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COMPUTATIONALLY EFFICIENT EXPRESSIONS OF THE DYADIC GREEN'S FUNCTION FOR RECTANGULAR ENCLOSURES

F. Marliani and A. Ciccolella

European Space Agency, D/TOS/EEE
Electromagnetic Compatibility & Antenna Measurement Section
Keplerlaan 1-2201 AZ NOORDWIJK, The Netherlands

Abstract–This paper considers expressions of the dyadic Green's function for rectangular enclosures, which are efficient from a computational point of view. The inherent application is to solve numerically electromagnetic scattering problems with an integral equation formulation, using the Method of Moments. The Green's dyadic is derived from an image-spectral approach, which has the flexibility to generate directly the expression with the fastest convergence once the locations of both the observation and the source points are given. When the observation point is at the source region, slowly converging sums arise that are overcome by extending the method of Ewald to the dyadic case. Numerical proofs are reported in tabular form to validate the technique developed.

1. INTRODUCTION

The formulation of the dyadic Green's function for perfectly conducting rectangular cavities has previously received considerable attention in literature. Classical methods, like the eigenfunction expansion [1–5] and the image theory [6–8], have been the basic devices to build the solution. Depending on the approach selected, they emphasize specific features of electromagnetic propagation in a cavity, as shown by the different mathematical morphology of the expressions that the two techniques return.

The eigenfunction method generates the Green's dyadic function through a modal formalism, which underlines the frequency selection operated by the rectangular enclosure and explicitly reveals that the dyadic is not bounded at the resonant frequencies. The image theory leads to a series representation of the Green's dyadic that points up the contribution of both the source and the multiple reflections on the conducting walls. Also, this approach shows directly the singular behavior when the observation point approaches the source point [9]. When this condition is anticipated, severe problems of convergence affect the series appearing in the Green's dyadic.

So far, the intrinsic problems of computational efficiency associated to Green's dyadic discouraged the use of the integral equation formulation to solve for electromagnetic interactions inside rectangular cavities.

This study presents a set of alternative procedures yielding the Green's dyadic function by means of mathematical expressions that are numerically advantageous for Method of Moments (MoM) applications.

After a brief overview of the canonical methods used to derive the electric-type Green's dyadic function for rectangular cavities (Section 2), we introduce a complementary formulation consisting of a joint application of the image principle and the plane wave expansion of the electromagnetic field (Section 3).

The image-spectral approach, built directly on the electric field rather than on the potential, is initially established by replacing an infinitesimal current source inside the enclosure with a lattice of images that satisfy the boundary conditions at the cavity's walls. Subsequently, the field contribution of any image is expressed as an expansion of plane waves. The resulting spectral representation involves both a double integral and a Dirac delta function accounting for the source region and leads to the Green's dyadic function as a double series that is consistent with the classical results.

We discuss the properties of convergence of the series involved in this representation and compare it with the classical formulation attainable in literature. We remark that the mutual location of the source point and the observation point plays a fundamental role to select the spectral variable of integration that gives the fastest convergence.

When the observation point is very close to the source point, the convergence is inherently critical. For this specific situation, we have adapted and extended to the dyadic case the method introduced by P. P. Ewald to accelerate the computation of electrodynamic potentials generated by a three-dimensional periodic lattice of sources (Section 4).

We finally present results on the calculation of the Green's dyadic function that proves de facto the numerical efficiency of the notions reported in the paper.

2. DYADIC GREEN'S FUNCTION FOR A PERFECTLY CONDUCTING CAVITY

The electric Green's dyadic function $\overline{\mathbf{G}}_E(\mathbf{r},\mathbf{r}')$ is the tensor kernel of the integral operator that transforms the boundary conditions and the electric current density $\mathbf{J}_e(\mathbf{r}')$ into the electric field $\mathbf{E}(\mathbf{r})$ as follows:

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \iiint_{V'} \overline{\mathbf{G}}_E\left(\mathbf{r},\mathbf{r}'\right)\mathbf{J}_e\left(\mathbf{r}'\right)\mathrm{d}\mathbf{r}' \tag{1}$$

For a lossless metallic enclosure, $\overline{\mathbf{G}}_E$ is the solution of the differential equation with boundary conditions

$$\begin{cases} \nabla\times\nabla\times\overline{\mathbf{G}}_E(\mathbf{r},\mathbf{r}') - k^2\overline{\mathbf{G}}_E(\mathbf{r},\mathbf{r}') = \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \\ \hat{\mathbf{n}}\times\overline{\mathbf{G}}_E(\mathbf{r},\mathbf{r}') = 0 \quad \mathbf{r}\in S \end{cases} \tag{2}$$

where $\overline{\mathbf{I}}$ is the unit dyadic, $\mathbf{r}$ and $\mathbf{r}'$ are, respectively, the coordinates of the observation and of the source points, k is the wave number related to the medium in the cavity and $\hat{\mathbf{n}}$ is the local normal to the surface S that bounds the volume of the cavity.

The classical method to represent the electric dyadic Green's function for lossless rectangular cavities consists of using an eigenfunction expansion as done by Morse and Feshback [1] and Tai [2, 3].

The dyadic is obtained as a triple series by expanding the generalised tensor function $\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}')$ in terms of an appropriate set of vector wave functions that are solutions of the homogeneous problem. Thereafter, the Ohm-Rayleigh method is applied. This technique leads to the Green's dyadic in terms of two solenoidal (transversal) sets $\mathbf{E}_S^{TE}$ and $\mathbf{E}_S^{TM}$ and one irrotational (longitudinal) component $\mathbf{E}_L$ as the Helmhotz theorem dictates, i.e.:

$$\overline{\mathbf{G}}_E(\mathbf{r},\mathbf{r}') = \sum_{r=0}^{+\infty}\sum_{s=0}^{+\infty}\sum_{t=0}^{+\infty}\left(\frac{\epsilon_{0r}\epsilon_{0s}\epsilon_{0t}}{abc}\right)\left[\frac{\mathbf{E}_S^{TE}(\mathbf{r})\,\mathbf{E}_S^{TE}(\mathbf{r}')}{\left(k_{rst}^2-k^2\right)k_{rs}^2} + \frac{\mathbf{E}_S^{TM}(\mathbf{r})\,\mathbf{E}_S^{TM}(\mathbf{r}')}{\left(k_{rst}^2-k^2\right)k_{rst}^2 k_{rs}^2} - \frac{\mathbf{E}_L(\mathbf{r})\,\mathbf{E}_L(\mathbf{r}')}{k_{rst}^2 k^2}\right] \tag{3}$$

where

$$\begin{aligned}
\mathbf{E}_S^{TE}(\mathbf{r}) &= \nabla\times\left[\cos\left(\frac{r\pi}{a}\right)\cos\left(\frac{s\pi}{b}\right)\sin\left(\frac{t\pi}{c}\right)\right]\\
\mathbf{E}_S^{TM}(\mathbf{r}) &= \nabla\times\nabla\times\left[\sin\left(\frac{r\pi}{a}\right)\sin\left(\frac{s\pi}{b}\right)\cos\left(\frac{t\pi}{c}\right)\right]\\
\mathbf{E}_L(\mathbf{r}) &= \nabla\left[\sin\left(\frac{r\pi}{a}\right)\sin\left(\frac{s\pi}{b}\right)\sin\left(\frac{t\pi}{c}\right)\right]\\
k_{rst}^2 &= \left(\frac{r\pi}{a}\right)^2+\left(\frac{s\pi}{b}\right)^2+\left(\frac{t\pi}{c}\right)^2\\
k_{rs}^2 &= \left(\frac{r\pi}{a}\right)^2+\left(\frac{s\pi}{b}\right)^2\\
\epsilon_{0p} &= \begin{cases}1 & \text{for } p=0\\ 2 & \text{for } p\neq 0\end{cases}
\end{aligned}$$

and a, b, c are the dimensions of the cavity along the x, y, z axes respectively.

Rahmat-Samii [4] proposed an equivalent method that derives the Green's dyadic function through the diagonal dyadic potential $\overline{\mathbf{G}}_A$ as an intermediary step. $\overline{\mathbf{G}}_A$ is solution of the following differential equation with the associated boundary conditions:

$$\begin{cases}\nabla^2\overline{\mathbf{G}}_A(\mathbf{r},\mathbf{r}')+k^2\overline{\mathbf{G}}_A(\mathbf{r},\mathbf{r}') = -\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}')\\ \hat{\mathbf{n}}\times\overline{\mathbf{G}}_A(\mathbf{r},\mathbf{r}')=0 \quad \mathbf{r}\in S\\ \nabla\cdot\overline{\mathbf{G}}_A(\mathbf{r},\mathbf{r}')=0 \quad \mathbf{r}\in S\end{cases} \tag{4}$$

Solving for $\overline{\mathbf{G}}_A$ and using the relation $\overline{\mathbf{G}}_E = \left[\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right] \overline{\mathbf{G}}_A$ we obtain the electric Green's dyadic as:

$$\overline{\mathbf{G}}_E\left(\mathbf{r},\mathbf{r}'\right) = \sum_{r=0}^{+\infty}\sum_{s=0}^{+\infty}\sum_{t=0}^{+\infty}\left(\frac{\epsilon_{0r}\epsilon_{0s}\epsilon_{0t}}{abc}\right)\frac{1}{\left(k_{rst}^2 - k^2\right)}\left[\Psi_{rst}^x\left(\mathbf{r}\right)\Psi_{rst}^x\left(\mathbf{r}'\right)\hat{\mathbf{x}}\hat{\mathbf{x}}\right.$$
$$\left. + \Psi_{rst}^y\left(\mathbf{r}\right)\Psi_{rst}^y\left(\mathbf{r}'\right)\hat{\mathbf{y}}\hat{\mathbf{y}} + \Psi_{rst}^z\left(\mathbf{r}\right)\Psi_{rst}^z\left(\mathbf{r}'\right)\hat{\mathbf{z}}\hat{\mathbf{z}}\right]$$
$$- \frac{1}{k^2}\sum_{r=0}^{+\infty}\sum_{s=0}^{+\infty}\sum_{t=0}^{+\infty}\left(\frac{\epsilon_{0r}\epsilon_{0s}\epsilon_{0t}}{abc}\right)\frac{1}{\left(k_{rst}^2 - k^2\right)}$$
$$\cdot \nabla\Psi_{rst}\left(\mathbf{r}\right)\nabla'\Psi_{rst}\left(\mathbf{r}'\right) \tag{5}$$

where

$$\Psi_{rst}^x\left(\mathbf{r}\right) = \cos\left(\frac{r\pi}{a}\right)\sin\left(\frac{s\pi}{b}\right)\sin\left(\frac{t\pi}{c}\right)$$
$$\Psi_{rst}^y\left(\mathbf{r}\right) = \sin\left(\frac{r\pi}{a}\right)\cos\left(\frac{s\pi}{b}\right)\sin\left(\frac{t\pi}{c}\right)$$
$$\Psi_{rst}^z\left(\mathbf{r}\right) = \sin\left(\frac{r\pi}{a}\right)\sin\left(\frac{s\pi}{b}\right)\cos\left(\frac{t\pi}{c}\right)$$
$$\Psi_{rst}\left(\mathbf{r}\right) = \sin\left(\frac{r\pi}{a}\right)\sin\left(\frac{s\pi}{b}\right)\sin\left(\frac{t\pi}{c}\right)$$

The triple sum appearing in both the (3) and the (5) can be reduced to a double one by resolving in closed form with respect to a selected index.

