Progress In Electromagnetics Research, PIER 32, 45–64, 2001

'GENERALIZED FINITE DIFFERENCES' IN COMPUTATIONAL ELECTROMAGNETICS

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Abstract—The geometrical approach to Maxwell's equations promotes a way to discretize them that can be dubbed "Generalized Finite Differences", which has been realized independently in several computing codes. The main features of this method are the use of two grids in duality, the "metric-free" formulation of the main equations (Ampère and Faraday), and the concentration of metric information in the discrete representation of the Hodge operator. The specific role that finite elements have to play in such an approach is emphasized, and a rationale for Whitney forms is proposed, showing why they are the finite elements of choice.

1. INTRODUCTION

Finite elements, brought to the attention of electrical engineers in the early Seventies [15], soon superseded finite differences in low frequency

applications, thanks to their ability to model complex shapes. Still, in wave propagation problems, Yee's classical FDTD method [22, 16] continues to reign. This and other facts like the efficiency of the MAFIA group of codes [19], Tonti's "elementless" demonstration software [18], the interest for "lattice oriented" methods [17], seem to suggest that finite elements, and the variational method, may not be that dominant a paradigm after all. Are finite differences back?

We purport to show that, when seeing things from a geometrical point of view, there is no such sharp contrast. Finite elements don't hold the foreground, but they are nonetheless essential.

The geometrical approach to computational electromagnetism can be traced back to Kottler, Cartan, and van Dantzig [6], who independently pointed out, Post [14] tells us, the metric-free character of the two main Maxwell equations, $\partial_t B + \mathrm{rot}\, E = 0$ (Faraday's law) and $-\partial_t D + \mathrm{rot}\, H = J$ (Ampère's theorem). These make sense without any metric structure in 3D space, whereas metric is needed to account for constitutive laws such as $B = \mu H$ and $D = \epsilon E$. (This is just as true of Ohm's law $J = \sigma E$, but we shall ignore it for simplicity, and assume all currents given.) The Hodge operator, by which they can be expressed, depends on the metric structure. More, it *brings in*, one may argue [5, 4], the metric structure.

This fact explains the close similarity of independently proposed schemes (see e.g., [9, 18, 19]) as regards the discretization of Faraday and Ampère: These "grid equations", to use Weiland's expression [20], are forced on us, so to speak, as soon as a discretization mesh is there. What makes the variety of methods and distinguishes numerical schemes is, mainly, their treatment of the Hodge operator. Each of them is characterized by a specific "discrete Hodge". Finding under which conditions the latter will produce a stable and convergent scheme appears, therefore, as the central problem of the theory, the one on which error analysis should focus.

Which is where finite elements enter the stage: not as a prerequisite, as in the Galerkin approach, but as a theoretical tool for what is, in this perspective, a task for professionals—error analysis. The whole approach takes on a different flavor, which "generalized finite differences" (GFD) seems to describe well. Moreover, from this vantage point, the Galerkin approach, as well as finite volume methods, do not appear so different from the Darmstadt school's Finite Integration Technique: the Galerkin method, as realized by WETD [10] in wave problems, is a case of GFD, with a particular dual mesh (the "barycentric" one) and a particular discrete Hodge.

I'd wish to show that, even at this late stage, geometry illuminates the way. Appropriate finite elements for differential forms (the

Whitney complex) can be *derived*, by geometric reasoning. They answer requirements about interpolation that almost dictate their specific form. I will point in particular to a unifying property of Whitney forms, that they constitute, in a precise sense, a *partition of unity*.

This makes Section 8 of the present paper. Before that, one recalls the differential-geometric way to state Maxwell's equations (Section 2) and to discretize them (Section 3), and convergence is examined in statics (Section 4) and transients (Section 6), within a formalism that can accommodate different discrete Hodge operators. A digression (Section 5) shows how so-called "mixed" methods fit in this picture. Section 7 introduces Whitney forms, with a new twist: they come in answer to the natural question "how to associate a p-chain with a p-manifold".

2. DIFFERENTIAL FORMS, AND THE EQUATIONS

Let's briefly recall why differential forms, not vector fields, are the right geometrical objects in electromagnetic theory. The main idea is that a *physical* entity, as identified via empirical phenomena, should be represented by the simplest possible mathematical entity. When the electric field, say, is represented by a vector field E, the electromotive force (emf, expressed in volts) along an oriented curve c is $\int_c \tau \cdot E$, where τ denotes the unit tangent vector. (Orienting c, here done by having chosen one of the two possible τs, is essential.) The physical field is entirely characterized (from the point of view of what can be known about it, what can be measured) by its emf's, which we denote by $\int_c e$, for all c's. In the equality $\int_c e = \int_c \tau \cdot E$, E must change if the dot product changes, in order to preserve the emf at the left-hand side. Therefore what is mathematically cogent is not E (called here a "proxy" field for this reason), but the *map* from curves to emf's, $c \to \int_c e \equiv \int_c \tau \cdot E$. Such a map (additive with respect to c and continuous with respect to deformations of c) is called a *differential form of degree* 1, or 1-*form*, and we reserve the notation e for it. One-forms add, $e + e'$ being the map $c \to \int_c e + \int_c e'$, and have scalar multiples, λe being $c \to \lambda \int_c e$.

There are p-forms for all values of p from 0 to 3. In particular, a mapping from oriented surfaces to real numbers, with similar properties as above, is a 2-form. The magnetic flux b is one, which assigns to an oriented surface S the flux $\int_S b$. Three-forms map oriented volumes to real numbers, and 0-forms map points to numbers.

Orientation, in each of these cases, is an attribute of the manifold on which integration is performed, not of ambient 3D space. We say

the manifold has *inner* orientation to stress that. (The orientation of a point is just a sign, + or −.) To each p-manifold its own orientation. In the case of a manifold and its boundary, however, orienting one of them canonically orients the other one, because "from inside to outside", relative to the manifold, is an intrinsic notion. For instance, inner orientation of the rim ∂S of a surface S derives from an orientation of the latter. We say that independently assigned orientations "match" when they conform to this canonical choice.

Faraday's experiment can at this stage be accounted for by

$$\partial_t \int_S b + \int_{\partial S} e = 0 \tag{1}$$

for all inner-oriented surfaces S, provided the orientation of (each piece of) ∂S matches that of S.

Note that we refrained to confuse inner orientation of S with a "crossing direction" through S: the two notions can indeed be linked, via orientation of ambient space, but the latter is physically irrelevant. Giving a crossing direction, however, is right if the concept one wants to capture is that of *flow* of something, electric charge for instance, *through* the surface. A surface Σ thus equipped is said to be *outer* oriented. Additive continuous maps from outer-oriented surfaces to reals are called *twisted* 2-forms. Current-density j is such a map, and the number $\int_\Sigma j$ it assigns to Σ is the intensity through Σ (expressed in ampères).

