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METHOD OF MOMENTS ANALYSIS OF ELECTRICALLY LARGE THIN SQUARE AND RECTANGULAR LOOP ANTENNAS: NEAR- AND FAR-ZONE FIELDS

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Abstract—This paper presents a method of moments (MoM) analysis, obtains the non-uniform current distribution in closed form, and computes the resulted radiated patterns in both near and far zones, of square and rectangular loop antennas with electrically larger perimeter. An oblique incident field in its general form is considered in the formulation of the non-uniform current distributions. In the Galerkin's MoM analysis, the Fourier exponential series is considered as the full-domain basis function series. As a result, the current distributions along the square and rectangular loops are expressed analytically in terms of the azimuth angle for various sizes of large loops. Finally, an alternative vector analysis of the electromagnetic (EM) fields radiated from thin rectangular loop antennas of arbitrary length $2a$ and width $2b$ is introduced. This method which employs the dyadic Green's function (DGF) in the derivation of the EM radiated fields makes the analysis general, compact and straightforward in both near- and far-zones. The EM radiated fields are expressed in terms of the vector wave eigenfunctions. Not only the exact solution of the EM fields in the near and far zones outside $\sqrt{a^2+b^2}$ are derived by the use of the spherical Bessel and Hankel functions of the first kind respectively,

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but also the inner regions between a and $\sqrt{a^2+b^2}$ are characterized by both the spherical Bessel and Hankel functions of the first kind. Validity of the numerical results is discussed and clarified.

1 Introduction

2 General Formulation of Current Distributions

3 General Formulation of EM Radiated Fields

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Appendix A. Analytical Expressions of Current Distributions

A.1 Case I: $\frac{\lambda}{8} \times \frac{\lambda}{4}$ rectangular loop

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1. INTRODUCTION

Thin circular loop antennas carrying different forms of the currents and their radiation characteristics have been investigated by many researchers over the last several decades. The recognized contributions to the field are made by many researchers, e.g., [1–19], to the authors' best knowledge. The radiation characteristics of the small circular loop antennas can also be readily found from antenna text books [20, 21]. However, literature on polygonal loop antennas located in free-space is limited [22–24].

In the literature [24], it has been cited that "theoretical analysis seems to be unsuccessful" for rectangular loops and therefore application of the rectangular loop antenna has received less attention. Previously, the rectangular loop antennas were modeled as a dipole [23] and the presentation was extremely complicated as compared to the analysis demonstrated here. And only experimental techniques and measurements were presented by [22]. In [20, 21], only an approximated current distribution is used to show the radiation patterns in far zone. Nowadays, the results in the far-zone can

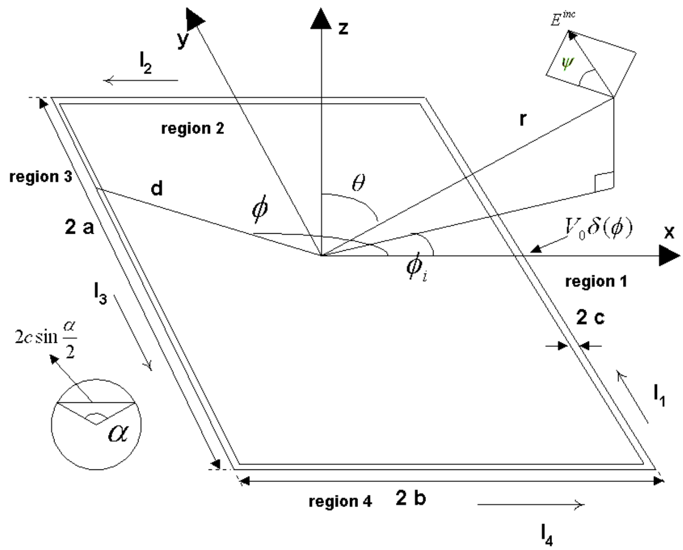

Figure 1. Geometry of a thin rectangular loop antenna.

be obtained easily from running the commercially available software packages. However, for the near-zone fields, none has been reported in the literature due to its calculational difficulty and the near-field results are not readily available in most of the commercial packages. Also, users who wish to design a loop antenna have to purchase the commercial packages.

