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DIFFRACTION OF SCALAR PULSES AT PLANE APERTURES: A DIFFERENT APPROACH

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Abstract—We generalize to scalar pulses with finite duration a previous work [1] in which a new approach to diffraction at plane apertures is developed for scalar harmonic waves. A particular attention is given to rectangular pulse modulated signals for which an exact solution to the diffraction problem is obtained. As an example, the diffraction of a truncated harmonic pulse is investigated and the numerical problems to be solved are discussed with an important simplification when one is only interested in the diffraction pattern far from the aperture. More works are needed for apertures with no simple geometrical form.

1. INTRODUCTION

We generalize to scalar pulses, solutions of the wave equation, an approach to diffraction by plane apertures previously developed [1] for harmonic scalar waves, solutions of the Helmholtz equation. We first present, for fields depending arbitrarily on time, the generalization of the integral formulation used [1] to discuss harmonic field scattering with boundary data given on some surface S, supposed here to be a perfectly conducting plane. We still note $\mathbf{x}$ and $\mathbf{x}'$ the action and source points, Σ and Σ' the surface S of the action and source points respectively.

2. SCATTERING INTEGRAL EQUATIONS: A DIFFERENT APPROACH

We consider the scattering of scalar pulses on a perfectly conducting plane so that the total field $\psi(\mathbf{x}, t)$ incident plus reflected is with $\mathbf{x} = (x, y, z)$

$$\psi(\mathbf{x}, t) = \psi_i(\mathbf{x}, t) + \psi_r(\mathbf{x}, t) \tag{1}$$

Then ψ and the Green's functions $g_{D,N}(\mathbf{x}, t; \mathbf{x}', t')$ in the integral equations, obtained from the wave equation $D\psi(\mathbf{x}, t) = 0$ in which D is the Dalembertian operator, are supposed to satisfy on $\Sigma(z = 0)$ the Dirichlet or Neumann boundary conditions

$$[\psi(\mathbf{x}, t)]_\Sigma = 0 \quad , \quad [g_D(\mathbf{x}, t; \mathbf{x}', t')]_\Sigma = 0 \tag{2a}$$

$$[\partial_z \psi(\mathbf{x}, t)]_\Sigma = 0 \quad , \quad [\partial_z g_N(\mathbf{x}, t; \mathbf{x}', t')]_\Sigma = 0 \tag{2b}$$

And for fields null at infinity, the integral equations for harmonic fields [1] are replaced by:

$$\psi(\mathbf{x}, t) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' [g_D(\mathbf{x}, t; \mathbf{x}', t') \partial_{z'} \psi(\mathbf{x}', t')]_{z'=0} \tag{3a}$$

$$\psi(\mathbf{x}, t) = -\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' [\psi(\mathbf{x}', t') \partial_{z'} g_N(\mathbf{x}, t; \mathbf{x}', t')]_{z'=0} \tag{3b}$$

It is a simple exercise to check that the solution (3a) (resp. (3b)) of the wave equation satisfies the boundary condition (2a) (resp. (2b)). Let $g(\mathbf{x}, t; \mathbf{x}', t')$ be the free space Green's function of the wave equation in the conventional formalism, g is the Dirac distribution

$$g(\mathbf{x}, t; \mathbf{x}', t') = R^{-1} \delta[R - c(t - t')] \ , \ R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 \tag{4}$$

while here we define g as the inverse Laplace transform with respect to the variable $s = ik$ of the free space Green's function $g(\mathbf{x}, \mathbf{x}'; k)$ of the Helmholtz equation (Eq. (12) of 11])

$$g(\mathbf{x}, \mathbf{x}'; k) = (i/8\pi^2) \iint_{-\infty}^{\infty} d\beta d\gamma k_z^{-1} \exp[i\beta(x - x') + i\gamma(y - y') + ik_z |z - z'|), \quad k_z^2 = k^2 - \beta^2 - \gamma^2 \tag{5}$$

so that with $s_z = (s^2 + \beta^2 + \gamma^2)^{1/2}$

$$g(\mathbf{x}, t; \mathbf{x}', t') = (ic/16\pi^3) \int_{Br} ds \iint_{-\infty}^{\infty} d\beta d\gamma s_z^{-1} \Phi(\beta, \gamma, s) \exp(s_z|z - z'|) \tag{6}$$

$$\Phi(\beta, \gamma, s) = \exp[s(ct - ct') + i\beta(x - x') + i\gamma(y - y')] \tag{6a}$$

The Bromwich contour Br in the integral (6) is made of a line L parallel to the imaginary axis with all the singularities of the integrand on its left. Then, the Green's functions g_D, g_N are obtained from g by the method of images with $\xi = (x, y, -z)$ being the Σ-mirrored point of $\mathbf{x}$

$$\begin{aligned} g_D(\mathbf{x}, t; \mathbf{x}', t') &= g(\mathbf{x}, t; \mathbf{x}', t') - g(\xi, t; \mathbf{x}', t'), \\ g_N(\mathbf{x}, t; \mathbf{x}', t') &= g(\mathbf{x}, t; \mathbf{x}', t') + g(\xi, t; \mathbf{x}', t'), \end{aligned} \tag{7}$$

(the conventional formalism uses the image point ξ' of x' with respect to Σ').

