Progress In Electromagnetics Research, PIER 34, 29–61, 2001

PLANE WAVE DIFFRACTION BY TANDEM IMPEDANCE SLITS

B. Polat

Oersted-DTU Electromagnetic Systems
Technical University of Denmark
Building 348, Oersteds Plads, DK-2800 Kgs. Lyngby, Denmark

A. Buyukaksoy

Gebze Institute of Technology
Faculty of Science
Department of Mathematics
P.O. Box 141, TR-41400 Gebze Kocaeli, Turkey

G. Cinar

Gebze Institute of Technology
Faculty of Engineering
Department of Electronics Engineering
P.O. Box 141. TR-41400 Gebze Kocaeli, Turkey

Abstract—The diffraction of E-polarized plane waves by a tandem impedance slit waveguide is investigated rigorously by using the Fourier transform technique in conjunction with the Mode Matching method. This mixed method of formulation gives rise to a *scalar* Wiener-Hopf equation of the second kind, the solution of which contains infinitely many constants satisfying an infinite system of linear algebraic equations. A numerical solution of this system is obtained for various values of the surface impedances, slit width and the distance between the slits, through which the effect of these parameters on the diffraction phenomenon are studied.

1 Introduction

2 Analysis

3 Even Excitation

1. INTRODUCTION

The diffraction of electromagnetic waves by tandem slits has received wide attention due to its importance in microwave and optical instrumentation (e.g., microwave passive filters, coupling structures, etc.). Alldredge [1] analysed the case when vertically polarized plane electromagnetic waves were falling normally upon a perfectly conducting tandem slit by using the variational procedure introduced by Levine and Schwinger [2] for planar diffraction problems. The same problem was later considered by Kashyap and Hamid [3] for oblique incidence case using the Wiener-Hopf and generalized scattering matrix techniques.

In the present work the diffraction of E_z-polarized plane waves by a tandem slit waveguide with different exterior and interior surface impedances will be analyzed rigorously by using the Wiener-Hopf technique in conjunction with the Mode Matching method. By using the classical Fourier transform technique, the related boundary value problem can generally be reduced into a modified *matrix* Wiener-Hopf equation. Except for a very restricted class of matrices no general method exists to accomplish the Wiener-Hopf factorization of an arbitrary matrix including the one related to the present problem. This mixed method of formulation, which is based on expanding the total field into a series of normal modes in the waveguide region and using the Fourier transform technique elsewhere, gives rise to a *scalar* modified Wiener-Hopf equation. Note that a variant of this method was used by Yoshidomi and Aoki [4] and Büyükaksoy and Polat [5, 6] in treating the scattering by impedance structures. The analysis can also be generalized to the case when the slits are of certain wall thickness in a straightforward manner by following the procedure given in [5, 6] for thick waveguides. The solution contains infinitely many constants satisfying an infinite system of linear algebraic equations. A numerical solution of this system is obtained for various values of the surface impedances, slit width and the distance between the plates, through which the effect of these parameters on the diffraction phenomenon are studied.

A time factor $e^{-i\omega t}$ with ω being the angular frequency is assumed and suppressed throughout the paper.

2. ANALYSIS

We consider the diffraction of an E_z-polarized plane wave by a tandem slit waveguide formed by four semi-infinite impedance plates defined by $\mathcal{S}_{1,2} = \{(x,y,z);\ x \in (-\infty,0),\ y = \pm d,\ z \in (-\infty,\infty)\}$ and $\mathcal{S}_{3,4} = \{(x,y,z);\ x \in (\ell,\infty),\ y = \pm d,\ z \in (-\infty,\infty)\}$ as depicted in Fig. 1.

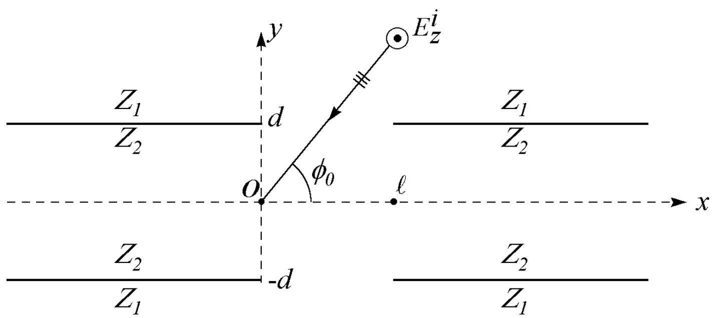

Figure 1. Tandem impedance slit waveguide.

The exterior and interior surface impedances of the waveguide are denoted by $Z_1 = \eta_1 Z_0$ and $Z_2 = \eta_2 Z_0$, respectively, with Z_0 being the characteristic impedance of the free space.

In order to determine the scattered field one can proceed by decomposing the incident wave into even and odd excitations as indicated in Fig. 2a and Fig. 2b. Relying upon the image bisection principle, it can be shown that the configurations shown in Fig. 2a and Fig. 2b are equivalent to those depicted in Fig. 2c and Fig. 2d, respectively.

In what follows the even and odd excitations will be treated separately.

3. EVEN EXCITATION

Let us consider first the configuration depicted in Fig. 2c which is equivalent to the even excitation case. Since in this case the total field is symmetrical about the plane $y = 0$, the normal derivative of the total electric field must vanish for $y = 0$, $x \in (-\infty,\infty)$ (magnetic wall).

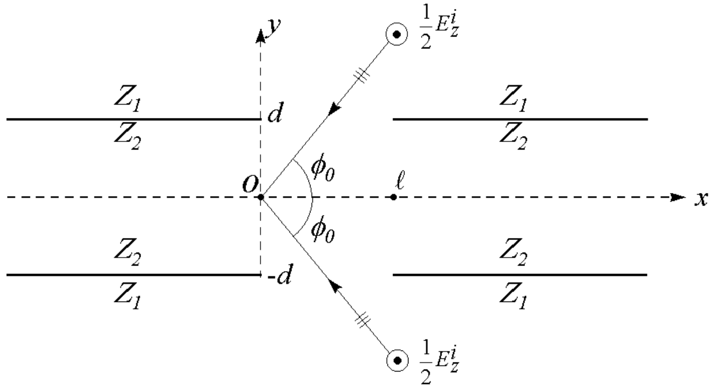

Figure 2a. Symmetric (even) excitation.

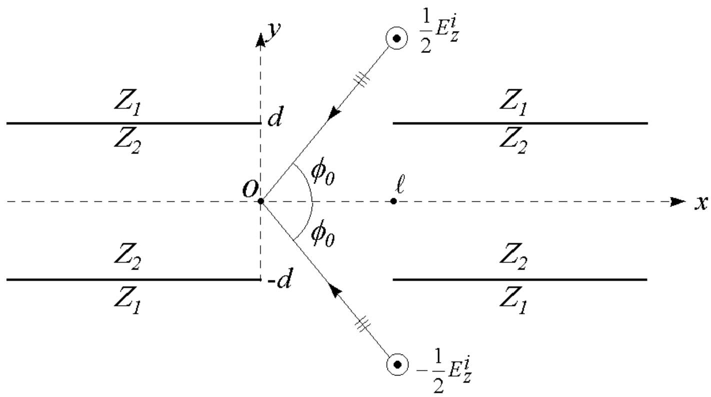

Figure 2b. Asymmetric (odd) excitation.

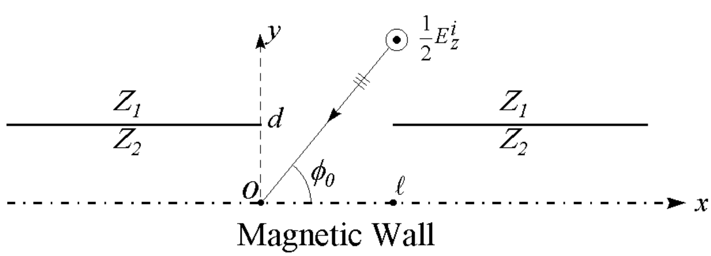

Figure 2c. Equivalence to Fig. 2a.

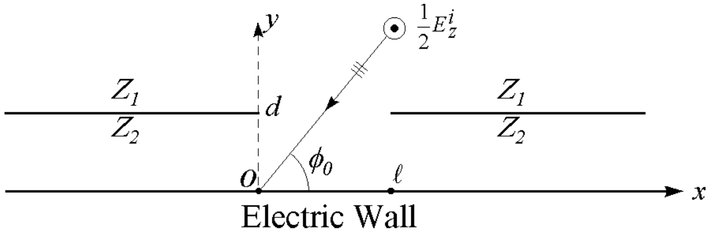

Figure 2d. Equivalence to Fig. 2b.

For analysis purposes, it is convenient to express the total field as follows:

$$u_T^{(e)}(x,y) = u^i + u^r + u_1^{(e)} \quad , \quad y > d$$
$$u_2^{(e)}\mathcal{H}(-x) + u_3^{(e)}[\mathcal{H}(x) - \mathcal{H}(x-\ell)] + u_4^{(e)}\mathcal{H}(x-\ell) \quad , \quad 0 < y < d \tag{1a}$$

Here, $\mathcal{H}$ is the unit step function, u^i is the incident field given by

$$E_z^i = u^i(x,y) = \exp\{-ik[x\cos\phi_0 + y\sin\phi_0]\} \tag{1b}$$

while u^r denotes the field reflected from the plane $y = d$, namely

$$u^r(x,y) = \frac{\eta_1 \sin\phi_0 - 1}{\eta_1 \sin\phi_0 + 1} \exp\{-ik[x\cos\phi_0 - (y-2d)\sin\phi_0]\} \tag{1c}$$

with k and ϕ_0 being the free space wave number and the angle of incidence, respectively. For the sake of analytical convenience we shall assume that k has a small imaginary part. The lossless case can then be obtained by making $\Im_m k \to 0$ at the end of the analysis. $u_j^{(e)}$, $j = 1,2,3,4$, which satisfy the Helmholtz equation, are to be determined with the aid of the following boundary and continuity relations:

$$\left(1 + \frac{\eta_1}{ik}\frac{\partial}{\partial y}\right) u_1^{(e)}(x,d) = 0 \quad , \quad x \in \{(-\infty,0) \cup (\ell,\infty)\} \tag{2a}$$
$$\left(1 - \frac{\eta_2}{ik}\frac{\partial}{\partial y}\right) u_2^{(e)}(x,d) = 0 \quad , \quad x < 0 \tag{2b}$$
$$\left(1 - \frac{\eta_2}{ik}\frac{\partial}{\partial y}\right) u_4^{(e)}(x,d) = 0 \quad , \quad x > \ell \tag{2c}$$
$$\frac{\partial}{\partial y} u_2^{(e)}(x,0) = 0 \quad , \quad x < 0 \tag{2d}$$

$$\begin{aligned}
\frac{\partial}{\partial y}u_3^{(e)}(x,0) &= 0 \quad , \quad 0<x<\ell && \text{(2e)}\\
\frac{\partial}{\partial y}u_4^{(e)}(x,0) &= 0 \quad , \quad x>\ell && \text{(2f)}\\
u_1^{(e)}(x,d)-u_3^{(e)}(x,d) &= -\frac{2\eta_1\sin\phi_0}{1+\eta_1\sin\phi_0}e^{-ikd\sin\phi_0}\,e^{-ikx\cos\phi_0}\\
&\qquad , \quad 0<x<\ell && \text{(2g)}\\
\frac{\partial}{\partial y}u_1^{(e)}(x,d)-\frac{\partial}{\partial y}u_3^{(e)}(x,d) &= \frac{2ik\sin\phi_0}{1+\eta_1\sin\phi_0}e^{-ikd\sin\phi_0}\,e^{-ikx\cos\phi_0}\\
&\qquad , \quad 0<x<\ell && \text{(2h)}\\
u_2^{(e)}(0,y) &= u_3^{(e)}(0,y) \quad , \quad 0<y<d && \text{(2i)}\\
\frac{\partial}{\partial x}u_2^{(e)}(0,y) &= \frac{\partial}{\partial x}u_3^{(e)}(0,y) \quad , \quad 0<y<d && \text{(2j)}\\
u_3^{(e)}(\ell,y) &= u_4^{(e)}(\ell,y) \quad , \quad 0<y<d && \text{(2k)}\\
\frac{\partial}{\partial x}u_3^{(e)}(\ell,y) &= \frac{\partial}{\partial x}u_4^{(e)}(\ell,y) \quad , \quad 0<y<d && \text{(2l)}
\end{aligned}$$