To properly define this kind of orientation, one begins with vector subspaces: Outer orientation of one is, by definition, inner orientation of any of its complements. The notion immediately extends to manifolds. In particular, an outer oriented line γ is one equipped with a sense of "which way to turn around it", i.e., an inner orientation of its transverse planes. Additive continuous maps from such lines to reals are called twisted 1-forms. They are the right objects to represent the magnetic field strength, here denoted h. The number $\int_\gamma h$ is called "the magnetomotive force (mmf) over γ", expressed in ampères. As for the displacement current d, it's a twisted 2-form like j, and Ampère's theorem is the statement

$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j, \tag{2}$$

for all outer-oriented surfaces Σ, provided the outer orientation of (each piece of) $\partial\Sigma$ matches that of Σ.

Our discretization strategy will consist in enforcing (1) and (2), which in theory must hold for all surfaces, inner- or outer-oriented as the case may be, for a selected family of such surfaces, those generated

by the faces of two interlocked meshes. But first, we need to deal with constitutive laws. It's now (but only now) that a metric is necessary.

Consider a small plane patch S and a small straight segment γ, at right angle at their intersection x. Assume an inner orientation of S (any of the two possible ones) and take it as outer orientation of γ. Now let a twisted 1-form h be given. Thus knowing the value of $\int_\gamma h$, we set

$$\int_S b = \mu_x \frac{\text{area}(S)}{\text{length}(\gamma)} \int_\gamma h, \tag{3}$$

where μ_x is the permeability at point x, if well-defined there. (Discontinuities of μ are a minor difficulty, to be resolved later, cf. (10) below.) This extends to any surface S (chop it into small patches, form Riemann sums, and go to the limit) and thus defines a straight 2-form b. Hence a (linear) map from twisted 1-forms to straight 2-forms, which we call the μ-related *Hodge operator,* and shall simply denote by μ, since there is no ambiguity about the status of μ as an operator, not a scalar, in such an expression as $b = \mu h$. (We'll use ν for the inverse of μ, such that $h = \nu b$.) There is also an ϵ-related Hodge operator, with similar definition.

Let's emphasize the importance of metric elements in (3), length, area, and especially, orthogonality: In terms of the vector proxies B and H, (3) would translate as $n \cdot B = \mu\, \tau \cdot H$, where n and τ are a normal and a tangent unit vector respectively, i.e., $n \cdot H = \tau \cdot H$ whatever H, i.e., $n = \tau$, which does require orthogonality between γ and S.

3. DISCRETIZATION

Now let's address the problem of waves emitted from an antenna and reflecting off the perfectly conductive walls of a bounded cavity D. This means solving (1) and (2), together with the constitutive laws

$$d = \epsilon\, e, \quad h = \nu b, \tag{4}$$

for a given current density j inside D, null up to $t = 0$, plus

$$\int_c e = 0 \ \text{ for all curves } c \text{ lying in } \partial D, \tag{5}$$

which expresses the boundary condition. Let's show how the foregoing considerations lead towards a natural extension of Yee's scheme.

Suppose a pair of meshes, both made of adjacent cells, with smooth, but not necessarily plane, common faces. (We'll say "p-cells", with $p = 0$ to 3, and refer to them as nodes, edges, faces, and volumes.

Note that edges can be curved, or crooked, and that 2-cells are not necessarily triangles or squares.) Arrange them so that p-cells of one (the "primal" mesh) are in 1–1 correspondence with $(n-p)$-cells of the other (called the "dual" mesh). As a rule, this is done by having edges of one traverse faces of the other. Provide each primal p-cell with its own (arbitrary) inner orientation, which automatically outer-orients the corresponding dual $(n-p)$-cell. (The 2-cell dual to e is $\tilde{e}$, the 1-cell dual to f is $\tilde{f}$.) Call $\mathbf{R}^e_f$ the incidence number of primal edge e to face f, defined as 0 if the boundary ∂f does not contain e, and ± 1 if it does, the sign depending on whether orientations match or not. Hence a rectangular incidence matrix $\mathbf{R}$, entries of which are ± 1 or 0.

Next, assign real time-dependent degrees of freedom (DoF) to edges and faces: $\mathbf{b}_f$ and $\mathbf{e}_e$ are meant to approximate the magnetic flux $\int_f b$ and the emf $\int_e e$, while $\mathbf{h}_f$ and $\mathbf{d}_e$, on the dual side, stand for dual-edge mmf's and dual-face electric fluxes (approximations of $\int_{\tilde{f}} h$ and $\int_{\tilde{e}} d$). Call **e, b, h, d**, as above, the arrays of such DoF's, indexed over sets of cells of the same kind. Similarly, $\mathbf{j}$ gathers the intensities $\mathbf{j}_e$ through dual faces $\tilde{e}$ (which are not unknown, but data, in the present restricted context).

Now let's apply the previously announced strategy: enforce (1) for all surfaces S made of an assembly of primal faces, which by additivity of the integrals means

$$\partial_t \int_f b + \int_{\partial f} e = 0 \ \text{ for each primal face } f.$$

By the very definition of $\mathbf{R}$, this results in $\partial_t \mathbf{b}_f + \sum_e \mathbf{R}^e_f \mathbf{e}_e = 0$ for each f. Hence, putting that in matrix form, and by similar considerations applied to surfaces Σ made of *dual* faces,

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0, \quad -\partial_t \mathbf{d} + \mathbf{R}^t \mathbf{h} = \mathbf{j}, \tag{6}$$

where $\mathbf{R}^t$ is the transpose of $\mathbf{R}$.

These "network equations" reflect the topology of the interlocking networks, but they don't use any metric information. Neither do they require any notion of finite element.

By duality, components of $\mathbf{b}$ and $\mathbf{h}$, or $\mathbf{d}$ and $\mathbf{e}$, sit at the same place. Therefore, a natural way to discretize the constitutive laws is to set up *square* symmetric matrices $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ (boldface) and to enforce

$$\mathbf{d} = \boldsymbol{\epsilon}\,\mathbf{e}, \ \ \mathbf{h} = \boldsymbol{\nu}\mathbf{b}. \tag{7}$$

Then the "leapfrog scheme"

$$-\boldsymbol{\epsilon}(\mathbf{e}^{k+1/2} - \mathbf{e}^{k-1/2})/\delta t + \mathbf{R}^t \boldsymbol{\nu} \mathbf{b}^k = \mathbf{j}^k, \ \ (\mathbf{b}^{k+1} - \mathbf{b}^k)/\delta t + \mathbf{R}\mathbf{e}^{k+1/2} = 0 \tag{8}$$

(with $\mathbf{b}_0 = 0$ and $\mathbf{e}^{-1/2} = 0$ at the beginning) solves the approximation problem. How should $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ be constructed? *Sparsity*, which reflets the local character of laws (4), is an obvious requirement, but beyond this, *diagonality* of $\boldsymbol{\epsilon}$ is desirable, because it makes the scheme *explicit*. *Positive-definiteness* of $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ is required for stability. With all that, convergence of (7), (8) towards (6), (7) when the time-step δt tends to 0 is an easy matter. (The hard part is, of course, convergence of (6), (7) to (1), (2), (4). We shall face this task in Sections 4 and 6.)