We illustrate this integral formalism on the simple case of a truncated plane wave ψ_i incident from the region $z < 0$ on the perfectly conducting plane $z = 0$:

$$\psi_i(z, t) = f(t - z/c)[U(t - z/c) - U(t - \tau - z/c)] \tag{8}$$

in which U is the unit step function, f an arbitrary function with partial derivatives while τ is the duration of the incident pulse. According to the Descartes-Snell law, the reflected pulse $\psi_r(z, t)$ is deduced from (8) by changing z into $-z$ so that the total field is on the Σ plane

$$[\psi(z, t)]_\Sigma = 2f(ct)[U(ct) - U(ct - c\tau)] \tag{9}$$

$\psi(z, t)$ satisfies the Newmann boundary condition (2b), so this pulse must be a solution of the integral equation (3b) and we need $[\partial_{z'} g_N]_\Sigma$. Now, we get from (6) and (7)

$$\begin{aligned} g_N(\mathbf{x}, t; \mathbf{x}', t') = (-ic/16\pi^3) \int_{Br} ds \iint_{-\infty}^{\infty} d\beta d\gamma s_z^{-1} \Phi(\beta, \gamma, s) \\ \cdot \{\exp(s_z|z - z'|) + \exp(s_z|z + z'|)\} \end{aligned} \tag{10}$$

and using the relations

$$\begin{aligned} &[z - z']_{z'=0} = -z \quad , \quad [\partial_{z'}|z - z'|]_{z'=0} = 1 \qquad z < 0 \\ &[z + z']_{z'=0} = z \quad , \quad [\partial_{z'}|z + z'|]_{z'=0} = 1 \qquad z > 0 \end{aligned} \tag{11}$$

we get

$$[\partial_{z'} g_N(\mathbf{x}, t; \mathbf{x}', t')]_{z'=0} = (ic/8\pi^3) \int_{Br} ds \iint_{-\infty}^{\infty} d\beta d\gamma \Phi(\beta, \gamma, s) \cosh(s_z z) \tag{12}$$

Substituting (9) (written for the point source t') and (12) into (3b) and taking into account (6a) give

$$\begin{aligned} \psi(z,t) &= (c/4i\pi^3) \int_{Br} ds \exp(sct) \\ &\quad \cdot \iint_{-\infty}^{\infty} d\beta d\gamma \exp(i\beta x + i\gamma y) \cosh(s_z z) F(\beta, \gamma, s) \end{aligned} \tag{13}$$

in which $F(\beta, \gamma, s)$ is the integral

$$\begin{aligned} F(\beta, \gamma, s) &= \iint_{-\infty}^{\infty} dx' dy' \exp(-i\beta x' - i\gamma y') \int_0^{\tau} dt' f(ct') \exp(-sct') \\ &= 4\pi^2 \delta(\beta)\delta(\gamma) \int_0^{\tau} dt' f(ct') \exp(-sct') \end{aligned} \tag{13a}$$

so that changing the order of integration, the integral equation (13) becomes

$$\begin{aligned} \psi(z,t) &= (c/i\pi) \int_0^{\tau} dt' f(ct') \int_{Br} ds \exp(sct - sct') \\ &\quad \cdot \iint_{-\infty}^{\infty} d\beta d\gamma \delta(\beta)\delta(\gamma) \exp(i\beta x + i\gamma y) \cosh(s_z z) \\ &= \int_0^{\tau} dt' f(ct') [B_+(z,t') + B(z,t')] \end{aligned} \tag{14}$$

with the functions $B_\pm$ defined by the relations

$$\begin{aligned} B_\pm(z,t) &= (c/2i\pi) \int_{Br} ds \exp[sct - sc(t' \pm z/c)] \\ &= cL^{-1}[-s(ct' \pm z)] \end{aligned} \tag{14a}$$

in which L^{-1} is the inverse Laplace operator and using the well known Laplace transform formula $L^{-1}\{\exp(-as)\} = \delta(t-a)$ for $a > 0$ that we write $\delta(t-a)U(a)$ we get

$$B_\pm(z,t') = c\delta(ct - ct' \pm z')U(ct' \pm z) \tag{15}$$

Substituting (15) into (14) gives

$$\psi(z,t) = c \int_0^{\tau} dt' f(ct') [U(ct' - z)\delta(ct - ct' + z) + U(ct' + z)\delta(ct - ct' - z)] \tag{16}$$

