline{\overline{B}}^T\overline{\overline{A}}, \qquad \left(\overline{\overline{A}}\,\overline{\overline{B}}\right)^{\tau T} = \overline{\overline{B}}\left(\overline{\overline{A}}^T\right). \tag{163}$$

The transpose tetradic can be expressed as

$$\mathcal{T} = \mathcal{I}^T = {}^T\mathcal{I}. \tag{164}$$

The transpose operators can be expressed explicitly through the dyadic identity

$$\overline{\overline{A}} \,{}^{\times}_{\times}\, \overline{\overline{I}} = \left(\overline{\overline{A}} : \overline{\overline{I}}\right) \overline{\overline{I}} - \overline{\overline{A}}^T, \tag{165}$$

which applies for tetradics in the form

$$\mathcal{A} \,{}^{\times}_{\times}\, \overline{\overline{I}} = \left(\mathcal{A} : \overline{\overline{I}}\right) \overline{\overline{I}} - \mathcal{A}^T, \qquad \overline{\overline{I}} \,{}^{\times}_{\times}\, \mathcal{A} = \overline{\overline{I}} \left(\overline{\overline{I}} : \mathcal{A}\right) - {}^T\!\mathcal{A}. \tag{166}$$

These can be used to define the transpose operations:

$$\mathcal{A}^T = \left(\mathcal{A} : \overline{\overline{I}}\right) \overline{\overline{I}} - \mathcal{A} \,{}^{\times}_{\times}\, \overline{\overline{I}}, \qquad {}^T\!\mathcal{A} = \overline{\overline{I}} \left(\overline{\overline{I}} : \mathcal{A}\right) - \overline{\overline{I}} \,{}^{\times}_{\times}\, \mathcal{A}. \tag{167}$$

Choosing $\mathcal{A} = \mathcal{I}$ we have

$$\mathcal{T} = \mathcal{I}^T = \overline{\overline{I}}\,\overline{\overline{I}} - \mathcal{I} \,{}^{\times}_{\times}\, \overline{\overline{I}} = {}^T\mathcal{I} = \overline{\overline{I}}\,\overline{\overline{I}} - \overline{\overline{I}} \,{}^{\times}_{\times}\, \mathcal{I}, \tag{168}$$

which gives the identity

$$\overline{\overline{I}} \,{}^{\times}_{\times}\, \mathcal{I} = \mathcal{I} \,{}^{\times}_{\times}\, \overline{\overline{I}} = \overline{\overline{I}}\,\overline{\overline{I}} - \mathcal{T}. \tag{169}$$

Similarly, we have

$$\overline{\overline{I}} \,{}^{\times}_{\times}\, \mathcal{T} = \mathcal{T} \,{}^{\times}_{\times}\, \overline{\overline{I}} = \overline{\overline{I}}\,\overline{\overline{I}} - \mathcal{I}. \tag{170}$$

We can define the double-dot product for two tetradics as

$$(\mathbf{abcd}) : (\mathbf{efgh}) = (\mathbf{c} \cdot \mathbf{e})(\mathbf{d} \cdot \mathbf{f})(\mathbf{abgh}). \tag{171}$$

From this we see the identity

$$\mathcal{I} : \mathcal{A} = \mathcal{A} \tag{172}$$

for any tetradic $\mathcal{A}$. Also, we have

$$\mathcal{T} : \mathcal{A} = {}^T\!\mathcal{A}, \qquad \mathcal{A} : \mathcal{T} = \mathcal{A}^T. \tag{173}$$

A.2 Isotropic Tetradics

Elastodynamic media invariant to rotation of the medium are called isotropic. To find their most general expression, let us consider first the isotropic dyadic.

Isotropic Dyadics

Isotropic dyadics $\overline{\overline{E}}$ are dyadics which do not change the direction of a given vector $\mathbf{a}$ in the linear mapping

$$\overline{\overline{E}} \cdot \mathbf{a} = \epsilon \mathbf{a}. \tag{174}$$

Because $\left(\overline{\overline{E}} - \epsilon \overline{\overline{I}}\right) \cdot \mathbf{a} = 0$ must be valid for any vector $\mathbf{a}$, $\overline{\overline{E}}$ must be a multiple of the unit dyadic $\overline{\overline{I}}$, i.e., of the form

$$\overline{\overline{E}} = \epsilon \overline{\overline{I}}. \tag{175}$$

An isotropic dyadic is always symmetric: it is isotropic in multiplication from the left and from the right.