There are essentially two ways[1] to construct these matrices. A well-known one is the Galerkin method, that we shall revisit below (Section 8), which takes for $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ the so-called "mass matrices" of finite element theory. With this choice, (7) and (8) are the "WETD" scheme [10]. But since $\boldsymbol{\epsilon}$ here, though relatively sparse, is not diagonal, the explicit character of Yee's method is lost.

Hence the interest for the second way [19, 18], in which one designs the meshes to be mutually orthogonal, a concept[2] which goes back to Maxwell [11], meaning that all cells are straight and that a primal cell and its dual mate intersect at right angle. Then, in the case when ϵ and μ are uniform, one sets $\boldsymbol{\epsilon}^{ee'} = 0$ if $e \neq e'$, $\boldsymbol{\mu}^{ff'} = 0$ if $f \neq f'$, and (cf. (3))

$$\boldsymbol{\epsilon}^{ee} = \epsilon \, \frac{\text{area}(\tilde{e})}{\text{length}(e)}, \quad \boldsymbol{\mu}^{ff} = \mu \, \frac{\text{area}(f)}{\text{length}(\tilde{f})}, \tag{9}$$

which does provide diagonal matrices $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$. In the case of non-uniform coefficients, formulas such as

$$\boldsymbol{\mu}^{ff} = \frac{\mu_1 \mu_2 \, \text{area}(f)}{\mu_1 \text{length}(\tilde{f}_1) + \mu_2 \text{length}(\tilde{f}_2)}, \tag{10}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the parts of $\tilde{f}$ belonging to the two volumes adjacent to f, apply instead. Of course, $\boldsymbol{\nu}$ is the inverse of $\boldsymbol{\mu}$.

We note that metric is needed in both procedures, and hence appears in the "network constitutive laws" (7).

Such rules as (9) or (10), which closely mimic the very definition of the Hodge operator, are so natural that one might feel like claiming discretization without shape functions! Indeed, (7), (8) with (9) or (10) is the closest thing to the Yee scheme for staggered cellular grids, and we just derived it without using any shape function.

Yet, finite elements are needed. As we just saw, the construction of the "discrete Hodge operators" $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ is not forced on us, the way (6) were. So if a rule for setting up $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ is proposed, by which

[1] See [8] for an attempt to go beyond this crude dichotomy.

[2] Not to be confused with that of an orthogonal grid à la Yee, with all edges parallel to one of the Cartesian axes.

one obtains square symmetric positive-definite—though not necessarily diagonal—matrices, how do we assess the value of such a proposal? By proving the convergence of the scheme when both meshes are indefinitely refined, and this requires finite elements, as an analysis of the proof will show.

4. CONVERGENCE: STATICS

Let's first deal with *magnetostatics*, which consists in finding a pair $\{b, h\}$ such that

$$\int_S b = 0, \quad b = \mu h, \quad \int_{\partial\Sigma} h = \int_\Sigma j, \tag{11}$$

for all *closed* inner-oriented surfaces S and all outer-oriented surfaces Σ inside D. Since $\partial\Sigma$ reduces to 0 for closed such surfaces, the given j must satisfy $\int_\Sigma j = 0$ for all *closed* Σ. By the same discretization strategy as above, this becomes

$$\mathbf{Db} = 0, \quad \mathbf{b} = \boldsymbol{\mu}\mathbf{h}, \quad \mathbf{R}^t\mathbf{h} = \mathbf{j}, \tag{12}$$

where $\mathbf{D}$ is the face-to-volumes incidence matrix, defined the same way $\mathbf{R}$ was. (Since $\mathbf{DR} = 0$, a fact with straightforward proof, (12) stems from (6) in static situations.) Problem (12) has a unique solution $\{\mathbf{b}, \mathbf{h}\}$, provided $\mathbf{G}^t\mathbf{j} = 0$, where $\mathbf{G}$ is the nodes-to-edges incidence matrix. (Again, $\mathbf{RG} = 0$, therefore $\mathbf{G}^t\mathbf{R}^t = 0$, and hence the necessary condition about $\mathbf{j}$.)

We'll need some notation: $\mathcal{F}$ is the set of "active" primal faces (excluding those of the boundary, on which $\mathbf{b}_f = 0$) and an expression such as $\mathbf{(b, h)}$, with boldface parentheses, denotes the sum $\sum_{f\in\mathcal{F}} \mathbf{b}_f\mathbf{h}_f$. An expression such as (b, h) refers to what would be, in terms of vector proxies, $\int_D B \cdot H$, a quantity which actually does not depend on the metric (it's $\int_D b \wedge h$ in differential geometric language). We'll assume simple topology of D, in which case not only $\mathbf{RG} = 0$ and $\mathbf{DR} = 0$, but $\ker(\mathbf{R})$ and $\ker(\mathbf{D})$ coincide with the ranges of $\mathbf{G}$ and $\mathbf{R}$ (proofs of these well-known properties can be found in [1], Chap. 5). As a consequence, the orthogonality relation

$$\mathbf{Du} = 0 \text{ and } \mathbf{R}^t\mathbf{v} = 0 \Rightarrow \mathbf{(u, v)} = 0 \tag{13}$$

holds. Last, we denote by $\|\mathbf{u}\|_\nu$ and $\|\mathbf{v}\|_\mu$ the square roots of the quantities $\mathbf{(\nu u, u)}$ and $\mathbf{(\mu v, v)}$.

Let us denote by $\mathfrak{m}$ the pair of interlocked meshes, and call "grain", $\gamma_\mathfrak{m}$, the maximum diameter of all cells. Our purpose is to study $\{b_\mathfrak{m}, h_\mathfrak{m}\}$ when $\gamma_\mathfrak{m}$ tends to 0, while respecting the uniformity

condition that there be a *finite* number of element shapes in the *whole* family of refinements of the initial coarse mesh. We adopt "$\mathrm{m} \to 0$" as a shorthand for that. We shall compare the computed DoF arrays, $\mathbf{b}$ and $\mathbf{h}$, with arrays of the same kind, $r_\mathrm{m} b = \{\int_f b : f \in \mathcal{F}\}$ and $r_\mathrm{m} h = \{\int_{\tilde{f}} h : f \in \mathcal{F}\}$, composed of the fluxes and mmf's of the (unknown) solution of (11).