Therefore, this motivates the present work and the paper aims to present the formulations of the analytical accurate current distributions using the Galerkin's MoM and the EM fields in inner, outer and intermediate zones radiated by the square and rectangular loop antennas. This serves as an extension of the present works in [18, 19] where the expansion coefficients of current distribution are not obtained.

2. GENERAL FORMULATION OF CURRENT DISTRIBUTIONS

Fig. 1 shows the geometry of a thin rectangular loop antenna located at $z = 0$ and fed at $\phi_i = 0$ with the delta-function generator $V_0\delta(\phi)$ for an active antenna system. Also, an incident wave is considered here for a passive antenna system.

According to the boundary condition, the loop current carried by the transceiver antenna satisfies the integral equation [1, 5, 10, 25] as follows:

$$V_0\delta(\phi) + dE_\phi^{inc}(d,\phi) = \frac{j\eta_o}{4\pi}\sum_{t=1}^{4}\int_{\text{region t}} K_t(\phi-\phi')I_t(\phi')\,d\phi' \qquad (1)$$

where E_ϕ^{inc} denotes the incident field, d stands for the distance between origin and side of the loop and $\eta_o = 120\pi\Omega$ is the free-space wave impedance. The current $I_t(\phi)$ can be expressed by

$$I_t(\phi) = \sum_{m=-\infty}^{\infty} I_m^t e^{-jm\phi} \qquad (2)$$

where t takes the form of integers of 1 to 4 for square and rectangular loop, δ_{m0} denotes the Kronecker symbol and I_m^t stands for the series expansion coefficients. The integral kernel in (1) is represented by the following Fourier series expansion

$$\begin{aligned}
K_t(\phi-\phi') &= \sum_{m=-\infty}^{\infty} g_m^t e^{-jm(\phi-\phi')}, && (3)\\
g_m^t &= \int_{\text{region t}}\left(\frac{k_0 d}{2}(N_{m+1}^t + N_{m-1}^t) - \frac{m^2}{k_0 d}N_m^t\right)d\phi', && (4)\\
d &= \begin{cases} a/|\cos\phi'|, & \text{regions 1 \& 3}\\ b/|\sin\phi'|, & \text{regions 2 \& 4}\end{cases}, && (5)\\
N_m^t &= \frac{1}{4\pi^2}\int_{\text{region t}}\int_{-\pi}^{\pi}\cdot\frac{e^{jm(\phi-\phi')}e^{(-jk_0 r')}}{r'}\,d\alpha\,d\phi', && (6)\\
r' &= \begin{cases} \sqrt{\left(\dfrac{a}{|\cos\phi'|}\right)^2 + 4c^2\sin(\alpha/2)}, & \text{regions 1 \& 3}\\ \sqrt{\left(\dfrac{b}{|\sin\phi'|}\right)^2 + 4c^2\sin(\alpha/2)}, & \text{regions 2 \& 4}\end{cases}, && (7)
\end{aligned}$$

$$\begin{bmatrix}\text{region 1}\\ \text{region 2}\\ \text{region 3}\\ \text{region 4}\end{bmatrix} \Longrightarrow \begin{bmatrix} -\phi_0 < \phi' \le \phi_0\\ \phi_0 < \phi' \le \pi-\phi_0\\ \pi-\phi_0 < \phi' \le \pi+\phi_0\\ \pi+\phi_0 < \phi' \le 2\pi-\phi_0\end{bmatrix}, \qquad (8)$$

and

$$\phi_0 = \tan^{-1}(b/a). \qquad (9)$$

The incident field E_ϕ^{inc} at the angle ϕ is given by

$$E_\phi^{inc} = E_0^{inc}[\cos\psi\cos(\phi-\phi_i)+\sin\psi\sin(\phi-\phi_i)\cos\theta]e^{jk_0 d\cos(\phi-\phi_i)\sin\theta}. \tag{10}$$