Since $u_1^{(e)}(x,y)$ satisfies the Helmholtz equation in the range $x \in (-\infty,\infty)$, its Fourier transform with respect to x gives

$$\left[\frac{d^2}{dy^2}+(k^2-\alpha^2)\right]F^{(e)}(\alpha,y)=0 \tag{3a}$$

with

$$F^{(e)}(\alpha,y)=F_-^{(e)}(\alpha,y)+F_1^{(e)}(\alpha,y)+e^{i\alpha\ell}F_+^{(e)}(\alpha,y) \tag{3b}$$

where

$$\begin{aligned}
F_-^{(e)}(\alpha,y) &= \int_{-\infty}^{0} u_1^{(e)}(x,y)e^{i\alpha x}dx && \text{(3c)}\\
F_1^{(e)}(\alpha,y) &= \int_{0}^{\ell} u_1^{(e)}(x,y)e^{i\alpha x}dx && \text{(3d)}\\
F_+^{(e)}(\alpha,y) &= \int_{\ell}^{\infty} u_1^{(e)}(x,y)e^{i\alpha(x-\ell)}dx && \text{(3e)}
\end{aligned}$$

By taking into account the following asymptotic behaviours of $u_1^{(e)}$ for $x\to\pm\infty$

$$u_1^{(e)}(x,y)=O(e^{\pm ikx}) \quad , \quad x\to\pm\infty \tag{4}$$

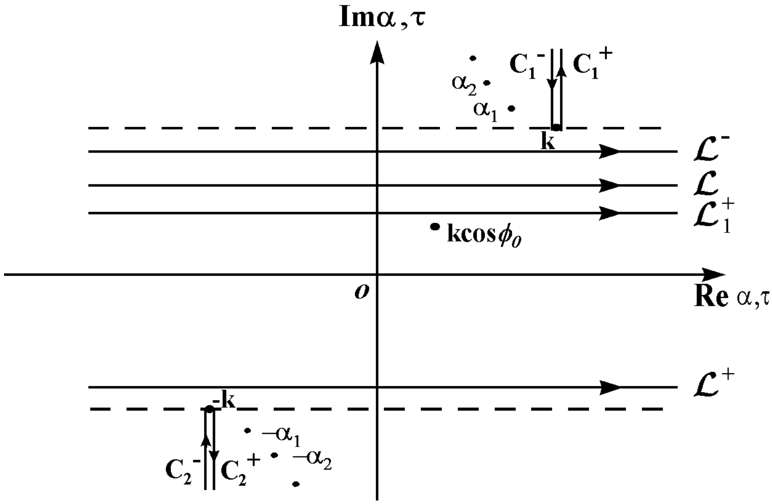

Figure 3. Complex α, τ-plane.

one can show that $F_+^{(e)}(\alpha, y)$ and $F_-^{(e)}(\alpha, y)$ are regular functions of α in the spectral half-planes $\Im m(\alpha) > \Im m(-k)$ and $\Im m(\alpha) < \Im m(k)$, respectively, while $F_1^{(e)}(\alpha, y)$ is an entire function. The general solution of (3a) satisfying the radiation condition for $y \to \infty$ reads

$$F_-^{(e)}(\alpha, y) + F_1^{(e)}(\alpha, y) + e^{i\alpha\ell} F_+^{(e)}(\alpha, y) = A^{(e)}(\alpha)\, e^{iK(\alpha)(y-d)} \tag{5a}$$

with

$$K(\alpha) = \sqrt{k^2 - \alpha^2} \tag{5b}$$

The square-root function is defined in the complex α-plane, which is denoted in Fig. 3 cut along $\alpha = k$ to $\alpha = k + i\infty$ and $\alpha = -k$ to $\alpha = -k - i\infty$, such that $K(0) = k$.

In the Fourier transform domain (2a) takes the form

$$\left[F_-^{(e)}(\alpha, d) + \frac{\eta_1}{ik}\dot{F}_-^{(e)}(\alpha, d)\right] + e^{i\alpha\ell}\left[F_+^{(e)}(\alpha, d) + \frac{\eta_1}{ik}\dot{F}_+^{(e)}(\alpha, d)\right] = 0 \tag{6}$$

where the dot specifies the derivative with respect to y. By using (5a), its derivative with respect to y, and (6), we get

$$P_1^{(e)}(\alpha) = \frac{K(\alpha)}{k\chi(\alpha)} A^{(e)}(\alpha) \tag{7a}$$

where

$$P_1^{(e)}(\alpha) = F_1^{(e)}(\alpha, d) + \frac{\eta_1}{ik}\dot{F}_1^{(e)}(\alpha, d) \tag{7b}$$

and

$$\chi(\alpha) = \left[\eta_1 + \frac{k}{K(\alpha)}\right]^{-1} \tag{7c}$$

In the region $0 < y < d$, $u_3^{(e)}(x, y)$ satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u_3^{(e)}(x, y) = 0 \tag{8}$$

in the range $0 < x < \ell$. Hence, multiplying (8) by $e^{i\alpha x}$ and integrating the resultant equation with respect to x from 0 to ℓ, we obtain

$$\left[\frac{d^2}{dy^2} + K^2(\alpha)\right] G_1^{(e)}(\alpha, y) = f^{(e)}(\alpha, y) + e^{i\alpha\ell} g^{(e)}(\alpha, y) \tag{9a}$$

with

$$f^{(e)}(\alpha, y) = \frac{\partial}{\partial x} u_3^{(e)}(0, y) - i\alpha u_3^{(e)}(0, y) \tag{9b}$$

$$g^{(e)}(\alpha, y) = -\frac{\partial}{\partial x} u_3^{(e)}(\ell, y) + i\alpha u_3^{(e)}(\ell, y) \tag{9c}$$

$G_1^{(e)}(\alpha, y)$, which is defined by

$$G_1^{(e)}(\alpha, y) = \int_0^\ell u_3^{(e)}(x, y) e^{i\alpha x} dx \tag{10}$$

is an entire function. The general solution of (9a) satisfying the Neumann boundary condition at $y = 0$ reads

$$\begin{aligned} G_1^{(e)}(\alpha, y) &= B^{(e)}(\alpha) \cos[Ky] + \frac{1}{K(\alpha)} \\ &\quad \cdot \int_0^y [f^{(e)}(\alpha, t) + e^{i\alpha\ell} g^{(e)}(\alpha, t)] \sin[K(y - t)] dt \end{aligned} \tag{11}$$

Combining (2g) and (2h), we get

$$P_1^{(e)}(\alpha) = G_1^{(e)}(\alpha, d) + \frac{\eta_1}{ik} \dot{G}_1^{(e)}(\alpha, d) \tag{12}$$

and $B^{(e)}(\alpha)$ can be solved uniquely to give

$$\begin{aligned} M^{(e)}(\alpha) B^{(e)}(\alpha) &= P_1^{(e)}(\alpha) - \int_0^d [f^{(e)}(\alpha, t) + e^{i\alpha\ell} g^{(e)}(\alpha, t)] \\ &\quad \cdot \left[\frac{\sin[K(d - t)]}{K} + \frac{\eta_1}{ik} \cos[K(d - t)]\right] dt \end{aligned} \tag{13a}$$

with

$$M^{(e)}(\alpha) = \cos[Kd] - \frac{\eta_1}{ik} K \sin[Kd] \tag{13b}$$

Replacing (13a) into (11) we get

$$\begin{aligned} G_1^{(e)}(\alpha, y) \;=\; & \frac{\cos[Ky]}{M^{(e)}(\alpha)} \left\{ P_1^{(e)}(\alpha) - \int_0^d \left[f^{(e)}(\alpha, t) + e^{i\alpha\ell} g^{(e)}(\alpha, t) \right] \right. \\ & \left. \cdot \left[\frac{\sin[K(d-t)]}{K} + \frac{\eta_1}{ik} \cos[K(d-t)] \right] dt \right\} \\ & + \frac{1}{K} \int_0^y \left[f^{(e)}(\alpha, t) + e^{i\alpha\ell} g^{(e)}(\alpha, t) \right] \sin[K(y-t)] dt \end{aligned} \tag{14}$$

Although the left-hand side of (14) is regular in the upper half-plane $\Im m(\alpha) > \Im m(-k)$, the regularity of the right-hand side is violated by the presence of simple poles occurring at the zeros of $M^{(e)}(\alpha)$, namely at $\alpha = \pm\alpha_m^e$ satisfying

$$M^{(e)}(\pm\alpha_m^e) = 0 \quad , \quad \Im m(\alpha_m^e) > \Im m(k) \quad , \quad m = 1, 2, \cdots \tag{15}$$

These poles can be eliminated by imposing that their residues are zero. This gives

$$\begin{aligned} P_1^{(e)}(\alpha_m^e) \;=\; & \frac{\sin[K_m^e d]}{2K_m^e} \left[1 - \frac{\eta_1^2}{k^2} (K_m^e)^2 \right] \\ & \cdot \nu_m^e \left[f_m^e(\alpha_m^e) + e^{i\alpha_m^e \ell} g_m^e(\alpha_m^e) \right] \end{aligned} \tag{16a}$$

$$\begin{aligned} P_1^{(e)}(-\alpha_m^e) \;=\; & \frac{\sin[K_m^e d]}{2K_m^e} \left[1 - \frac{\eta_1^2}{k^2} (K_m^e)^2 \right] \\ & \cdot \nu_m^e \left[f_m^e(-\alpha_m^e) + e^{-i\alpha_m^e \ell} g_m^e(-\alpha_m^e) \right] \end{aligned} \tag{16b}$$

where K_m^e, ν_m^e, $f_m^e(\pm\alpha_m^e)$ and $g_m^e(\pm\alpha_m^e)$ specify

$$K_m^e = K(\pm\alpha_m^e) \tag{16c}$$

$$\nu_m^e = d + \frac{\eta_1}{ik} \sin^2[K_m^e d] \tag{16d}$$

$$\begin{bmatrix} f_m^e(\pm\alpha_m^e) \\ g_m^e(\pm\alpha_m^e) \end{bmatrix} = \frac{2}{\nu_m^e} \int_0^d \begin{bmatrix} f^{(e)}(\pm\alpha_m^e, t) \\ g^{(e)}(\pm\alpha_m^e, t) \end{bmatrix} \cos[K_m^e t] dt. \tag{16e}$$