Since $\int_S b = 0$ for all closed surfaces S, this applies to the boundary of any primal 3-cell v (v for volume), hence the sum of face fluxes $\sum_f \mathbf{D}_v^f \int_f b$ must vanish for all v. Similarly, $\int_{\partial\Sigma} h = \int_\Sigma j$ applies when Σ is a dual 2-cell, which yields the relation $\sum_f \mathbf{R}_{\tilde{f}}^e \int_{\tilde{f}} h = \int_{\tilde{e}} j$. In matrix form, all this results in $\mathbf{D} r_\mathrm{m} b = 0$, $\mathbf{R}^t r_\mathrm{m} h = r_\mathrm{m} j \equiv \mathbf{j}$. Comparing with (12), we obtain

$$\mathbf{D}(\mathbf{b} - r_\mathrm{m} b) = 0, \quad \mathbf{R}^t(\mathbf{h} - r_\mathrm{m} h) = 0, \tag{14}$$

and

$$(\mathbf{b} - r_\mathrm{m} b) - \boldsymbol{\mu}(\mathbf{h} - r_\mathrm{m} h) = (\boldsymbol{\mu} r_\mathrm{m} - r_\mathrm{m}\mu)h \equiv \boldsymbol{\mu}(r_\mathrm{m}\nu - \boldsymbol{\nu} r_\mathrm{m})b. \tag{15}$$

Now, let's premultiply (15) by $\boldsymbol{\nu}$ and take the $()$-scalar product of this with (15) itself, side by side. Some algebra, using the orthogonality (13) that (14) implies, leads to

$$(\boldsymbol{\nu}(\mathbf{b}-r_\mathrm{m} b), \mathbf{b}-r_\mathrm{m} b) + (\boldsymbol{\mu}(\mathbf{h}-r_\mathrm{m} h), \mathbf{h}-r_\mathrm{m} h) = ((\boldsymbol{\mu} r_\mathrm{m} - r_\mathrm{m}\mu)h, (r_\mathrm{m}\nu - \boldsymbol{\nu} r_\mathrm{m})b), \tag{16}$$

the cornerstone of the convergence proof, which we recast as

$$\|\mathbf{b} - r_\mathrm{m} b\|_\nu^2 + \|\mathbf{h} - r_\mathrm{m} h\|_\mu^2 = \|(\boldsymbol{\nu} r_\mathrm{m} - r_\mathrm{m}\nu)b\|_\mu^2 \equiv \|(\boldsymbol{\mu} r_\mathrm{m} - r_\mathrm{m}\mu)h\|_\nu^2 \tag{17}$$

to better suggest the geometric interpretation of this equality, and also its physical meaning: If we call "discrete energy" of $\mathbf{b}$ [resp. "discrete coenergy" of $\mathbf{h}$] the quantity $\frac{1}{2}\|\mathbf{b}\|_\nu^2$ [resp. $\frac{1}{2}\|\mathbf{h}\|_\mu^2$], we see that the "distance in energy" between the computed array of fluxes [resp. mmf's] and the actual one tends to zero, provided the so-called *inconsistency error* at the right-hand side of (17) tends to 0 with m. We rewrite the latter requirement, in less formal but more suggestive fashion, as

$$(\boldsymbol{\nu} r_\mathrm{m} - r_\mathrm{m}\nu)b \to 0 \text{ when } \mathrm{m} \to 0, \quad (\boldsymbol{\mu} r_\mathrm{m} - r_\mathrm{m}\mu)h \to 0 \text{ when } \mathrm{m} \to 0, \tag{18}$$

for any 2-form b or twisted 1-form h, two equivalent properties which express the *consistency* of the approximation scheme.

This is the core of the matter: a good discrete Hodge operator must satisfy (18). Do the ones in (9) comply?

Yes, as easily verified thanks to a local computation, for which we revert again to vector proxies. First suppose that H is uniform in the vicinity of face f. The flux $[(\boldsymbol{\mu} r_{\mathrm{m}} - r_{\mathrm{m}}\mu)h]_f$ then vanishes, *because so was the rationale behind the definition of* $\boldsymbol{\mu}$, back in (9). For a non-uniform H, now, a Taylor expansion about the point $f \cap \tilde{f}$ allows us to bound this quantity by $C\gamma_{\mathrm{m}}\,\mathrm{area}(f)$, where C depends on H but not on m, hence (same symbol C for all constants, with possibly different values)

$$\|(\boldsymbol{\mu} r_{\mathrm{m}} - r_{\mathrm{m}}\mu)h\|_\nu^2 \leq C \sum_f \frac{\mathrm{length}(\tilde{f})}{\mathrm{area}(f)} \gamma_{\mathrm{m}}^2 \mathrm{area}(f)^2 \leq C\gamma_{\mathrm{m}}^2 \mathrm{volume}(D).$$

This very rough estimate could be improved upon, by more finely taking into account the local geometry, and by exploiting the uniformity of the mesh sequence, but it suffices to show that, indeed, (18) holds.

The same argument works for all reasonable ways to define $\boldsymbol{\mu}$, including the Galerkin method, to which we shall return, and "finite volume" approaches [13]. The important thing is to have the inconsistency $(\boldsymbol{\mu} r_{\mathrm{m}} - r_{\mathrm{m}}\mu)h$ vanish for locally uniform fields[3].

Going back to (17), we see that $\|\mathbf{b} - r_{\mathrm{m}} b\|_\nu^2$ behaves like γ_{m}^2 when m is refined. But this is hardly enough to prove convergence in any definite way. For this, one should build from the DoF arrays $\mathbf{b}$ and $\mathbf{h}$ an approximation $\{b_{\mathrm{m}}, h_{\mathrm{m}}\}$ of the field, and prove that both $b_{\mathrm{m}} - b$ and $h_{\mathrm{m}} - h$ tend to 0 with m, in an appropriate sense. Setting $b_{\mathrm{m}} = p_{\mathrm{m}}\mathbf{b}$ and $h_{\mathrm{m}} = p_{\mathrm{m}}\mathbf{h}$, where each p_{m} is a linear map, not yet specified, we may for instance require that both $(\nu(b - p_{\mathrm{m}}\mathbf{b}), b - p_{\mathrm{m}}\mathbf{b})$ and $(\mu(h - p_{\mathrm{m}}\mathbf{h}), h - p_{\mathrm{m}}\mathbf{h})$ tend to 0 with m (convergence "in energy"). Sufficient conditions on p_{m} to this effect are the consistency condition:

