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COVARIANT ISOTROPIC CONSTITUTIVE RELATIONS IN CLIFFORD'S GEOMETRIC ALGEBRA

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Abstract—Constitutive relations for isotropic material media are formulated in a manifestly covariant manner. Clifford's geometric algebra is used throughout. Polarisable, chiral and Tellegen medium are investigated. The investigation leads to the discovery of an underlying algebraic structure that completely classifies isotropic media. Variational properties are reviewed, special attention is paid to the imposed constraints on material parameters. Covariant reciprocity condition is given. Finally, duality transformations and their relevance to constitutive relations are investigated. Duality is shown to characterise 'well-behavedness' of medium which has an interesting metric tensor related implication.

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1. INTRODUCTION

Maxwell equations are usually supplemented by constitutive relations, for these provide the information about the surrounding medium. In the Minkowski space constitutive relations take the bivector (or 2-form) $\mathrm{F} \sim (\mathbf{E}, \mathbf{B})$, and map it to the bivector (2-form) $\mathrm{G} \sim (\mathbf{H}, \mathbf{D})$.

However, given their importance in practical problem solving, constitutive relations as well as their representations are less frequently explained in physics textbooks than what one might expect. Constitutive relations appear to be more or less a engineer's cup of tea, for it is usually antenna and microwave engineering oriented works that discuss them (see for example the numerous references listed in [1]). As these works describe constitutive relations within the Gibbs–Heaviside vector algebra requiring an inertial frame to be specified, physicists have not adopted them widely. Therefore we look for a covariant description that satisfies the engineer and the physicist alike. The Minkowski space is of course the space to be used, for constitutive relations suggest that Maxwell equations are covariant under Lorentz transformations [2, 3]. In this study we give manifestly covariant representations of a few important isotropic media and discuss their implications. We also hope that the geometric substance of constitutive relations becomes more apparent with our treatment. To convey this 'geometricity' of constitutive relations more efficiently we adopt the language of Clifford's geometric algebra, which we will introduce in Section 2. The practical results are given in the examples in Section 3. For the benefit of more theoretically inclined readers, we discuss briefly variational aspects in Section 4 and duality properties in Section 5. The underlying algebraic structure is exposed in the end of Section 3, and the reciprocity conditions are reviewed in the latter part of Section 4.

2. ELECTROMAGNETISM IN CLIFFORD'S GEOMETRIC ALGEBRA

Constitutive relations are much easier to write down explicitly, if they are assumed to be linear, and this is normally done in the literature [3]. We make that assumption too. Furthermore, we discard the distinction between vectors and forms, for we work entirely within a known metric. This identification of vectors with their duals makes the Clifford's geometric algebra –or Clifford algebra for short– an obvious choice for our working environment. We work within an algebra generated by basis vectors $\mathbf{e}_1 \ldots \mathbf{e}_4$ with the relations [1]

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_2^2 = 1, \qquad \mathbf{e}_4 = -1$$
$$\mathbf{e}_\lambda \mathbf{e}_\nu = -\mathbf{e}_\nu \mathbf{e}_\lambda \qquad \text{for} \quad \lambda \neq \nu.$$

This algebra bears the label $\mathcal{C}\ell_{3,1}$, which means that three of the basis vectors have norm 1 and the remaining one has -1. For a quick reference, we have explained in Appendix A the different products of Clifford algebra. Some reading suggestions are also listed there.

The electromagnetic field can be written as two bivectors[2] F and G

$$\mathrm{F} = \frac{1}{2} F^{\lambda\nu} \mathbf{e}_{\lambda\nu}, \qquad \mathrm{G} = \frac{1}{2} G^{\lambda\nu} \mathbf{e}_{\lambda\nu}, \tag{1}$$

where we have used the summation convention. F and G satisfy Maxwell equations

$$\boldsymbol{\partial} \wedge \mathrm{F} = 0, \tag{2}$$
$$\boldsymbol{\partial} \wedge \mathrm{G} = \mathrm{J} \tag{3}$$

where Dirac operator $\boldsymbol{\partial}$, when given in its space–and–time-component form, reads $\boldsymbol{\partial} = (\nabla - c^{-1}\mathbf{e}_4\, \partial_t)$. J is a trivector representing electric charges and currents.

Constitutive relations in the context of Clifford algebra have been discussed earlier by Hillion in ref. [6], but his approach differs from ours, being concerned with the well-posedness of the problem. We will focus on the algebraic properties of the constitutive operator.

The constitutive operator $\breve{\chi}$ maps F to G:

$$\mathrm{G} = \breve{\chi}\ \mathrm{F}. \tag{4}$$

The reason why we mark χ with the accent $\breve{}$ will become clear in a moment.