Consider now the waveguide region $0 < y < d$, $x \in \{(-\infty, 0) \cup (\ell, \infty)\}$ where the total field can be expressed in terms of Fourier Cosine series as

$$u_2^{(e)}(x,y) = \sum_{n=1}^{\infty} a_n^e \cos[\xi_n^e y]\, e^{-i\beta_n^e x} \quad , \quad x < 0 \tag{17a}$$

$$u_4^{(e)}(x,y) = \sum_{n=1}^{\infty} b_n^e \cos[\xi_n^e y]\, e^{i\beta_n^e (x-\ell)} \quad , \quad x > \ell \tag{17b}$$

with

$$\cos[\xi_n^e d] + \frac{\eta_2}{ik}\xi_n^e \sin[\xi_n^e d] = 0 \quad , \quad n = 1, 2, \cdots \tag{17c}$$

and

$$\beta_n^e = \sqrt{k^2 - (\xi_n^e)^2} \quad , \quad \Im m(\beta_n^e) > \Im m(k) \tag{17d}$$

From the continuity relations (2i-l) and (9b,c) we get

$$\frac{\partial}{\partial x} u_2^{(e)}(0,y) - i\alpha u_2^{(e)}(0,y) = f^{(e)}(\alpha, y) \quad , \quad 0 < y < d \tag{18a}$$

$$-\frac{\partial}{\partial x} u_4^{(e)}(\ell,y) + i\alpha u_4^{(e)}(\ell,y) = g^{(e)}(\alpha, y) \quad , \quad 0 < y < d \tag{18b}$$

Owing to (16e), $f^{(e)}(\alpha, y)$ and $g^{(e)}(\alpha, y)$ can be expanded into Fourier Cosine series as follows:

$$\begin{bmatrix} f^{(e)}(\alpha, y) \\ g^{(e)}(\alpha, y) \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} f_m^{(e)}(\alpha) \\ g_m^{(e)}(\alpha) \end{bmatrix} \cos[K_m^e y] \tag{19}$$

Substituting (17a,b) and (19) into (18a,b) we obtain

$$\sum_{m=1}^{\infty} f_m^e(\alpha) \cos[K_m^e y] = -i \sum_{n=1}^{\infty} a_n^e (\beta_n^e + \alpha) \cos[\xi_n^e y] \quad , \quad 0 < y < d \tag{20}$$

and

$$\sum_{m=1}^{\infty} g_m^e(\alpha) \cos[K_m^e y] = -i \sum_{n=1}^{\infty} b_n^e (\beta_n^e - \alpha) \cos[\xi_n^e y] \quad , \quad 0 < y < d \tag{21}$$

The multiplication of both sides of (20) and (21) by $\cos[K_j^e y]$ and integration from $y = 0$ to $y = d$ gives

$$f_j^e(\alpha) = -\frac{2i}{\nu_j^e} \sum_{n=1}^{\infty} \Delta_{n,j}^e (\beta_n^e + \alpha) a_n^e \quad , \quad j = 1, 2, \cdots \tag{22a}$$

$$g_j^e(\alpha) = -\frac{2i}{\nu_j^e} \sum_{n=1}^{\infty} \Delta_{n,j}^e (\beta_n^e - \alpha) b_n^e \quad , \quad j = 1, 2, \cdots \tag{22b}$$

with $\Delta^e_{n,j}$ being defined by

$$\Delta^e_{n,j} = \frac{\eta_1+\eta_2}{ik}\frac{K^e_j\xi^e_n}{(\xi^e_n)^2-(K^e_j)^2}\sin[K^e_jd]\sin[\xi^e_nd] \tag{22c}$$

Consider the continuity relation (2h) which reads, in the Fourier transform domain

$$\dot{F}^{(e)}_1(\alpha,d)-\dot{G}^{(e)}_1(\alpha,d) = \frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}[e^{i\ell(\alpha-k\cos\phi_0)}-1] \tag{23}$$

Taking into account (5a), (7a) and (14) one obtains

$$\begin{aligned} &ik\frac{e^{-iKd}\chi(\alpha)}{M^{(e)}(\alpha)}P^{(e)}_1(\alpha)-\dot{F}^{(e)}_-(\alpha,d)-e^{i\alpha\ell}\dot{F}^{(e)}_+(\alpha,d)\\ =\;& \frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}[e^{i\ell(\alpha-k\cos\phi_0)}-1]\\ &+\frac{1}{M^{(e)}(\alpha)}\int_0^d[f^{(e)}(\alpha,t)+e^{i\alpha\ell}g^{(e)}(\alpha,t)]\cos[Kt]dt \end{aligned} \tag{24}$$

Substituting (19) in (24) and evaluating the resultant integral, one obtains the following modified Wiener-Hopf equation of the third kind

$$\begin{aligned} &ik\frac{\chi(\alpha)}{N^{(e)}(\alpha)}P^{(e)}_1(\alpha)-\dot{F}^{(e)}_-(\alpha,d)-e^{i\alpha\ell}\dot{F}^{(e)}_+(\alpha,d)\\ =\;& \frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}[e^{i\ell(\alpha-k\cos\phi_0)}-1]\\ &+\sum_{m=1}^{\infty}\frac{K^e_m\sin[K^e_md]}{[\alpha^2-(\alpha^e_m)^2]}(f^e_m(\alpha)+e^{i\alpha\ell}g^e_m(\alpha)) \end{aligned} \tag{25a}$$

with

$$N^{(e)}(\alpha)=e^{iK(\alpha)d}M^{(e)}(\alpha) \tag{25b}$$

Although this modified Wiener-Hopf equation is valid for $\Im m(-k) < \Im m(\alpha) < \Im m(k)$, for the sake of analytical convenience, we shall restrict the strip of regularity to $\Im m(k\cos\phi_0) < \Im m(\alpha) < \Im m(k)$.

The first step in solving (25a) is to factorize the kernel function $\chi(\alpha)/N^{(e)}(\alpha)$ in the Wiener-Hopf sense, that is,

$$\frac{\chi(\alpha)}{N^{(e)}(\alpha)} = \frac{\chi_+(\alpha)}{N^{(e)}_+(\alpha)}\frac{\chi_-(\alpha)}{N^{(e)}_-(\alpha)} \tag{26}$$

Here $N_+^{(e)}(\alpha)$, $\chi_+(\alpha)$ and $N_-^{(e)}(\alpha)$, $\chi_-(\alpha)$ are the split functions, regular and free of zeros in the half-planes $\Im m(\alpha) > \Im m(-k)$ and $\Im m(\alpha) < \Im m(k)$ respectively. The explicit expression of $N_\pm^{(e)}(\alpha)$ can be obtained by following the procedure outlined in [7]:

$$\begin{aligned} N_+^{(e)}(\alpha) &= \left[\cos[kd] - \frac{\eta_1}{i}\sin[kd]\right]^{1/2} \exp\left\{\frac{Kd}{\pi}\ln\left(\frac{\alpha + iK}{k}\right)\right\} \\ &\quad \times \exp\left\{\frac{i\alpha d}{\pi}\left(1 - \mathcal{C} + \ln\left(\frac{2\pi}{kd}\right) + i\frac{\pi}{2}\right)\right\} \\ &\quad \cdot \prod_{m=1}^{\infty}\left(1 + \frac{\alpha}{\alpha_m^e}\right)\exp\left(\frac{i\alpha d}{m\pi}\right) \qquad (27a) \\ N_-^{(e)}(\alpha) &= N_+^{(e)}(-\alpha) \qquad (27b) \end{aligned}$$

In (27a) $\mathcal{C}$ is the Euler's constant given by $\mathcal{C} = 0.57721...$ As to the split functions $\chi_\pm(\alpha)$, they can be expressed explicitly in terms of the Maluizhinets function [8] as follows:

$$\begin{aligned} &\chi_+(k\cos\phi) \\ &= 2^{3/2}\sqrt{\frac{2}{\eta_1}}\sin\frac{\phi}{2}\ \left\{\frac{\mathcal{M}_\pi(3\pi/2 - \phi - \theta)\mathcal{M}_\pi(\pi/2 - \phi + \theta)}{\mathcal{M}_\pi^2(\pi/2)}\right\}^2 \\ &\quad \times\left\{\left[1 + \sqrt{2}\cos\left(\frac{\pi/2 - \phi - \theta}{2}\right)\right]\right. \\ &\quad \left.\cdot\left[1 + \sqrt{2}\cos\left(\frac{3\pi/2 - \phi - \theta}{2}\right)\right]\right\}^{-1} \qquad (28a) \end{aligned}$$

with

$$\sin\theta = \frac{1}{\eta_1} \qquad (28b)$$

and

$$\mathcal{M}_\pi(z) = \exp\left\{-\frac{1}{8\pi}\int_0^z \frac{\pi\sin u - 2\sqrt{2}\pi\sin(u/2) + 2u}{\cos u}du\right\} \qquad (28c)$$

Now, let us multiply both sides of (25a) by $N_-^{(e)}(\alpha)/\chi_-(\alpha)$ to get

$$\begin{aligned} &ik\frac{\chi_+(\alpha)}{N_+^{(e)}(\alpha)}P_1^{(e)}(\alpha) - \frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\dot{F}_-^{(e)}(\alpha, d) - e^{i\alpha\ell}\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)} \\ &\times\left[\dot{F}_+^{(e)}(\alpha, d) + \frac{2k\sin\phi_0}{1 + \eta_1\sin\phi_0}\times\frac{e^{-ik(d\sin\phi_0 + \ell\cos\phi_0)}}{\alpha - k\cos\phi_0}\right. \end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{[\alpha^2-(\alpha_m^e)^2]}g_m^e(\alpha)\Bigg]\\
=\quad &-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\\
&+\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{[\alpha^2-(\alpha_m^e)^2]}f_m^e(\alpha)\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}
\end{aligned}
\tag{29}
$$

The first and the second terms appearing at the left-hand side of (29) are evidently regular in the upper and lower spectral half-planes, respectively, whereas the other terms in the remaining part of the equation have singularities in both spectral half-planes. Hence, one has necessarily to apply the Wiener-Hopf decomposition procedure to these terms. A decomposition of the first term at the right-hand side of (29) can readily be achieved by isolating its pole at $\alpha = k\cos\phi_0$ and yields:

$$
\begin{aligned}
&-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\\
=\quad &-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\left[\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}-\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\right]\\
&-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}
\end{aligned}
\tag{30}
$$

A similar decomposition for the second term at the right-hand side can be written by isolating the poles at $\alpha = -\alpha_m^e$, that is

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{[\alpha^2-(\alpha_m^e)^2]}f_m^e(\alpha)\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\\
=\quad &\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{\alpha+\alpha_m^e}\left[\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\frac{f_m^e(\alpha)}{\alpha-\alpha_m^e}+\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{f_m^e(-\alpha_m^e)}{2\alpha_m^e}\right]\\
&-\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{\alpha+\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{f_m^e(-\alpha_m^e)}{2\alpha_m^e}
\end{aligned}
\tag{31}
$$

As to the third term at the left-hand side of (29) which involves also the unknown function $\dot{F}_+^{(e)}(\alpha, d)$, only a formal decomposition is possible.