$$p_{\mathrm{m}} r_{\mathrm{m}} b \to b,\ p_{\mathrm{m}} r_{\mathrm{m}} h \to h, \text{ in energy, when } \mathrm{m} \to 0, \tag{19}$$

and the following inequalities:

$$\alpha(\nu p_{\mathrm{m}}\mathbf{b}, p_{\mathrm{m}}\mathbf{b}) \leq (\boldsymbol{\nu}\mathbf{b}, \mathbf{b}), \quad \alpha(\mu p_{\mathrm{m}}\mathbf{h}, p_{\mathrm{m}}\mathbf{h}) \leq (\boldsymbol{\mu}\mathbf{h}, \mathbf{h}), \tag{20}$$

for all $\mathbf{b}$ and $\mathbf{h}$, where the constant $\alpha > 0$ does not depend on m. This expresses the *stability* of the approximation scheme. Convergence, now, is straightforward (Lax's theorem): First, $p_{\mathrm{m}}(\mathbf{b} - r_{\mathrm{m}} b) \to 0$, by (20), then $p_{\mathrm{m}}\mathbf{b} \to b$, thanks to (19). Same argument about h. Note that only L^2 estimates came into play.

So what about the map p_{m}? We shall return to that in general, with Whitney forms. But here, by sheer luck, there happens to be an

[3] As far as I can judge, credit for this remark should go to Tonti [18].

easy way to interpolate from $\mathbf{b}$ in the case of a tetrahedral primal mesh: build a piecewise constant field B, the flux of which at face f is $\mathbf{b}_f$. This makes three unknowns per volume, subject to four equations, but $\mathbf{Db} = \mathbf{0}$ makes one of them dependent, hence there is such a field. By a general result on Whitney forms to which we shall return, $p_\mathrm{m} r_\mathrm{m} b \to b$ does hold, then. As for (20), note that $(\nu p_\mathrm{m}\mathbf{b}, p_\mathrm{m}\mathbf{b})$, being a quadratic form of $\mathbf{b}$, equals $(\mathbf{M}_2(\nu)\mathbf{b}, \mathbf{b})$, where $\mathbf{M}_2$ is some positive-definite matrix. (It's the "mass matrix" for Whitney face elements.) Suppose first a *single* tetrahedron in the mesh $\mathfrak{m}$, and consider the quotient $(\boldsymbol{\nu}\mathbf{b}, \mathbf{b})/(\mathbf{M}_2(\nu)\mathbf{b}, \mathbf{b})$. Its lower bound with respect to $\mathbf{b}$, strictly positive, depends only on the *shape* of the tetrahedron, not on its size. Uniformity of the mesh sequence, then, allows us to take for α in (20) the smallest of these lower bounds, which is strictly positive and independent of $\mathfrak{m}$. We may thereby conclude that $p_\mathrm{m}\mathbf{b}$ converges towards b in energy, and hence $\nu p_\mathrm{m}\mathbf{b} \to h$. (There is here, implicit, a lazy way to build a p_m for h: Set $p_\mathrm{m}\mathbf{h} = \nu p_\mathrm{m}\boldsymbol{\mu}\mathbf{h}$. It's somewhat unsavory, because "non-conformal", and $r_\mathrm{m} p_\mathrm{m} \neq 1$.)

5. OTHER EQUIVALENT SCHEMES IN STATICS

The proof thus carried out applies to all numerical schemes algebraically equivalent to (12). Let's review them.

First, a "scalar potential" one. As we have assumed, the kernels $\ker(\mathbf{R})$ and $\ker(\mathbf{D})$ coincide with the ranges of $\mathbf{G}$ and $\mathbf{R}$. By transposition, $\ker(\mathbf{G}^t)$ is the range of $\mathbf{R}^t$, and $\ker(\mathbf{R}^t)$ is the range of $\mathbf{D}^t$. So if $\mathbf{G}^t\mathbf{j} = 0$, there exists an $\mathcal{F}$-indexed array $\mathbf{h}^\mathbf{j}$ such that $\mathbf{R}^t\mathbf{h}^\mathbf{j} = \mathbf{j}$. (It's not unique, and need not be explicitly constructed, though that would be a trivial task. That there is one is all we need.) Now, $\mathbf{R}^t(\mathbf{h} - \mathbf{h}^\mathbf{j}) = 0$, so there is a DoF array $\boldsymbol{\phi}$, indexed over volumes, such that $\mathbf{h} = \mathbf{h}^\mathbf{j} + \mathbf{D}^t\boldsymbol{\phi}$, and (12) reduces to

$$\mathbf{D}\boldsymbol{\mu}\mathbf{D}^t\boldsymbol{\phi} = -\mathbf{D}\boldsymbol{\mu}\mathbf{h}^\mathbf{j}. \tag{21}$$

This is a square symmetric linear system, with respect to $\boldsymbol{\phi}$, the matrix of which can be singular, with a kernel equal, since $\boldsymbol{\mu}$ is regular, to $\ker(\mathbf{D}^t)$. There are solutions, because the right-hand side is automatically in $\mathrm{ran}(\mathbf{D})$, and $\mathbf{D}^t\boldsymbol{\phi}$, which represents the magnetic field, is unique, though $\boldsymbol{\phi}$ may not be. Equation (21) thus appears as a way to solve (12): having $\boldsymbol{\phi}$, we set $\mathbf{h} = \mathbf{h}^\mathbf{j} + \mathbf{D}^t\boldsymbol{\phi}$, and $\mathbf{b} = \boldsymbol{\mu}\mathbf{h}$. This is a typical *finite volume* approach to magnetostatics, with one degree of freedom per volume of the (primal) mesh, which one may of course interpret as the value of a magnetic potential at the dual node contained by this volume.

Symmetrically, we also have a "vector potential" scheme. Since $\ker(\mathbf{D})$ is the range of $\mathbf{R}$, one may look for $\mathbf{b}$, which has to be in it, in the form $\mathbf{b} = \mathbf{R}\mathbf{a}$, where the DoF array $\mathbf{a}$ is indexed over the set $\mathcal{E}$ of active edges. Then (12) is equivalent to

$$\mathbf{R}^t \boldsymbol{\mu}^{-1} \mathbf{R}\mathbf{a} = \mathbf{j}. \tag{22}$$

No uniqueness, again, but there are solutions, thanks to the condition $\mathbf{G}^t\mathbf{j} = 0$, which guarantees that $\mathbf{j}$ lies in the range of $\mathbf{R}^t$, and $\mathbf{b} = \mathbf{R}\mathbf{a}$ is the same for all these solutions. So solving (22), thus getting a unique $\mathbf{b}$, and setting $\mathbf{h} = \boldsymbol{\mu}^{-1}\mathbf{b}$, is equivalent to solving (12).