It can also be expressed in terms of the Fourier series as follows:

$$E_\phi^{inc} = \sum_{m=-\infty}^{\infty}\sum_{t=1}^{4} f_m^t e^{-jm\phi} \tag{11}$$

where

$$\begin{aligned} f_m^t = \frac{1}{2\pi}\int_{\text{region t}} & E_0^{inc}\left\{ j^{m-1}\cos\psi e^{jm\phi_0} J'_m(k_0 d\sin\theta) \right. \\ & \left. + j^m \sin\psi\cos\theta e^{jm\phi_0}\frac{mJ_m(k_0 d\sin\theta)}{k_0 d\sin\theta}\right\} d\phi'. \end{aligned} \tag{12}$$

To obtain maximum electric and magnetic responses, the loop orientation is made at $\psi = 0$, $\theta = \frac{\pi}{2}$, and $\phi_0 = 0$ which simplifies f_m^t and I_m^t to

$$f_m^t = \frac{1}{2\pi}\int_{\text{region t}} \frac{j^{m-1}}{2}\left[J_{m-1}(k_0 d) - J_{m+1}(k_0 d)\right] d\phi' \tag{13}$$

and

$$I_m^t = -j\frac{V(0)+2\pi f_m^t}{g_m^t \pi\eta_o}. \tag{14}$$

3. GENERAL FORMULATION OF EM RADIATED FIELDS

The detailed formulation of the EM radiated fields of circular loop antennas has been shown in [19, (1)–(7)]. In view of this, the authors will not repeat the procedure. The inner, outer and intermediate regions defined in this paper are depicted in Fig. 2. With the current distribution given in (2), we can obtain the EM field expressions for the four zones as follows:

$$\begin{bmatrix} E_r^0 \\ E_r^1 \\ E_r^2 \\ E_r^3 \end{bmatrix} = -\frac{\eta_0 k_0^2}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n} D_{mn} P_n^m(\cos\theta) \begin{matrix}\cos \\ \sin\end{matrix}\ (m\phi)$$

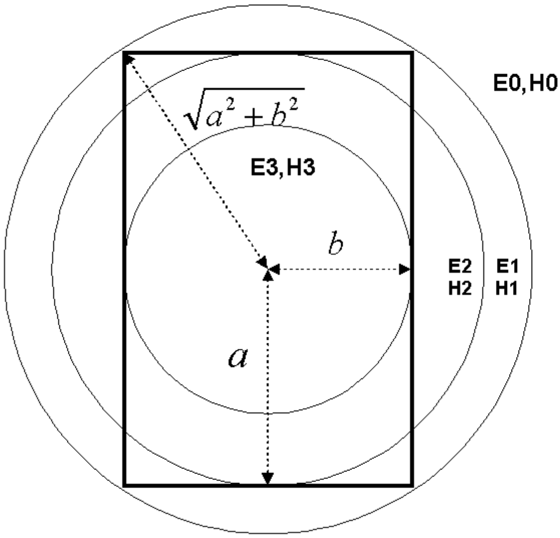

Figure 2. Intermediated zones.

$$\cdot\frac{n(n+1)}{k_0 r}\left\{ j_n(k_0 r)\begin{bmatrix} 0 \\ \left(\Phi^{N,r,1>}_{\substack{e\\o}mn}+\Phi^{N,\phi,1>}_{\substack{e\\o}mn}\right) \\ \left(\Phi^{N,r,2>}_{\substack{e\\o}mn}+\Phi^{N,\phi,2>}_{\substack{e\\o}mn}\right) \\ \left(\Phi^{N,r>}_{\substack{e\\o}mn}+\Phi^{N,\phi>}_{\substack{e\\o}mn}\right) \end{bmatrix} \right.$$