Isotropic Tetradics

An isotropic tetradic $\mathcal{E}$ defines such a linear mapping from a dyadic $\overline{\overline{A}}$ to another dyadic $\overline{\overline{B}}$ that does not carry along any preferred directions. Taking the dyad $\overline{\overline{A}} = \mathbf{ab}$ as an example, this means that it must be mapped to a linear combination of the dyad $\mathbf{ab}$ itself, the transposed dyad $\mathbf{ba}$ and a multiple of the unit dyadic $\overline{\overline{I}}$. Since the scalar factor of $\overline{\overline{I}}$ must be a linear function of both $\mathbf{a}$ and $\mathbf{b}$ and there must be no other vectors involved, the mapping must be of the form

$$\mathcal{E} : \mathbf{ab} = \mu \mathbf{ab} + \eta \mathbf{ba} + \lambda (\mathbf{a} \cdot \mathbf{b}) \overline{\overline{I}}. \tag{176}$$

Mapping each term of the dyadic $\overline{\overline{A}} = \sum \mathbf{a}_i \mathbf{b}_i$ we obtain

$$\mathcal{E} : \overline{\overline{A}} = \mu \overline{\overline{A}} + \eta \overline{\overline{A}}^T + \lambda \overline{\overline{I}} \left(\overline{\overline{I}} : \overline{\overline{A}} \right) = \left(\mu \mathcal{I} + \eta \mathcal{T} + \lambda \overline{\overline{I}}\,\overline{\overline{I}} \right) : \overline{\overline{A}}, \tag{177}$$

to be satisfied by any dyadic $\overline{\overline{A}}$, whence the isotropic tetradic has the general form

$$\mathcal{E} = \mu \mathcal{I} + \eta \mathcal{T} + \lambda \overline{\overline{I}}\,\overline{\overline{I}}. \tag{178}$$

Requiring $\mathcal{E}$ to be symmetric, i.e., satisfy $\mathcal{E}^\tau = \mathcal{E}$, we must further have $\mu = \eta$, which reduces the number of free parameters from three to two. The symmetric isotropic tetradic thus has the form

$$\mathcal{E} = \mu (\mathcal{I} + \mathcal{T}) + \lambda \overline{\overline{I}}\,\overline{\overline{I}}. \tag{179}$$

For an arbitrary tetradic $\mathcal{A}$ the isotropic tetradic satisfies

$$\mathcal{E} : \mathcal{A} = \mu\mathcal{A} + \eta\left({}^T\mathcal{A}\right) + \lambda\overline{\overline{I}}\left(\overline{\overline{I}} : \mathcal{A}\right), \tag{180}$$

$$\mathcal{A} : \mathcal{E} = \mu\mathcal{A} + \eta\mathcal{A}^T + \lambda\left(\overline{\overline{I}} : \mathcal{A}\right)\overline{\overline{I}}. \tag{181}$$

For two isotropic tetradics we obtain the rule

$$\begin{aligned}\mathcal{E}_1 : \mathcal{E}_2 = &(\mu_1\mu_2 + \eta_1\eta_2)\mathcal{I} + (\mu_1\eta_2 + \eta_1\mu_2)\mathcal{T} \\ &+ (\mu_1\lambda_2 + \eta_1\lambda_2 + \mu_2\lambda_1 + \eta_2\lambda_1 + 3\lambda_1\lambda_2)\overline{\overline{I}}\,\overline{\overline{I}},\end{aligned} \tag{182}$$

which again is an isotropic tetradic.

Eigendyadics

Let us consider the eigenproblem for the isotropic tetradic:

$$\mathcal{E} : \overline{\overline{X}} = q\overline{\overline{X}}, \qquad \overline{\overline{X}} \neq 0. \tag{183}$$

This can be expressed as the dyadic equation

$$(\mu - q)\overline{\overline{X}} + \eta\overline{\overline{X}}^T + \lambda\overline{\overline{I}}\left(\overline{\overline{X}} : \overline{\overline{I}}\right) = 0. \tag{184}$$

It can be seen that the left-hand eigenproblem

$$\overline{\overline{X}} : \mathcal{E} = q\overline{\overline{X}} \tag{185}$$

leads to the same eigendyadics and eigenvalues. Taking the trace of (184) gives

$$(\mu + \eta + 3\lambda - q)\left(\overline{\overline{X}} : \overline{\overline{I}}\right) = 0, \tag{186}$$

whence either $\overline{\overline{X}}$ is trace-free or $q = \mu + \eta + 3\lambda$. Let us consider the first case: $\overline{\overline{X}} : \overline{\overline{I}} = 0$, whence (184) reduces to

$$(\mu - q)\overline{\overline{X}} = -\eta\overline{\overline{X}}^T. \tag{187}$$

Taking the sum and difference of this and its transpose gives two equations

$$(\mu - \eta - q)\left(\overline{\overline{X}} - \overline{\overline{X}}^T\right) = 0, \qquad (\mu + \eta - q)\left(\overline{\overline{X}} + \overline{\overline{X}}^T\right) = 0. \tag{188}$$

These have two eigenvalues and trace-free dyadics $\overline{\overline{X}}$ as solutions

$$q_1 = \mu + \eta, \qquad \overline{\overline{X}}_1 = \overline{\overline{X}}_1^T, \tag{189}$$

$$q_2 = \mu - \eta, \qquad \overline{\overline{X}}_2 = -\overline{\overline{X}}_2^T. \tag{190}$$