We note that the singular term of the dyadic is embedded, respectively, in the longitudinal term of the (3) and in the second sum of the (5), whereas the just a poxition of the operators ∇ and ∇' is performed.

The last alternative approach we mention was proposed by Hamid [6]. The method derives the potential Green's function using a ray-optical technique for large wave numbers. After replacing the infinitesimal primary source with a lattice of images, the vector potential is obtained by adding up the ray contributions from each source and applying the geometrical optics approximation. This asymptotic technique requires that the electrical length of each ray be greater than a few radians so that the principle of stationary phase can be applied.

3. IMAGES EXPANSION AND PLANE-WAVES REPRESENTATION

3.1 Scheme of the Solution

We reduce the boundary value problem defined by (2) to the equivalent problem of a periodical infinite lattice of time-varying sources in free space by applying the image theory. Consider that the volume V of the rectangular cavity is confined in the region of space $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. Both the observation point and the primary source point belong to V. We build the infinite lattice of sources starting with the basic cell of eight sources which are generated by mirroring the primary source on the three planes $x = 0$, $y = 0$, $z = 0$ with the appropriate sign. The basic cell is then repeated with periods $2a$, $2b$ and $2c$ along the x, y and z directions, respectively, to produce the equivalent free space problem.

The resulting field at a generic observation point inside the cavity can be interpreted as the superposition of the contributions from infinite images, whose locations depend on both the size of the cavity and the position of the primary source. This formulation yields explicitly the singular behavior of the Green's dyadic in the primary source region.

In mathematical terms, we can write the global Green's function as an infinite three-dimensional array of dyadics:

$$\overline{\mathbf{G}}_E\left(x,y,z,x',y',z'\right) = \sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty} \overline{\mathbf{G}}_E^C\left(x,y,z,x'+2ra,y'+2bs,z'+2tc\right) \quad (6)$$

where $\overline{\mathbf{G}}_E^C(x,y,z,x',y',z')$ is the Green's function of the basic cell.

To calculate $\overline{\mathbf{G}}_E^C$, we must account for the orientation of the primary electric current source with respect to the walls for each principal direction since this influences the sign of the relevant image. This implies that we will build the dyadic $\overline{\mathbf{G}}_E^C$ through a column-by-column process. We begin to construct $\overline{\mathbf{G}}_E^C$ from the spectral representation of the electric Green's dyadic function in free space $\widetilde{\mathbf{G}}_E^{FS}$ in cartesian coordinates for a single point source:

$$\widetilde{\mathbf{G}}_E^{FS}\left(k_x,k_y,k_z,x',y',z'\right)=\frac{-\exp\left[j\left(k_xx'+k_yy'+k_zz'\right)\right]}{k_0^2\left(k_0^2-k_x^2-k_y^2-k_z^2\right)}\cdot\begin{bmatrix}k_0^2-k_x^2 & -k_xk_y & -k_xk_z\\ -k_yk_x & k_0^2-k_y^2 & -k_yk_z\\ -k_zk_x & -k_zk_y & k_0^2-k_z^2\end{bmatrix} \tag{7}$$

We start the process by considering a z-oriented delta source located in $\mathbf{r}'$, which leads us to work on the third column of the spectral dyadic. By summing the eight elements of the basic cell with their appropriate sign we obtain the third column of $\overline{\mathbf{G}}_E^C$ as follows:

$$\begin{aligned}&\widetilde{\mathbf{G}}_E^{C(3)}\left(k_x,k_y,k_z,x',y',z'\right)\\ &=\widetilde{\mathbf{G}}_E^{FS(3)}\left(k_x,k_y,k_z,x',y',z'\right)-\widetilde{\mathbf{G}}_E^{FS(3)}\left(k_x,k_y,k_z,-x',y',z'\right)\\ &\quad-\widetilde{\mathbf{G}}_E^{FS(3)}\left(k_x,k_y,k_z,x',-y',z'\right)+\widetilde{\mathbf{G}}_E^{FS(3)}\left(k_x,k_y,k_z,-x',-y',z'\right)\\ &\quad+\widetilde{\mathbf{G}}_E^{FS(3)}\left(k_x,k_y,k_z,x',y',-z'\right)-\widetilde{\mathbf{G}}_E^{FS(3)}\left(k_x,k_y,k_z,-x',y',-z'\right)\\ &\quad-\widetilde{\mathbf{G}}_E^{FS(3)}\left(k_x,k_y,k_z,x',-y',-z'\right)+\widetilde{\mathbf{G}}_E^{FS(3)}\left(k_x,k_y,k_z,-x',-y',-z'\right)\\ &=\frac{8\sin\left(k_xx'\right)\sin\left(k_yy'\right)\cos\left(k_zz'\right)}{k_0^2\left(k_0^2-k_x^2-k_y^2-k_z^2\right)}\cdot\begin{bmatrix}-k_xk_z\\ -k_yk_z\\ k_0^2-k_z^2\end{bmatrix}\end{aligned} \tag{8}$$

This representation of the third column of $\overline{\mathbf{G}}_E^C$ can be inserted in (6) yielding the corresponding column of the Green's dyadic function for the rectangular cavity:

$$\begin{aligned}\overline{\mathbf{G}}_E^{(3)}\left(\mathbf{r},\mathbf{r}'\right)=&-\sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty}\frac{1}{8\pi^3}\iiint_{-\infty}^{+\infty}\\ &\cdot\frac{8\sin\left(k_xx'\right)\sin\left(k_yy'\right)\cos\left(k_zz'\right)}{k_0^2\left(k_0^2-k_x^2-k_y^2-k_z^2\right)}\begin{bmatrix}k_xk_z\\ k_yk_z\\ k_z^2-k_0^2\end{bmatrix}\\ &\cdot\exp\left[j\left(2ra-x\right)k_x\right]\exp\left[j\left(2bs-y\right)k_y\right]\\ &\cdot\exp\left[j\left(2tc-z\right)k_z\right]\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z\end{aligned} \tag{9}$$

We can now reduce the triple integral in (9) to a double integral by resolving in closed form with respect to one of the variables k_x, k_y, k_z

with the contour integration method. Without loss of generality, we select to close the integral with respect to k_z variable, which yields:

$$\overline{\mathbf{G}}_E^{(3)}\left(\mathbf{r},\mathbf{r}'\right) = \sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty}\frac{1}{4\pi^2}\iint_{-\infty}^{+\infty}\widetilde{\mathbf{G}}_E^{(3)}\left(k_x,k_y,x',y',z',z\right) \cdot \exp\left[-j\left(k_x x+k_y y\right)\right]\mathrm{d}k_x\mathrm{d}k_y - \frac{\hat{\mathbf{z}}\hat{\mathbf{z}}}{k_0^2}\delta\left(\mathbf{r}-\mathbf{r}'\right) \tag{10}$$

where

$$\begin{aligned}&\widetilde{\mathbf{G}}_E^{(3)}\left(k_x,k_y,x',y',z',z\right)\\&=\frac{2j}{k_0^2}\frac{\sin\left(k_x x'\right)\sin\left(k_y y'\right)}{\sqrt{k_0^2-k_x^2-k_y^2}}\exp\left(j2rak_x\right)\exp\left(j2bsk_y\right)\\&\cdot\left\{\exp\left(-jk_{zp}\left|z+2ct-z'\right|\right)\begin{bmatrix}-\mathrm{sign}\left(z+2ct-z'\right)k_xk_{zp}\\-\mathrm{sign}\left(z+2ct-z'\right)k_yk_{zp}\\k_x^2+k_y^2\end{bmatrix}\right.\\&\left.+\exp\left(-jk_{zp}\left|z+2ct+z'\right|\right)\begin{bmatrix}-\mathrm{sign}\left(z+2ct+z'\right)k_xk_{zp}\\-\mathrm{sign}\left(z+2ct+z'\right)k_yk_{zp}\\k_x^2+k_y^2\end{bmatrix}\right\}\end{aligned}$$

and $k_{zp}=\sqrt{k_0^2-k_x^2-k_y^2}$. We notice that $\sqrt{k_0^2-k_x^2-k_y^2}$ is a branch singularity and its sign must be selected to guarantee the convergence of the integrand, i.e., $\mathrm{Im}(k_{zp})<0$ and $\mathrm{Re}(k_{zp})>0$.