Now the relation $\int_a^b dx f(x)\delta(x-x_0) = f(x_0)$ requires x_0 in the interval $[a,b]$

$$\int_a^b dx f(x)\delta(x-x_0) = f(x_0)[U(x_0-a) - U(x_0-b)] \tag{17}$$

So, taking into account (17), the integral (16) becomes

$$\begin{aligned}\psi(z,t) &= f(ct-z)U(t)[U(ct-z) - U(ct-c\tau-z)] \\ &\quad + f(ct+z)[U(ct+z) - U(ct-c\tau+z)] \\ &= \psi_i(z,t) + \psi_r(z,t)\end{aligned} \tag{18}$$

which is the expected result. In this relation, the step function $U(t)$ implies that (18) makes sense only for $t > 0$ in agreement with the fact that the pulse (8) reaches the plane $z = 0$ at the time $t = 0$ so that no reflected field can exist before this time. One has a similar result for a pulse (8) impinging with an arbitrary angle and incidently, this proves the validity of the Descartes-Snell law for reflection of pulses on perfectly conducting planes.

3. PULSE DIFFRACTION AT PLANE APERTURES

We now show how this integral formulation can be used to deal with diffraction of pulses at a hole in a perfectly conducting plane assuming that a hole with a finite area in an infinite plane does not change the Green's functions. We suppose that ψ_i impinges from the region $z < 0$ on the plane $z = 0$ punctured by the aperture A. The total field is given for $z < 0$ by

$$\psi(\mathbf{x},t) = \psi_i(\mathbf{x},t) + \psi_s(\mathbf{x},t) \ , \ \psi_s(\mathbf{x},t) = \psi_r(\mathbf{x},t) + \psi_d(\mathbf{x},t) \quad z < 0 \tag{19}$$

in which ψ_s, ψ_r, ψ_d, are the scattered, reflected and diffracted fields while $\psi_+(\mathbf{x},t)$ is the field in the region $z > 0$. We impose on the plane $z = 0$ the boundary conditions

$$[\partial_z \psi(\mathbf{x},t)]_{\Sigma - A} = 0 \tag{20a}$$

$$[\psi(\mathbf{x},t) - \psi_+(\mathbf{x},t)]_A = 0 \tag{20b}$$

The Neumann condition (20a) outside the aperture is also valid for ψ_+ while (20b) implies the continuity of the total field through the aperture. So, we use the integral equation (3b) that we write in the illuminated region $z < 0$

$$\psi(\mathbf{x},t) = \psi_\Sigma(\mathbf{x},t) + \psi_A(\mathbf{x},t) \qquad , \qquad z < 0 \tag{21}$$

$$\begin{aligned}\psi_\Sigma(\mathbf{x},t) &= \psi_i(\mathbf{x},t)+\psi_r(\mathbf{x},t)\\ &= -\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'[\psi(\mathbf{x}',t')\partial_{z'}g_N(\mathbf{x},t;\mathbf{x}',t')]_{z'=0} \qquad (21a)\\ \psi_A(\mathbf{x},t) &= -\int_{-\infty}^{\infty}dt'\int_A dx'dy'[\psi_+(\mathbf{x}',t')-\psi_i(\mathbf{x}',t')-\psi_r(\mathbf{x}',t')]_{z'=0}\\ &\quad\cdot[\partial_{z'}g_N(\mathbf{x},t;\mathbf{x}',t')]_{z'=0} \qquad (21b)\end{aligned}$$

$[\psi(\mathbf{x}',t')]_{z'=0}$ in (21a) is the total field on the perfectly conducting plane in absence of aperture.

Let us prove that the boundary conditions (20a,b) are fulfilled by (21). First, according to (6a) and (12)

$$\begin{aligned}\{[\partial_{z'}g_N(\mathbf{x},t;\mathbf{x}',t')]_{z'=0}\}_{z=0} &= (ic/8\pi^3)\int_{Br}ds\iint_{-\infty}^{\infty}d\beta d\gamma\Phi(\beta,\gamma,s)\\ &= -\delta(t-t')\delta(x-x')\delta(y-y') \qquad (22a)\end{aligned}$$

while we still get from (12)

$$\{\partial_z[\partial_{z'}g_N(\mathbf{x},t;\mathbf{x}',t')]_{z'=0}\}_{z=0}=0 \qquad (22b)$$