Let us denote $\overline{\overline{X}}_i = \overline{\overline{S}}^\circ$ for any trace-free symmetric dyadic and $\overline{\overline{X}} = \mathbf{x}\times\overline{\overline{I}}$ for any antisymmetric dyadic (which is automatically trace-free). Corresponding to $q_3 = \mu + \eta + 3\lambda$ we have a third eigendyadic $\overline{\overline{X}}_3$ satisfying $\overline{\overline{X}}_3 : \overline{\overline{I}} \neq 0$. Inserted in (184) and taking the difference with its transpose gives us

$$-(2\eta + 3\lambda)\left(\overline{\overline{X}}_3 - \overline{\overline{X}}_3^T\right) = 0, \tag{191}$$

which in general $(2\eta + 3\lambda \neq 0)$ leads to $\overline{\overline{X}}_3^T = \overline{\overline{X}}_3$, i.e., a symmetric eigendyadic. Substituting again in (184) leads to

$$\overline{\overline{X}}_3 = \frac{1}{3}\left(\overline{\overline{X}}_3 : \overline{\overline{I}}\right)\overline{\overline{I}}, \tag{192}$$

which is satisfied by any multiple of the unit dyadic $\xi\overline{\overline{I}}$. The three eigendyadics can be seen to satisfy the orthogonality condition

$$\overline{\overline{X}}_i : \overline{\overline{X}}_j = \delta_{ij}\overline{\overline{X}}_i : \overline{\overline{X}}_i. \tag{193}$$

Any given dyadic $\overline{\overline{X}}$ can be expanded in terms of the orthogonal eigendyadics as

$$\overline{\overline{X}} = \sum_{i=1}^{3} \frac{\overline{\overline{X}}_i\overline{\overline{X}}_i}{\overline{\overline{X}}_i : \overline{\overline{X}}_i} : \overline{\overline{X}} = \overline{\overline{S}}^\circ + \mathbf{x}\times\overline{\overline{I}} + \xi\overline{\overline{I}}, \tag{194}$$

which mapped by the isotropic tetradic gives

$$\begin{aligned}\mathcal{E} : \overline{\overline{X}} &= q_1\overline{\overline{X}}_1 + q_2\overline{\overline{X}}_2 + q_3\overline{\overline{X}}_3 \\ &= (\mu+\eta)\overline{\overline{S}}^\circ + (\mu-\eta)\mathbf{x}\times\overline{\overline{I}} + (\mu+\eta+3\lambda)\xi\overline{\overline{I}}. \end{aligned} \tag{195}$$

This prompts us to write the following decomposition for the isotropic tetradic in the respective three terms:

$$\begin{aligned}\mathcal{E} &= q_1\mathcal{P}_1 + q_2\mathcal{P}_2 + q_3\mathcal{P}_3 \\ &= \frac{\mu+\eta}{2}\left(\mathcal{I} + \mathcal{T} - \frac{2}{3}\overline{\overline{I}}\,\overline{\overline{I}}\right) + \frac{\mu-\eta}{2}(\mathcal{I} - \mathcal{T}) + \frac{\mu+\eta+3\lambda}{3}\overline{\overline{I}}\,\overline{\overline{I}}. \quad (196)\end{aligned}$$

The three projector tetradics $\mathcal{P}_i$ defined by

$$\mathcal{P}_1 = \frac{1}{2}(\mathcal{I}+\mathcal{T}) - \frac{1}{3}\overline{\overline{I}}\,\overline{\overline{I}}, \quad \mathcal{P}_2 = \frac{1}{2}(\mathcal{I}-\mathcal{T}), \quad \mathcal{P}_3 = \frac{1}{3}\overline{\overline{I}}\,\overline{\overline{I}}, \qquad (197)$$

satisfy the following orthogonality and completeness conditions:

$$\mathcal{P}_i : \mathcal{P}_j = \delta_{ij}\mathcal{P}_i, \qquad \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 = \mathcal{I}. \qquad (198)$$

As an example, $\mathcal{P}_1$ operating on a dyadic gives its trace-free symmetric component. $\mathcal{P}_2$ and $\mathcal{P}_3$ project similarly any dyadic onto the respective subspaces of antisymmetric dyadics and multiples of the unit dyadic.

Corresponding to the symmetric isotropic tetradic (179) there is no antisymmetric eigendyadic $\overline{\overline{X}}_2$ and projector tetradic $\mathcal{P}_2$.