Indeed, in accordance with a very well-known general procedure,

$$\begin{aligned}
& \cdot\, e^{i\alpha\ell}\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\left[\dot{F}_+^{(e)}(\alpha,d)+\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\right.\\
& \cdot\left.\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{\alpha-k\cos\phi_0}+\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{[\alpha^2-(\alpha_m^e)^2]}g_m^e(\alpha)\right]\\
= & \frac{1}{2\pi i}\int\limits_{\mathcal{L}_1^+}\frac{N_-^{(e)}(\tau)}{\chi_-(\tau)}U^{(e)}(\tau)\frac{e^{i\tau\ell}}{\tau-\alpha}d\tau\\
& -\frac{1}{2\pi i}\int\limits_{\mathcal{L}^-}\frac{N_-^{(e)}(\tau)}{\chi_-(\tau)}U^{(e)}(\tau)\frac{e^{i\tau\ell}}{\tau-\alpha}d\tau
\end{aligned}\tag{32a}$$

where $\mathcal{L}_1^+$ and $\mathcal{L}^-$ stand for the integration lines shown in Fig. 3, while $U^{(e)}(\alpha)$ stands for

$$\begin{aligned}
U^{(e)}(\alpha) = & \dot{F}_+^{(e)}(\alpha,d)+\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{\alpha-k\cos\phi_0}\\
& +\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{[\alpha^2-(\alpha_m^e)^2]}g_m^e(\alpha)
\end{aligned}\tag{32b}$$

By substituting (30), (31) and (32a) in (29), and after some simple manipulations, one obtains

$$\begin{aligned}
& ik\frac{\chi_+(\alpha)}{N_+^{(e)}(\alpha)}P_1^{(e)}(\alpha)-\frac{1}{2\pi i}\int\limits_{\mathcal{L}_1^+}\frac{N_-^{(e)}(\tau)}{\chi_-(\tau)}U^{(e)}(\tau)\frac{e^{i\tau\ell}}{\tau-\alpha}d\tau\\
& +\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\\
& +\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{\alpha+\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{f_m^e(-\alpha_m^e)}{2\alpha_m^e}\\
= & \frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\dot{F}_-^{(e)}(\alpha,d)-\frac{1}{2\pi i}\int\limits_{\mathcal{L}^-}\frac{N_-^{(e)}(\tau)}{\chi_-(\tau)}U^{(e)}(\tau)\\
& \cdot\frac{e^{i\tau\ell}}{\tau-\alpha}d\tau-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\\
& \times\left[\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}-\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\right]+\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{\alpha+\alpha_m^e}
\end{aligned}$$

$$\cdot\left[\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\frac{f_m^e(\alpha)}{\alpha-\alpha_m^e}+\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{f_m^e(-\alpha_m^e)}{2\alpha_m^e}\right] \tag{33}$$

This equation is now ready to apply the well-known Liouville theorem. Indeed, all functions at the left-hand side are regular in the upper spectral half-plane $\Im m(\alpha) > \Im m(k\cos\phi_0)$, while those at the right-hand side are regular in the lower spectral half-plane $\Im m(\alpha) < \Im m(k)$. Therefore, by analytical continuation principle, they define an entire function. By using the following order relations

$$\frac{\partial}{\partial y}u_T^{(e)}(0,d)\ ,\ \frac{\partial}{\partial y}u_T^{(e)}(\ell,d)=O(\rho^{-1/2})\quad,\quad \rho\to 0 \tag{34a}$$

with ρ being the distance from the edges to the observation point, and by virtue of (34a), it can be shown that $\dot{F}_\pm^{(e)}(\alpha,d)$ behaves like

$$\dot{F}_\pm^{(e)}(\alpha,d)=O\left((\pm\alpha)^{-1/2}\right) \tag{34b}$$

as $|\alpha|\to\infty$ in their respective regions of regularity. Now, by taking into account (34b) and the fact that

$$\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\to O(1) \tag{35}$$

it can be shown that the entire function is merely zero. Hence, one has

$$\begin{aligned}\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}L^{(e)}(\alpha) &= \frac{1}{2\pi i}\int\limits_{\mathcal{L}^-}\frac{N_-^{(e)}(\tau)}{\chi_-(\tau)}U^{(e)}(\tau)\frac{e^{i\tau\ell}}{\tau-\alpha}d\tau\\ &\quad-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\\ &\quad-\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{\alpha+\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{f_m^e(-\alpha_m^e)}{2\alpha_m^e}\end{aligned} \tag{36a}$$

where

$$\begin{aligned}L^{(e)}(\alpha) &= \dot{F}_-^{(e)}(\alpha,d)-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\\ &\quad+\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{[\alpha^2-(\alpha_m^e)^2]}f_m^e(\alpha)\end{aligned} \tag{36b}$$

An integral equation similar to (36a), with the roles of $U^{(e)}(\alpha)$ and $L^{(e)}(\alpha)$ interchanged, can be obtained by multiplying both sides of (25a) by $e^{-i\alpha\ell}N_+^{(e)}(\alpha)/\chi_+(\alpha)$. The result is:

$$\begin{aligned}
&\frac{N_+^{(e)}(\alpha)}{\chi_+(\alpha)}U^{(e)}(\alpha)\\
&= -\frac{1}{2\pi i}\int_{\mathcal{L}^+}\frac{N_+^{(e)}(\tau)}{\chi_+(\tau)}L^{(e)}(\tau)\frac{e^{-i\tau\ell}}{\tau-\alpha}d\tau + \frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_+^{(e)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)}\\
&\quad\times\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{\alpha-k\cos\phi_0} - \sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{\alpha-\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{g_m^e(\alpha_m^e)}{2\alpha_m^e}
\end{aligned}\tag{37}$$

In obtaining this integral equation, the integration line $\mathcal{L}_1^+$ is replaced by $\mathcal{L}^+$ as shown in Fig.3, and the residue contribution related to the pole $\tau = k\cos\phi_0$ has been taken into account.

The coupled integral equations given in (36a) and (37) can be solved by using an iterative procedure. When $k\ell$ (slit width) is large, the free terms lying at the right-hand side of (36a) and (37) give the first order solutions. Second order solutions can then be obtained by replacing the unknown functions appearing in the integrands by their first order approximations. Thus, one can write

$$L^{(e)}(\alpha) \simeq L_{[1]}^{(e)}(\alpha) + L_{[2]}^{(e)}(\alpha) \tag{38a}$$

$$U^{(e)}(\alpha) \simeq U_{[1]}^{(e)}(\alpha) + U_{[2]}^{(e)}(\alpha) \tag{38b}$$

with

$$\begin{aligned}
L_{[1]}^{(e)}(\alpha) &= -\frac{\chi_-(\alpha)}{N_-^{(e)}(\alpha)}\Bigg[\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\\
&\quad+\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{\alpha+\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{f_m^e(-\alpha_m^e)}{2\alpha_m^e}\Bigg]
\end{aligned}\tag{39a}$$

$$\begin{aligned}
U_{[1]}^{(e)}(\alpha) &= \frac{\chi_+(\alpha)}{N_+^{(e)}(\alpha)}\Bigg[\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_+^{(e)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)}\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{\alpha-k\cos\phi_0}\\
&\quad-\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{\alpha-\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{g_m^e(\alpha_m^e)}{2\alpha_m^e}\Bigg]
\end{aligned}\tag{39b}$$

and

$$L_{[2]}^{(e)}(\alpha) = \frac{\chi_-(\alpha)}{N_-^{(e)}(\alpha)} I^{(e)}(\alpha) \tag{40a}$$

$$U_{[2]}^{(e)}(\alpha) = \frac{\chi_+(\alpha)}{N_+^{(e)}(\alpha)} J^{(e)}(\alpha) \tag{40b}$$

where $I^{(e)}(\alpha)$ and $J^{(e)}(\alpha)$ stand for

$$I^{(e)}(\alpha) = \frac{1}{2\pi i}\int_{\mathcal{L}^-} \frac{N_-^{(e)}(\tau)}{\chi_-(\tau)} U_{[1]}^{(e)}(\tau)\frac{e^{i\tau\ell}}{\tau-\alpha}d\tau \tag{41a}$$

$$J^{(e)}(\alpha) = -\frac{1}{2\pi i}\int_{\mathcal{L}^+} \frac{N_+^{(e)}(\tau)}{\chi_+(\tau)} L_{[1]}^{(e)}(\tau)\frac{e^{-i\tau\ell}}{\tau-\alpha}d\tau \tag{41b}$$

Consider first the integral in (41a) and rearrange it as follows:

$$I^{(e)}(\alpha) = I_1^{(e)}(\alpha) + I_2^{(e)}(\alpha) \tag{42a}$$

with

$$\begin{aligned} I_1^{(e)}(\alpha) &= \frac{k\sin\phi_0 e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{i\pi(1+\eta_1\sin\phi_0)}\frac{N_+^{(e)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)} \\ &\quad\cdot\int_{\mathcal{L}^-}\frac{N_-^{(e)}(\tau)}{N_+^{(e)}(\tau)}\frac{\chi_+(\tau)}{\chi_-(\tau)}\frac{e^{i\tau\ell}}{(\tau-k\cos\phi_0)}\frac{d\tau}{(\tau-\alpha)} \end{aligned} \tag{42b}$$

$$\begin{aligned} I_2^{(e)}(\alpha) &= -\frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{2\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}g_m^e(\alpha_m^e) \\ &\quad\cdot\int_{\mathcal{L}^-}\frac{N_-^{(e)}(\tau)}{N_+^{(e)}(\tau)}\frac{\chi_+(\tau)}{\chi_-(\tau)}\frac{e^{i\tau\ell}}{(\tau-\alpha_m^e)}\frac{d\tau}{(\tau-\alpha)} \end{aligned} \tag{42c}$$

According to Jordan's Lemma, the integration line $\mathcal{L}^-$ in (42b) can be deformed onto the branch-cut $C_1^+ + C_1^-$ lying in the upper spectral half-plane. The resulting integral can be written in the following form:

$$\begin{aligned} I_1^{(e)}(\alpha) &= \frac{k\sin\phi_0}{i\pi(1+\eta_1\sin\phi_0)}\frac{N_+^{(e)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)}\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{\alpha-k\cos\phi_0} \\ &\quad\times\int_{C_1^+}\frac{\mathcal{T}_+(\tau)}{\sqrt{\tau-k}}\left(\frac{1}{\tau-\alpha}-\frac{1}{\tau-k\cos\phi_0}\right)e^{i\tau\ell}d\tau \end{aligned} \tag{43a}$$

with

$$\mathcal{T}_+(\tau) = -\frac{2ik}{\sqrt{\tau+k}}\left[\frac{\chi_+(\tau)M^{(e)}(\tau)}{N_+^{(e)}(\tau)}\right]^2 \tag{43b}$$

Changing the variable of integration from τ to t via

$$\tau - k = te^{i\pi/2}, \quad t > 0,$$

(43a) reduces to

$$\begin{aligned} I_1^{(e)}(\alpha) \; &= \; \frac{k\sin\phi_0}{i\pi(1+\eta_1\sin\phi_0)}\frac{N_+^{(e)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)}\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}e^{ik\ell}}{\alpha-k\cos\phi_0} \\ &\times\int_0^\infty \frac{\mathcal{T}_+(k+it)}{(it)^{1/2}}\left(\frac{1}{t-i(k-\alpha)}-\frac{1}{t-ik(1-\cos\phi_0)}\right)e^{-t\ell}dt \end{aligned} \tag{44}$$

When $k\ell$ is large (wide slit), the main contribution to this integral comes from the end point $t = 0$. Hence, one may replace $\mathcal{T}_+(k+it)$ by $\mathcal{T}_+(k)$ and take $\mathcal{T}_+(k)$ outside of the integral. The resulting integral can be expressed in terms of the modified Fresnel integral

$$F(z) = -2i\sqrt{z}e^{-iz}\int_{\sqrt{z}}^\infty e^{it^2}dt \tag{45a}$$

satisfying the equality

$$\int_0^\infty \frac{1}{t+z}\frac{e^{-t\ell}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{\ell}}\frac{F(iz\ell)}{z} \tag{45b}$$

to give

$$\begin{aligned} I_1^{(e)}(\alpha) \;\; &= \;\; e^{-i3\pi/4}\sqrt{\frac{2}{\pi}}\,k^2\,\frac{e^{ik\ell}}{\sqrt{k\ell}}\left[\frac{\chi_+(k)}{N_+^{(e)}(k)}\right]^2\frac{\sin\phi_0}{1+\eta_1\sin\phi_0} \\ &\cdot\frac{N_+^{(e)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)}\times\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{\alpha-k\cos\phi_0} \\ &\cdot\left[\frac{F[k\ell(1-\alpha/k)]}{k-\alpha}-\frac{F[k\ell(1-\cos\phi_0)]}{k(1-\cos\phi_0)}\right] \end{aligned} \tag{46}$$