So $[\partial_z\psi(\mathbf{x},t)]_{z=0}=0$ according to (21a,b) and (22b) which implies the condition (20a). Now, substituting (22a) into (21a) gives

$$[\psi_\Sigma(\mathbf{x},t)]_{z=0}=[\psi_i(\mathbf{x},t)+\psi_r(\mathbf{x},t)]_{z=0} \qquad (23a)$$

while substituting (22a) into (21b) and using the 2D-generalization of (17), we get

$$[\psi(\mathbf{x},t)]_{z=0}=[\psi_+(\mathbf{x},t)-\psi_i(\mathbf{x},t)-\psi_r(\mathbf{x},t)]_{z=0}U(x,y\in A) \qquad (23b)$$

nonnull only in the aperture A. So, from (21) and (23a,b):

$$\begin{aligned}[\psi(\mathbf{x},t)]_{z=0} &= [\psi_i(\mathbf{x},t)+\psi_r(\mathbf{x},t)]_{z=0}\\ &\quad+[\psi_+(\mathbf{x},t)-\psi_i(\mathbf{x},t)-\psi_r(\mathbf{x},t)]_{z=0}U(x,y\in A) \qquad (24)\end{aligned}$$

which is the continuity condition (20b) through the aperture. Then, the solution of the wave equation supplied by the integral equation (21) satisfies the boundary conditions (20a,b).

Now the Babinet principle applied to (21b) gives for the total field $\psi_+(\mathbf{x},t)$ in the shadow region $z>0$ the integral equation

$$\begin{aligned}\psi_+(\mathbf{x},t) &= \int_{-\infty}^{\infty}dt'\int_A dx'dy'[\psi_+(\mathbf{x}',t')-\psi_i(\mathbf{x}',t')-\psi_r(\mathbf{x}',t')]_{z'=0}\\ &\quad\cdot[\partial_{z'}g_N(\mathbf{x},t;\mathbf{x}',t')]_{z'=0} \qquad z>0 \qquad (25)\end{aligned}$$

so that to get the diffracted pulse one has just to solve (25). Substituting the solution of (25) into (21b) gives the component ψ_A of ψ in the illuminated region $z < 0$.

Remark. One could use the integral equation (3a) with the boundary conditions

$$[\psi(\mathbf{x}, t)]_{\Sigma - A} = 0 \qquad , \qquad [\partial_z\{\psi(\mathbf{x}, t) - \psi_+(\mathbf{x}, t)\}]_A = 0 \tag{26}$$

and, in this case, the Green's function g_D on $z' = 0$ has the form

$$[g_D(\mathbf{x}, t; \mathbf{x}', t')]_{z'=0} = (ic/8\pi^3)\int_{Br} ds \iint_{-\infty}^{\infty} d\beta d\gamma s_z^{-1}\Phi(\beta, \gamma, s)\sinh(s_z z) \tag{27}$$

with $\Phi(\beta, \gamma, s)$ still given by (6a) but the situation is a bit harder to handle because the integral equations in the regions $z < 0$ and $z > 0$ are coupled.

4. DIFFRACTION OF RECTANGULAR PULSE MODULATED SIGNALS

4.1. Solution of the Integral Equation

We apply the approach discussed in the previous section to the diffraction by an aperture A of a rectangular pulse modulated signal ψ_i incident from the region $z < 0$

$$\psi_i(\mathbf{x}, t) = f(ct - Z_i)[U(ct - Z_i) - U(ct - c\tau - Z_i)] \tag{28}$$

$$Z_i = x\sin\theta\sin\phi + y\sin\theta\cos\phi + z\cos\theta \tag{28a}$$

The reflected field is according to the Descanes-Snell law

$$\psi_r(\mathbf{x}, t) = f(ct - Z_r)[U(ct - Z_r) - U(ct - c\tau - Z_r)] \tag{29}$$

$$Z_r = x\sin\theta\sin\phi + y\sin\theta\cos\phi - z\cos\theta \tag{29a}$$

so that on the plane $\Sigma'(z' = 0)$, we get

$$\begin{aligned}\psi_0(\mathbf{x}', t') &= [\psi_i(\mathbf{x}', t') + \psi_r(\mathbf{x}', t')]_{z'=0} \\ &= 2f(ct' - Z')[U(ct' - Z') - U(ct' - c\tau - Z')]\end{aligned} \tag{30}$$

with

$$Z' = x'\sin\theta\sin\phi + y'\sin\theta\cos\phi \tag{30a}$$

We now prove that the solution of the equation (25) is obtained by taking inside the aperture

$$2[\psi_+(\mathbf{x}', t')]_A = \psi_0(\mathbf{x}', t')U(x, y \in A) \tag{31}$$

Substituting (31) into (25) and taking into account (30) give

$$2\psi_+(\mathbf{x},t) \;=\; -\int_{-\infty}^{\infty} dt' \int_A dx'dy'[\psi_i(\mathbf{x}',t') + \psi_r(\mathbf{x}',t')]_{z'=0} \cdot [\partial_{z'} g_N(\mathbf{x},t;\mathbf{x}',t')]_{z'=0} \qquad z>0 \tag{32}$$

so that one has just to prove that the condition (31) is satisfied by the integral (32): substituting (12) and (30a) into (32), we get

$$\psi_+(\mathbf{x},t) = \int_A dx'dy' \int_{Z'}^{Z'+c\tau} dt' f(ct'-Z')H(\mathbf{x},t;\mathbf{x}',t') \tag{33}$$

$$H(\mathbf{x},t;\mathbf{x}',t') \;=\; (c/8i\pi^3)\int_{Br} ds\,\exp(sct-sct') \cdot \iint_{-\infty}^{\infty} d\beta d\gamma\,\exp[i\beta(x-x')+i\gamma(y-y')]\cosh(s_z z) \tag{33a}$$

and this last expression becomes for $z=0$

$$[H(\mathbf{x},t;\mathbf{x}',t')]_{z=0} = \delta(t-t')\delta(x-x')\delta(y-y') \tag{34}$$