A.3 Uniaxial Tetradics

Uniaxial Dyadics

Symmetric[1] uniaxial dyadics $\overline{\overline{U}} = \overline{\overline{U}}^T$ prefer one direction in space which is defined by a unit vector $\mathbf{v}$. Such dyadics are also called transversely isotropic because they do not change the direction of the component transverse to $\mathbf{v}$ (denoted by the subscript $_t$) of a given vector $\mathbf{a}$ in the linear mapping from the left,

$$\overline{\overline{U}} \cdot \mathbf{a} = \alpha\mathbf{a}_t + \gamma(\mathbf{a})\mathbf{v}. \qquad (199)$$

Since the scalar $\gamma(\mathbf{a})$ is a linear function of $\mathbf{a}$ and there is only one special vector direction, we must have $\gamma(\mathbf{a}) = \beta\mathbf{v}\cdot\mathbf{a}$ and, thus, the uniaxial dyadic must be of the form

$$\overline{\overline{U}} = \alpha\overline{\overline{I}} + \beta\mathbf{v}\mathbf{v}. \qquad (200)$$

[1] More generally, the axial direction of a uniaxial dyadic may be different when operating from the left and from the right [2]. In this case a uniaxial dyadic is defined by two directions in space.

Uniaxial Tetradics

Let us now consider only symmetric uniaxial tetradics $\mathcal{U}$ operating on symmetric dyadics for simplicity. This means that there is again only one special direction, defined by the unit vector $\mathbf{v}$. Mapping of a symmetric dyad $\mathbf{aa}$ must then give a result which obviously is a linear combination of the symmetric dyadics $\mathbf{a}_t\mathbf{a}_t$, $\mathbf{a}_t\mathbf{v}+\mathbf{v}\mathbf{a}_t$, $\mathbf{vv}$ and $\overline{\overline{I}}_t$:

$$\mathcal{A}:\mathbf{aa}=\alpha\mathbf{a}_t\mathbf{a}_t+\beta(\mathbf{a})(\mathbf{a}_t\mathbf{v}+\mathbf{v}\mathbf{a}_t)+\gamma(\mathbf{a})\mathbf{vv}+\delta(\mathbf{a})\overline{\overline{I}}_t. \tag{201}$$

The scalars must be functions of the type (note that there are no preferred transverse vectors)

$$\beta(\mathbf{a})=\beta\mathbf{v}\cdot\mathbf{a}, \tag{202}$$

$$\gamma(\mathbf{a})=\gamma_t(\mathbf{a}_t\cdot\mathbf{a}_t)+\gamma_v(\mathbf{v}\cdot\mathbf{a})^2, \tag{203}$$

$$\delta(\mathbf{a})=\delta_t(\mathbf{a}_t\cdot\mathbf{a}_t)+\delta_v(\mathbf{v}\cdot\mathbf{a})^2. \tag{204}$$

Denoting a transverse orthonormal base by $\mathbf{u}_1$, $\mathbf{u}_2$, the mapping is of the form (summation over indices assumed)

$$\mathcal{U}:\overline{\overline{A}}=\alpha\mathbf{u}_m\mathbf{u}_n\left(\mathbf{u}_m\mathbf{u}_n:\overline{\overline{A}}\right)+\beta(\mathbf{u}_m\mathbf{v}\mathbf{u}_m\mathbf{v}+\mathbf{v}\mathbf{u}_m\mathbf{v}\mathbf{u}_m):\overline{\overline{A}} \\ +\gamma_t\mathbf{vv}\overline{\overline{I}}_t:\overline{\overline{A}}+\gamma_v\mathbf{vvvv}:\overline{\overline{A}}+\delta_t\overline{\overline{I}}_t\overline{\overline{I}}_t:\overline{\overline{A}}+\delta_v\overline{\overline{I}}_t\mathbf{vv}:\overline{\overline{A}}, \tag{205}$$

which is valid for $\overline{\overline{A}}=\mathbf{aa}$ and, thus, for any symmetric dyadic $\overline{\overline{A}}$. From this we see that the uniaxial tetradic must have the form

$$\mathcal{U}=\alpha\mathbf{u}_m\mathbf{u}_n\mathbf{u}_m\mathbf{u}_n+\beta(\mathbf{u}_m\mathbf{v}\mathbf{u}_m\mathbf{v}+\mathbf{v}\mathbf{u}_m\mathbf{v}\mathbf{u}_m) \\ +\gamma_t\mathbf{vv}\overline{\overline{I}}_t+\gamma_v\mathbf{vvvv}+\delta_t\overline{\overline{I}}_t\overline{\overline{I}}_t+\delta_v\overline{\overline{I}}_t\mathbf{vv}. \tag{206}$$

This is defined by six parameters. If we require that the uniaxial tetradic be also symmetric so that $\mathcal{U}:\overline{\overline{A}}=\overline{\overline{A}}:\mathcal{U}$ for all symmetric dyadics $\overline{\overline{A}}$, we must have $\gamma_t=\delta_v$, which reduces the number of free parameters from six to five and the general form of the symmetric uniaxial tetradic thus becomes

$$\mathcal{U}=\alpha\sum_{m,n=1}^{2}\mathbf{u}_m\mathbf{u}_n\mathbf{u}_m\mathbf{u}_n+\beta\sum_{m=1}^{2}(\mathbf{u}_m\mathbf{v}\mathbf{u}_m\mathbf{v}+\mathbf{v}\mathbf{u}_m\mathbf{v}\mathbf{u}_m) \\ +\gamma_t\left(\mathbf{vv}\overline{\overline{I}}_t+\overline{\overline{I}}_t\mathbf{vv}\right)+\gamma_v\mathbf{vvvv}+\delta_t\overline{\overline{I}}_t\overline{\overline{I}}_t. \tag{207}$$

In this study we consider only the simple special case when the uniaxial tetradic is a linear combination of an isotropic tetradic $\mathcal{E}$ and the tetrad $\mathbf{vvvv}$.