The case of isotropic media may look simple, but there is in fact more than one way to connect F to G. Therefore $\breve{\chi}$ must be constructed in a manner that allows such a diverse media as the familiar dielectric materials and the exotic chiral and Tellegen materials to be equally well represented. Even in the absence of polarisable materials there remains a constitutive map, namely the vacuum constitutive map. The origin

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[1] The notation we use is mainly borrowed from ref. [9]: Vectors and basis multivectors, e.g. $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_{12}$, are in boldface, but we deviate from ref. [9] by denoting all other multivectors with unslanted caps, e.g. F, $\mathfrak{G}$, I. Scalars and unspecified elements of algebra are both in lowercase, e.g. c, u, but it should be clear from the context which ones are meant. Ref. [5] uses identical notation.

[2] In this paper we do not observe the difference between inner and outer orientations. For nice pictorial expositions, see [4].

of this map might be the metric itself [3, 7], but in our case it suffices to say that the metric transforms F to G via the Hodge mapping $\star$ [8, p. 28]:

$$\mathrm{G} = \sqrt{\frac{\epsilon_0}{\mu_0}} \star \mathrm{F}, \tag{5}$$

where the Hodge mapping has the explicit form

$$\star\mathrm{F} = \widetilde{\mathrm{F}}\,\mathrm{I}. \tag{6}$$

The factor I gives the orientation of the space in a form of oriented volume element, in $\mathcal{C}\ell_{3,1}$ $\mathrm{I} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$. The 'tilde' operation reverses the order of elements in Clifford products [9, pp. 14–15], e.g. $(\mathbf{e}_{123} + \mathbf{e}_1)^{\sim} = \mathbf{e}_{321} + \mathbf{e}_1$. In the Minkowski space the Hodge operation has the closure property $\star\star = -\mathrm{id}$ when operating on bivectors.

We find the Hodge mapping to be convenient in relating different formulations of Maxwell equations. The formulation that Post in ref. [3, pp. 53–56] adopts can translated to our notation as follows:

$$\boldsymbol{\partial} \wedge \mathrm{F} = 0, \qquad \boldsymbol{\partial} \lrcorner\, \mathfrak{G} = \mathfrak{J},$$

where $\mathfrak{G} = -\star\mathrm{G}$ and $\mathrm{G} = \star\mathfrak{G}$. For sources, $\mathfrak{J} = -\star\mathrm{J}$ and $\mathrm{J} = -\star\mathfrak{J}$. The constitutive mapping χ takes the bivector F to the bivector $\mathfrak{G}$:

$$\mathfrak{G} = \chi\mathrm{F}.$$

Comparison with (4) gives the relation

$$\breve{\chi} = \star\chi.$$

Hodge mapping makes also an appearance in the so-called duality rotation

$$\mathrm{F}' \quad = \quad \cos\alpha\ \mathrm{F} + \sin\alpha\ \star\mathrm{F}, \tag{7}$$

which leaves the source-less vacuum Maxwell equations as well as the vacuum energy–momentum tensor invariant. Here α has no simple geometric interpretation, it gives just a measure of the amount of the rotation that the field has undergone in the abstract space of $(\mathrm{F}, \star\mathrm{F})$. The importance of duality rotation was probably first appreciated in the 20's by Rainich [10], and later elaborated and put in to a broader context by Misner and Wheeler [11]. In addition to the invariance property above, the duality rotation comes in handy when characterising media.

3. CLASSIFYING MEDIA

We are of course aware that an isotropic polarisable medium becomes bi-anisotropic when observed in a frame where the medium appears to be moving [19, pp. 594–595], and in order that we can give a covariant description of isotropy, we *define* an isotropic medium to be a medium, where the only discriminate direction is the medium's four-velocity. The definition works well in a macroscopic scale, but becomes rather impractical in the other end of the scale, where we have difficulties to distinguish between random thermal movements and externally induced displacements.