The double integral in (10) can be also solved in closed form if we recall the relation $\sum_n e^{-j2nL\xi}=\frac{\pi}{a}\sum_n\delta\left(\xi-\frac{n\pi}{L}\right)$. After some algebra, we can cast the third column of the dyadic as follows:

$$\begin{aligned}\overline{\mathbf{G}}_E^{(3)}(\mathbf{r},\mathbf{r}') = \frac{j}{2k_0^2}\frac{1}{ab}\sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty}\\&\cdot\frac{\sin\left(\frac{r\pi}{a}x'\right)\sin\left(\frac{s\pi}{b}y'\right)}{\sqrt{k_0^2-\left(\frac{r\pi}{a}\right)^2-\left(\frac{s\pi}{b}\right)^2}}\exp\left[-j\left(\frac{r\pi}{a}x+\frac{s\pi}{b}y\right)\right]\\&\cdot\left\{\exp\left(-jk_{rsp}\left|z+2ct-z'\right|\right)\begin{bmatrix}-\mathrm{sign}\left(z+2ct-z'\right)\frac{r\pi}{a}k_{rsp}\\-\mathrm{sign}\left(z+2ct-z'\right)\frac{s\pi}{b}k_{rsp}\\\left(\frac{r\pi}{a}\right)^2+\left(\frac{s\pi}{b}\right)^2\end{bmatrix}\right.\end{aligned}$$

$$+\exp\left(-jk_{rsp}\left|z+2ct+z'\right|\right)$$
$$\cdot\left[\begin{array}{c}-\text{sign}\left(z+2ct+z'\right)\frac{r\pi}{a}k_{rsp}\\ -\text{sign}\left(z+2ct+z'\right)\frac{s\pi}{b}k_{rsp}\\ \left(\frac{r\pi}{a}\right)^2+\left(\frac{s\pi}{b}\right)^2\end{array}\right]\Bigg\}-\frac{\hat{\mathbf{z}}\hat{\mathbf{z}}}{k_0^2}\delta\left(\mathbf{r}-\mathbf{r}'\right) \tag{11}$$

where $k_{rsp}=\sqrt{k_0^2-\left(\frac{r\pi}{a}\right)^2-\left(\frac{s\pi}{b}\right)^2}$.

It is worth emphasizing that the primary source is the origin of the Dirac delta contribution in (11) and this is the only one retained since the observation point lies inside the cavity by definition.

We can further simplify the sums involved in (11) by working out the individual components $\hat{\mathbf{x}}\hat{\mathbf{z}}$, $\hat{\mathbf{y}}\hat{\mathbf{z}}$ and $\hat{\mathbf{z}}\hat{\mathbf{z}}$ of the dyadic.

Let us consider the first row of expression (11), which corresponds to the element G_E^{xz}. We observe that the sum in t is an even function of the integers r and s, so we can write the equivalent expression:

$$G_E^{xz}(\mathbf{r},\mathbf{r}')=\frac{1}{k_0^2}\frac{2}{ab}\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{r\pi}{a}\cos\left(\frac{r\pi}{a}x\right)\sin\left(\frac{r\pi}{a}x'\right)\sin\left(\frac{s\pi}{b}y'\right)\sin\left(\frac{s\pi}{b}y\right)$$
$$\cdot\sum_{t=-\infty}^{+\infty}\left[\exp\left(-jk_{rsp}\left|z+2ct-z'\right|\right)\text{sign}\left(z+2ct-z'\right)\right.$$
$$\left.+\exp\left(-jk_{rsp}\left|z+2ct+z'\right|\right)\text{sign}\left(z+2ct+z'\right)\right] \tag{12}$$

Recalling the identity $\sum_{n=1}^{+\infty}e^{-j2\alpha nx}=-\frac{1}{2}\left[j\cot\left(\alpha x\right)+1\right]$, the sum t appearing in (12) can be calculated after some mathematical manipulations and it is reduced to the function:

$$\begin{cases}2j\sin\left(k_{rsp}z\right)\cos\left(k_{rsp}z'\right)\left(j\cot\left(k_{rsp}c\right)+1\right) & \text{for } t\neq 0\\ 2\cos\left(k_{rsp}z'\right)\exp\left(-jk_{rsp}z\right) \quad z>z' & \text{for } t=0\\ -2j\sin\left(k_{rsp}z\right)\exp\left(-jk_{rsp}z'\right) \quad z<z' & \end{cases}$$

Finally, the expression for G_E^{xz} takes the form

$$G_E^{xz}\left(\mathbf{r},\mathbf{r}'\right)$$
$$=-\frac{1}{k_0^2}\frac{4}{ab}\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{r\pi}{a}\cos\left(\frac{r\pi}{a}x\right)\sin\left(\frac{r\pi}{a}x'\right)\sin\left(\frac{s\pi}{b}y\right)\sin\left(\frac{s\pi}{b}y'\right)$$
$$\cdot\begin{cases}\cos\left(k_{rsp}z'\right)\left[\cos\left(k_{rsp}z\right)-\sin\left(k_{rsp}z\right)\cot\left(k_{rsp}c\right)\right] & z>z'\\ -\sin\left(k_{rsp}z\right)\left[\sin\left(k_{rsp}z'\right)+\cos\left(k_{rsp}z'\right)\cot\left(k_{rsp}c\right)\right] & z<z'\end{cases} \tag{13}$$

It is important to underline that G_E^{xz} is continuous at $z = z'$. This can be easily inferred by taking the limit for $z \to z'$ from both the left and the right and using the relation $\sum_{n=1}^{+\infty} n \sin(\alpha n) = 0$ (see Appendix A.6 of [5]).

Applying the identical scheme to the second and third row of expression (11), we obtain the result for G_E^{yz} and G_E^{zz} that we report here for completeness:

$$\begin{aligned}
&G_E^{yz}\left(\mathbf{r},\mathbf{r}'\right)\\
&= -\frac{1}{k_0^2}\frac{4}{ab}\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{s\pi}{b}\sin\left(\frac{r\pi}{a}x\right)\sin\left(\frac{r\pi}{a}x'\right)\cos\left(\frac{s\pi}{b}y\right)\sin\left(\frac{s\pi}{b}y'\right)\\
&\quad\cdot\begin{cases}\cos\left(k_{rsp}z'\right)\left[\cos\left(k_{rsp}z\right)-\sin\left(k_{rsp}z\right)\cot\left(k_{rsp}c\right)\right] & z>z'\\ -\sin\left(k_{rsp}z\right)\left[\sin\left(k_{rsp}z'\right)+\cos\left(k_{rsp}z'\right)\cot\left(k_{rsp}c\right)\right] & z<z'\end{cases}
\end{aligned}\tag{14}$$

$$\begin{aligned}
&G_E^{zz}\left(\mathbf{r},\mathbf{r}'\right)\\
&= -\frac{1}{k_0^2}\frac{4}{ab}\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{\left(\frac{r\pi}{a}\right)^2+\left(\frac{s\pi}{b}\right)^2}{k_{rsp}}\sin\left(\frac{r\pi}{a}x\right)\\
&\quad\cdot\sin\left(\frac{r\pi}{a}x'\right)\sin\left(\frac{s\pi}{b}y'\right)\sin\left(\frac{s\pi}{b}y\right)\\
&\quad\cdot\begin{cases}\cos\left(k_{rsp}z'\right)\left[\sin\left(k_{rsp}z\right)+\cos\left(k_{rsp}z\right)\cot\left(k_{rsp}c\right)\right] & z>z'\\ \cos\left(k_{rsp}z\right)\left[\sin\left(k_{rsp}z'\right)+\cos\left(k_{rsp}z'\right)\cot\left(k_{rsp}c\right)\right] & z<z'\end{cases}\\
&\quad-\frac{\hat{\mathbf{z}}\hat{\mathbf{z}}}{k_0^2}\delta\left(\mathbf{r}-\mathbf{r}'\right)
\end{aligned}\tag{15}$$

This concludes the derivation of the third column of the dyadic Green's function for a lossless rectangular cavity.

Equivalent mathematical formulations would have resulted if we had closed the integral (9) with respect to the k_x or the k_y variable.

We can apply the identical procedure to build the other two columns, considering both the x-oriented and the y-oriented sources associated with the relevant images and, finally, complete the dyadic Green's function. Symmetry is a useful property for the overall process since it allows us to solve for six elements instead of nine to fill the dyadic.

The results shown here are consistent with those reported in [3] and [5], which also consider a double sum in z.

3.2 Consideration on the Convergence

The dyadic Green's function exhibits, in general, well-behaved convergence when the observation point is outside the source region. In particular, considering for example the results (13)–(15), the asymptotic behavior of those expression (i.e., for large r and s) is $\exp(-\alpha |z-z'|)$ where $\alpha = \sqrt{\left(\frac{r\pi}{a}\right)^2 + \left(\frac{s\pi}{b}\right)^2 - k_0^2}$. Therefore, the larger the distance along z between the coordinates of the source and the observation point, the faster the convergence of the summations is anticipated.

Generalizing this consideration, we can cast the dyadic Green's function by closing the triple integrals for columns, similar to (9), with respect to the spectral variable revealing the larger separation of the individual cartesian components between the observation and the source point.

To illustrate the numerical efficiency of this scheme, let $E_z(x, y, z)$ be the z-component of the electric field observed in (x, y, z) generated by a z-oriented hertzian dipole of length l centered in (x_0, y_0, z_0) which supports a constant current I.

We calculate $E_z(x, y, z)$ by:

$$\begin{aligned} E_z(x,y,z) = j\omega\mu \iiint_{V'} \sum_{r=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} \sum_{t=-\infty}^{+\infty} \frac{1}{8\pi^3} \\ \iiint_{-\infty}^{+\infty} \frac{8\sin(k_x x')\sin(k_y y')\cos(k_z z')}{k_0^2\left(k_0^2 - k_x^2 - k_y^2 - k_z^2\right)} \\ \cdot \left(k_z^2 - k_0^2\right)\exp\left[j\left(2ra - k_x x\right)\right]\exp\left[j\left(2bs - k_y y\right)\right] \\ \cdot \exp\left[j\left(2tc - k_z z\right)\right] \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}k_z \delta\left(x' - x_0\right) \\ \cdot \delta\left(y' - y_0\right) \operatorname{rect}\left(\frac{z' - z_0}{l/2}\right) dx'dy'dz' \end{aligned} \tag{16}$$