So, according to (33), (34) and using (17):

$$\begin{aligned}[\psi_+(\mathbf{x},t)]_{z=0} &= \int_A dx'dy'\delta(x-x')\delta(y-y')\int_{Z'}^{Z'+c\tau} dt' f(ct'-Z')\delta(t-t') \\ &= \int_A dx'dy'\delta(x-x')\delta(y-y')f(ct-Z') \\ &\quad\cdot [U(ct-Z')-U(ct-c\tau-Z')] \\ &= f(ct-Z)[U(ct-Z)-U(ct-c\tau-Z)]U(x,y\in A)\end{aligned} \tag{35}$$

which is the relation (31) written on the action plane. Thus, the integral (32) gives the field diffracted by the plane aperture in the shadow region $z>0$ and an expression somewhat easier to handle than (33) is obtained by exchanging the order of integration

$$\psi_+(\mathbf{x},t) \;=\; (c/8i\pi^3)\int_{Br} ds\,\exp(sct) \cdot \iint_{-\infty}^{\infty} d\beta d\gamma\,\exp[i\beta x + i\gamma y)\cosh(s_z z)J(\beta,\gamma,s) \qquad z>0 \tag{36}$$

$$J(\beta,\gamma,s) = \int_A dx'dy' \int_{Z'}^{Z'+c\tau} dt' f(ct'-Z')\exp(-i\beta x' - i\gamma y' - sct') \tag{36a}$$

Some approximations of this analytical solution are needed to obtain expressions simple enough to make numerical calculations practicable. We show in the next section on a simple example the kind of difficulties to be met in calculations.

4.2. A Simple Example

We consider a rectangular pulse modulated harmonic wave incident normally on a slit $x \in (-a, a)$ in the plane $z = 0$

$$\psi_i(z,t) = \exp(ikct - z)[U(ct - z) - U(ct - c\tau - z)] \tag{37}$$

ψ_r is obtained by changing z into $-z$ in (37) and according to (30)

$$[\psi_0(z',t')]_{z'=0} = 2\exp(ikct')[U(ct') - U(ct' - c\tau)] \tag{38}$$

We suppose in addition that (z, x) is the plane of incidence so that one has to deal with a 2D-problem and the integral (36) becomes with $s_z = (s^2 + \beta^2)^{1/2}$

$$\psi_+(z,x,t) = (c/4i\pi^2)\int_{Br} ds\exp(sct)\int_{-\infty}^{\infty} d\beta\exp(i\beta x)\cosh(s_z z)J(\beta,s) \tag{39}$$

$$\begin{aligned} J(\beta,s) &= \int_{-a}^{a} dx'\exp(-i\beta x')\int_0^{\tau} dt'\exp(-sct' + ikx') \\ &= -2[\beta(s - ik)]^{-1}\sin(\beta a)[1 - \exp(ikc\tau - sc\tau)] \end{aligned} \tag{39a}$$

Taking into account (39a), we may write (39)

$$\psi_+(z,x,t) = \chi(z,x,t) - \chi(z,x,t;\tau) \tag{40}$$

$$\chi(z,x,t) = (-1/2i\pi^2)\int_{Br} ds(s - ik)^{-1}I(z,x,s) \tag{40a}$$

$$\chi(z,x,t;\tau) = (-1/4\pi^2)\exp(ikc\tau)\int_{Br} ds(s-ik)^{-1}\exp(sct-sc\tau)I(z,x,s) \tag{40b}$$

within these integrals

$$I(z,x,s) = \int_{-\infty}^{\infty} d\beta\beta^{-1}\sin(\beta a)\exp(i\beta x)\cosh(s_z z) \tag{41}$$