The role of four-velocity is best explained by studying a few examples, which also serve as a basis of our classification of media:

Magneto–dielectric medium

Basic isotropic magneto–dielectric medium has a constitutive relation of the form

$$\mathrm{G} = \frac{1}{2}(c\,\epsilon + c^{-1}\mu^{-1}) \star \mathrm{F} + \frac{1}{2}\,(c\,\epsilon - c^{-1}\mu^{-1})\mathbf{w}^{-1} \star \mathrm{F}\ \mathbf{w}, \qquad (8)$$

where $\mathbf{w} = \gamma\,\vec{v} + \gamma\,c\,\mathbf{e}_4$ is the four-velocity of the medium, relative velocity $\vec{v} \in \mathbb{R}^3$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$. When the medium appears to be stationary, the presently co-moving observer sets $\mathbf{w} = c\,\mathbf{e}_4$ in (8) and thus recovers the usual magneto–dielectric constitutive relations

$$\mathrm{G} = \frac{1}{2}(c\,\epsilon + c^{-1}\mu^{-1}) \star \mathrm{F} + \frac{1}{2}\,(c\,\epsilon - c^{-1}\mu^{-1})\mathbf{e}_4^{-1} \star \mathrm{F}\ \mathbf{e}_4, \qquad (9)$$

or in a traditional form (cf. Appendix 6) $\mathbf{D} = \epsilon\,\mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu}\,\mathbf{B}$.

In this particular case the term $\mathbf{e}_4^{-1} \star \mathrm{F}\mathbf{e}_4$, or reflection of $\star\mathrm{F}$ with respect to $\mathbf{e}_4$, on the right hand side of (9) can be interpreted as a space conjugation (inversion) sending

$$\mathbf{e}_i\mathbf{e}_j \to (-\mathbf{e}_i)(-\mathbf{e}_j) = \mathbf{e}_i\mathbf{e}_j, \qquad \mathbf{e}_i\mathbf{e}_4 \to (-\mathbf{e}_i)\mathbf{e}_4 = -\mathbf{e}_i\mathbf{e}_4$$

where $i, j \in \{1, 2, 3\}$.

In passing, we note that it was probably Jauch and Watson in 1948 who first wrote magneto–dielectric medium's constitutive relations in the form where the medium's four-velocity appears explicitly, eqn. 9 in ref. [12]. Thus our (8) can be viewed as a Clifford algebra version of their equation. Their constitutive relations, written in index notation, have appeared in some textbooks, e.g. [13].

We can bring relations (8) into an even neater form, for

$$\mathbf{w}^{-1}(\mathbf{w}^{-1}\mathrm{F}\mathbf{w})\mathbf{w} = \mathrm{r}_{\mathbf{w}}(\mathrm{r}_{\mathbf{w}}\mathrm{F}) = \mathrm{r}_{\mathbf{w}}\mathrm{r}_{\mathbf{w}}\mathrm{F} = \mathrm{F},$$

which means that algebraically reflections with respect to $\mathbf{w}$, denoted as $\mathrm{r}_{\mathbf{w}}$, behave like the so-called unipodal numbers [15], viz. $\mathrm{r}_{\mathbf{w}}\mathrm{r}_{\mathbf{w}} = 1$. We can then immediately conceive functions with unipodal arguments, especially exponential ones:

$$\exp(\mathrm{r}_{\mathbf{w}}\phi) = \cosh\phi + \mathrm{r}_{\mathbf{w}}\sinh\phi. \tag{10}$$

As a natural consequence of these observations, we define

$$\cosh\phi = \frac{1}{2}(\epsilon_r + \mu_r^{-1}), \qquad \sinh\phi = \frac{1}{2}(\epsilon_r - \mu_r^{-1}), \tag{11}$$

and thus (8) becomes

$$\mathrm{G} = \frac{1}{\eta_0}\, e^{\mathrm{r}_{\mathbf{w}}\phi} \star \mathrm{F}, \quad \eta_0 = \sqrt{\mu_0/\epsilon_0} \tag{12}$$

Chiral medium

Isotropic chiral medium contains handed inclusions of similar handedness mixed randomly in a host material (racemic mixtures are not considered here). These inclusions effect a coupling between dynamic electric and magnetic fields. We write the corresponding constitutive relations as

$$\mathrm{G} = \frac{1}{\eta_0}\, e^{\mathrm{r}_{\mathbf{w}}\phi}\mathrm{F} + \Xi\,\mathrm{r}_{\mathbf{w}}\mathrm{F}. \tag{13}$$

Here Ξ is an operator, which in 3D formulation involves a time derivative [14]. What kind of time derivative should Ξ contain, if all the observers are to concur with it? Luckily, we do have a common time standard, the proper time τ. We write

$$\Xi = \xi\frac{\partial}{\partial\tau} = \xi\partial_\tau, \tag{14}$$

where ξ is a parameter measuring the strength of chirality.