Similarly, one obtains for $I_2^{(e)}(\alpha)$, the following result:

$$
\begin{aligned}
I_2^{(e)}(\alpha) &= \frac{e^{i\pi/4}}{\sqrt{2\pi}}\,k\,\frac{e^{ik\ell}}{\sqrt{k\ell}}\left[\frac{\chi_+(k)}{N_+^{(e)}(k)}\right]^2\frac{F[k\ell(1-\alpha/k)]}{k-\alpha}\\
&\quad\cdot\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{k-\alpha_m^e}\,\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\,\frac{g_m^e(\alpha_m^e)}{2\alpha_m^e}
\end{aligned}
\tag{47}
$$

The integral in (41b) can now be evaluated in a similar way to (41a) and by rearranging it one gets

$$
J^{(e)}(\alpha) = J_1^{(e)}(\alpha) + J_2^{(e)}(\alpha)
\tag{48a}
$$

Now, deforming the contour $\mathcal{L}^+$ in (41b) onto the branch cut $C_2^+ + C_2^-$ lying in the lower spectral half-plane, the resulting branch cut integrals can be evaluated asymptotically in a way similar to that described above and one obtains

$$
\begin{aligned}
J_1^{(e)}(\alpha) &= e^{-i3\pi/4}\sqrt{\frac{2}{\pi}}\,k^2\,\frac{e^{ik\ell}}{\sqrt{k\ell}}\left[\frac{\chi_+(k)}{N_+^{(e)}(k)}\right]^2\frac{\sin\phi_0}{1+\eta_1\sin\phi_0}\\
&\quad\cdot\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\times\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}\\
&\quad\cdot\left[\frac{F[k\ell(1+\alpha/k)]}{k+\alpha}-\frac{F[k\ell(1+\cos\phi_0)]}{k(1+\cos\phi_0)}\right]
\end{aligned}
\tag{48b}
$$

$$
\begin{aligned}
J_2^{(e)}(\alpha) &= \frac{e^{i\pi/4}}{\sqrt{2\pi}}\,k\,\frac{e^{ik\ell}}{\sqrt{k\ell}}\left[\frac{\chi_+(k)}{N_+^{(e)}(k)}\right]^2\frac{F[k\ell(1+\alpha/k)]}{k+\alpha}\\
&\quad\cdot\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e d]}{k-\alpha_m^e}\,\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\,\frac{f_m^e(-\alpha_m^e)}{2\alpha_m^e}
\end{aligned}
\tag{48c}
$$

Since $I^{(e)}(\alpha)$ and $J^{(e)}(\alpha)$ are now determined, the approximate solution of the MWHE in (25a) reads:

$$
\begin{aligned}
P_1^{(e)}(\alpha) &= \frac{1}{ik}\left\{\frac{N_+^{(e)}(\alpha)}{\chi_+(\alpha)}\left[I_1^{(e)}(\alpha)+I_2^{(e)}(\alpha)+\tilde{L}_{[1]}^{(e)}(\alpha)\right]\right.\\
&\quad\left.+e^{i\alpha\ell}\frac{N_-^{(e)}(\alpha)}{\chi_-(\alpha)}\left[J_1^{(e)}(\alpha)+J_2^{(e)}(\alpha)+\tilde{U}_{[1]}^{(e)}(\alpha)\right]\right\}
\end{aligned}
\tag{49a}
$$

where we define $\tilde{L}^{(e)}_{[1]}(\alpha)$ and $\tilde{U}^{(e)}_{[1]}(\alpha)$ as

$$\tilde{L}^{(e)}_{[1]}(\alpha) = \frac{N^{(e)}_{-}(\alpha)}{\chi_{-}(\alpha)} L^{(e)}_{[1]}(\alpha) \tag{49b}$$

$$\tilde{U}^{(e)}_{[1]}(\alpha) = \frac{N^{(e)}_{+}(\alpha)}{\chi_{+}(\alpha)} U^{(e)}_{[1]}(\alpha) \tag{49c}$$

The expression of $P^{(e)}_1(\alpha)$ in (49a) involves the unknown constants $f^e_j(-\alpha^e_j)$ and $g^e_j(\alpha^e_j)$ which are related to $P^{(e)}_1(\pm\alpha^e_j)$ through (16a,b). Using the expressions of $f^e_j(\pm\alpha^e_j)$ and $g^e_j(\pm\alpha^e_j)$ in terms of a^e_j and b^e_j,

$$f^e_j\left(\pm\alpha^e_j\right) = -\frac{2i}{\nu^e_j}\sum_{n=1}^{\infty} \Delta^e_{n,j}\left(\beta^e_n \pm \alpha^e_j\right) a^e_n \quad , \quad j = 1, 2, \cdots \tag{50a}$$

$$g^e_j\left(\pm\alpha^e_j\right) = -\frac{2i}{\nu^e_j}\sum_{n=1}^{\infty} \Delta^e_{n,j}\left(\beta^e_n \mp \alpha^e_j\right) b^e_n \quad , \quad j = 1, 2, \cdots \tag{50b}$$

in (49a) and (16a,b) yields the following infinite systems of linear algebraic equations for a^e_j and b^e_j:

$$\sum_{n=1}^{\infty} a^e_n \mathcal{A}^e_n\left(\alpha^e_j\right) - \sum_{n=1}^{\infty} b^e_n \mathcal{B}^e_n\left(\alpha^e_j\right) = C^e\left(\alpha^e_j\right) \quad , \quad j = 1, 2, \cdots \tag{51a}$$

$$\sum_{n=1}^{\infty} a^e_n \mathcal{A}^e_n\left(-\alpha^e_j\right) - \sum_{n=1}^{\infty} b^e_n \mathcal{B}^e_n\left(-\alpha^e_j\right) = C^e\left(-\alpha^e_j\right) \quad , \quad j = 1, 2, \cdots \tag{51b}$$

with

$$\begin{aligned} \mathcal{A}^e_n(\alpha^e_j) = & -i\frac{\sin[K^e_j d]}{K^e_j}\left[1 - \frac{\eta_1^2}{k^2}(K^e_j)^2\right]\Delta^e_{n,j}(\beta^e_n + \alpha^e_j) \\ & + \sum_{m=1}^{\infty} \Delta^e_{n,m} K^e_m \sin[K^e_m d] \frac{(\beta^e_n - \alpha^e_m)}{\nu^e_m \alpha^e_m} \frac{N^{(e)}_{+}(\alpha^e_m)}{\chi_{+}(\alpha^e_m)} \\ & \cdot \left\{ -\frac{1}{k}\frac{N^{(e)}_{+}(\alpha^e_j)}{\chi_{+}(\alpha^e_j)} \frac{1}{\alpha^e_j + \alpha^e_m} + e^{i\alpha^e_j \ell} \frac{N^{(e)}_{-}(\alpha^e_j)}{\chi_{-}(\alpha^e_j)} \frac{e^{i\pi/4}}{\sqrt{2\pi}} \frac{e^{ik\ell}}{\sqrt{k\ell}} \right. \\ & \left. \cdot \left[\frac{\chi_{+}(k)}{N^{(e)}_{+}(k)}\right]^2 \frac{F[k\ell(1 + \alpha^e_j/k)]}{k + \alpha^e_j} \frac{1}{k - \alpha^e_m} \right\} \end{aligned} \tag{51c}$$

$$\mathcal{B}^e_n(\alpha^e_j) = i\frac{\sin[K^e_j d]}{K^e_j} e^{i\alpha^e_j \ell}\left[1 - \frac{\eta_1^2}{k^2}(K^e_j)^2\right]\Delta^e_{n,j}(\beta^e_n + \alpha^e_j)$$

$$+\sum_{m=1}^{\infty}\Delta_{n,m}^{e}K_{m}^{e}\sin[K_{m}^{e}d]\frac{(\beta_{n}^{e}-\alpha_{m}^{e})}{\nu_{m}^{e}\alpha_{m}^{e}}\frac{N_{+}^{(e)}(\alpha_{m}^{e})}{\chi_{+}(\alpha_{m}^{e})}$$

$$\cdot\left\{\frac{1}{k}e^{i\alpha_{j}^{e}\ell}\frac{N_{-}^{(e)}(\alpha_{j}^{e})}{\chi_{-}(\alpha_{j}^{e})}\frac{1}{\alpha_{j}^{e}-\alpha_{m}^{e}}-\frac{N_{+}^{(e)}(\alpha_{j}^{e})}{\chi_{+}(\alpha_{j}^{e})}\frac{e^{i\pi/4}}{\sqrt{2\pi}}\frac{e^{ik\ell}}{\sqrt{k\ell}}\right.$$

$$\left.\cdot\left[\frac{\chi_{+}(k)}{N_{+}^{(e)}(k)}\right]^{2}\frac{F[k\ell(1-\alpha_{j}^{e}/k)]}{k-\alpha_{j}^{e}}\frac{1}{k-\alpha_{m}^{e}}\right\}\tag{51d}$$

$$\begin{aligned}\mathcal{C}_{n}^{e}(\alpha_{j}^{e}) &= \frac{1}{ik}\frac{N_{+}^{(e)}(\alpha_{j}^{e})}{\chi_{+}(\alpha_{j}^{e})}\left[I_{1}^{(e)}(\alpha_{j}^{e})-\frac{2k\sin\phi_{0}}{1+\eta_{1}\sin\phi_{0}}\right.\\ &\quad\left.\cdot\frac{N_{-}^{(e)}(k\cos\phi_{0})}{\chi_{-}(k\cos\phi_{0})}\frac{e^{-ikd\sin\phi_{0}}}{\alpha_{j}^{e}-k\cos\phi_{0}}\right]\\ &\quad+\frac{e^{i\alpha_{j}^{e}\ell}}{ik}\frac{N_{-}^{(e)}(\alpha_{j}^{e})}{\chi_{-}(\alpha_{j}^{e})}\left[J_{1}^{(e)}(\alpha_{j}^{e})+\frac{2k\sin\phi_{0}}{1+\eta_{1}\sin\phi_{0}}\right.\\ &\quad\left.\cdot\frac{N_{+}^{(e)}(k\cos\phi_{0})}{\chi_{+}(k\cos\phi_{0})}\frac{e^{-ik(d\sin\phi_{0}+\ell\cos\phi_{0})}}{\alpha_{j}^{e}-k\cos\phi_{0}}\right]\end{aligned}\tag{51e}$$