$$\left. +h_n^{(1)}(k_0 r)\begin{bmatrix} \left(\Phi^{N,r<}_{\substack{e\\o}mn}+\Phi^{N,\phi<}_{\substack{e\\o}mn}\right) \\ \left(\Phi^{N,r,1<}_{\substack{e\\o}mn}+\Phi^{N,\phi,1<}_{\substack{e\\o}mn}\right) \\ \left(\Phi^{N,r,2<}_{\substack{e\\o}mn}+\Phi^{N,\phi,2<}_{\substack{e\\o}mn}\right) \\ 0 \end{bmatrix} \right\}, \tag{15a}$$

$$\begin{bmatrix} E_\theta^0 \\ E_\theta^1 \\ E_\theta^2 \\ E_\theta^3 \end{bmatrix} = -\frac{\eta_0 k_0^2}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\mp\frac{mD_{mn}}{\sin\theta}P_n^m(\cos\theta)$$

$$\cdot\left\{j_n(k_0r)\begin{bmatrix}0\\ \Phi^{M,1>}_{\substack{e\\o}mn}\\ \Phi^{M,2>}_{\substack{e\\o}mn}\\ \Phi^{M>}_{\substack{e\\o}mn}\end{bmatrix}+h_n^{(1)}(k_0r)\begin{bmatrix}\Phi^{M<}_{\substack{e\\o}mn}\\ \Phi^{M,1<}_{\substack{e\\o}mn}\\ \Phi^{M,2<}_{\substack{e\\o}mn}\\ 0\end{bmatrix}\right\}$$

$$\cdot\ \begin{matrix}\sin\\ \cos\end{matrix}\ (m\phi)+D_{mn}\frac{dP_n^m(\cos\theta)}{d\theta}\ \begin{matrix}\cos\\ \sin\end{matrix}\ (m\phi)$$

$$\cdot\left\{\frac{d[rj_n(k_0r)]}{k_0rdr}\begin{bmatrix}0\\ \left(\Phi^{N,r,1>}_{\substack{e\\o}mn}+\Phi^{N,\phi,1>}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,2>}_{\substack{e\\o}mn}+\Phi^{N,\phi,2>}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r>}_{\substack{e\\o}mn}+\Phi^{N,\phi>}_{\substack{e\\o}mn}\right)\end{bmatrix}\right.$$

$$\left.+\frac{d\left[rh_n^{(1)}(k_0r)\right]}{k_0rdr}\begin{bmatrix}\left(\Phi^{N,r<}_{\substack{e\\o}mn}+\Phi^{N,\phi<}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,1<}_{\substack{e\\o}mn}+\Phi^{N,\phi,1<}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,2<}_{\substack{e\\o}mn}+\Phi^{N,\phi,2<}_{\substack{e\\o}mn}\right)\\ 0\end{bmatrix}\right\}, \tag{15b}$$

$$\begin{bmatrix}E_\phi^0\\ E_\phi^1\\ E_\phi^2\\ E_\phi^3\end{bmatrix} = -\frac{\eta_0k_0^2}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}-D_{mn}\frac{dP_n^m(\cos\theta)}{d\theta}$$

$$\cdot\left\{j_n(k_0r)\begin{bmatrix}0\\ \Phi^{M,1>}_{\substack{e\\o}mn}\\ \Phi^{M,2>}_{\substack{e\\o}mn}\\ \Phi^{M>}_{\substack{e\\o}mn}\end{bmatrix}+h_n^{(1)}(k_0r)\begin{bmatrix}\Phi^{M<}_{\substack{e\\o}mn}\\ \Phi^{M,1<}_{\substack{e\\o}mn}\\ \Phi^{M,2<}_{\substack{e\\o}mn}\\ 0\end{bmatrix}\right\}$$

$$\cdot\ \begin{matrix}\cos\\ \sin\end{matrix}\ (m\phi)\mp D_{mn}\frac{m}{\sin\theta}P_n^m(\cos\theta)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (m\phi)$$