In traditional 3D formulations of electromagnetism we have tacitly assumed a harmonic time variance, which suggests the use of quantities such as $j\omega$ or j in (14), but here we are working in a geometric algebra, and hence every quantity must have a geometric content. We could consider a structure like $\mathbb{C} \otimes \mathcal{C}\ell_{3,1}$, and thus readily adopt j from 3D theory, but the imaginary numbers introduced in this way have no clear geometric meaning. On the other hand, $\mathcal{C}\ell_{3,1}$ itself contains several copies of complex fields, i.e. fields with generators

$$\{1, \mathbf{e}_{23}\}, \quad \{1, \mathbf{e}_{31}\}, \quad \{1, \mathbf{e}_{23}\}, \quad \{1, \mathbf{e}_{123}\}, \quad \{1, \mathbf{e}_{1234}\},$$

and these should be considered as possible substitutes for the traditional complex numbers. The choice turns out to be particularly simple in the case of circularly polarised waves. Suppose that $\mathbf{x}$ is a space–time position and $\mathbf{k}$ a (constant) wave four-vector, then the circularly polarised plane wave solution of homogeneous Maxwell equations becomes

$$\mathrm{F}(\mathbf{x} \cdot \mathbf{k}) = e^{\pm \mathrm{I}(\mathbf{k} \cdot \mathbf{x})} \mathrm{F}_0 \tag{15}$$

where F_0 is a constant bivector, $\mathrm{I} = \mathbf{e}_{1234}$, and $\pm$ signifies the two possible helicities. Hence

$$\begin{aligned} \partial_\tau \mathrm{F}(\mathbf{x} \cdot \mathbf{k}) &= \pm e^{\pm \mathrm{I}(\mathbf{k} \cdot \mathbf{x})} \mathrm{I}\, \partial_\tau (\mathbf{x} \cdot \mathbf{k}) \mathrm{F}_0 = \pm e^{\pm \mathrm{I}(\mathbf{k} \cdot \mathbf{x})} \mathrm{I}\, (\mathbf{w} \cdot \mathbf{k}) \mathrm{F}_0 \\ &= \pm \mathrm{I}\, (\mathbf{w} \cdot \mathbf{k})\, e^{\pm \mathrm{I}(\mathbf{k} \cdot \mathbf{x})} \mathrm{F}_0 = \pm \mathrm{I}\, (\mathbf{w} \cdot \mathbf{k})\, \mathrm{F}(\mathbf{x} \cdot \mathbf{k}). \end{aligned} \tag{16}$$

Therefore we find it convenient to set $\partial_\tau = \pm(\mathbf{w} \cdot \mathbf{k})\, \mathrm{I}$. The sign should be chosen to reflect the correct handedness. $\mathbf{e}_{1234}$ is a good candidate for an 'imaginary number' in view of the fact it is also a Lorentz-invariant.

Comparing now (6) and (14) shows that for circularly polarised waves Hodge and chirality operator are almost the same operator. This affects the Euler–Lagrange equations as we shall see later.

Tellegen medium

In 1948 Tellegen [16] introduced a new device for electric circuits: the gyrator, which 'gyrated' a current into a voltage and vice versa. Tellegen also suggested that such a device could be made of material with relations

$$\mathbf{D} = \epsilon' \mathbf{E} + \gamma \mathbf{H}, \qquad \mathbf{B} = \mu \mathbf{H} + \gamma \mathbf{E}, \quad \gamma^2 \approx \epsilon\mu,$$

which translates to

$$\mathrm{G} = \frac{1}{\eta_0}\, e^{\mathrm{r}_{\mathbf{w}} \phi} \star \mathrm{F} + \psi\, \mathrm{F}, \tag{17}$$

in our covariant notation. Here $\epsilon = \epsilon' - \gamma^2/\mu$ and $\psi = \gamma/\mu$ [1, p. 309]. We will discover in Section 4 how the 'Post-constraint' [3, eqn. (6.18)] results from (17).

Algebraic structure

When we look at the forms of (8), (13) and (17) a pattern begins to emerge: a most general isotropic medium has the form

$$\mathrm{G} = a_1\,\mathrm{F} + a_2\,\mathrm{r}_{\mathbf{w}}\mathrm{F} + a_3\,\star\mathrm{F} + a_4\,\mathrm{r}_{\mathbf{w}}\star\mathrm{F}. \tag{18}$$

Clearly, we have exhausted all possible combinations of $\star$ and $\mathrm{r}_{\mathbf{w}}$, which we take to be basis vectors of a new, induced algebra. Not surprisingly, this new algebra is of Clifford variety, since it is generated by $\{\star, \mathrm{r}_{\mathbf{w}}\}$ with relations

$$\star\,\mathrm{r}_{\mathbf{w}} = -\mathrm{r}_{\mathbf{w}}\,\star, \qquad \mathrm{r}_{\mathbf{w}}^2 = 1, \qquad \star^2 = -1,$$

i.e. we have here $\mathcal{C}\ell_{1,1}$ (the unipodal number system mentioned above is in fact $\mathcal{C}\ell_{1,0}$ and is contained in $\mathcal{C}\ell_{1,1}$). Thus all isotropic media can be classified by the four-dimensional algebra $\mathcal{C}\ell_{1,1}$.