4. ODD EXCITATION

The solution for odd excitation is similar to that of even excitation. Indeed, by assuming a representation similar to (1a) with the superscript (e) being replaced by (o); it can be seen that all the boundary and continuity relations in (2a–l) remain valid for the odd excitation case also, except (2d–f) which are to be changed as

$$u_{2}^{(o)}(x,0)=0\quad,\quad x<0\tag{52a}$$

$$u_{3}^{(o)}(x,0)=0\quad,\quad 0<x<\ell\tag{52b}$$

$$u_{4}^{(o)}(x,0)=0\quad,\quad x>\ell\tag{52c}$$

In this case the Wiener-Hopf equation reads

$$\begin{aligned}&k\frac{\chi(\alpha)}{N^{(o)}(\alpha)}P_{1}^{(o)}(\alpha)-\dot{F}_{-}^{(o)}(\alpha,d)-e^{i\alpha\ell}\dot{F}_{+}^{(o)}(\alpha,d)\\ &= \frac{2k\sin\phi_{0}}{1+\eta_{1}\sin\phi_{0}}\frac{e^{-ikd\sin\phi_{0}}}{\alpha-k\cos\phi_{0}}[e^{i\ell(\alpha-k\cos\phi_{0})}-1]\end{aligned}$$

$$-\sum_{m=1}^{\infty}\frac{K_m^o\cos[K_m^o d]}{[\alpha^2-(\alpha_m^o)^2]}(f_m^o(\alpha)+e^{i\alpha\ell}g_m^o(\alpha)) \tag{53a}$$

with

$$N^{(o)}(\alpha) = e^{iKd}\left[\sin[Kd]+\frac{\eta_1}{ik}K\cos[Kd]\right] \tag{53b}$$

$$K_m^o = \sqrt{k^2-(\alpha_m^o)^2} \tag{53c}$$

$$\begin{bmatrix} f_m^o \\ g_m^o \end{bmatrix} = \frac{2}{\nu_m^o}\int_0^d \begin{bmatrix} f^{(o)}(t) \\ g^{(o)}(t) \end{bmatrix}\sin[K_m^o t]dt$$

$$\nu_m^o = d+\frac{\eta_1}{ik}\cos^2[K_m^o d] \tag{53d}$$

where α_m^o are the roots of

$$\sin[Kd]+\frac{\eta_1}{ik}K\cos[Kd]=0 \quad , \quad \alpha=\alpha_m^o \quad , \quad \Im m(\alpha_m^o)>\Im m(k) \tag{53e}$$

The application of the Wiener-Hopf procedure to (53a) yields the following pair of coupled Wiener-Hopf equations:

$$\begin{aligned}\frac{N_-^{(o)}(\alpha)}{\chi_-(\alpha)}L^{(o)}(\alpha) &= \frac{1}{2\pi i}\int_{\mathcal{L}^-}\frac{N_-^{(o)}(\tau)}{\chi_-(\tau)}U^{(o)}(\tau)\frac{e^{i\tau\ell}}{\tau-\alpha}d\tau \\ &-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_-^{(o)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\times\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0} \\ &+\sum_{m=1}^{\infty}\frac{K_m^o\cos[K_m^o d]}{\alpha+\alpha_m^o}\frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)}\frac{f_m^o(-\alpha_m^o)}{2\alpha_m^o} \end{aligned} \tag{54a}$$

$$\begin{aligned}\frac{N_+^{(o)}(\alpha)}{\chi_+(\alpha)}U^{(o)}(\alpha) &= -\frac{1}{2\pi i}\int_{\mathcal{L}^+}\frac{N_+^{(o)}(\tau)}{\chi_+(\tau)}L^{(o)}(\tau)\frac{e^{-i\tau\ell}}{\tau-\alpha}d\tau \\ &+\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_+^{(o)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)}\times\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{\alpha-k\cos\phi_0} \\ &-\sum_{m=1}^{\infty}\frac{K_m^o\cos[K_m^o d]}{\alpha-\alpha_m^o}\frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)}\frac{g_m^o(\alpha_m^o)}{2\alpha_m^o} \end{aligned} \tag{54b}$$

with

$$L^{(o)}(\alpha) = \dot{F}_-^{(o)}(\alpha,d)-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikd\sin\phi_0}}{\alpha-k\cos\phi_0}$$

$$- \sum_{m=1}^{\infty} \frac{K_m^o \cos[K_m^o d]}{[\alpha^2 - (\alpha_m^o)^2]} f_m^o(\alpha) \tag{54c}$$

$$\begin{aligned} U^{(o)}(\alpha) &= \dot{F}_+^{(o)}(\alpha, d) + \frac{2k \sin \phi_0}{1 + \eta_1 \sin \phi_0} \frac{e^{-ik(d \sin \phi_0 + \ell \cos \phi_0)}}{\alpha - k \cos \phi_0} \\ &\quad - \sum_{m=1}^{\infty} \frac{K_m^o \cos[K_m^o d]}{[\alpha^2 - (\alpha_m^o)^2]} g_m^o(\alpha) \end{aligned} \tag{54d}$$

$N_+^{(o)}(\alpha)$ and $N_-^{(o)}(\alpha)$ are the split functions resulting from the Wiener-Hopf factorization of (53b) as

$$N^{(o)}(\alpha) = N_+^{(o)}(\alpha) N_-^{(o)}(\alpha) \tag{55a}$$

The explicit expressions of $N_\pm^{(o)}(\alpha)$ are [7]

$$\begin{aligned} N_+^{(o)}(\alpha) &= \sqrt{\alpha + k} \left[\frac{\sin[kd]}{k} + \frac{\eta_1}{ik} \cos[kd] \right]^{1/2} \exp\left\{ \frac{Kd}{\pi} \ln\left(\frac{\alpha + iK}{k} \right) \right\} \\ &\quad \times \exp\left\{ \frac{i\alpha d}{\pi} \left(1 - \mathcal{C} + \ln\left(\frac{2\pi}{kd} \right) + i\frac{\pi}{2} \right) \right\} \\ &\quad \cdot \prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{\alpha_m^o} \right) \exp\left(\frac{i\alpha d}{m\pi} \right) \end{aligned} \tag{55b}$$

$$N_-^{(o)}(\alpha) = N_+^{(o)}(-\alpha) \tag{55c}$$

By applying the iterative procedure, the solution of the coupled integral equations can be obtained as follows:

$$L^{(o)}(\alpha) \simeq L_{[1]}^{(o)}(\alpha) + L_{[2]}^{(o)}(\alpha) \tag{56a}$$

$$U^{(o)}(\alpha) \simeq U_{[1]}^{(o)}(\alpha) + U_{[2]}^{(o)}(\alpha) \tag{56b}$$

with

$$\begin{aligned} L_{[1]}^{(o)}(\alpha) &= -\frac{\chi_-(\alpha)}{N_-^{(o)}(\alpha)} \left[\frac{2k \sin \phi_0}{1 + \eta_1 \sin \phi_0} \frac{N_-^{(o)}(k \cos \phi_0)}{\chi_-(k \cos \phi_0)} \frac{e^{-ikd \sin \phi_0}}{\alpha - k \cos \phi_0} \right. \\ &\quad \left. - \sum_{m=1}^{\infty} \frac{K_m^o \cos[K_m^o d]}{\alpha + \alpha_m^o} \frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)} \frac{f_m^o(-\alpha_m^o)}{2\alpha_m^o} \right] \end{aligned} \tag{56c}$$

$$U_{[1]}^{(o)}(\alpha) = \frac{\chi_+(\alpha)}{N_+^{(o)}(\alpha)} \left[\frac{2k \sin \phi_0}{1 + \eta_1 \sin \phi_0} \frac{N_+^{(o)}(k \cos \phi_0)}{\chi_+(k \cos \phi_0)} \frac{e^{-ik(d \sin \phi_0 + \ell \cos \phi_0)}}{\alpha - k \cos \phi_0} \right.$$

$$
- \sum_{m=1}^{\infty} \frac{K_m^o \cos[K_m^o d]}{\alpha - \alpha_m^o} \frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)} \frac{g_m^o(\alpha_m^o)}{2\alpha_m^o} \Bigg] \tag{56d}
$$

$$
L_{[2]}^{(o)}(\alpha) = \frac{\chi_-(\alpha)}{N_-^{(o)}(\alpha)} \left[I_1^{(o)}(\alpha) + I_2^{(o)}(\alpha) \right] \tag{56e}
$$

$$
U_{[2]}^{(o)}(\alpha) = \frac{\chi_+(\alpha)}{N_+^{(o)}(\alpha)} \left[J_1^{(o)}(\alpha) + J_2^{(o)}(\alpha) \right] \tag{56f}
$$

where

$$
\begin{aligned}
I_1^{(o)}(\alpha) = {} & e^{-i3\pi/4} \sqrt{\frac{2}{\pi}}\, k^2 \frac{e^{ik\ell}}{\sqrt{k\ell}} \left[\frac{\chi_+(k)}{N_+^{(o)}(k)} \right]^2 \frac{\sin\phi_0}{1 + \eta_1 \sin\phi_0} \\
& \cdot \frac{N_+^{(o)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)} \times \frac{e^{-ik(d\sin\phi_0 + \ell\cos\phi_0)}}{\alpha - k\cos\phi_0} \\
& \cdot \left[\frac{F[k\ell(1-\alpha/k)]}{k-\alpha} - \frac{F[k\ell(1-\cos\phi_0)]}{k(1-\cos\phi_0)} \right]
\end{aligned} \tag{56g}
$$

$$
\begin{aligned}
I_2^{(o)}(\alpha) = {} & \frac{e^{i\pi/4}}{\sqrt{2\pi}}\, k \frac{e^{ik\ell}}{\sqrt{k\ell}} \left[\frac{\chi_+(k)}{N_+^{(o)}(k)} \right]^2 \frac{F[k\ell(1-\alpha/k)]}{k-\alpha} \\
& \cdot \sum_{m=1}^{\infty} \frac{K_m^o \cos[K_m^o d]}{k - \alpha_m^o} \frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)} \frac{g_m^o(\alpha_m^o)}{2\alpha_m^o}
\end{aligned} \tag{56h}
$$

$$
\begin{aligned}
J_1^{(o)}(\alpha) = {} & e^{-i3\pi/4} \sqrt{\frac{2}{\pi}}\, k^2 \frac{e^{ik\ell}}{\sqrt{k\ell}} \left[\frac{\chi_+(k)}{N_+^{(o)}(k)} \right]^2 \frac{\sin\phi_0}{1 + \eta_1 \sin\phi_0} \\
& \cdot \frac{N_-^{(o)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)} \times \frac{e^{-ikd\sin\phi_0}}{\alpha - k\cos\phi_0} \\
& \cdot \left[\frac{F[k\ell(1+\alpha/k)]}{k+\alpha} - \frac{F[k\ell(1+\cos\phi_0)]}{k(1+\cos\phi_0)} \right]
\end{aligned} \tag{56i}
$$

$$
\begin{aligned}
J_2^{(o)}(\alpha) = {} & -\frac{e^{i\pi/4}}{\sqrt{2\pi}}\, k \frac{e^{ik\ell}}{\sqrt{k\ell}} \left[\frac{\chi_+(k)}{N_+^{(o)}(k)} \right]^2 \frac{F[k\ell(1+\alpha/k)]}{k+\alpha} \\
& \cdot \sum_{m=1}^{\infty} \frac{K_m^o \sin[K_m^o d]}{k - \alpha_m^o} \frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)} \frac{f_m^o(-\alpha_m^o)}{2\alpha_m^o}
\end{aligned} \tag{56j}
$$

The approximate solution of the MWHE in (53a) is given as

$$\begin{aligned} P_1^{(o)}(\alpha) &= \frac{1}{k}\Bigg\{\frac{N_+^{(o)}(\alpha)}{\chi_+(\alpha)}\left[I_1^{(o)}(\alpha)+I_2^{(o)}(\alpha)+\tilde{L}_{[1]}^{(o)}(\alpha)\right] \\ &\quad + e^{i\alpha\ell}\frac{N_-^{(o)}(\alpha)}{\chi_-(\alpha)}\left[J_1^{(o)}(\alpha)+J_2^{(o)}(\alpha)+\tilde{U}_{[1]}^{(o)}(\alpha)\right]\Bigg\} \end{aligned} \tag{57a}$$

where we define $\tilde{L}_{[1]}^{(o)}(\alpha)$ and $\tilde{U}_{[1]}^{(o)}(\alpha)$ as

$$\tilde{L}_{[1]}^{(o)}(\alpha) = \frac{N_-^{(o)}(\alpha)}{\chi_-(\alpha)}L_{[1]}^{(o)}(\alpha) \tag{57b}$$

$$\tilde{U}_{[1]}^{(o)}(\alpha) = \frac{N_+^{(o)}(\alpha)}{\chi_+(\alpha)}U_{[1]}^{(o)}(\alpha) \tag{57c}$$