We close the inner triple integral with respect to the variables k_x, k_y and k_z and consider, without loss of generality, the case $x > x_0$, $y > y_0$, $z > l/2 + z_0$. After some mathematical manipulations we obtain, respectively, the following three equivalent expressions for $E_z(x,y,z)$:

$$E_z(x,y,z) = -\frac{4j}{\varepsilon\omega}\frac{\mathrm{I}}{bc}\sum_{s=1}^{+\infty}\sum_{t=0}^{+\infty}\epsilon_{0t}\frac{k_0^2-\left(\frac{t\pi}{c}\right)^2}{k_{pst}\cdot\frac{t\pi}{c}}\sin\left(\frac{s\pi}{b}y\right)\sin\left(\frac{s\pi}{b}y_0\right)$$
$$\cdot\cos\left(\frac{t\pi}{c}z\right)\cos\left(\frac{t\pi}{c}z_0\right)\sin\left(\frac{t\pi}{c}\frac{l}{2}\right)\sinh\left(k_{pst}x_0\right)$$
$$\cdot\exp\left(-k_{pst}x\right)\frac{1-\exp\left[-2k_{pst}\left(a-x\right)\right]}{1-\exp\left(2k_{pst}x\right)} \quad (17)$$

$$E_z(x,y,z) = -\frac{4j}{\varepsilon\omega}\frac{\mathrm{I}}{ac}\sum_{r=1}^{+\infty}\sum_{t=0}^{+\infty}\epsilon_{0t}\frac{k_0^2-\left(\frac{t\pi}{c}\right)^2}{k_{prt}\cdot\frac{t\pi}{c}}\sin\left(\frac{r\pi}{a}x\right)\sin\left(\frac{r\pi}{a}x_0\right)$$
$$\cdot\cos\left(\frac{t\pi}{c}z\right)\cos\left(\frac{t\pi}{c}z_0\right)\sin\left(\frac{t\pi}{c}\frac{l}{2}\right)\sinh\left(k_{prt}y_0\right)$$
$$\cdot\exp\left(-k_{prt}y\right)\frac{1-\exp\left[-2k_{prt}\left(b-y\right)\right]}{1-\exp\left(2k_{prt}b\right)} \quad (18)$$

$$E_z(x,y,z) = -\frac{8j}{\varepsilon\omega}\frac{\mathrm{I}}{ab}\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{\left(\frac{r\pi}{a}\right)^2+\left(\frac{s\pi}{b}\right)^2}{k_{prs}^2}\sin\left(\frac{r\pi}{a}x\right)\sin\left(\frac{r\pi}{a}x_0\right)$$
$$\cdot\sin\left(\frac{s\pi}{b}y\right)\sin\left(\frac{s\pi}{b}y_0\right)\cosh\left(k_{prs}z_0\right)\sinh\left(k_{prs}\frac{l}{2}\right)$$
$$\cdot\exp\left(-k_{prs}z\right)\frac{1+\exp\left[-2k_{prs}\left(c-z\right)\right]}{1-\exp\left(2k_{prs}c\right)} \quad (19)$$

where $k_{pst} = \sqrt{\left(\frac{s\pi}{b}\right)^2+\left(\frac{t\pi}{c}\right)^2-k_0^2}$, $k_{prt} = \sqrt{\left(\frac{r\pi}{a}\right)^2+\left(\frac{t\pi}{c}\right)^2-k_0^2}$, $k_{prs} = \sqrt{\left(\frac{r\pi}{a}\right)^2+\left(\frac{s\pi}{b}\right)^2-k_0^2}$.

We can now compare the numerical convergence of (17)–(19) by setting $a = 3\,\mathrm{m}$, $b = 4\,\mathrm{m}$, $c = 5\,\mathrm{m}$, $l = 10\,\mathrm{cm}$, $\mathrm{I} = 1\,\mathrm{mA}$ and $f = 155\,\mathrm{MHz}$. The centre of the dipole source is placed at (1,1.5,2.5) meter. We assume that convergence is reached when the fourth decimal digit of E_z, expressed in mV/m and calculated following (17)–(19), is stable.

In Table 1 we show the number of terms required by the sums (17), (18) and (19), respectively, to achieve convergence as a function of the observation point. The distance $\Delta x = (x - x_0)$ is kept constant and

Table 1. Number of terms required to achieve convergence (to the fourth decimal place) as a function of the mutual location of the source point and observation point. The distance along the x-direction is kept constant ($\Delta x = x - x_0 = 1.5\,\mathrm{m}$) whilst the distances along y ($\Delta y = y - y_0$) and z ($\Delta z = z - z_0 - l/2$) are parametric. Source position: $(x_0, y_0, z_0) = (1, 1.5, 2.5)\,\mathrm{m}$. Parameters: $f = 155\,\mathrm{MHz}$, $a = 3\,\mathrm{m}$, $b = 4\,\mathrm{m}$, $c = 5\,\mathrm{m}$, $l = 10\,\mathrm{cm}$ and $\mathrm{I} = 1\,\mathrm{mA}$.

Δx	Δy	Δz	N. of terms for (17)	N. of terms for(18)	N. of terms for(19)	E (mV/m)
1.5	1.45	1.5	110	110	50	−3.4097
1.5	0.95	1	156	156	100	−2.8473
1.5	0.45	0.5	156	306	289	+2.2901
1.5	0.05	0.05	156	53130	57600	+5.0578
1.5	0.01	0.01	156	1441200	1440000	+5.1227

Table 2. Number of terms required to achieve convergence (to the fourth decimal place) as a function of the mutual location of the source point and observation point. The distance along the y-direction is kept constant ($\Delta y = y - y_0 = 1.5\,\mathrm{m}$) whilst the distances along x ($\Delta x = x - x_0$) and z ($\Delta z = z - z_0 - l/2$) are parametric. Source position: $(x_0, y_0, z_0) = (1, 1.5, 2.5)\,\mathrm{m}$. Parameters: $f = 155\,\mathrm{MHz}$, $a = 3\,\mathrm{m}$, $b = 4\,\mathrm{m}$, $c = 5\,\mathrm{m}$, $l = 10\,\mathrm{cm}$ and $\mathrm{I} = 1\,\mathrm{mA}$.

Δx	Δy	Δz	N. of terms for (17)	N. of terms for(18)	N. of terms for(19)	E (mV/m)
1.45	1.5	1.5	50	110	110	−3.2496
0.95	1.5	1	121	110	132	−2.3060
0.45	1.5	0.5	676	156	552	+4.9935
0.05	1.5	0.05	78400	210	78680	+11.8769
0.01	1.5	0.01	1960000	156	3064250	+12.1339

equal to $1.5\,\mathrm{m}$, while $\Delta y = (y - y_0)$ and $\Delta z = (z - z_0 - l/2)$ are parametric.

Similarly this is done for Tables 2 and 3, where the distances $\Delta y = (y - y_0)$ and $\Delta z = (z - z_0 - l/2)$ are, respectively, fixed to $1.5\,\mathrm{m}$.

The results shown in Tables 1–3 clearly confirm the initial postulate that, in order to get the fastest convergence, the most convenient integration variable is the one leading to the largest separation distance among the individual coordinates of the observation point and the source. When the distance between observation point and source is

Table 3. Number of terms required to achieve convergence (to the fourth decimal place) as a function of the mutual location of the source point and observation point. The distance along the z-direction is kept constant ($\Delta z = z - z_0 - l/2 = 1.5\,\text{m}$) whilst the distances along x ($\Delta x = x - x_0$) and y ($\Delta y = y - y_0$) are parametric. Source position: $(x_0, y_0, z_0) = (1, 1.5, 2.5)\,\text{m}$. Parameters: $f = 155\,\text{MHz}$, $a = 3\,\text{m}$, $b = 4\,\text{m}$, $c = 5\,\text{m}$, $l = 10\,\text{cm}$ and $\text{I} = 1\,\text{mA}$.

Δx	Δy	Δz	N. of terms for (17)	N. of terms for(18)	N. of terms for(19)	E (mV/m)
1.45	1.45	1.5	110	110	81	−3.6923
0.95	0.95	1.5	272	272	81	−7.6928
0.45	0.45	1.5	1332	1332	100	+1.2534
0.05	0.05	1.5	144780	144780	144	+8.4100
0.01	0.01	1.5	3241800	3241800	121	+8.5043

small along a particular direction, we notice that the number of terms required to achieve convergence may differ in four orders of magnitude depending on the selected formulation.

However, when the observation point tends to the source point in all three directions, the asymptotic exponential term tends to unity causing severe convergence problems to the series (17)–(19).

This situation always occurs when the Green's function is used for numerical solutions with the Method of Moments and therefore alternative expressions must be considered.

Several techniques [10–12] have been proposed to accelerate the convergence of the series involved in the calculation of the Green function in the source region. In 1921 P. P. Ewald developed an extremely efficient technique for calculating the electrostatic (electrodynamic) scalar potential in a three-dimensional periodic system of point charges [13, 14] which is still regarded as the state-of-art for this specific application.

His mathematical process is physically equivalent to first neutralising each point source by the superposition of a spherical gaussian distributions of opposite charge, centered on the original source. This aggregate of the sources is the forcing term of the Poisson (Helmhotz) equation, whose solution yields the space domain part of the potential. Subsequently, a second identical set of gaussian distribution with opposite sign, still centered on the point charges, is further superimposed to cancel out the effect of the first set. The potential due to these sources is obtained by solving the corresponding Poisson (Helmhotz) equation as a Fourier series. This yields the reciprocal space part of the potential. Both the real space and the reciprocal space contributions of the potential exhibit fast convergence, in particular when the observation point is in the source region.

Having previously reduced our original problem to a three-dimensional lattice of current sources in free space, it is now natural to extend the Ewald Method to the dyadic case.

4. IMAGES EXPANSION AND EWALD'S METHOD

In principle, the Green's dyadic function for a lattice of infinitesimal current sources could be calculated either by adding up the free space dyadic of all the sources of the lattice in the spatial domain or by using the modal theory yielding the Floquet's representation in the reciprocal domain. These two approaches are a Fourier-transform pair and each of them exhibits convergence problems inherently connected with the singularities of the transformed domain. Namely, the singularities in the spectral domain (i.e., the cavity's resonances) are reflected in the slowly convergent series of the spatial domain, while the singularity of the source region in the spatial domain slows down the convergence of the series in the reciprocal domain.

Here we aim to replace a slowly converging sum for the dyadic with two rapidly converging sums, one in the real space and the other in the reciprocal space with a hybrid spatial-spectral representation, as done by Ewald.

This is accomplished by adding and subtracting to the infinitesimal current sources of the lattice, the current distribution with the following density

$$\mathcal{I}\left(\mathbf{r},\mathbf{r}'\right)=\frac{\exp\left(\dfrac{\eta^2 k_0^2}{4}\right)}{\eta^3\pi^{\frac{3}{2}}}\exp\left(-\frac{|\mathbf{r}-\mathbf{r}'|^2}{\eta^2}\right) \tag{20}$$

where $\mathbf{r}'$ is the position of the generic lattice source and η is a positive parameter that adjusts the width of the gaussian distribution.