4. VARIATIONAL ASPECTS

Let us briefly investigate variational implications of our constitutive relations. For a moment, assume that Ξ behaves like a scalar (i.e. commutes with every element of algebra). Moreover, assume that (2) implies

$$\mathrm{F} = \boldsymbol{\partial}\wedge\mathbf{A}, \tag{19}$$

where $\mathbf{A}$ is a four-potential and thus the Lagrangian of the electromagnetic field with sources is a quadrivector

$$\mathrm{L} = \frac{1}{2}(\boldsymbol{\partial}\wedge\mathbf{A})\wedge\mathrm{G}(\mathbf{A}) - \mathbf{A}\wedge\mathrm{J}, \tag{20}$$

where G assumes the form (we use Thirring's Lagrangian [17, p. 46] with suitable modifications)

$$\mathrm{G}(\mathbf{A}) = \psi\,\boldsymbol{\partial}\wedge\mathbf{A} + \Xi\,\mathrm{r}_{\mathbf{w}}\boldsymbol{\partial}\wedge\mathbf{A} + \frac{1}{\eta_0}\,e^{\mathrm{r}_{\mathbf{w}}\phi}\star\boldsymbol{\partial}\wedge\mathbf{A}. \tag{21}$$

To be precise, our assumption (19) is true only for star-like regions [18, pp. 38–46], but this should not be too restricting, for we can consider regions that are locally star-like.

Variation $\mathbf{A}\to\mathbf{A}+\delta\mathbf{A}$ gives

$$\delta W = \int d\mathrm{V}[\;\delta\mathbf{A}\wedge\;\boldsymbol{\partial}\wedge(\frac{1}{\eta_0}\,e^{\mathrm{r}_{\mathbf{w}}\phi}\star\boldsymbol{\partial}\wedge\mathbf{A}) - \delta\mathbf{A}\wedge\mathrm{J}$$

$$
\begin{aligned}
&+\boldsymbol{\partial}\wedge(\delta\mathbf{A}\frac{1}{\eta_0}\,e^{\mathrm{r}_{\mathbf{w}}\phi}\star\boldsymbol{\partial}\wedge\mathbf{A})+\psi\,\boldsymbol{\partial}\wedge(\mathbf{A}\wedge\partial\wedge\delta\mathbf{A})\,]\\
&=\int_V d\mathrm{V}\;\delta\mathbf{A}\wedge\,(\boldsymbol{\partial}\wedge\frac{1}{\eta_0}\,e^{\mathrm{r}_{\mathbf{w}}\phi}\star\boldsymbol{\partial}\wedge\mathbf{A}-\mathrm{J})\\
&+\oint_{\partial V} d\mathrm{S}\,\delta\mathbf{A}\frac{1}{\eta_0}\,e^{\mathrm{r}_{\mathbf{w}}\phi}\star\boldsymbol{\partial}\wedge\mathbf{A}+\oint_{\partial V} d\mathrm{S}\,\psi\,\delta\mathbf{A}\wedge\boldsymbol{\partial}\wedge\mathbf{A}\,.
\end{aligned}
$$

In above we made use of Stokes' theorem

$$\int_V d\mathrm{V}\,\boldsymbol{\partial}\wedge f=\oint_{\partial V} d\mathrm{S}\,f, \tag{22}$$

where $d\mathrm{V}$ is of grade n and f is any multivector valued function of grade $n-1$. Here $n=4$, $d\mathrm{V}=dx\,dy\,dz\,cdt\;\mathrm{I}=dx\,dy\,dz\,cdt\;\mathbf{e}_{1234}$, and $d\mathrm{S}$ is an oriented differential surface element of grade $n-1=3$ and of compatible orientation [9, p. 261]. Note that δW is a scalar quantity. Furthermore, we have used identities

$$
\begin{aligned}
\mathrm{A}\wedge\star\mathrm{B} &= \mathrm{B}\wedge\star\mathrm{A}, & (23)\\
\mathrm{A}\wedge\mathrm{r}_{\mathbf{w}}\mathrm{B} &= -\mathrm{B}\wedge\mathrm{r}_{\mathbf{w}}\mathrm{A}, & (24)\\
\mathrm{A}\wedge\mathrm{r}_{\mathbf{w}}\star\mathrm{B} &= \mathrm{B}\wedge\mathrm{r}_{\mathbf{w}}\star\mathrm{A}, & (25)
\end{aligned}
$$

whenever A and B are any bivectors in $\mathcal{C}\ell_{3,1}$. The second and third relation are peculiarities of dimension $n=4$, while the first is valid for any pair of multivectors of similar, homogeneous degree and for any dimension [20, pp. 121–122].