Eigendyadics

For the special uniaxial tetradic of the form

$$\mathcal{U} = \mathcal{E} + \alpha\mu\mathbf{vvvv} = \mu\mathcal{I} + \eta\mathcal{T} + \lambda\overline{\overline{I}}\,\overline{\overline{I}} + \alpha\mu\mathbf{vvvv}, \tag{208}$$

following the tracks of the isotropic tetradic, let us consider the eigendyadic problem

$$\mathcal{U} : \overline{\overline{X}} = \mu\overline{\overline{X}} + \eta\overline{\overline{X}}^T + \lambda\overline{\overline{I}}\left(\overline{\overline{X}} : \overline{\overline{I}}\right) + \alpha\mu\mathbf{vv}\left(\overline{\overline{X}} : \mathbf{vv}\right) = q\overline{\overline{X}}. \tag{209}$$

Writing this and its transpose together in matrix form

$$\begin{pmatrix} \mu - q & \eta \\ \eta & \mu - q \end{pmatrix} \begin{pmatrix} \overline{\overline{X}} \\ \overline{\overline{X}}^T \end{pmatrix} = -\begin{pmatrix} \lambda\overline{\overline{I}} & \alpha\mu\mathbf{vv} \\ \lambda\overline{\overline{I}} & \alpha\mu\mathbf{vv} \end{pmatrix} \begin{pmatrix} \overline{\overline{X}} : \overline{\overline{I}} \\ \overline{\overline{X}} : \mathbf{vv} \end{pmatrix}, \tag{210}$$

we see that if the determinant of the matrix on the left vanishes, $(\mu - q)^2 - \eta^2 = 0$, the matrix has no inverse and in this case also the right-hand side must vanish. This possibility gives us two solutions,

$$q_{1,2} = \mu \pm \eta, \qquad \overline{\overline{X}} : \overline{\overline{I}} = 0, \qquad \overline{\overline{X}} : \mathbf{vv} = 0. \tag{211}$$

It is easy to see that the two eigendyadics and their eigenvalues are defined by

$$\overline{\overline{X}}_1 = \overline{\overline{X}}_1^T, \qquad \overline{\overline{X}}_1 : \overline{\overline{I}} = 0, \qquad \overline{\overline{X}}_1 : \mathbf{vv} = 0, \qquad q_1 = \mu + \eta, \tag{212}$$

$$\overline{\overline{X}}_2 = -\overline{\overline{X}}_2^T, \qquad q_2 = \mu - \eta. \tag{213}$$

The eigendyadic $\overline{\overline{X}}_2$ can be any antisymmetric dyadic, which satisfies $\overline{\overline{X}}_2 : \overline{\overline{I}} = \overline{\overline{X}}_2 : \mathbf{vv} = 0$ automatically. The eigendyadic $\overline{\overline{X}}_1$ can be any symmetric dyadic which is trace free and satisfies $\overline{\overline{X}}_1 : \mathbf{vv} = 0$.

There are two more eigendyadics (actually the total number coincides with that of the free parameters μ, η, λ, α of the tetradic). They satisfy the condition opposite to that above, $(\mu - q)^2 \neq \eta^2$, whence the matrix in (210) can now be inverted. Doing that we see that the right-hand side is the same for $\overline{\overline{X}}$ and $\overline{\overline{X}}^T$, whence the two remaining eigendyadics $\overline{\overline{X}}_{3,4}$ must be symmetric. Starting again from (209) shows us that the eigendyadics must be linear combinations of $\overline{\overline{I}}$ and $\mathbf{vv}$:

$$(\mu+\eta-q)\overline{\overline{X}} = -\lambda\overline{\overline{I}}\left(\overline{\overline{X}}:\overline{\overline{I}}\right) - \alpha\mu\mathbf{vv}\left(\overline{\overline{X}}:\mathbf{vv}\right). \qquad (214)$$

Multiplying this by $\overline{\overline{I}}$: and $\mathbf{vv}$: we obtain two equations for $\overline{\overline{X}}:\overline{\overline{I}}$ and $\overline{\overline{X}}:\mathbf{vv}$:

$$\begin{pmatrix} \mu+\eta+3\lambda-q & \alpha\mu \\ \lambda & \mu+\eta+\alpha\mu-q \end{pmatrix}\begin{pmatrix} \overline{\overline{X}}:\overline{\overline{I}} \\ \overline{\overline{X}}:\mathbf{vv} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (215)$$