Using the expressions of $f_j^o(\pm\alpha_j^o)$ and $g_j^o(\alpha_j^o)$ in terms of a_j^o and b_j^o,

$$f_j^o(\pm\alpha_j^o) = -\frac{2i}{\nu_j^o}\sum_{n=1}^{\infty}\Delta_{n,j}^o(\beta_n^o\pm\alpha_j^o)a_n^o \quad , \quad j=1,2,\cdots \tag{58a}$$

$$g_j^o(\pm\alpha_j^o) = -\frac{2i}{\nu_j^o}\sum_{n=1}^{\infty}\Delta_{n,j}^o(\beta_n^o\mp\alpha_j^o)b_n^o \quad , \quad j=1,2,\cdots \tag{58b}$$

in (57a) yields the following infinite systems of linear algebraic equations for a_j^o and b_j^o:

$$\sum_{n=1}^{\infty}a_n^o\mathcal{A}_n^o(\alpha_j^o)+\sum_{n=1}^{\infty}b_n^o\mathcal{B}_n^o(\alpha_j^o) = C^o(\alpha_j^o) \quad , \quad j=1,2,\cdots \tag{59a}$$

$$\sum_{n=1}^{\infty}a_n^o\mathcal{A}_n^o(-\alpha_j^o)+\sum_{n=1}^{\infty}b_n^o\mathcal{B}_n^o(-\alpha_j^o) = C^o(-\alpha_j^o) \quad , \quad j=1,2,\cdots \tag{59b}$$

with

$$\begin{aligned} \mathcal{A}_n^o(\alpha_j^o) &= \frac{\cos[K_j^o d]}{K_j^o}\left[1-\frac{\eta_1^2}{k^2}(K_j^o)^2\right]\Delta_{n,j}^o(\beta_n^o+\alpha_j^o) \\ &\quad + \sum_{m=1}^{\infty}\Delta_{n,m}^o K_m^o\cos[K_m^o d]\frac{(\beta_n^o-\alpha_m^o)}{\nu_m^o\alpha_m^o}\frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)} \\ &\quad \cdot\Bigg\{\frac{1}{k}\frac{N_+^{(o)}(\alpha_j^o)}{\chi_+(\alpha_j^o)}\frac{1}{\alpha_j^o+\alpha_m^o}-e^{i\alpha_j^o\ell}\frac{N_-^{(o)}(\alpha_j^o)}{\chi_-(\alpha_j^o)}\frac{e^{i\pi/4}}{\sqrt{2\pi}}\frac{e^{ik\ell}}{\sqrt{k\ell}} \end{aligned}$$

$$\left[\frac{\chi_+(k)}{N_+^{(o)}(k)}\right]^2 \frac{F[k\ell(1+\alpha_j^o/k)]}{k+\alpha_j^o}\frac{1}{k-\alpha_m^o}\Bigg\} \tag{59c}$$

$$\begin{aligned}
\mathcal{B}_n^o(\alpha_j^o) \;=\;& \frac{\cos[K_j^o d]}{K_j^o} e^{i\alpha_j^o \ell}\left[1-\frac{\eta_1^2}{k^2}(K_j^o)^2\right]\Delta_{n,j}^o(\beta_n^o-\alpha_j^o) \\
& + \sum_{m=1}^{\infty} \Delta_{n,m}^o K_m^o \cos[K_m^o d]\frac{(\beta_n^o-\alpha_m^o)}{\nu_m^o \alpha_m^o}\frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)} \\
& \cdot \Bigg\{ -\frac{1}{k} e^{i\alpha_j^o \ell}\frac{N_-^{(o)}(\alpha_j^o)}{\chi_-(\alpha_j^o)}\frac{1}{\alpha_j^o-\alpha_m^o} \\
& + \left[\frac{\chi_+(k)}{N_+^{(o)}(k)}\right]^2 \frac{F[k\ell(1-\alpha_j^o/k)]}{k-\alpha_j^o}\frac{1}{k-\alpha_m^o}\Bigg\}
\end{aligned} \tag{59d}$$

$$\begin{aligned}
\mathcal{C}_n^o(\alpha_j^o) \;=\;& \frac{1}{ik}\frac{N_+^{(o)}(\alpha_j^o)}{\chi_+(\alpha_j^o)}\Bigg[I_1^{(o)}(\alpha_j^o) + \frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_-^{(o)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)} \\
& \cdot \frac{e^{-ikd\sin\phi_0}}{\alpha_j^o - k\cos\phi_0}\Bigg] + \frac{e^{i\alpha_j^o\ell}}{ik}\frac{N_-^{(o)}(\alpha_j^o)}{\chi_-(\alpha_j^o)}\Bigg[J_1^{(o)}(\alpha_j^o) + \frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0} \\
& \cdot \frac{N_+^{(o)}(k\cos\phi_0)}{\chi_+(k\cos\phi_0)}\frac{e^{-ik(d\sin\phi_0+\ell\cos\phi_0)}}{\alpha_j^o - k\cos\phi_0}\Bigg]
\end{aligned} \tag{59e}$$

where

$$\Delta_{n,j}^o = \frac{\eta_1+\eta_2}{ik}\frac{K_j^o \xi_n^o \cos[K_j^o d]\cos[\xi_n^o d]}{(\xi_n^o)^2-(K_j^o)^2} \tag{59f}$$

and

$$\beta_n^o = \sqrt{k^2-(\xi_n^o)^2} \quad , \quad \Im m(\beta_n^o) > \Im m(k) \tag{59g}$$

with ξ_n^o being the roots of

$$\sin[\xi_n^o d] - \frac{\eta_2}{ik}\xi_n^o\cos[\xi_n^o d] = 0 \quad , \quad n = 1, 2, \cdots \tag{59h}$$

Note that in the series terms of $\mathcal{A}_n^e(-\alpha_j^e)$, $\mathcal{B}_n^e(\alpha_j^e)$ and $\mathcal{A}_n^o(-\alpha_j^o)$, $\mathcal{B}_n^o(\alpha_j^o)$ given in (51c,d) and (59c,d) there occurs an indeterminate form $\frac{0}{0}$ for $m = j$ which should be removed by using L'Hospital's rule.

5. ANALYSIS OF THE DIFFRACTED FIELD

The scattered field in the region $y > d$ for even and odd excitations can be obtained by taking the inverse Fourier transforms

$$u_1^{(e)}(x,y) = \frac{1}{2\pi}\int_{\mathcal{L}} k\frac{\chi(\alpha)}{K(\alpha)}P_1^{(e)}(\alpha)\, e^{iK(\alpha)(y-d)}\, e^{-i\alpha x}d\alpha \qquad (60)$$

$$u_1^{(o)}(x,y) = \frac{1}{2\pi}\int_{\mathcal{L}} k\frac{\chi(\alpha)}{K(\alpha)}P_1^{(o)}(\alpha)\, e^{iK(\alpha)(y-d)}\, e^{-i\alpha x}d\alpha \qquad (61)$$

Here $\mathcal{L}$ is a straight line parallel to the real α-axis lying in the strip $\Im m(k\cos\phi_0) < \Im m(\alpha) < \Im m(k)$. By using (49a) and (57a), (60a,b) can be put into the following form

$$u_1^{(e,o)}(x,y) = u_{11}^{(e,o)}(x,y) + u_{12}^{(e,o)}(x,y) \qquad (62)$$

The asymptotic evaluation of the integrals in (60a) and (60b) through the saddle point technique enables us to write for the diffracted field

$$u_1(x,y) = \frac{u_1^{(e)}(x,y) + u_1^{(o)}(x,y)}{2} \qquad (63)$$

with

$$u_{11}^{(e)}(\rho,\phi) = \frac{e^{-i3\pi/4}}{\sqrt{2\pi}}\frac{e^{ik\rho}}{\sqrt{k\rho}}\frac{\sin\phi}{1+\eta_1\sin\phi}\frac{N_-^{(e)}(k\cos\phi)}{\chi_-(k\cos\phi)} \times\left[I_1^{(e)}(-k\cos\phi)+I_2^{(e)}(-k\cos\phi)+\tilde{L}_{[1]}^{(e)}(-k\cos\phi)\right] \qquad (64)$$

$$u_{12}^{(e)}(r,\psi) = \frac{e^{-i3\pi/4}}{\sqrt{2\pi}}\frac{e^{ikr}}{\sqrt{kr}}\frac{\sin\psi}{1+\eta_1\sin\psi}\frac{N_+^{(e)}(k\cos\psi)}{\chi_+(k\cos\psi)} \times\left[J_1^{(e)}(-k\cos\psi)+J_2^{(e)}(-k\cos\psi)+\tilde{U}_{[1]}^{(e)}(-k\cos\psi)\right] \qquad (65)$$

$$u_{11}^{(o)}(\rho,\phi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\frac{e^{ik\rho}}{\sqrt{k\rho}}\frac{\sin\phi}{1+\eta_1\sin\phi}\frac{N_-^{(o)}(k\cos\phi)}{\chi_-(k\cos\phi)} \times\left[I_1^{(o)}(-k\cos\phi)+I_2^{(o)}(-k\cos\phi)+\tilde{L}_{[1]}^{(o)}(-k\cos\phi)\right] \qquad (66)$$

$$u_{12}^{(o)}(r,\psi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\frac{e^{ikr}}{\sqrt{kr}}\frac{\sin\psi}{1+\eta_1\sin\psi}\frac{N_+^{(o)}(k\cos\psi)}{\chi_+(k\cos\psi)} \times\left[J_1^{(o)}(-k\cos\psi)+J_2^{(o)}(-k\cos\psi)+\tilde{U}_{[1]}^{(o)}(-k\cos\psi)\right] \qquad (67)$$

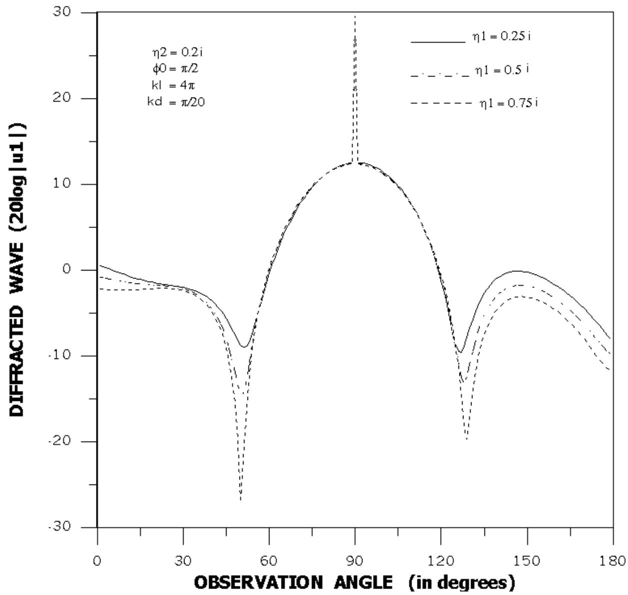

Figure 4. Diffracted field for different positive values of η_1.

where (ρ, ϕ) and (r, ψ) are the cylindrical polar coordinates defined by

$$\begin{aligned} x &= \rho\cos\phi \quad , \quad y-d=\rho\sin\phi \\ x-\ell &= r\cos\psi \quad , \quad y-d=r\sin\psi \end{aligned}$$

6. NUMERICAL RESULTS

In this section some numerical results displaying the effects of various parameters such as the surface impedances of the plates, slit width and the separation distance between the parallel-plates are presented. From Fig. 4 and Fig. 5 we can see that the diffracted field amplitude decreases with the increasing values of $|\eta_1|$.