We immediately notice that chirality does not contribute to the variation of the action W, and in fact this can be seen already from (21), because $(\boldsymbol{\partial}\wedge\mathbf{A})\wedge\mathrm{r}_{\mathbf{w}}(\boldsymbol{\partial}\wedge\mathbf{A})=0$ for any vectors $\mathbf{A},\mathbf{w}\in\mathcal{C}\ell_{3,1}$. Thus Ξ has to more be than just a scalar and chirality –as is well-known– a non-instantaneous effect. Making a substitution $\Xi\to\pm\xi(\mathbf{w}\cdot\mathbf{k})\mathbf{e}_{1234}$ would immediately correct the situation. The chiral term would then give a non-vanishing contribution to the volume integral.

If the variation is to vanish for all $\delta\mathbf{A}$ with a boundary value $\delta\mathbf{A}|_{\partial V}=0$, the integrand of the volume integral must satisfy

$$\boldsymbol{\partial}\wedge\frac{1}{\eta_0}\,e^{\mathrm{r}_{\mathbf{w}}\phi}\star\mathrm{F}-\mathrm{J}=0, \tag{26}$$

and thus (3) is recovered (the surface integrals over ∂V do not contribute to (26) due to the condition $\delta\mathbf{A}|_{\partial V}=0$). As the chiral and Tellegen parameter do not appear in (26) Post concluded that for instantaneous and local effects it is economical to demand that $\chi^{[\lambda\nu\sigma\kappa]}=0$, or to put it differently, it does no harm to set $\Xi=\psi=0$, when Ξ and ψ are scalar quantities.[3, Ch. VI, §2]

However, this may not be the complete story. It should be noted that from the set of electromagnetic Lagrangians we picked just one possible Lagrangian, other choices could have been made. Our choice was dictated by convenience and tradition. Furthermore, we have not discussed Noether currents. There might still be contribution to the conserved observables as noted by Thirring in Remarks (2.1.6) and (2.1.8) of [17]. The contribution to the canonical energy–momentum tensor is proportional to $\boldsymbol{\partial} \wedge ((\mathbf{e} \lrcorner \mathbf{A}) \wedge \mathrm{F})$, where $\mathbf{e}$ is a vector field generating the Lie derivatives. The term is unfortunately gauge–dependent. The meaning and significance of such a term should be duly investigated.

Reciprocity

We conclude this section by briefly checking the reciprocity properties of our constitutive mapping $\breve{\chi}$. The covariant form of reciprocity condition for fields F and G is

$$\int_V d\mathrm{V}(\mathrm{F}_a \wedge \mathrm{r}_{\mathbf{w}} \mathrm{G}_b - \mathrm{F}_b \wedge \mathrm{r}_{\mathbf{w}} \mathrm{G}_a) = 0, \tag{27}$$

which corresponds to the condition (15) in [19, p. 402]. As before, $\mathbf{w}$ is the medium's four-velocity. Now set

$$\mathrm{G}_{a,b} = \psi\, \mathrm{F}_{a,b} + \Xi\, \mathrm{r}_{\mathbf{w}} \mathrm{F}_{a,b} + \frac{1}{\eta_0}\, e^{\mathrm{r}_{\mathbf{w}} \phi} \star \mathrm{F}_{a,b}, \tag{28}$$

where Ξ may be a scalar or a scalar multiple of I, and use the conditions (23)–(25). The integrand reduces to

$$\mathrm{F}_a \wedge \mathrm{r}_{\mathbf{w}} \psi \mathrm{F}_b - \mathrm{F}_b \wedge \mathrm{r}_{\mathbf{w}} \psi F_a = 2\psi\, \mathrm{F}_a \wedge \mathrm{r}_{\mathbf{w}} \psi \mathrm{F}_b \neq 0,$$

therefore a medium with a Tellegen parameter present is not reciprocal.

5. DUALITY ROTATION

We return now to the duality rotation mentioned earlier. The general duality transformation is of the form

$$\begin{pmatrix} \mathrm{F}' \\ \mathrm{G}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{F} \\ \mathrm{G} \end{pmatrix}, \tag{29}$$

where a, b, c, d are just scalars. The associated general linear duality group can be restricted with the vacuum constitutive relations [2, Ch.