Requiring the determinant of the matrix to vanish gives us equation for the two remaining eigenvalues:

$$(\mu+\eta-q)^2 + (\mu+\eta-q)(3\lambda+\alpha\mu) + 2\alpha\mu\lambda = 0. \qquad (216)$$

The solutions are

$$q_{3,4} = \mu+\eta+\frac{3\lambda+\alpha\eta}{2} \pm \frac{1}{2}\sqrt{(3\lambda-\alpha\eta)^2+4\alpha\mu\lambda}. \qquad (217)$$

The ratio of $\overline{\overline{X}}:\overline{\overline{I}}$ and $\overline{\overline{X}}:\mathbf{vv}$ is different for the two eigendyadics and can now be solved from (215):

$$(\mu+\eta+3\lambda-q_{3,4})\,\overline{\overline{X}}_{3,4}:\overline{\overline{I}} + \alpha\mu\overline{\overline{X}}_{3,4}:\mathbf{vv} = 0. \qquad (218)$$

This finally gives the two eigendyadics as

$$\begin{aligned} \overline{\overline{X}}_{3,4} &= \frac{1}{\mu+\eta-q_{3,4}}\left[-\lambda\overline{\overline{I}}\left(\overline{\overline{X}}_{3,4}:\overline{\overline{I}}\right) - \alpha\mu\mathbf{vv}\left(\overline{\overline{X}}_{3,4}:\mathbf{vv}\right)\right] \\ &= \frac{1}{\mu+\eta-q_{3,4}}\left[-\lambda\left(\overline{\overline{X}}_{3,4}:\overline{\overline{I}}\right)\overline{\overline{I}} + (\mu+\eta+3\lambda-q_{3,4})\left(\overline{\overline{X}}_{3,4}:\overline{\overline{I}}\right)\mathbf{vv}\right] \\ &= A_{3,4}\left[\overline{\overline{I}} - Q_{3,4}\mathbf{vv}\right], \qquad (219) \end{aligned}$$

$$Q_{3,4} = \frac{1}{\lambda}(\mu+\eta+3\lambda-q_{3,4}) = \frac{3\lambda-\alpha\eta}{2\lambda} \mp \frac{1}{2\lambda}\sqrt{(3\lambda-\alpha\eta)^2+4\alpha\mu\lambda}. \qquad (220)$$

Here A_3 and A_4 are arbitrary coefficients. From (219) and (217) it is easy to show that these eigendyadics satisfy the orthogonality

$\overline{\overline{X}}_3 : \overline{\overline{X}}_4 = 0$. The other orthogonalities are obvious and we can in general write

$$\overline{\overline{X}}_i : \overline{\overline{X}}_j = \delta_{ij}\overline{\overline{X}}_i : \overline{\overline{X}}_i, \quad i = 1, \ldots, 4. \tag{221}$$

Finally, we can express the uniaxial tetradic in terms of its eigendyadics as

$$\mathcal{U} = \sum_{i=1}^{4} q_i \mathcal{P}_i = \sum_{i=1}^{4} q_i \frac{\overline{\overline{X}}_i \overline{\overline{X}}_i}{\overline{\overline{X}}_i : \overline{\overline{X}}_i}. \tag{222}$$

This is not a very useful decomposition since the dyadics $\overline{\overline{X}}_1$ and $\overline{\overline{X}}_2$ involve arbitrary dyadics. However, from the analogy of (196) we can write a more explicit form for the projector tetradics $\mathcal{P}_i$ as

$$\mathcal{P}_1 = \frac{1}{2}\left[\mathcal{I} + \mathcal{T} - \left(\overline{\overline{I}} - \mathbf{vv}\right)\overline{\overline{I}} + \left(\overline{\overline{I}} - 3\mathbf{vv}\right)\mathbf{vv}\right], \tag{223}$$

$$\mathcal{P}_2 = \frac{1}{2}\left[\mathcal{I} - \mathcal{T}\right], \tag{224}$$

$$\mathcal{P}_{3,4} = \frac{\overline{\overline{X}}_{3,4}\overline{\overline{X}}_{3,4}}{\overline{\overline{X}}_{3,4} : \overline{\overline{X}}_{3,4}} = \frac{\left(\overline{\overline{I}} - Q_{3,4}\mathbf{vv}\right)\left(\overline{\overline{I}} - Q_{3,4}\mathbf{vv}\right)}{(Q_{3,4} - 1)^2 + 2}. \tag{225}$$

The projector tetradics satisfy the orthogonality and completeness conditions as

$$\mathcal{P}_i : \mathcal{P}_j = \delta_{ij}\mathcal{P}_i, \qquad \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 = \mathcal{I}. \tag{226}$$

It is interesting to consider the limiting case $\alpha \to 0$, whence the uniaxial tetradic $\mathcal{U}$ becomes isotropic. It is easy to see from $Q_3 \to 0$, $Q_4 \to 3$ that the sum of $\mathcal{P}_1$ and $\mathcal{P}_4$ now becomes the projection tetradic $\mathcal{P}_1$ of the isotropic case while the projection tetradics $\mathcal{P}_2$ and $\mathcal{P}_3$ become the corresponding tetradics of the isotropic case.