All these expressions are exact but, to discuss their physical content, we need some approximation of (41). In fact, one is generally interested in the diffraction pattern far from the slit for large positive z, so neglecting $\exp(-s_z z)$ that tends to zero reduces the integral (41) to

$$2I(z,x,s) = \int_{-\infty}^{\infty} d\beta\beta^{-1}\sin(\beta a)\exp(i\beta x)\exp(s_z z) \tag{41a}$$

with an asymptotic approximation supplied by the method of the steepest descent [2]: let $I_1(z)$ be the integral in which $f(\beta)$ is analytic and positive

$$I_1(z) = \int_{-\infty}^{\infty} d\beta h(\beta)\exp[zf(\beta)] \tag{42}$$

then, for large $z > 0$

$$I_1(z) \sim 2\exp[zf(\beta_0)] \sum_{m=0}^{\infty} \Gamma(m+1/2)a_{2m}/z^{m+1/2} \qquad (42a)$$

the saddle point β_0 is the root of $f'(\beta) = 0$ and Γ the gamma function. Formulas for the first two coefficients are [2]

$$\begin{aligned} a_0 &= h/(2f'')^{1/2}, \\ a_2 &= \{2h'' - 2f^{iii}h'/f'' + [(f^{iii}\sqrt{5}/f''\sqrt{6})^2 - h^{iv}/2f'']h\}(2h'')^{-3/2} \end{aligned} \qquad (42b)$$

where f, h and their derivatives are evaluated at $\beta = \beta_0$.

In the integral (41a) $f(\beta) = s_z = (s^2 + \beta^2)^{1/2}$ so that $\beta_0 = 0$ and applying (42a) limited to its first term, we get according to (42b)

$$I(z,x,s) \sim a(\pi s/2z)^{1/2}\exp(sz) \qquad z \Rightarrow \infty \qquad (43)$$

This approximation which does not depend on x has the drawback to give no information on the trans- verse spreading of the diffracted pulse which would require the second term of the expansion (42a). Then, substituting (43) into (40a,b) gives

$$\begin{aligned} \chi(z,x,t) &= -a(1/2\pi z)^{1/2}(1/2\pi i) \\ &\quad \cdot \int_{Br} ds\, s^{1/2}(s-ik)^{-1}\exp(sct+sz) \qquad (44a) \\ \chi(z,x,t;\tau) &= -a(1/2\pi z)^{1/2}\exp(ikc\tau)(1/2i\pi) \\ &\quad \cdot \int_{Br} ds\, s^{1/2}(s-ik)^{-1}\exp(sct-sc\tau+sz) \quad (44b) \end{aligned}$$

Assuming $ct + z > 0$ and $ct - c\tau + z > 0$, we perform the integration of (44a,b) on the contour of Fig. 1, both integrals having a pole at $s = ik$ and a branch point at $s = 0$. The pole contribution to (44a) is given by the method of residues

$$\chi_1 = -a(ick/2\pi z)^{1/2}\exp(ikct + ikz)U(ct+z) \qquad (45a)$$

in which the unit step function reminds that this result requires $ct + z > 0$. To get the branch point contribution, we first note that on the small circle δ of radius ε at the origin $|s| \ll |k|$ and the Bromwich integral vanishes when $|s| \Rightarrow 0$. Then, on the upper (resp.lower) line of the barrier along the negative side of the real axis, we write $s = \xi\exp(-i\pi)$ (resp. $s = \xi\exp(i\pi)$) yielding for the Bromwich integral

(44a) the two contributions

$$-(1/2\pi)\int_{\infty}^{0} d\xi\,\xi^{1/2}(\xi+ik)^{-1}\exp(-\xi ct-\xi z),$$
$$-(1/2\pi)\int_{0}^{\infty} d\xi\,\xi^{1/2}(\xi+ik)^{-1}\exp(-\xi ct-\xi z)$$

summing these two integrals gives the branch point contribution to (44a)

$$\chi_2 = -a(1/2\pi^3 z)^{1/2}\varpi(z,t)U(ct+z) \tag{45b}$$

with

$$\varpi(z,t) = \int_{0}^{\infty} d\xi\,\xi^{1/2}(\xi+ik)^{-1}\exp(-i\xi ct-\xi z) \tag{46}$$

Then, according to (45a) and (45b)

$$\chi(z,x,t) = -a(1/2\pi^3 z)^{1/2}[(2i\pi k)^{1/2}\exp(ikct+ikz)+\varpi(z,t)]U(ct+z) \tag{47a}$$

A similar calculation gives for (44b)

$$\begin{aligned}\chi(z,x,t;\tau) \;=\; & -a(1/2\pi^3 z)^{1/2}[(2i\pi k)^{1/2}\exp(ikct+ikz)\\ & +\exp(ikc\tau)\varpi(z,t-\tau)]U(ct-c\tau+z)\end{aligned} \tag{47b}$$

Substituting (47a,b) into (40) gives the field diffracted by the slit in which $\exp(ikct+ikz)$ represents the propagation of the incident field in the shadow region $z>0$ while ϖ is the diffracted pulse.