We then generate an equivalent scheme of solution:

$$\begin{cases}\nabla^2\overline{\mathbf{D}}_A^{SD}+k_0^2\overline{\mathbf{D}}_A^{SD}=-\overline{\mathbf{S}}_1\left(\mathbf{r},\mathbf{r}'\right)\\ \overline{\mathbf{D}}_E^{SD}=\left[\overline{\mathbf{I}}+\dfrac{\nabla\nabla}{k_0^2}\right]\overline{\mathbf{D}}_A^{SD}\end{cases} \tag{21}$$

$$\nabla\times\nabla\times\overline{\mathbf{D}}_E^{RS}\left(\mathbf{r},\mathbf{r}'\right)-k_0^2\overline{\mathbf{D}}_E^{RS}\left(\mathbf{r},\mathbf{r}'\right)=\overline{\mathbf{S}}_2\left(\mathbf{r},\mathbf{r}'\right) \tag{22}$$

in such a way that the global Green's function $\overline{\mathbf{G}}_E$ is given by

$$\overline{\mathbf{G}}_E\left(\mathbf{r},\mathbf{r}'\right)=\overline{\mathbf{D}}_E^{SD}\left(\mathbf{r},\mathbf{r}'\right)+\overline{\mathbf{D}}_E^{RS}\left(\mathbf{r},\mathbf{r}'\right). \tag{23}$$

In (21) and (22)

$$\begin{aligned}\overline{\mathbf{S}}_1&\left(\mathbf{r},\mathbf{r}'\right)\\ &=\hat{\mathbf{x}}\hat{\mathbf{x}}\sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty}\mathrm{SX}\left[\delta\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)-\mathcal{I}\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)\right]\\ &\quad+\hat{\mathbf{y}}\hat{\mathbf{y}}\sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty}\mathrm{SY}\left[\delta\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)-\mathcal{I}\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)\right]\\ &\quad+\hat{\mathbf{z}}\hat{\mathbf{z}}\sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty}\mathrm{SZ}\left[\delta\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)-\mathcal{I}\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)\right]\end{aligned} \tag{24}$$

$$\begin{aligned}\overline{\mathbf{S}}_2&\left(\mathbf{r},\mathbf{r}'\right)\\ &=\sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}\ \mathrm{SX}\ \mathcal{I}\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)+\right.\\ &\quad\left.+\hat{\mathbf{y}}\hat{\mathbf{y}}\ \mathrm{SY}\ \mathcal{I}\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)+\hat{\mathbf{z}}\hat{\mathbf{z}}\ \mathrm{SZ}\ \mathcal{I}\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)\right]\end{aligned} \tag{25}$$

where $X_r^{\pm}=x\pm x'+2\,ra$, $Y_s^{\pm}=y\pm y'+2\,bs$ and $Z_t^{\pm}=z\pm z'+2\,tc$ are shorthand notations that will be used throughout the remainder of the paper and identify the mutual position between the observation point and the generic source of the lattice. Furthermore, each triplet $\left(X_r^{\pm},Y_s^{\pm},Z_t^{\pm}\right)$ must be intended as the superposition of eight terms corresponding to all the possible combinations ($+++,++-,\cdots$ etc.).

In (24) and (25), SX, SY and SZ are defined as follows:

$$SX=\begin{cases}+1\ \text{for}\ \left(X_r^{\pm},Y_s^{-},Z_t^{-}\right)\ \text{or}\ \left(X_r^{\pm},Y_s^{+},Z_t^{+}\right)\\ -1\ \text{for}\ \left(X_r^{\pm},Y_s^{+},Z_t^{-}\right)\ \text{or}\ \left(X_r^{\pm},Y_s^{-},Z_t^{+}\right)\end{cases}$$

$$SY=\begin{cases}+1\ \text{for}\ \left(X_r^{-},Y_r^{\pm},Z_r^{-}\right)\ \text{or}\ \left(X_r^{+},Y_r^{\pm},Z_r^{+}\right)\\ -1\ \text{for}\ \left(X_r^{-},Y_r^{\pm},Z_r^{+}\right)\ \text{or}\ \left(X_r^{+},Y_r^{\pm},Z_r^{-}\right)\end{cases}$$

$$SZ=\begin{cases}+1\ \text{for}\ \left(X_r^{-},Y_s^{-},Z_t^{\pm}\right)\ \text{or}\ \left(X_r^{+},Y_s^{+},Z_t^{\pm}\right)\\ -1\ \text{for}\ \left(X_r^{-},Y_s^{+},Z_t^{\pm}\right)\ \text{or}\ \left(X_r^{+},Y_s^{-},Z_t^{\pm}\right)\end{cases}$$

The dyadic operators $\overline{\mathbf{D}}_E^{SD}$ and $\overline{\mathbf{D}}_E^{RS}$ in equations (21) and (22) represent the electromagnetic response in free space to a lattice of dyadic source expressed respectively by $\overline{\mathbf{S}}_1$ and $\overline{\mathbf{S}}_2$.

The parameter η determines the balance between the contributions of $\overline{\mathbf{D}}_E^{SD}$ and $\overline{\mathbf{D}}_E^{RS}$. A small η places the major computational effort on $\overline{\mathbf{D}}_E^{RS}$, while a large η moves the computational burden on $\overline{\mathbf{D}}_E^{SD}$. By construction, the final result will be necessarily independent of η.

4.1 Calculation of the Spatial Contribution

The equation (21) is solved in the spatial domain since the original impulsive currents, counteracted by the gaussian current distributions, give rise to a fast decaying field at long range.

Given the linearity and the space invariance of equation (21), the problem is then reduced to solving the equation:

$$\nabla^2\Psi\left(\mathbf{r},\mathbf{r}'\right)+k_0^2\Psi\left(\mathbf{r},\mathbf{r}'\right)=-\delta\left(\mathbf{r}-\mathbf{r}'\right)+\frac{e^{\frac{\eta^2k_0^2}{4}}}{\eta^3\pi^{\frac{3}{2}}}e^{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{\eta^2}} \tag{26}$$

which yields

$$\begin{aligned}\Psi\left(\mathbf{r},\mathbf{r}'\right)=&\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\cos\left(k_0|\mathbf{r}-\mathbf{r}'|\right)\\&-\frac{1}{8\pi|\mathbf{r}-\mathbf{r}'|}e^{+jk_0|\mathbf{r}-\mathbf{r}'|}\Phi\left(|\mathbf{r}-\mathbf{r}'|/\eta+jk_0\eta/2\right)\\&-\frac{1}{8\pi|\mathbf{r}-\mathbf{r}'|}e^{-jk_0|\mathbf{r}-\mathbf{r}'|}\Phi\left(|\mathbf{r}-\mathbf{r}'|/\eta-jk_0\eta/2\right)\end{aligned} \tag{27}$$

where Φ is the error function. The mathematical details of the solution are given in Appendix A.

We notice that the contributions of the furthermost sources (i.e., when $|\mathbf{r}-\mathbf{r}'|$ is large) can be approximated as:

$$\Psi\left(\mathbf{r},\mathbf{r}'\right)\simeq\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\cos\left(k_0|\mathbf{r}-\mathbf{r}'|\right)-\Phi\left(|\mathbf{r}-\mathbf{r}'|/\eta\right)\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\cos\left(k_0|\mathbf{r}-\mathbf{r}'|\right) \tag{28}$$

The fast convergence of the error function to 1, for sufficiently large arguments, corresponds to a fast convergence of Ψ to 0. Consequently, the series representation of the lattice contributions achieves it final value rapidly.

Finally the spatial domain component of the Green's dyadic function can be cast as:

$$\overline{\mathbf{D}}_E^{SD}(\mathbf{r},\mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k_0^2}\right) \cdot \sum_{r=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} \sum_{t=-\infty}^{+\infty} \Big[\hat{\mathbf{x}}\hat{\mathbf{x}}\ \mathrm{SX}\ \Psi_{rst}\Big(X_r^{\pm}, Y_s^{\pm}, Z_t^{\pm}\Big)$$
$$+ \hat{\mathbf{y}}\hat{\mathbf{y}}\ \mathrm{SY}\ \Psi_{rst}\Big(X_r^{\pm}, Y_s^{\pm}, Z_t^{\pm}\Big) + \hat{\mathbf{z}}\hat{\mathbf{z}}\ \mathrm{SZ}\ \Psi_{rst}\Big(X_r^{\pm}, Y_s^{\pm}, Z_t^{\pm}\Big)\Big] \quad (29a)$$

where

$$\Psi_{rst}\Big(X_r^{\pm}, Y_s^{\pm}, Z_t^{\pm}\Big)$$
$$= \frac{1}{8\pi\sqrt{\left(X_r^{\pm}\right)^2+\left(Y_s^{\pm}\right)^2+\left(Z_t^{\pm}\right)^2}} \left[2\cos\left(k_0\sqrt{\left(X_r^{\pm}\right)^2+\left(Y_s^{\pm}\right)^2+\left(Z_t^{\pm}\right)^2}\right)\right.$$
$$- e^{jk_0\sqrt{\left(X_r^{\pm}\right)^2+\left(Y_s^{\pm}\right)^2+\left(Z_t^{\pm}\right)^2}}\,\Phi\left(\frac{\sqrt{\left(X_r^{\pm}\right)^2+\left(Y_s^{\pm}\right)^2+\left(Z_t^{\pm}\right)^2}}{\eta}+j\frac{k_0}{2}\eta\right)$$
$$\left.- e^{-jk_0\sqrt{\left(X_r^{\pm}\right)^2+\left(Y_s^{\pm}\right)^2+\left(Z_t^{\pm}\right)^2}}\,\Phi\left(\frac{\sqrt{\left(X_r^{\pm}\right)^2+\left(Y_s^{\pm}\right)^2+\left(Z_t^{\pm}\right)^2}}{\eta}-j\frac{k_0}{2}\eta\right)\right] \quad (29b)$$

4.2 Spectral Domain Contribution

The reciprocal domain contribution, $\overline{\mathbf{D}}_E^{RS}$, can be computed by making use of the eigenfunctions representation of the dyadic Green's function (Section 2). The expression (22), where $\overline{\mathbf{S}}_2$ has arisen from the application of the image theorem in a stepwise fashion, can be recast as follows:

$$\begin{cases} \nabla \times \nabla \times \overline{\mathbf{D}}_E^{RS}(\mathbf{r},\mathbf{r}') - k_0^2 \overline{\mathbf{D}}_E^{RS}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{I}}\mathcal{I}(\mathbf{r}-\mathbf{r}') \\ \hat{\mathbf{n}} \times \overline{\mathbf{D}}_E^{RS}(\mathbf{r},\mathbf{r}') = 0 \quad \mathbf{r} \in S \end{cases} \quad (30)$$

where $\overline{\mathbf{I}}$ is the unit dyadic, $\mathbf{r}$ and $\mathbf{r}'$ are, respectively, the coordinates of the observation and of the gaussian current points and $\hat{\mathbf{n}}$ is the local normal to the surface S that bounds the volume of the cavity.