This is not all. If we refrain to eliminate $\mathbf{h}$ in the reduction of (12) to (22), but still use $\mathbf{b} = \mathbf{R}\mathbf{a}$, we get an intermediate two-equation system,

$$\begin{pmatrix} -\boldsymbol{\mu} & \mathbf{R} \\ \mathbf{R}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{h} \\ \mathbf{a} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{j} \end{pmatrix}, \tag{23}$$

often called a *mixed* algebraic system. The same manipulation in the other direction (eliminating $\mathbf{h}$ by $\mathbf{h} = \mathbf{h}^{\mathbf{j}} + \mathbf{D}^t\boldsymbol{\phi}$, but keeping $\mathbf{b}$) gives

$$\begin{pmatrix} -\boldsymbol{\mu}^{-1} & \mathbf{D}^t \\ \mathbf{D} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} -\mathbf{h}^{\mathbf{j}} \\ 0 \end{pmatrix}. \tag{24}$$

Systems (21), (22), (23), and (24) differ in size and in sparsity, but give the same solution pair $\{\mathbf{b}, \mathbf{h}\}$, so which one effectively to solve is a matter of algorithmics that need not concern us here. (Assigning b to *dual* faces and h to *primal* edges would generate a similar family of equivalent systems, but not equivalent to (21)–(24), thus yielding *complementary* information. See [1], Chap. 6.) The important point is, the error analysis we have performed applies to all of them.

In this approach, therefore, there is no need to invoke so-called "mixed variational principles" and "inf-sup conditions" to prove the convergence of mixed schemes—a welcome simplification. What we have is a *discretizing machinery*, which we apply systematically: replace fields by DoF arrays, on the primal or dual mesh according to their nature as differential forms, replace curl by $\mathbf{R}$ on the primal side and $\mathbf{R}^t$ on the dual side, etc., and most importantly, ϵ and μ by $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$. These "discrete Hodges" are the only non-standardized parts of the machine, and hence the only ones that call for "certification", i.e., verification of (18), (19), and (20). We just did that for the Hodges of (9) and (10), and will soon deal with the Galerkin-induced ones.

6. CONVERGENCE: TRANSIENTS

Meanwhile, let us sketch the convergence proof in transient situations.

First, linear interpolation in time between the values of the DoF arrays, as output by the Yee scheme (8), provides DoF-array-valued functions of time which converge, when δt tends to zero, towards the solution of the "spatially discretized" equations (6). This is standard.

Next, taking advantage of linearity, let's Laplace-transform, and examine the behavior of the solution of

$$-p\,\mathbf{D} + \mathbf{R}^t\mathbf{H} = \mathbf{J}, \quad p\,\mathbf{B} + \mathbf{R}\mathbf{E} = 0, \tag{25}$$

$$\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu}\mathbf{H}, \tag{26}$$

when $\mathrm{m} \to 0$. Here, $p = \xi + \mathrm{i}\omega$, with $\xi > 0$, and small capitals denote Laplace transforms. If one can prove *uniform* convergence with respect to ω (which the requirement $\xi > 0$ makes possible), convergence of the solution of (6) will ensue, by inverse Laplace transformation. The main problem, therefore, is to compare $\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}$, as given by (25) and (26), with $r_\mathrm{m}\mathrm{E}, r_\mathrm{m}\mathrm{B}, r_\mathrm{m}\mathrm{H}, r_\mathrm{m}\mathrm{D}$.

The approach is similar to what we did in statics. First establish that

$$p\boldsymbol{\mu}(\mathbf{H} - r_\mathrm{m}\mathrm{H}) + \mathbf{R}(\mathbf{E} - r_\mathrm{m}\mathrm{E}) = p(r_\mathrm{m}\mu - \boldsymbol{\mu}r_\mathrm{m})\mathrm{H}, \tag{27}$$

$$-p\boldsymbol{\epsilon}(\mathbf{E} - r_\mathrm{m}\mathrm{E}) + \mathbf{R}^t(\mathbf{H} - r_\mathrm{m}\mathrm{H}) = -p(r_\mathrm{m}\epsilon - \boldsymbol{\epsilon}r_\mathrm{m})\mathrm{E}. \tag{28}$$

Then, right-multiply (27) by $(\mathbf{H} - r_\mathrm{m}\mathrm{H})^*$ and the conjugate of (28) by $-(\mathbf{E} - r_\mathrm{m}\mathrm{E})$, add, see the middle terms (in $\mathbf{R}$ and $\mathbf{R}^t$) cancel out, and get the energy estimates. The similarity between the right-hand sides of (15), on the one hand, and (27), (28), on the other hand, shows that no further consistency requirements emerge. Stability, thanks to $\xi > 0$, holds there if it held in statics. What is a good Hodge discrete operator in statics, therefore, is a good one in transient situations.

7. INTERPOLATION: WHITNEY FORMS

So for each kind of DoF array, we need an operator, generically denoted by p_m, which maps it to a differential form (DF) of the appropriate kind: $p_\mathrm{m}\mathbf{e}$, starting from an edge-based DoF array $\mathbf{e}$, should be a 1-form; $p_\mathrm{m}\mathbf{b}$, obtained from a face-based $\mathbf{b}$, should be a 2-form; and so forth. It should satisfy $r_\mathrm{m}p_\mathrm{m} = 1$ and $p_\mathrm{m}r_\mathrm{m} \to 1$ when $\mathrm{m} \to 0$. We'll present the solution to this problem that Whitney forms provide [21], in the case of a *simplicial* primal mesh, in a way that gives a rationale for these forms.

We know a solution in the case of (primal-) *node-based* DoF arrays. Such arrays correspond to straight DF's of degree 0, i.e., to

functions, and interpolating a function from its nodal values is done via the well known "hat functions" of finite element theory, that we shall denote by w^n, one for each node n of the mesh. We note that hat functions serve two purposes. One is to represent a point x in terms of the nodal positions, that is, with mild notational abuse, as the barycenter

$$x = \sum_{n \in \mathcal{N}} w^n(x)\, x_n, \tag{29}$$

where $\mathcal{N}$ is the set of nodes. The other is to interpolate from an array $\{\boldsymbol{\psi}_n : n \in \mathcal{N}\}$ of nodal values:

$$(p_{\mathrm{m}}\boldsymbol{\psi})(x) = \sum_{n \in \mathcal{N}} w^n(x)\boldsymbol{\psi}_n, \tag{30}$$

yielding a piecewise affine function whose value at x_n is $\boldsymbol{\psi}_n$. Hence a p_{m} for which $r_{\mathrm{m}}p_{\mathrm{m}} = 1$ holds. Compare these expressions: The value we attribute to $p_{\mathrm{m}}\boldsymbol{\psi}$ at point x is the weighted sum of the nodal values $\boldsymbol{\psi}_n$, *the weights being the same as those by which x is expressed as a sum of nodes*. That's our clue: if we were somehow able to express a line as a weighted sum of mesh-edges, a surface as a weighted sum of mesh-faces, etc., we could generate a 1-form from an edge-based DoF array, a 2-form from a face-based array, and so on. Such weighted sums, written as $\sum_s \mathbf{c}^s\, s$, where s spans the set of p-simplices, have status in Homology, where they are called *chains*. In short, we need to be able to *assign a p-chain to a p-manifold*.