To analyse more closely the diffraction pattern, we look for an asymptotic approximation of $\varpi(z,t)$ and of $\varpi(z,t-\tau)$ when $ct+z$ and $ct-c\tau+z$ are large. In this case, the major part of the integral (46) will be obtained for values of ξ in the neighborhood of the origin then, expanding the denominator in ascending powers of ξ, we get

$$\varpi(z,t) = (1/ik)\int_{0}^{\infty} d\xi \sum_{n=0}^{\infty}(-1/ik)^n \xi^{n+1/2}\exp[-\xi(ct+z)] \tag{48}$$

and assuming the uniform convergence of the series making possible to exchange integration and summation, the term by term integration gives

$$\varpi(z,t) = (1/ik)\sum_{n=0}^{\infty}(-1/ik)^n\Gamma(n+3/2)(ct+z)^{-n-3/2} \tag{48a}$$

Taking the first term only of (48a) the expressions (47a) and (47b) become

$$\chi(z,x,t) = -a(ik/\pi^2 z)^{1/2}[\exp(ikct+ikz)+\{2ik(ct+z)\}^{-2/3}]U(ct+z) \tag{49a}$$

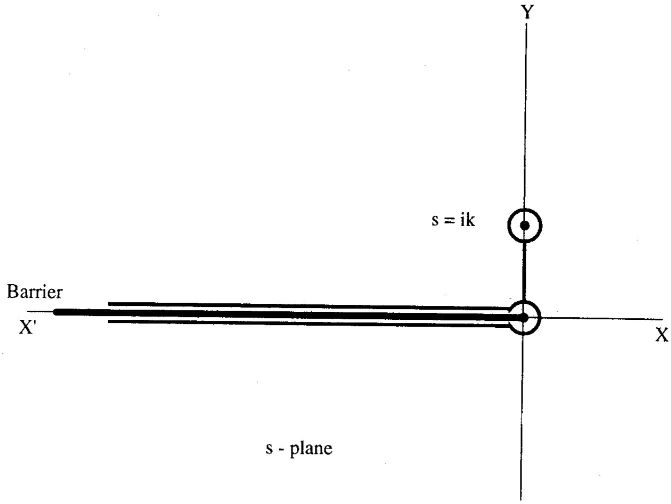

Figure 1. Bromwich contour on the s-plane.

$$\begin{aligned}\chi(z,x,t;\tau) \;=\; & -a(ik/\pi^2 z)^{1/2}[\exp(ikct+ikz)+\exp(ikc\tau) \\ & \cdot\{2ik(ct-c\tau+z)^{-2/3}\}]U(ct-c\tau+z) \qquad (49b)\end{aligned}$$

For $ct+z$ and $ct-c\tau+z$ large enough, the second term in the square bracket of (49a,b) is very small and the contribution of the diffracted field to the scalar pulse becomes negligible so that the pulse appears as conveying the image of the slit with a duration τ and a $z^{-1/2}$ fading.

At the expense of some more calculations, one would obtain in the same way the diffraction by a slit of a truncated harmonic pulse incident with an angle θ. To illustrate what happens in this case, we consider the diffraction of a Dirac pulse

$$\psi_i(x,z,t) = \delta(ct - x\sin\theta - z\cos\theta) \qquad (50)$$

so that according to (30)

$$[\psi_0(x',z',t']_{z'=0} = 2\delta(ct' - x'\sin\theta) \qquad (51)$$

The integral (39) is still valid but the function $J(\beta,s)$ becomes

$$\begin{aligned}J(\beta,s) \;=\; & \int_{-a}^{a} dx' \exp(-i\beta x') \int_{-\infty}^{\infty} dt'\delta(ct'-x'\sin\theta)\exp(-sct') \\ =\; & 2c^{-1}(i\beta + s\sin\theta)^{-1}\sinh(ia\beta + as\sin\theta) \qquad (52)\end{aligned}$$

Substituting (52) into (39) gives

$$\begin{aligned}\psi_+(x,z,t) &= (1/2i\pi^2)\int_{Br} ds\exp(sct)\int_{-\infty}^{\infty} d\beta(i\beta+s\sin\theta)^{-1} \\ &\quad\cdot\sinh(ia\beta+as\sin\theta)\exp(i\beta x)\cosh(s_z z) \\ &= \chi_a(x,z,t)-\chi_a(x,z,t) \end{aligned} \tag{53}$$

with

$$\chi_{\pm a}(x,z,t) = (1/4i\pi^2)\int_{Br} ds\exp(sct\pm as\sin\theta)I_{\pm a}(x,z,s) \tag{53a}$$

and

$$I_{\pm a}(x,z,s) = \int_{Br} d\beta(i\beta+s\sin\theta)^{-1}\exp[i\beta(x\pm a)\cosh(s_z z) \tag{54}$$

For large positive z, one may neglect $\exp(-s_z z)$ and using the first term of the approximation (42a), we get

$$I_{\pm a}(x,z,s) \sim \mathrm{cosec}\theta s^{-1/2}(2\pi/z)^{1/2}\exp(sz) \qquad z\Rightarrow\infty \tag{55}$$