Fig. 6 and Fig. 7 display the effects of η_2 to the diffracted field. It is observed that the diffracted field amplitude is insensitive to the variation of η_2 for angles of incidence near $\pi/2$, as expected

In Fig. 8 we see that the diffracted field amplitude becomes more oscillating when the slit width l increases. Finally, Fig. 9 show the variation of the diffracted field amplitude with respect to the observation angle for different values of the distance d, separating the parallel-plates.

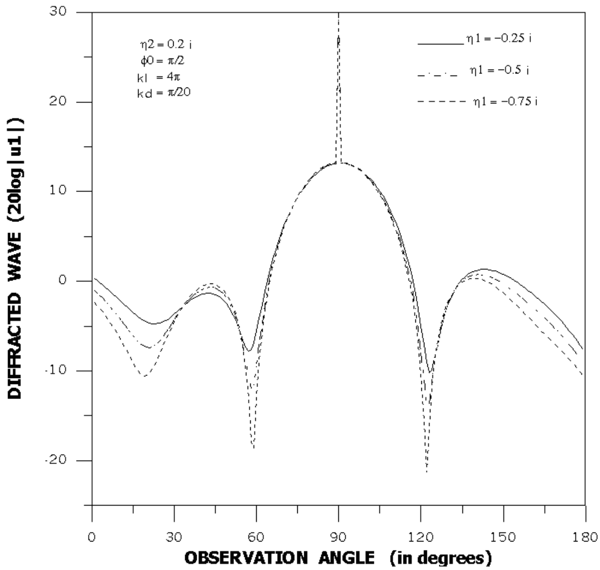

Figure 5. Diffracted field for different negative values of η_1.

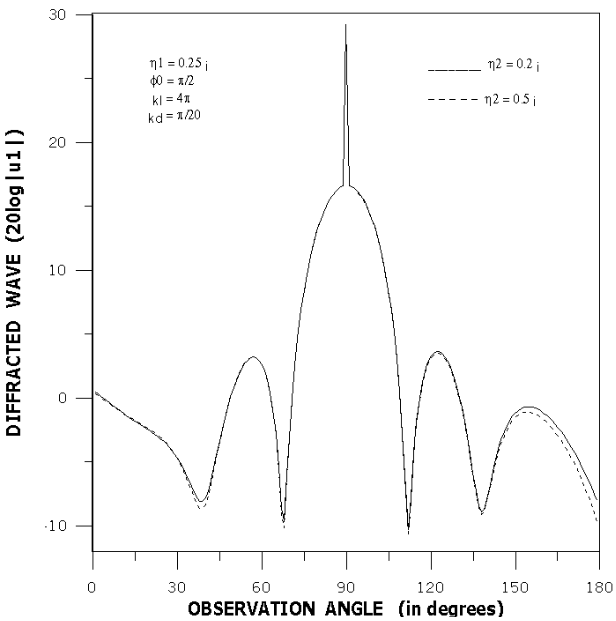

Figure 6. Diffracted field for different positive values of η_2 for angle of incidence $\pi/2$.

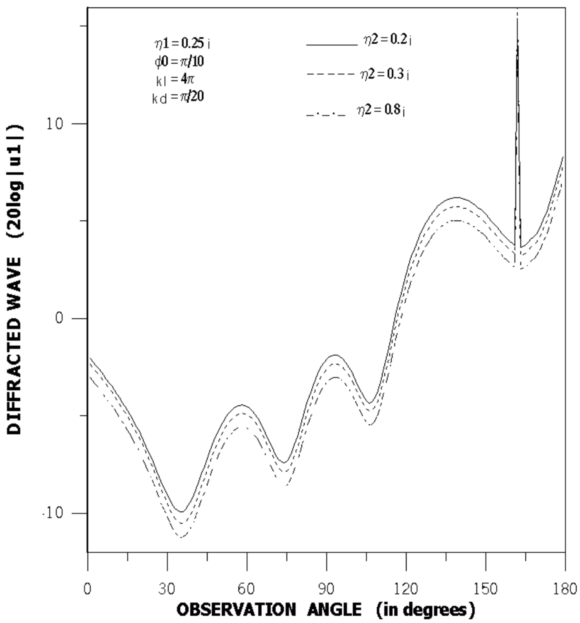

Figure 7. Diffracted field for different negative values of η_2 for angle of incidence $\pi/10$.

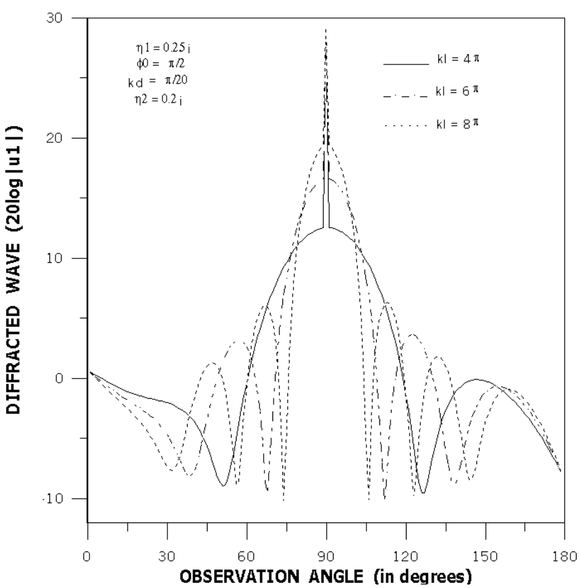

Figure 8. Diffracted field for different values of kl.

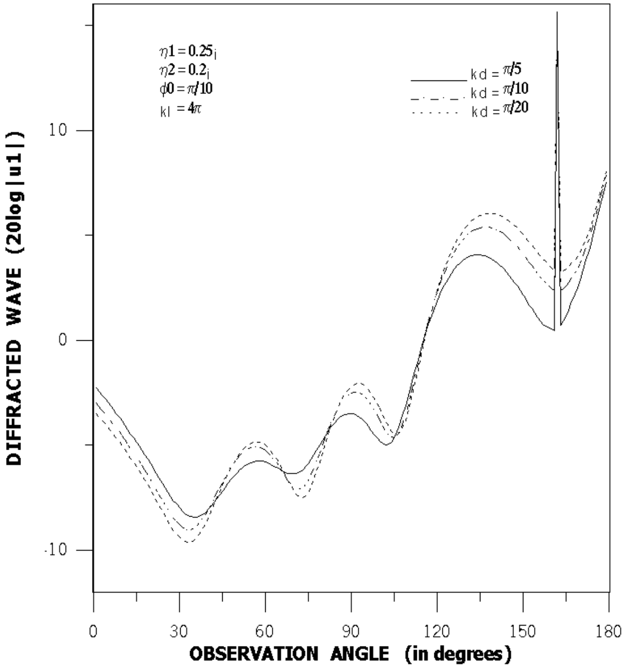

Figure 9. Diffracted field for different values of kl.

7. CONCLUDING REMARKS

In this work the diffraction of E-polarized plane waves by a tandem impedance slit waveguide is investigated rigorously by using the Fourier transform technique in conjunction with the mode-matching method. The approximate scattered field expressions (61c–f) are uniformly valid for all angles of incidence and observation except for the grazing incidence.

By applying the duality principle, the results related to the H-polarized incident plane wave can be obtained from (61c–f) on replacing $\eta_{1,2}$ by $1/\eta_{1,2}$.

For the special case $Z_1 + Z_2 = 0$ we get

$$\alpha_m^e = \beta_m^e \quad , \quad K_m^e = \xi_m^e \quad m = 1, 2, \cdots$$

so that (20) and (21) are identically satisfied for

$$f_m^e = -i a_m^e (\beta_m^e + \alpha) \quad , \quad g_m^e = -i b_m^e (\beta_m^e - \alpha)$$

This gives

$$f_m^e(-\alpha_m^e) = 0 \quad , \quad g_m^e(\alpha_m^e) = 0$$

implying the series contributions to the total diffracted field in (61c,d) be zero. Similar considerations are also valid for the odd excitation case.

For $\ell \to \infty$, it can be checked easily that the solution of the Wiener-Hopf equations in (49a) and (57a) reduce to the results related to a parallel plate impedance waveguide (see [6] formulas (29a) and (37) for zero plate thickness).

Furthermore for $d \to 0$, $\eta_1 \to 0$ we have

$$N_{\pm}^{(e)}(\alpha) \to 1 \quad , \quad N_{\pm}^{(o)}(\alpha) \to 0 \quad , \quad \chi_{\pm}(\alpha) \to \sqrt{1 \pm \alpha/k}$$

and the expression of the total diffracted field from the origin reduces to

$$\begin{aligned} u_1(\rho,\phi) \quad = \quad & -\frac{e^{i\pi/4}}{\sqrt{2\pi}} \frac{\sqrt{1+\cos\phi_0}\sqrt{1+\cos\phi}}{\cos\phi_0+\cos\phi} \frac{e^{ik\rho}}{\sqrt{k\rho}} \\ & + \frac{i}{\pi} e^{-ik\ell\cos\phi_0} \frac{\sqrt{1-\cos\phi_0}\sqrt{1+\cos\phi}}{\cos\phi_0+\cos\phi} \\ & \times \left[\frac{F[k\ell(1-\cos\phi_0)]}{1-\cos\phi_0} - \frac{F[k\ell(1+\cos\phi)]}{1+\cos\phi} \right] \frac{e^{ik\ell}}{\sqrt{k\ell}} \frac{e^{ik\rho}}{\sqrt{k\rho}} \end{aligned} \tag{68}$$

which is nothing but the well-known result for a slit in a perfectly conducting plane [9].

REFERENCES

1. Alldredge, L. R., "Diffraction of microwaves by tandem slits," *IRE Trans. on Antennas and Propagat.,* Vol. AP-4, 640–649, Oct. 1956.
2. Levine, H. and J. Schwinger, "On the theory of electromagnetic wave diffraction by an aperture in an infinite plane conducting screen," *Comm. on Pure and Appl. Math.,* Vol. 3, 355–391, December 1950.
3. Kashyap, S. C. and M. A. K. Hamid, "Diffraction characteristics of a slit in a thick conducting screen," *IEEE Trans. on Antennas and Propagat.,* Vol. AP-19, No. 4, 499–507, July 1971.
4. Yoshidomi, K. and K. Aoki, "Scattering of an E-polarized plane wave by two parallel rectangular impedance cylinders," *Radio Sci.,* Vol. 23, 471–480, 1988.
5. Büyükaksoy, A. and B. Polat, "Diffraction of acoustic waves by a semi-infinite cylindrical impedance pipe of certain wall thickness," *J. Engng. Math.,* Vol. 33, 333–352, 1998.

6. Büyükaksoy, A. and B. Polat, "Plane wave diffraction by a thick walled impedance waveguide," *IEEE Trans. on Antennas and Propagation,* Vol. 46, 1692–1699, 1998.
7. Mittra, R. and S. W. Lee, *Analytical Techniques in the Theory of Guided Waves*, The Macmillan Company, New York, 1971.
8. Senior, T. B. A., "Half-plane edge diffraction," *Radio Sci.,* Vol. 10, 645–650, 1975.
9. Karp, S. N. and A. Russek, "Diffraction by a wide slit," *J. Appl. Phys.,* Vol. 37, 886–894, 1956.