Now, by comparing (30) with (2), it follows that

$$\overline{\mathbf{D}}_E^{RS}(\mathbf{r},\mathbf{r}') = \iiint_{-\infty}^{+\infty} \overline{\mathbf{G}}_E(\mathbf{r},\xi)\,\mathcal{I}(\xi-\mathbf{r}')\,\mathrm{d}\xi \quad (31)$$

In (31), the integrand involves terms like those shown in (32) and corresponding to the first, second and third column of the dyadic respectively, that can be calculated in closed form:

$$\frac{e^{\frac{\eta^2 k_0^2}{4}}}{\eta^3 \pi^{\frac{3}{2}}} \iiint\limits_{-\infty}^{+\infty} \begin{bmatrix} \cos \\ \sin \\ \sin \end{bmatrix} \left(\frac{r\pi}{a}\xi_1\right) \begin{bmatrix} \sin \\ \cos \\ \sin \end{bmatrix} \left(\frac{s\pi}{b}\xi_2\right) \begin{bmatrix} \sin \\ \sin \\ \cos \end{bmatrix}$$

$$\cdot \left(\frac{t\pi}{c}\xi_3\right) e^{-\frac{\sqrt{(\xi_1-x')^2+(\xi_2-y')^2+(\xi_3-z')^2}}{\eta^2}} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$

$$= e^{-\frac{\eta^2}{4}\left[\left(\frac{r\pi}{a}\right)^2+\left(\frac{s\pi}{b}\right)^2+\left(\frac{t\pi}{c}\right)^2-k_0^2\right]} \begin{bmatrix} \cos \\ \sin \\ \sin \end{bmatrix} \left(\frac{r\pi}{a}x'\right) \begin{bmatrix} \sin \\ \cos \\ \sin \end{bmatrix} \left(\frac{s\pi}{b}y'\right) \begin{bmatrix} \sin \\ \sin \\ \cos \end{bmatrix} \left(\frac{t\pi}{c}z'\right) \quad (32)$$

Finally, the spectral domain contribution to the Green's function assumes the following expression:

$$\overline{\mathbf{D}}_E^{RS}(\mathbf{r}, \mathbf{r}') = \sum_{r=0}^{+\infty} \sum_{s=0}^{+\infty} \sum_{t=0}^{+\infty} \left(\frac{\epsilon_{0r}\epsilon_{0s}\epsilon_{0t}}{abc}\right)$$

$$\cdot \exp\left\{-\frac{\eta^2}{4}\left[\left(\frac{r\pi}{a}\right)^2 + \left(\frac{s\pi}{b}\right)^2 + \left(\frac{t\pi}{c}\right)^2 - k_0^2\right]\right\}$$

$$\cdot \left[\frac{\mathbf{E}_S^{TE}(\mathbf{r})\,\mathbf{E}_S^{TE}(\mathbf{r}')}{\left(k_{rst}^2 - k^2\right) k_{rs}^2} + \frac{\mathbf{E}_S^{TM}(\mathbf{r})\,\mathbf{E}_S^{TM}(\mathbf{r}')}{\left(k_{rst}^2 - k^2\right) k_{rst}^2 k_{rs}^2} \frac{\mathbf{E}_L(\mathbf{r})\,\mathbf{E}_L(\mathbf{r}')}{k_{rst}^2 k^2}\right] \quad (33)$$

where $\mathbf{E}_S^{TE}$, $\mathbf{E}_S^{TM}$ and $\mathbf{E}_L$ are reported in Section 2.

4.3 Demonstration of the Effectiveness of the Ewald's Approach

To validate the effectiveness of the dyadic Ewald approach, we consider again the case of a z-oriented dipole inside a cavity. This time, we assume that the dipole supports a piecewise sinusoidal (PWS) current as follows:

$$I(x, y, z, x_0, y_0, z_0) = \mathrm{I} \cdot \frac{\sin\left[k_0\left(\frac{l}{2} - |z - z_0|\right)\right]}{\sin\left(k_0 \frac{l}{2}\right)} \delta(x - x_0)$$

$$\cdot \delta(y - y_0)\,\mathrm{rect}\left(\frac{z - z_0}{l/2}\right) \quad (34)$$

Our choice of the linear PWS current distribution is dictated by its importance in practical problems. Interaction and radiation of electromagnetic fields in the presence of wire structures are widely treated with the "thin-wire approximation" (i.e., length to radius ratio greater than 10), which assumes a filament distribution of current along the wire axis. Furthermore, the PWS function allows us to solve the radiation integral in closed form whatever the complexity of kernel involved. In this light, the PWS distribution is regarded as the most convenient basis function for wire-oriented Method of Moments built on the electric type Green's function, rather than on the potential vector, inside rectangular enclosures.

For illustrative purposes only, we compute the z-component of the electric field radiated by the PWS current by invoking both the Green's function computed in Section 3 and the Green's function as expressed by (23) and performing a convolution between the $\hat{\mathbf{z}}\hat{\mathbf{z}}$ term of the corresponding dyadic and the PWS source. Applying the Green's function computed in Section 3 in its most advantageous form to a PWS current distribution we get:

$$\begin{aligned} E_z(x,y,z) = & -\frac{4j}{\varepsilon\omega}\frac{\mathrm{I}}{ac}\sum_{r=1}^{+\infty}\sum_{t=0}^{+\infty}\epsilon_{0t}\frac{k_0\left[\cos\left(\frac{t\pi}{c}l/2\right)-\cos\left(k_0 l/2\right)\right]}{k_{prt}} \\ & \cdot\sin\left(\frac{r\pi}{a}x\right)\sin\left(\frac{r\pi}{a}x_0\right)\cos\left(\frac{t\pi}{c}z\right)\cos\left(\frac{t\pi}{c}z_0\right)\frac{1}{\sin\left(k_0\frac{l}{2}\right)} \\ & \cdot\sinh\left(k_{prt}y_0\right)\exp\left(-k_{prt}y\right)\frac{1-\exp\left[-2k_{prt}\left(b-y\right)\right]}{1-\exp\left(2k_{prt}b\right)} \end{aligned} \tag{35}$$

Then, to compute the field throughout the Ewald's method, we invoke the general formula for the PWS current distribution [15]:

$$\begin{aligned} & \int_{z'-l/2}^{z'+l/2} I(\zeta)\left(1+\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)\mathcal{F}(R)\,\mathrm{d}\zeta \\ & = \frac{1}{k_0}\left[\mathcal{F}(R_+)+\mathcal{F}(R_-)-2\cos\left(k_0\frac{l}{2}\right)\mathcal{F}(R)\right] \end{aligned}$$

where $I(\zeta)$ is a PWS function, $\mathcal{F}$ is a $C^{(2)}$ function, $R_\pm = \sqrt{(x-x')^2+(y-y')^2+(z-z'\pm l/2)^2}$, and $R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$.

By identifying $\mathcal{F}$ with Ψ_{rst} in (29), also the spatial domain contribution can be cast in closed form as we did for the reciprocal space contribution. Summarising, the overall field (E_z^{EW}) takes the form:

$$E_z^{EW} = E_z^{SD} + E_z^{RS} \tag{36}$$

where

$$\begin{aligned} E_z^{SD}(x,y,z) = & -j\omega\mu \frac{\mathrm{I}}{k_0 \sin(k_0 l/2)} \sum_{r=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} \sum_{t=-\infty}^{+\infty} \\ & \cdot \mathrm{SZ} \left[\Psi_{rst} \left(X_r^{\pm}, Y_s^{\pm}, Z_t^{\pm} + \frac{l}{2} \right) \Psi_{rst} \left(X_r^{\pm}, Y_s^{\pm}, Z_t^{\pm} - \frac{l}{2} \right) \right. \\ & \left. -2\cos\left(\frac{k_0 l}{2}\right) \Psi_{rst} \left(X_r^{\pm}, Y_s^{\pm}, Z_t^{\pm} \right) \right] \end{aligned} \tag{36a}$$

and

$$\begin{aligned} & E_z^{RS}(x,y,z) \\ & = -j\omega\mu \frac{8\mathrm{I}}{k_0 \sin\left(k_0 \frac{1}{2}\right)} \sum_{r=1}^{+\infty} \sum_{s=1}^{+\infty} \sum_{t=0}^{+\infty} \frac{\epsilon_{0t}}{abc} \frac{\left[\cos(k_0 l/2) - \cos\left(\frac{t\pi}{c} l/2\right)\right]}{\left[\left(\frac{r\pi}{a}\right)^2 + \left(\frac{s\pi}{b}\right)^2 + \left(\frac{t\pi}{c}\right)^2 - k_0^2\right]} \\ & \quad \cdot \exp\left\{ -\frac{\eta^2}{4} \left[\left(\frac{r\pi}{a}\right)^2 + \left(\frac{s\pi}{b}\right)^2 + \left(\frac{t\pi}{c}\right)^2 - k_0^2 \right] \right\} \sin\left(\frac{r\pi}{a} x\right) \\ & \quad \cdot \sin\left(\frac{r\pi}{a} x_0\right) \sin\left(\frac{s\pi}{b} y\right) \sin\left(\frac{s\pi}{b} y_0\right) \cos\left(\frac{t\pi}{c} z\right) \cos\left(\frac{t\pi}{c} z_0\right) \end{aligned} \tag{36b}$$

We now compare the computational effort required by (35) and (36) in terms of the CPU time necessary to achieve a relative error less than 1.E-8. The physical dimension of the cavity and the length of the dipole are identical to the numerical case described in section 3. In Tables 4 and 5, the position (x_0, y_0, z_0) of the center of the z-oriented dipole coincides with the center of the metallic enclosure. The cases of $f = 155\,\mathrm{MHz}$ with $\eta = 1$ and $f = 316\,\mathrm{MHz}$ with $\eta = 1.5$ are considered, respectively. The observation point is variable in the y direction only and it lies on the straight line which results from the intersection of the planes $x = x_0,\ z = z_0$.