Substituting (55) into (54) gives

$$\chi_{\pm a}(x,z,s) = \mathrm{cosec}\theta(1/2\pi z)^{1/2}(1/2\pi i)\int_{Br} s^{-1/2}\exp(sct+sz\pm as\sin\theta) \tag{56}$$

The only singularity of the integrand is a branch point at $s=0$ so, still using the contour of Fig, 1, we get easily

$$\chi_{\pm a}(x,z,s) = \mathrm{cosec}\theta(1/2\pi z)^{1/2}(ct+z\pm a\sin\theta)^{-1/2}U(ct+z\pm a\sin\theta) \tag{56a}$$

In this case also one should have to use the second term of (42a) to get the transverse spreading of the Dirac pulse.

These results show that the asymptotic approximation (42a) simplifies notably calculations; it should be possible for instance to get with only some extra work the diffraction pattern for a rectangular pulse modulated harmonic wave incident on a rectangular aperture.

5. DISCUSSION

Signals used in communication technology have a finite duration and they can be described by expressions similar to (18). Then, their diffraction at plane apertures resorts in the present approach to Eq.

(36). And, one has to handle the calculation of this integral that we write

$$\psi_+(\mathbf{x},t) = (c/2i\pi)\int_{Br} ds\exp(sct)\phi(\mathbf{x},s), \qquad z>0 \tag{57}$$

with

$$\phi(\mathbf{x},s) = \iint_{-\infty}^{\infty} d\beta d\gamma \exp(i\beta x + i\gamma y)\cosh(s_z z)J(\beta,\gamma,s) \tag{58a}$$

$$J(\beta,\gamma,s) = (1/4\pi^2)\int_A dx'dy'\int_{Z'}^{Z'+c\tau} dt' f(ct'-Z')\exp(-i\beta x' - i\gamma y' - sct') \tag{58b}$$

so that the calculation of ψ_+ can be made in three steps. One has first to perform the integral (58b) whose evaluation depends strongly on the function f and on the form of the aperture A. Fortunately, in present day technology, f can be represented by a truncated harmonic function or by a sequence of step functions making rather easy the calculation of the t'-integral. So, one is left with the integral on A and calculations are simplified when curvilinear coordinates can be chosen so that one of the coordinate lines coincides with the boundary of the aperture. In addition, when f is a step function, one has a result similar to that obtained for Fraunhofer diffraction [3]: let x', y' denote the curvilinear coordinates and A_1 be an aperture such that the extension of A_1 in a particular direction is μ times that of A, then:

$$J_1(\beta,\gamma,s) = \mu J(\mu\beta,\gamma,s) \tag{59}$$

in which J_1 is the function J for the aperture A_1. So the integral (58b) for an aperture A with the form of an ellipse or a parallelogram may be obtained from that of a circle or a rectangle respectively.

As a second step, one has to perform the integral (58a) and if one is only interested in the diffraction pattern far from the aperture, one may use an asymptotic approximation for large $z > 0$. Neglecting $\exp(-s_z z)$ and applying for instance (42a) limited to the first term of (42b) successively to the integrals on γ and on β, we get since the saddle points are at $\beta_0 = 0,\ \gamma_0 = 0$

$$\begin{aligned} 2\phi(\mathbf{x},s) &\sim (2\pi/z)^{1/2}\int_{-\infty}^{\infty} d\beta(s^2+\beta^2)^{1/2}J(\beta,0,s)\exp(i\beta x) \\ &\quad \cdot \exp[z(s^2+\beta^2)^{1/2}] \\ &\sim (\pi s/z)J(0,0,s)\exp(sz) \qquad z \Rightarrow +\infty \end{aligned} \tag{60}$$

but, as previously noticed, this approximation gives no information on the transverse spreading of the diffracted pulse which would require the second term of (42b) with more intricate calculations.

Finally, one is left with the Bromwich integral (57) which becomes with the approximation (60)

$$\psi_+(\mathbf{x},t) = (c/2\pi z)\int_{Br} sdsJ(0,0,s)\exp(sct+sz) \qquad z \Rightarrow +\infty \quad (61)$$

which presents no particular difficulty using a convenient contour as discussed in Section 4.

The integrals (57) and (58a,b) could also be numerically approximate which seems to be the only possibility for an arbitrary aperture A but this numerical approach has still to be made. One must note the existence of powerful codes to perform the inverse of the Laplace transform [4].

This work is made in the spirit of the traditional approach of deriving, sometime approximate, solutions for idealized problems rather than producing numerical solutions for more realistic problems. Such a position could seem to be out of fashion at a time when to run and assess the accuracy of computer codes is a large part of the duties of most engineers, however running software is not necessarily the best way of understanding the basics of any subject. But now that theoretical formulae exist, the important work to implement numerical computations has still to be made.

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