9]: Set $\mathrm{G} = \eta_0^{-1} \star \mathrm{F}$ and assume that the relation holds for duality transformed fields, $\mathrm{G}' = \eta_0^{-1} \star \mathrm{F}'$, hence

$$\begin{pmatrix} \mathrm{F}' \\ \eta_0^{-1} \star \mathrm{F}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{F} \\ \eta_0^{-1} \star \mathrm{F} \end{pmatrix}, \tag{30}$$

substitute F' in $\eta_0^{-1}\mathrm{F}'$ and use the closure property $\star\star = -\mathrm{id}$. In addition to that, we want the energy–momentum tensor to be invariant [11], so we choose the determinant of the matrix to be $= 1$. After re-parametrisation we get

$$\begin{pmatrix} \mathrm{F}' \\ \mathrm{G}' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \eta_0 \sin\alpha \\ -\eta_0^{-1} \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathrm{F} \\ \mathrm{G} \end{pmatrix}. \tag{31}$$

Upon examination of (8), (13) and (17) we note that chiral and magneto–dielectric constitutive relations behave essentially like the Hodge mapping, that is, they have a similar closure property:

$$\breve{\chi}\breve{\chi} = -a^2 \,\mathrm{id}. \tag{32}$$

where a is a scalar factor with a dimension of admittance. However, Tellegen medium does not behave so nicely: application of $\breve{\chi}$ twice does not lead to identity (modulo a scalar factor). Thus chiral medium and magneto–dielectric medium can always be incorporated in a new dual operator, denote it with $\star'$, and a general isotropic constitutive mapping $\breve{\chi}$ can be split in two

$$\breve{\chi} = a \star' + \psi, \tag{33}$$

where $\star'\star' = -\mathrm{id.}$, and ψ is the Tellegen parameter. In other words, chiral parameter can be duality rotated away, whereas Tellegen parameter not. This can be demonstrated by a straightforward calculation: Transform both the fields and constitutive operator of the Tellegen medium

$$\mathrm{G}' = (\cos\alpha + \sin\alpha\star)(\frac{1}{\eta_0}\, e^{\mathrm{r}_{\mathbf{w}}\phi} \star + \psi)(\cos\alpha - \sin\alpha\star)\ \mathrm{F}',$$

after some easy manipulations (remember that we can now use $\mathcal{C}\ell_{1,1}$)

$$\mathrm{G}' = \frac{1}{\eta_0}\cosh\phi\star\mathrm{F}' + \frac{1}{\eta_0}\sinh\phi\sin 2\alpha\, \mathrm{r}_{\mathbf{w}}\mathrm{F}' + \frac{1}{\eta_0}\sinh\phi\cos 2\alpha\, \mathrm{r}_{\mathbf{w}}\star\mathrm{F}' + \psi\,\mathrm{F}',$$

whence it follows that there is no α which makes the Tellegen parameter disappear.

6. DISCUSSION

We saw that constitutive relations of isotropic medium give rise to an induced Clifford algebra $\mathcal{C}\ell_{1,1}$. It is tempting to think that other more complicated media induce these kind of algebraic structures. As a consequence, it would be possible to classify media by the algebras they induce. It is probably obvious but we nevertheless point out that the classification of more general media involves higher dimensional algebras. On the other hand, we noticed that constitutive mappings of reciprocal and non-reciprocal media do not behave similarly, which suggests that we could use this as an alternative way of classifying media. Remembering that the reciprocity was related to the closure property of constitutive mapping we can divide general isotropic media in two classes, those that satisfy this property and those that do not. This observation has some implications, for there is interesting theory originating from the study of Yang–Mills-fields that says that given a duality operator we can find a corresponding conformal class of metric [21, 22]. The duality operator has to be sufficiently well-behaved, among the requirements we find that the closure property should be satisfied. Therefore magneto–dielectric medium as well as chiral medium can be embedded in the (symmetric) metric tensor, but Tellegen medium does not fit in.

Doubtless an alert reader has noticed that Tellegen medium has been a subject of controversy for some time. Those who have missed this discussion are referred to paper [23] and correspondence [24], for instance. We hope that the variational derivation and the remarks presented above have given new ideas.