Table 4. Comparison between the classical Green's function and the Green's function derived throughout the Ewald's approach by means of the number of terms (N.) required to achieve convergence of the series involved and the relative CPU time (in sec). Parameters: $a = 3\,\mathrm{m}$, $b = 4\,\mathrm{m}$, $c = 5\,\mathrm{m}$, $l = 10\,\mathrm{cm}$ and $\mathrm{I} = 1\,\mathrm{mA}$. Variables: $f = 155\,\mathrm{MHz}$, $\eta = 1.5\,\mathrm{m}$, $(x_0, y_0, z_0) = (1.5, 2, 2.5)\,\mathrm{m}$ and $(x, y, z) = (1.5, y, 2.5)\,\mathrm{m}$.

	E_z^{RS}		E_z^{SD}		E_z^{EW}		E_z		
Δy	N.	E in mV/m	N.	E in mV/m	CPU	E in mV/m	N.	CPU	E in mV/m
0.3	576	216.5299539	27	-128.8420274	1.1E-2	87.6879265	13110	9.E-2	87.6879263
0.1	576	225.8282199	27	218.5355531	1.1E-2	444.363773	90300	0.65	444.363765
0.05	576	226.7183758	27	1968.457673	1.1E-2	2195.17604	289982	2.2	2195.17600
0.01	576	227.0039033	27	29288.21712	1.1E-2	29515.2210	6505050	48.5	29515.2214

Table 5. Comparison between the classical Green's function and the Green's function derived throughout the Ewald's approach by means of the number of terms (N.) required to achieve the convergence of the series involved and the relative CPU time (in sec). Parameters: $a = 3\,\mathrm{m}$, $b = 4\,\mathrm{m}$, $c = 5\,\mathrm{m}$, $l = 10\,\mathrm{cm}$ and $\mathrm{I} = 1\,\mathrm{mA}$. Variables: $f = 316\,\mathrm{MHz}$, $\eta = 1\,\mathrm{m}$, $(x_0, y_0, z_0) = (1.5, 2, 2.5)\,\mathrm{m}$ and $(x, y, z) = (1.5, y, 2.5)\,\mathrm{m}$.

	E_z^{RS}		E_z^{SD}		E_z^{EW}		E_z		
$\Delta y = \Delta z$	N.	E in mV/m	N.	E in mV/m	CPU	E in mV/m	N.	CPU	E in mV/m
0.3	2940	29823.4832369	27	-29796.9862372	1.8E-2	26.4969997	11342	7.5E-2	26.4969995
0.1	2940	32691.6121369	27	-32539.3857827	1.8E-2	152.226354	8556	0.62	152.226348
0.05	2940	32974.6499389	27	-32008.4222887	1.8E-2	966.227650	303050	2.24	966.227614
0.01	2940	33065.7531026	27	-19063.1534993	1.8E-2	14002.5996	7325142	54.5	14002.5996

In Tables 6 and 7, the position (x_0, y_0, z_0) of the center of the z-oriented dipole is located at (1.5,0.1,0.1) to enhance the effect of the nearest images. Again, $f = 155\,\mathrm{MHz}$ with $\eta = 1$ and $f = 316\,\mathrm{MHz}$ with $\eta = 1.5$ are considered, respectively. The observation point lies on the plane $x = x_0$ and moves along the straight line of equation $y = z - l/2$.

We remark that when the observation point approaches the source region, the relative CPU time required by the classical Green's function severely increases, whereas it remains constant for the case of the Ewald's approach. The computational effort necessary to achieve numerical convergence with (35) exceeds that required by (36), ranging

Table 6. Comparison between the classical Green's function and the Green's function derived throughout the Ewald's approach by means of the number of terms (N.) required to achieve convergence of the series involved and the relative CPU time (in sec). Parameters: $a = 3\,\mathrm{m}$, $b = 4\,\mathrm{m}$, $c = 5\,\mathrm{m}$, $l = 10\,\mathrm{cm}$ and $\mathrm{I} = 1\,\mathrm{mA}$. Variables: $f = 155\,\mathrm{MHz}$, $\eta = 1.5\,\mathrm{m}$, $(x_0, y_0, z_0) = (1.5, 0.1, 0.1)\,\mathrm{m}$ and $(x, y, z) = (1.5, y, z)\,\mathrm{m}$.

	E_z^{RS}		E_z^{SD}		E_z^{EW}		E_z		
Δy	N.	E in mV/m	N.	E in mV/m	CPU	E in mV/m	N.	CPU	E in mV/m
0.3	576	21.3813221	27	-35.3132958	1.1E-2	-13.9319737	12210	8.6E-2	-139319739
0.1	576	9.34607523	27	-567.964411	1.1E-2	-558.618335	281430	2	-558.618302
0.05	576	6.94513530	27	-8866.47472	1.1E-2	-8859.52958	6252500	45	-8859.52989
0.01	576	6.63894640	27	-21475.1326	1.1E-2	-21468.4936	24014900	169.8	-21468.4948

Table 7. Comparison between the classical Green's function and the Green's function derived throughout the Ewald's approach by means of the number of terms (N.) required to achieve convergence of the series involved and the relative CPU time (in sec). Parameters: $a = 3\,\mathrm{m}$, $b = 4\,\mathrm{m}$, $c = 5\,\mathrm{m}$, $l = 10\,\mathrm{cm}$ and $\mathrm{I} = 1\,\mathrm{mA}$. Variables: $f = 316\,\mathrm{MHz}$, $\eta = 1\,\mathrm{m}$, $(x_0, y_0, z_0) = (1.5, 0.1, 0.1)\,\mathrm{m}$ and $(x, y, z) = (1.5, y, z)\,\mathrm{m}$.

	E_z^{RS}		E_z^{SD}		E_z^{EW}		E_z		
$\Delta y = \Delta z$	N.	E in mV/m	N.	E in mV/m	CPU	E in mV/m	N.	CPU	E in mV/m
0.3	2940	8086.0375890	27	-8035.8353636	1.8E-2	50.2022254	8190	5.76E-2	50.2022246
0.1	2940	4217.6435698	27	-4532.6287487	1.8E-2	-314.985178	260640	1.88	-314.985153
0.05	2940	3178.2341554	27	-7695.1454892	1.8E-2	-4516.91133	5762400	42.51	-4516.91161
0.01	2940	3042.5838980	27	-13846.779104	1.8E-2	-10804.1952	23044800	163	-10804.1958

from one to four orders of magnitude, depending on the mutual distance between source and observation points. Furthermore, even more advantageous is the Ewald's technique when the observation point is in the source region. However, for the treatment of this singular case, extreme caution must be exercised. We recall that the starting point of the Ewald's technique is the application of the image theory that gives us the ability of isolating the singular part of the dyadics that can be treated apart analytically.

5. CONCLUSION

This paper has reviewed the groundwork to solve for electromagnetic interactions inside metallic enclosures using an integral equation formulation. The useful application of this method prescribes to have at hand expressions of the dyadic Green's function that are numerically efficient, which has been the main purpose of the study. After a brief summary of the classical methods to derive the dyadic Green's function for rectangular cavities, we have proposed a complementary formulation based on both the image theory and the plane wave expansion of the electromagnetic field. This method enhances the flexibility to select directly the most advantageous numerical representation of the Green's function in the light of the mutual position of the source and the observation point. The results are consistent with those obtained with classical methods. When the observation point is very close to the source, the series composing the dyadic Green's function exhibits inherently slow convergence. The above situation is unavoidable in the Method of Moments, especially where the calculation of the self-admittance elements is concerned. We showed this problem could be overcome by extending the method of Ewald, which accelerates the series of the scalar electrodynamic potential for a lattice of infinitesimal source points, to the dyadic case. Basic examples of calculations have been reported in tables to prove the efficiency and the reliability of the notions described in the paper. This work proposes new perspectives for the application of the Moment Method to the scattering of electromagnetic structures inside rectangular enclosures. In particular, we anticipate that the Method of Moments will surely be competitive for steady-state electromagnetic interaction of wire structures inside rectangular cavities with respect to the Finite Difference Method, which is currently the dominant one in closed domains.

APPENDIX A

In this appendix we describe the process to derive (27) from (26).

We recall the Fourier transform pair

$$\widetilde{\Psi}(\mathbf{k}) = \int\int\int_{\mathbf{r}} \Psi\left(\mathbf{r}\right) e^{+j\mathbf{k}\cdot\mathbf{r}} \mathrm{d}\mathbf{r}$$
$$\Psi(\mathbf{r}) = \frac{1}{8\pi^3} \int\int\int_{\mathbf{k}} \widetilde{\Psi}\left(\mathbf{k}\right) e^{-j\mathbf{k}\cdot\mathbf{r}} \mathrm{d}\mathbf{k}$$

where $\Psi(\mathbf{r})$ is a function in the tri-dimensional space domain and $\widetilde{\Psi}(\mathbf{k})$ is its Fourier transform. The spectral transform of the function

$$\mathcal{I}(\mathbf{r}) = \frac{\exp\left(\dfrac{\eta^2 k_0^2}{4}\right)}{\eta^3 \pi^{\frac{3}{2}}} \exp\left(-\frac{|\mathbf{r}|^2}{\eta^2}\right) \tag{A1}$$

is calculated as follows:

$$\begin{aligned}
\mathcal{I}(\mathbf{k}) &= \frac{e^{\frac{\eta^2 k_0^2}{4}}}{\eta^3 \pi^{\frac{3}{2}}} \iiint_{-\infty}^{+\infty} e^{-(x^2+y^2+z^2)/\eta^2} e^{j(k_x x + k_y y + k_z z)} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{e^{\frac{\eta^2 k_0^2}{4}}}{\eta^3 \pi^{\frac{3}{2}}} \int_0^{+\infty} \int_0^{\pi} \int_0^{2\pi} e^{-r^2/\eta^2} e^{jkr\cos\theta} r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\
&= \frac{2e^{\frac{\eta^2 k_0^2}{4}}}{\eta^3 \sqrt{\pi}} \int_0^{+\infty} r^2 e^{-r^2/\eta^2} \int_0^{\pi} e^{jkr\cos\theta} \sin\theta \mathrm{d}\theta\mathrm{d}r \\
&= \frac{2e^{\frac{\eta^2 k_0^2}{4}}}{\eta^3 \sqrt{\pi}} \int_0^{+\infty} r e^{-r^2/\eta^2} \sin(kr)\, \mathrm{d}r = e^{\frac{\eta^2\left(k_0^2 - k^2\right)}{4}}
\end{aligned} \tag{A2}$$