Let's tackle the case $p = 1$, a line. To the price of chopping the line into small segments, and doing sums, we can solve the problem if we know how to deal with a segment xy, which we may assume lies entirely in one of the tetrahedra. Since $x = \sum_n w^n(x)\, x_n$ and $y = \sum_n w^n(y)\, x_n$, one has, by deliberately confusing oriented segments such as xy with vectors such as $y - x$,

$$y - x = y - \sum_n w^n(x)\, x_n = \sum_n w^n(x)\, (y - x_n)$$

(since $\sum_n w^n(x) = 1$), hence

$$xy = \sum_n w^n(x)\, (x_n - \sum_m w^m(y)\, x_m) = \sum_n w^n(x) \sum_m w^m(y)\; x_m x_n, \tag{31}$$

and that's a step forward: (31) represents the segment xy as a weighted sum of segments which coincide with edges, but either with one or the other orientation. So each of the relevant edges appears twice, with

opposite signs, in the above sum. Grouping these pairs of terms, we find, assuming an edge oriented from n to m,

$$xy = (w^n(x)w^m(y) - w^m(x)w^n(y))\, x_n x_m + \ldots,$$

where the dots stand for five similar expressions for the other edges of the tetrahedron containing xy—hence our weights, and a 1-chain which represents xy. Now, owing to the fact that $w^n(y) - w^n(x) = \langle \mathrm{d}w^n, y - x\rangle$, by which we denote the effect of the 1-form $\mathrm{d}w^n$ on the vector $y - x$, we find that

$$xy = \sum_e \langle w^e(x), y - x\rangle\, e, \tag{32}$$

where

$$w^e = w^n \mathrm{d}w^m - w^m \mathrm{d}w^n \tag{33}$$

is the *Whitney form* of edge $e = \{n, m\}$. (More familiar, of course, is its proxy vector field W^e, which reads

$$W^e = w^n \nabla w^m - w^m \nabla w^n, \tag{33$'$}$$

the well known "edge element".) The chain $p^t_{\mathrm{m}}(xy)$ to be associated with xy (this notation will soon be justified) is thus $\sum_e \langle w^e(x), y - x\rangle\, e = \sum_e (\int_{xy} w^e)\, e$.

This shows how to assign a chain to an oriented curve c: by linearity of the integral, this chain must be (compare with (29))

$$p^t_{\mathrm{m}} c = \sum_e (\int_c w^e)\, e. \tag{34}$$

Correlatively, we have solved the problem of interpolating from edge values: what interpolates from the edge-DoF array $\mathbf{a}$ is the (straight) 1-form

$$p_{\mathrm{m}}\mathbf{a} = \sum_{e \in \mathcal{E}} \mathbf{a}_e w^e \tag{35}$$

(compare with (30)). The symbol p^t_{m} is appropriate, for if we denote the integral $\int_c a$ of a 1-form by $\langle a, c\rangle$ and the duality pairing between a 1-chain $\mathbf{c}$ and a 1-cochain $\mathbf{a}$ by $(\mathbf{a}, \mathbf{c}) = \sum_{e \in \mathcal{E}} \mathbf{a}_e \mathbf{c}^e$, then $\langle p_{\mathrm{m}}\mathbf{a}, c\rangle = (\mathbf{a}, p^t_{\mathrm{m}} c)$ —a transposition. Note that $\int_{e'} w_e = 0$ for $e' \neq e$, because both w^n and w^m vanish on other edges than $e = \{m, n\}$. This shows (integrate both sides of (35) along e') that $r_{\mathrm{m}} p_{\mathrm{m}} = 1$ holds, again in this case. Last, the convergence property, $p_{\mathrm{m}} r_{\mathrm{m}} \to 1$ when $\mathrm{m} \to 0$, also holds, under the condition of mesh uniformity [7, 12].

This is true not only for $p = 1$, but for all Whitney forms up the dimensional scale, which can now be found by easy induction. On

the model of (31), we have the representation of a small triangle xyz as $\sum_{l,m,n} \lambda^l(x)\lambda^m(y)\lambda^n(z)\, x_l x_m x_n$, from which stems the 2-form $w^f = 2(w^l \mathrm{d}w^m \wedge \mathrm{d}w^n + w^m \mathrm{d}w^n \wedge \mathrm{d}w^l + w^n \mathrm{d}w^l \wedge \mathrm{d}w^m)$, associated with the face $f = \{l, m, n\}$, and so on. (The proxy of w^f is the "face element" $W^f = 2(w^l \nabla w^m \times \nabla w^n + \ldots)$, with the same circular permutation of terms.) The generic Whitney form for the p-simplex $s = \{n_0, n_1, \ldots, n_p\}$ is thus seen to be, as in [21],

$$w^s = (-1)^i\, p! \sum_{i=0,\ldots,p} w^{n_i} \mathrm{d}w^{n_0} \wedge \ldots \langle i \rangle \ldots \wedge \mathrm{d}w^{n_p}, \tag{36}$$

where $\langle i \rangle$ means "omit the term $\mathrm{d}w^{n_i}$".

An automatic feature of this way of representing small p-simplices as p-chains is that $p^t_{\mathrm{m}} \partial = \boldsymbol{\partial} p^t_{\mathrm{m}}$, where $\boldsymbol{\partial}$ is the boundary operator for chains. By simple transposition, this gives us the all important structural property of Whitney forms, $\mathrm{d}p_{\mathrm{m}} = p_{\mathrm{m}}\mathbf{d}$, where $\mathbf{d}$ is the generic incidence matrix. From this we derive, denoting by $W^p(D)$ the finite dimensional space generated by Whitney p-forms,

$$\mathrm{d}\, W^{p-1} \subset W^p, \quad \ker(\mathrm{d}\,; W^p) = \mathrm{d}\, W^{p-1}.$$

8. THE GALERKIN HODGE

To finish, let's see in which manner the Galerkin method can be understood as a realization of GFD. (Proofs, easy to carry out, are only sketched. Details can be found in [3].) We restrict consideration to a simplicial primal mesh. The key decision is to use the *barycentric* dual mesh: So, for instance, the dual edge $\tilde{e}$ is obtained by joining the barycenters of all faces and volumes adjacent to edge e, as encountered when turning around e.