A.4 Inverse Tetradic

The inverse of a tetradic solves a linear dyadic equation as

$$\overline{\overline{Y}} = \mathcal{A} : \overline{\overline{X}} \quad \Rightarrow \quad \overline{\overline{X}} = \mathcal{A}^{-1} : \overline{\overline{Y}}, \tag{227}$$

or

$$\mathcal{A} : \mathcal{A}^{-1} = \mathcal{A}^{-1} : \mathcal{A} = \mathcal{I}. \tag{228}$$

At this point it does not seem possible to obtain an analytic formula for the inverse of the general tetradic. Thus, we have to consider special cases. For example, because of $\mathcal{I}:\mathcal{I}=\mathcal{T}:\mathcal{T}=\mathcal{I}$, the inverse of the unit tetradic or the transpose tetradic equals the tetradic itself:

$$\mathcal{I}^{-1}=\mathcal{I}, \qquad \mathcal{T}^{-1}=\mathcal{T}. \tag{229}$$

Isotropic Tetradic

The inverse of an isotropic tetradic is obtained directly from (196) by writing the expansion with inverse eigenvalues:

$$\mathcal{E}^{-1}=\frac{1}{\alpha_1}\mathcal{P}_1+\frac{1}{\alpha_2}\mathcal{P}_2+\frac{1}{\alpha_3}\mathcal{P}_3. \tag{230}$$

Inserting the eigenvalues and projector tetradics we obtain

$$\left(\mu\mathcal{I}+\eta\mathcal{T}+\lambda\overline{\overline{I}}\,\overline{\overline{I}}\right)^{-1}=\frac{1}{\mu^2-\eta^2}\left(\mu\mathcal{I}-\eta\mathcal{T}-\frac{\lambda(\mu-\eta)}{\mu+\eta+3\lambda}\overline{\overline{I}}\,\overline{\overline{I}}\right). \tag{231}$$

For vanishing eigenvalues $\mu=\pm\eta$, $\mu+\eta+3\lambda=0$, this expression breaks down and there is no inverse. This is understandable because the corresponding eigendyadic is mapped to zero and there is no way to recover the dyadic. We may define the inverse of the tetradic in some dyadic subspace by excluding the component corresponding to the zero eigenvalue.

For example, working with symmetric dyadics only, we can consider isotropic tetradics with $\mu=\eta$, whence only two of the projector tetradics are present:

$$\mathcal{E}_s=\alpha_1\mathcal{P}_1+\alpha_3\mathcal{P}_3=\mu\left(\mathcal{I}+\mathcal{T}-\frac{2}{3}\overline{\overline{I}}\,\overline{\overline{I}}\right)+\frac{2\mu+3\lambda}{3}\overline{\overline{I}}\,\overline{\overline{I}}=\mu(\mathcal{I}+\mathcal{T})+\lambda\overline{\overline{I}}\,\overline{\overline{I}}. \tag{232}$$

Its inverse can then be expressed as

$$\mathcal{E}_s^{-1}=\frac{1}{\alpha_1}\mathcal{P}_1+\frac{1}{\alpha_3}\mathcal{P}_3=\frac{1}{2\mu}\left(\frac{1}{2}(\mathcal{I}+\mathcal{T})-\frac{\lambda}{2\mu+3\lambda}\overline{\overline{I}}\,\overline{\overline{I}}\right). \tag{233}$$

Extension Method

Let us assume that we know the inverse of a tetradic $\mathcal{A}$ and wish to find the inverse of the extended tetradic $\mathcal{A}+\overline{\overline{A}}\,\overline{\overline{B}}$, where $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are general dyadics. The equation to be solved is

$$\overline{\overline{Y}}=\mathcal{A}:\overline{\overline{X}}+\overline{\overline{A}}\left(\overline{\overline{B}}:\overline{\overline{X}}\right) \quad\Rightarrow\quad \overline{\overline{X}}=\mathcal{A}^{-1}:\left(\overline{\overline{Y}}-\overline{\overline{A}}\left(\overline{\overline{B}}:\overline{\overline{X}}\right)\right). \tag{234}$$

Let us solve $\overline{\overline{B}}:\overline{\overline{X}}$ by operating through $\overline{\overline{B}}:$

$$\overline{\overline{B}}:\overline{\overline{X}}=\frac{\overline{\overline{B}}:\mathcal{A}^{-1}:\overline{\overline{Y}}}{1+\overline{\overline{B}}:\mathcal{A}^{-1}:\overline{\overline{A}}}. \tag{235}$$