Now, we transform in the spectral domain equation (26) obtaining:

$$\left(-k^2 + k_0^2\right) \widetilde{\Psi}(\mathbf{k}) = -e^{j\mathbf{k}\cdot\mathbf{r}'} + e^{\frac{\eta^2\left(k_0^2 - k^2\right)}{4}} e^{j\mathbf{k}\cdot\mathbf{r}'} \tag{A3}$$

Thus

$$\widetilde{\Psi}(\mathbf{k}) = \frac{e^{j\mathbf{k}\cdot\mathbf{r}'}}{k^2 - k_0^2} - \frac{e^{\frac{-\eta^2\left(k_0^2 - k^2\right)}{4}}}{k^2 - k_0^2} e^{j\mathbf{k}\cdot\mathbf{r}'} \tag{A4}$$

In the expression above we identify

$$\widetilde{\Psi}_A(\mathbf{k}) = \frac{e^{j\mathbf{k}\cdot\mathbf{r}'}}{k^2 - k_0^2} \qquad \text{(delta source)} \tag{A5}$$

$$\widetilde{\Psi}_B(\mathbf{k}) = \frac{e^{-\eta^2\left(k^2 - k_0^2\right)\frac{\eta^2}{4}}}{k^2 - k_0^2} e^{j\mathbf{k}\cdot\mathbf{r}'} \qquad \text{(Gaussian source)} \tag{A6}$$

By anti-transforming we obtain:

$$\Psi_A\left(\mathbf{r}, \mathbf{r}'\right) = \frac{1}{8\pi^3} \int_0^{+\infty} \int_0^{\pi} \int_0^{2\pi} \frac{k^2}{k^2 - k_0^2} e^{-jk|\mathbf{r}-\mathbf{r}'|\cos\beta} \sin\beta \mathrm{d}k\mathrm{d}\beta\mathrm{d}\alpha$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2}\int_0^{+\infty}\int_0^{\pi}\frac{k^2}{k^2-k_0^2}e^{-jk|\mathbf{r}-\mathbf{r}'|\cos\beta}\sin\beta \mathrm{d}k\mathrm{d}\beta \\
&= \frac{1}{2\pi^2|\mathbf{r}-\mathbf{r}'|}\int_0^{+\infty}\frac{k}{k^2-k_0^2}\sin k|\mathbf{r}-\mathbf{r}'|\mathrm{d}\mathbf{k} \\
&= \frac{-j}{4\pi^2|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{+\infty}\frac{k}{k^2-k_0^2}e^{jk|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k \\
&= \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}
\end{aligned}
\tag{A7}
$$

where the last integral has been closed using the classical residue theory.

$$
\begin{aligned}
\Psi_B\left(\mathbf{r},\mathbf{r}'\right) &= \frac{1}{8\pi^3}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\frac{k^2}{k^2-k_0^2}e^{\frac{-\eta^2\left(k^2-k_0^2\right)}{4}} \\
&\quad\cdot e^{-jk|\mathbf{r}-\mathbf{r}'|\cos\beta}\sin\beta\mathrm{d}k\mathrm{d}\beta\mathrm{d}\alpha \\
&= \frac{-j}{4\pi^2|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{+\infty}\frac{k}{k^2-k_0^2}e^{\frac{-\eta^2\left(k^2-k_0^2\right)}{4}}e^{jk|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k
\end{aligned}
\tag{A8}
$$

Substituting the relation

$$
\frac{e^{-\frac{\eta^2}{4}\left(k^2-k_0^2\right)}}{k^2-k_0^2} = -\int_0^{\eta}\frac{\xi}{2}e^{-\frac{\xi^2}{4}\left(k^2-k_0^2\right)}\mathrm{d}\xi + \frac{1}{k^2-k_0^2}
\tag{A9}
$$

in (A8), $\Psi_B(\mathbf{r},\mathbf{r}')$ can be cast as:

$$
\Psi_B\left(\mathbf{r},\mathbf{r}'\right) = \underbrace{\frac{j}{4\pi^2|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{+\infty}\int_0^{\eta}\frac{k\xi}{2}e^{-\frac{\xi^2}{4}\left(k^2-k_0^2\right)}e^{jk|\mathbf{r}-\mathbf{r}'|}\mathrm{d}k\mathrm{d}\xi}_{I} + \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}
\tag{A10}
$$

Since the integrand of (A10) is continuos and bounded in the integration domain, the conditions to swap the order of integration are met. I is integrable in closed form with respect to k yielding:

$$
I = -\frac{1}{2\pi^{3/2}}\int_0^{\eta}e^{\left(\frac{\xi^2k_0^2}{4}-\frac{|\mathbf{r}-\mathbf{r}'|^2}{\xi^2}\right)}\frac{\mathrm{d}\xi}{\xi^2}
\tag{A11}
$$

Substituting $\mu = \dfrac{1}{\xi}$ we get

$$
I = -\frac{1}{4\pi}\cdot\frac{2}{\sqrt{\pi}}\int_{1/\eta}^{+\infty}e^{\frac{k_0^2}{4\mu^2}-|\mathbf{r}-\mathbf{r}'|^2\mu^2}\mathrm{d}\mu
$$

$$= -\frac{e^{jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi}\cdot\frac{2}{\sqrt{\pi}}\int_{1/\eta}^{+\infty} e^{-\left(\frac{jk_0}{2\mu}+|\mathbf{r}-\mathbf{r}'|\mu\right)^2}\mathrm{d}\mu \qquad \text{(A12)}$$

We make the substitution $\tau = |\mathbf{r}-\mathbf{r}'|\mu + jk_0/\,2\,\mu$; $\mathrm{d}\mu = \dfrac{1}{2|\mathbf{r}-\mathbf{r}'|}\left(1\pm\dfrac{\tau}{\sqrt{\tau^2-2jk_0|\mathbf{r}-\mathbf{r}'|}}\right)\mathrm{d}\tau$ so that I reduces to

$$I = -\frac{1}{4\pi}\frac{e^{jk_0|\mathbf{r}-\mathbf{r}'|}}{2|\mathbf{r}-\mathbf{r}'|}\cdot\frac{2}{\sqrt{\pi}}\int_{|\mathbf{r}-\mathbf{r}'|/\eta+jk_0\eta/2}^{+\infty} e^{-\tau^2}\mathrm{d}\tau + \cdot\underbrace{\pm\frac{1}{4\pi}\frac{e^{jk_0|\mathbf{r}-\mathbf{r}'|}}{2|\mathbf{r}-\mathbf{r}'|}\cdot\frac{2}{\sqrt{\pi}}\int_{|\mathbf{r}-\mathbf{r}'|/\eta+jk_0\eta/2}^{+\infty}\frac{e^{-\tau^2}\tau}{\sqrt{\tau^2-2jk_0|\mathbf{r}-\mathbf{r}'|}}\mathrm{d}\tau}_{I_1} \qquad \text{(A13)}$$

By definition $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_z^{+\infty} e^{-t^2}\mathrm{d}t = 1-\Phi(z)$, being Φ the error function. We have:

$$\frac{2}{\sqrt{\pi}}\int_{|\mathbf{r}-\mathbf{r}'|/\eta+jk_0\eta/2}^{+\infty} e^{-\tau^2}d\tau = 1-\Phi\Big(|\mathbf{r}-\mathbf{r}'|/\eta+jk_0\eta/2\Big)\,. \qquad \text{(A14)}$$

To calculate the integral I_1 we make the additional substitution $\tau^2 - 2jk_0|\mathbf{r}-\mathbf{r}'| = \Omega^2$; $\tau\mathrm{d}\tau = \Omega\mathrm{d}\Omega$ that yields

$$I_1 = \pm\frac{1}{4\pi}\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{2|\mathbf{r}-\mathbf{r}'|}\cdot\frac{2}{\sqrt{\pi}}\int_\alpha^\beta e^{-\Omega^2}\mathrm{d}\Omega \qquad \text{(A15)}$$

where (α,β) can be either $(-\infty, -|\mathbf{r}-\mathbf{r}'|/\eta + jk_0\eta/2)$ or $(|\mathbf{r}-\mathbf{r}'|/\eta - jk_0\eta/2, +\infty)$ and the integral will give the same result, $1-\Phi(|\mathbf{r}-\mathbf{r}'|/\eta - jk_0\eta/2)$. Finally I takes the expression

$$\begin{aligned} I = &-\frac{1}{4\pi}\frac{e^{+jk_0|\mathbf{r}-\mathbf{r}'|}}{2|\mathbf{r}-\mathbf{r}'|}\left[1-\Phi\big(|\mathbf{r}-\mathbf{r}'|/\eta+jk_0\eta/2\big)\right] \\ &\pm\frac{1}{4\pi}\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{2|\mathbf{r}-\mathbf{r}'|}\left[1-\Phi\big(|\mathbf{r}-\mathbf{r}'|/\eta-jk_0\eta/2\big)\right] \end{aligned} \qquad \text{(A16)}$$

In order to solve for the sign ambiguity, we note that for $k_0 \to 0$ the sign "+" leads to the trivial solution $(I = 0)$. The sign "−" produces the solution $-\dfrac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\left[1-\Phi\big(|\mathbf{r}-\mathbf{r}'|/\eta\big)\right]$ which is consistent with the result of the (A11) when $k_0 \to 0$. Finally the solution Ψ can be cast as follows

$$\begin{aligned}\Psi(\mathbf{r}) = \Psi_A(\mathbf{r}) - \Psi_B(\mathbf{r}) = &-\frac{1}{8\pi|\mathbf{r}-\mathbf{r}'|}e^{+jk_0|\mathbf{r}-\mathbf{r}'|}\Phi\big(|\mathbf{r}-\mathbf{r}'|/\eta + jk_0\eta/2\big)\\ &-\frac{1}{8\pi|\mathbf{r}-\mathbf{r}'|}e^{-jk_0|\mathbf{r}-\mathbf{r}'|}\Phi\big(|\mathbf{r}-\mathbf{r}'|/\eta - jk_0\eta/2\big)\\ &+\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\cos\big(k_0|\mathbf{r}-\mathbf{r}'|\big) \qquad \text{(A17)}\end{aligned}$$

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