First, a new notion: "dyadic products", or simply "dyadics", obtained by taking the tensor product of a vector and a covector. As a convenient metaphor for such objects, imagine a machine with two slots, one on the left that can receive a covector, one on the right that can receive a vector, and a central dial that displays a real number, with the usual linearity properties with respect to both arguments. Denoting by M the machine, we shall write the dial's reading in the form $\langle \omega, M, v \rangle$.

A way to build such a machine is to use as its inner components a fixed vector w and a fixed covector η, and to make the dial indicate $\langle \omega, w \rangle \langle \eta, v \rangle$ when ω and v are slipped into the slots. Let's denote the machine thus obtained by the symbol $w\rangle\langle\eta$, and call that the *dyadic product* of w and η. So if we substitute $w\rangle\langle\eta$ for M in the above

expression $\langle \omega, M, v \rangle$, we get this:

$$\langle \omega, w \rangle\langle \eta, v \rangle = \langle \omega, w \rangle\langle \eta, v \rangle \ \forall\, \omega, v,$$

which in spite of looking like a notational joke is a bona-fide *definition* of $w\rangle\langle\eta$! The general machine is a sum of dyadics. Among them, one is special, the *unity*, denoted by 1, and defined by $\langle \omega, 1, v \rangle = \langle \omega, v \rangle$ for all ω and v.

Now, given a manifold, one may envision such a machine standing at each point, and hence, introduce *fields* of them (in which one will recognize so-called "(1-1)-tensors"). In particular, a dyadic field is a smooth assignment of a dyadic to each point, which one can build from a vector field and a 1-form by following the above recipe.

With this, we can rewrite (32) as

$$\sum_{e \in \mathcal{E}} e\rangle\langle w^e(x) = 1, \tag{37}$$

where symbol e, up to this point a mere label for edge e, now doubles for a vector, the vector along edge e. One may argue that the very definition of Whitney 1-forms, or edge elements, was engineered in order to obtain this formula.

Edges here have no privilege. There is a similar identity for simplices of all dimensions: For faces, one has $\sum_{f \in \mathcal{F}} f\rangle\langle w^f(x) = 1$, where f is interpreted as a 2-vector. For nodes, $\sum_{n \in \mathcal{N}} n\rangle\langle w^n(x) = 1$. This one, like all formulas of the family, says two things: that $x = \sum_n w^n(x)\, x_n$ (formula (29)), and that $\sum_n w^n(x) = 1$, whatever x, the "partition of unity" property of hat functions. Formula (37) and similar ones, therefore, say that *Whitney forms* of any given degree *make a partition of unity*.

Next, let us consider the ϵ-related *mass matrix* of edge elements, defined by by $(\mathbf{M}_1(\epsilon))^{ee'} = \int_D w^e \wedge \epsilon w^{e'}$, or alternatively, if one uses vector proxies, $(\mathbf{M}_1(\epsilon))^{ee'} = \int_D \epsilon\, W^e \cdot W^{e'}$. From (37), or more easily from (32), if one uses vector proxies, one gets $\sum_{e'} (\mathbf{M}_1(\epsilon))^{ee'}\, e' = \int_D \epsilon W^e$: substitute $W^{e'}$ for xy, hence $W^{e'}(x) = \sum_e (W^e(x) \cdot W^{e'}(x))\, e$. Then permute e and e', and integrate over D.

On the other hand, one can prove, by elementary geometry, that the vector-valued integral $\int_D W^e$ equals the *vectorial area* of $\tilde{e}$. (This metric concept applies to outer-oriented polyhedral surfaces: For a plane surface, take its area times its normal vector. For polyhedra, just add contributions.) Using (33′), and assuming ϵ uniform, one gets $\int_D \epsilon\, W^e = \epsilon\, \tilde{e}$, where $\tilde{e}$ now stands for the vectorial area of the dual face. This rewrites, in differential geometric language (and, this way, ϵ need not be uniform, cf. [3]), as $\int_D \epsilon w^e = \epsilon \tilde{e}$, if ϵ is properly

understood: on the left, the Hodge operator acting on 1-forms, on the right, a Hodge-like operator, defined by duality, which sends 2-vectors to vectors.

Putting all this together, we get the vectorial identity

$$\sum_{e'} (\mathbf{M}_1(\epsilon))^{ee'}\, e' = \epsilon \tilde{e}. \tag{38}$$

Now, let a *uniform* 1-form u act on both sides of this vector equality. This gives $\sum_{e'} (\mathbf{M}_1(\epsilon))^{ee'}\, \mathbf{u}_{e'} = \int_{\tilde{e}} \epsilon u$, which it's more illuminating to read as $\mathbf{M}_1(\epsilon) r_{\mathrm{m}} u = r_{\mathrm{m}} \epsilon u$ for a constant u, the very equality from which we were able to derive (18), the consistency property, in the case of the diagonal Hodge $\boldsymbol{\mu}$. So as a consequence of the partition-of-unity property, we may set $\boldsymbol{\epsilon} = \mathbf{M}_1(\epsilon)$, and thus have a "good Hodge" (though not a diagonal one—hence the problematic of *mass lumping*, evoked at more length in [2]). Stability is here a built-in property, because discrete energy and continuous energy coincide, by construction.

Let's conclude with some heuristic considerations. The point in what precedes was not to prove anew the convergence of the Galerkin method, only to present it as one realization, among others, of the GFD paradigm. But now, one realizes that the argument may be turned the other way: the geometric equality

$$\sum_{e'} \boldsymbol{\epsilon}^{ee'}\, e' = \epsilon \tilde{e}, \tag{39}$$

and its analogue for faces,

$$\sum_{f'} \boldsymbol{\nu}^{ff'}\, f' = \nu \tilde{f},$$

are *consistency criteria* that a proposed discrete Hodge, $\boldsymbol{\epsilon}$ or $\boldsymbol{\nu}$, should meet. More precisely, given a primal mesh and a matrix $\boldsymbol{\epsilon}$, there should be a way to draw the dual mesh that makes (39) hold for all[4] edges e. As we saw, the diagonal Hodges in (9) and (10) meet the criterion if an orthogonal dual mesh exists. (Otherwise, "cheat", as suggested in Note 4.) The Galerkin Hodge, as built from Whitney elements on a simplicial mesh, is computationally more expensive, but has the advantage that a suitable dual mesh always exists: the barycentric one.

[4] *Almost* all, actually—one can cheat on some of them, provided their proportion vanishes fast enough, with refinement, to still satisfy (18). For instance, when orthogonality of e and $\tilde{e}$ could not be achieved, area($\tilde{e}$) in (9) can be replaced by the area of its orthogonal projection along the direction of e.

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