This substituted in the above expression solves $\overline{\overline{X}}$ as

$$\overline{\overline{X}}=\mathcal{A}^{-1}:\left(\overline{\overline{Y}}-\overline{\overline{A}}\frac{\overline{\overline{B}}:\mathcal{A}^{-1}:\overline{\overline{Y}}}{1+\overline{\overline{B}}:\mathcal{A}^{-1}:\overline{\overline{A}}}\right). \tag{236}$$

From this we can identify the inverse tetradic:

$$\left(\mathcal{A}+\overline{\overline{A}}\,\overline{\overline{B}}\right)^{-1}=\mathcal{A}^{-1}-\frac{\mathcal{A}^{-1}:\overline{\overline{A}}\,\overline{\overline{B}}:\mathcal{A}^{-1}}{1+\overline{\overline{B}}:\mathcal{A}^{-1}:\overline{\overline{A}}}. \tag{237}$$

As two special cases we may write

$$\left(\mathcal{I}+\overline{\overline{A}}\,\overline{\overline{B}}\right)^{-1}=\mathcal{I}-\frac{\overline{\overline{A}}\,\overline{\overline{B}}}{1+\overline{\overline{B}}:\overline{\overline{A}}}, \tag{238}$$

$$\left(\mathcal{T}+\overline{\overline{A}}\,\overline{\overline{B}}\right)^{-1}=\mathcal{T}-\frac{\overline{\overline{A}}^T\overline{\overline{B}}^T}{1+\overline{\overline{B}}^T:\overline{\overline{A}}}. \tag{239}$$

Let us check the latter expression:

$$\begin{aligned}&\left(\mathcal{T}-\frac{\overline{\overline{A}}^T\overline{\overline{B}}^T}{1+\overline{\overline{B}}^T:\overline{\overline{A}}}\right):\left(\mathcal{T}+\overline{\overline{A}}\,\overline{\overline{B}}\right)\\ &\quad=\mathcal{I}-\frac{\overline{\overline{A}}^T\overline{\overline{B}}}{1+\overline{\overline{B}}^T:\overline{\overline{A}}}+\overline{\overline{A}}^T\overline{\overline{B}}-\frac{\overline{\overline{A}}^T\overline{\overline{B}}^T}{1+\overline{\overline{B}}^T:\overline{\overline{A}}}:\overline{\overline{A}}\,\overline{\overline{B}}=\mathcal{I}.\end{aligned} \tag{240}$$

This method can be extended further by adding more terms, but the analytic expressions become quite involved as is seen from the next generalization:

$$\begin{aligned}&\left(\mathcal{A}+\overline{\overline{A}}\,\overline{\overline{B}}+\overline{\overline{C}}\,\overline{\overline{D}}\right)^{-1}\\ &\quad=\left(\mathcal{A}+\overline{\overline{A}}\,\overline{\overline{B}}\right)^{-1}-\frac{\left(\mathcal{A}+\overline{\overline{A}}\,\overline{\overline{B}}\right)^{-1}:\overline{\overline{C}}\,\overline{\overline{D}}:\left(\mathcal{A}+\overline{\overline{A}}\,\overline{\overline{B}}\right)^{-1}}{1+\overline{\overline{D}}:\left(\mathcal{A}+\overline{\overline{A}}\,\overline{\overline{B}}\right)^{-1}:\overline{\overline{C}}}.\end{aligned} \tag{241}$$

Here we should insert the expression (237).

Uniaxial Tetradic

The inverse of the special uniaxial tetradic could be obtained from the expansion (222) by replacing the eigenvalues through their inverses. However, it is easier to apply the formula (237) for

$$\mathcal{U} = \mathcal{E} + \alpha\mu\mathbf{vvvv}, \tag{242}$$

where $\mathcal{E}$ is an isotropic tetradic whose inverse was given above. Applying (237) we can write

$$\mathcal{U}^{-1} = \mathcal{E}^{-1} - \frac{\alpha\mu\left(\mathcal{E}^{-1} : \mathbf{vv}\right)\left(\mathcal{E}^{-1} : \mathbf{vv}\right)}{1 + \alpha\mu\mathbf{vv} : \mathcal{E}^{-1} : \mathbf{vv}}. \tag{243}$$

Inserting the expression (231) for $\mathcal{E}^{-1}$ the inverse will have the form

$$\mathcal{U}^{-1} = \mathcal{E}^{-1} - \frac{\alpha\mu}{1 + \dfrac{\alpha\mu}{\mu+\eta}\dfrac{\mu+\eta+2\lambda}{\mu+\eta+3\lambda}}\overline{\overline{A}}\,\overline{\overline{A}}, \tag{244}$$

with the dyadic

$$\overline{\overline{A}} = \frac{1}{\mu+\eta}\left(\mathbf{vv} - \frac{\lambda}{\mu+\eta+3\lambda}\overline{\overline{I}}\right). \tag{245}$$

It is easy to check that this really leads to the correct inverse tetradic.

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