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APPENDIX A. PRODUCTS OF CLIFFORD'S GEOMETRIC ALGEBRA

The Clifford product of a vector $\mathbf{x}$ and any element u of Clifford algebra can be written as

$$\mathbf{x}\,u = \mathbf{x} \rfloor u + \mathbf{x} \wedge u,$$

where $\rfloor$ denotes the left contraction and $\wedge$ the exterior product. These decrease and increase the grade, respectively. The left contraction can be defined by its characteristic properties [9, Ch. 3, Ch. 22]

$$\begin{aligned} \mathbf{x} \rfloor \mathbf{y} &= \langle \mathbf{x}, \mathbf{y} \rangle, \\ \mathbf{x} \rfloor (u \wedge v) &= (\mathbf{x} \rfloor u) \wedge v + \hat{u} \wedge (\mathbf{x} \rfloor v), \\ (u \wedge v) \rfloor w &= u \rfloor (v \rfloor w), \end{aligned}$$

where $\langle \, , \rangle$ denotes an inner product which in this work is chosen to be the usual dot product. Here we have also introduced a 'hat'-operation, grade involution, which is an automorphism that reverses the direction of every vector, for instance

$$(\widehat{\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3}) = \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3 = (-\mathbf{e}_1)(-\mathbf{e}_2)(-\mathbf{e}_3) = -\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3.$$

Contraction has an important 'duality' property relating it to the exterior product:

$$u \rfloor v = (u \wedge (v \, \mathrm{I})) \, \mathrm{I}^{-1},$$

where I is the highest grade element of the algebra, e.g. in $\mathcal{C}\ell_{3,1}$ $\mathrm{I} = \mathbf{e}_{1234}$.

Of course, there exist a right contraction $\llcorner$ as well, but for this work the left contraction is quite sufficient. See [9, Ch. 3] for the properties of $\llcorner$.

The exterior product and the left contraction can be reconstructed from the Clifford product

$$\mathbf{x} \wedge u = \frac{1}{2} \left(\mathbf{x} u + \hat{u} \mathbf{x} \right), \qquad \mathbf{x} \rfloor u = \frac{1}{2} \left(\mathbf{x} u - \hat{u} \mathbf{x} \right).$$

These reconstructions are from Riesz's lecture notes, [25, pp. 61–67], which is also a good classic introduction (albeit notationally old-fashioned) to Clifford algebra. For a modern introduction, one might want to check Lounesto's review [9]. Clifford algebra in electrodynamics is developed carefully in refs. [26, 27]. Standard references for Clifford algebra are [28, 29].

It is sometimes of interest to translate Clifford algebra products to Gibbs–Heaviside vector calculus products. In order to do that, we have to single out an inertial system, as explained in 6. Then we can find the following relations between the cross product and the exterior product (now $\mathrm{I} = \mathbf{e}_{123}$):

$$\mathbf{x} \times \mathbf{y} = (\mathbf{x} \wedge \mathbf{y}) \, \mathrm{I}^{-1}, \qquad \mathbf{x} \wedge \mathbf{y} = (\mathbf{x} \times \mathbf{y}) \, \mathrm{I}.$$

APPENDIX B. FIELD AND FLUX VECTORS

The electric field vector $\mathbf{E}$ can be found from F by choosing a particular frame and measuring the electric field in that frame. As an algebraic procedure this amounts to choosing a basis V', identification of $\mathbf{e}_4 \in V'$ with the observer, and left contracting F by $c\,\mathbf{e}_4$:

$$\mathbf{E} = c\,\mathbf{e}_4 \rfloor \mathrm{F}.$$

The procedure can be repeated to yield **H** from G

$$\mathbf{H} = c\,\mathbf{e}_4^{-1} \rfloor \mathrm{G} = -c\,\mathbf{e}_4 \rfloor \mathrm{G}.$$

Electric and magnetic flux densities are a bit trickier, here we also have to invoke the duality of vectors and trivectors:

$$\mathbf{B} = \star(\mathbf{e}_4^{-1} \wedge \mathrm{F}), \qquad \mathbf{D} = \star(\mathbf{e}_4^{-1} \wedge \mathrm{G}).$$

Thus in an inertial frame F and G can be decomposed as[3]

$$\mathrm{F} = \frac{\mathbf{E}}{c}\mathbf{e}_4 - \mathbf{B}\mathbf{e}_{123}, \qquad \mathrm{G} = -\frac{\mathbf{H}}{c}\mathbf{e}_4 - \mathbf{D}\mathbf{e}_{123}.$$

We can say that we have split F and G in space and time components. Similarly, for $\mathfrak{G}$ we can write

$$\mathfrak{G} = \mathbf{D}\mathbf{e}_4 - \frac{\mathbf{H}}{c}\mathbf{e}_{123}.$$

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[3] Note that there is sign error in eqn. (19) in [5].

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