$$\cdot\left\{\frac{d[rj_n(k_0r)]}{k_0rdr}\begin{bmatrix}0\\ \left(\Phi^{N,r,1>}_{\substack{e\\o}mn}+\Phi^{N,\phi,1>}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,2>}_{\substack{e\\o}mn}+\Phi^{N,\phi,2>}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r>}_{\substack{e\\o}mn}+\Phi^{N,\phi>}_{\substack{e\\o}mn}\right)\end{bmatrix}\right.$$

$$\left.+\frac{d[rh^{(1)}_n(k_0r)]}{k_0rdr}\begin{bmatrix}\left(\Phi^{N,r<}_{\substack{e\\o}mn}+\Phi^{N,\phi<}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,1<}_{\substack{e\\o}mn}+\Phi^{N,\phi,1<}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,2<}_{\substack{e\\o}mn}+\Phi^{N,\phi,2<}_{\substack{e\\o}mn}\right)\\ 0\end{bmatrix}\right\}; \tag{15c}$$

and

$$\begin{bmatrix}H^0_r\\ H^1_r\\ H^2_r\\ H^3_r\end{bmatrix}=\frac{ik_0^2}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}D_{mn}P^m_n(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)\frac{n(n+1)}{k_0r}$$

$$\cdot\left\{j_n(k_0r)\begin{bmatrix}0\\ \Phi^{M,1>}_{\substack{e\\o}mn}\\ \Phi^{M,2>}_{\substack{e\\o}mn}\\ \Phi^{M>}_{\substack{e\\o}mn}\end{bmatrix}+h^{(1)}_n(k_0r)\begin{bmatrix}\Phi^{M<}_{\substack{e\\o}mn}\\ \Phi^{M,1<}_{\substack{e\\o}mn}\\ \Phi^{M,2<}_{\substack{e\\o}mn}\\ 0\end{bmatrix}\right\}, \tag{16a}$$

$$\begin{bmatrix}H^0_\theta\\ H^1_\theta\\ H^2_\theta\\ H^3_\theta\end{bmatrix}=\frac{ik_0^2}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}D_{mn}\frac{m}{\sin\theta}P^m_n(\cos\theta)\begin{matrix}\sin\\ \cos\end{matrix}(m\phi)$$

$$\cdot\left\{\mp j_n(k_0r)\begin{bmatrix}0\\ \left(\Phi^{N,r,1>}_{\substack{e\\o}mn}+\Phi^{N,\phi,1>}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,2>}_{\substack{e\\o}mn}+\Phi^{N,\phi,2>}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r>}_{\substack{e\\o}mn}+\Phi^{N,\phi>}_{\substack{e\\o}mn}\right)\end{bmatrix}\right.$$

$$
\mp h_n^{(1)}(k_0 r)\begin{bmatrix}\left(\Phi^{N,r<}_{\substack{e\\o}mn}+\Phi^{N,\phi<}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,1<}_{\substack{e\\o}mn}+\Phi^{N,\phi,1<}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,2<}_{\substack{e\\o}mn}+\Phi^{N,\phi,2<}_{\substack{e\\o}mn}\right)\\ 0\end{bmatrix}\Bigg]\Bigg\}
$$

$$
+D_{mn}\frac{dP_n^m(\cos\theta)}{d\theta}\ \begin{matrix}\cos\\ \sin\end{matrix}\ (m\phi)
$$

$$
\cdot\left\{\frac{d[rj_n(k_0r)]}{k_0 r dr}\begin{bmatrix}0\\ \Phi^{M,1>}_{\substack{e\\o}mn}\\ \Phi^{M,2>}_{\substack{e\\o}mn}\\ \Phi^{M>}_{\substack{e\\o}mn}\end{bmatrix}+\frac{d[rh_n^{(1)}(k_0r)]}{k_0 r dr}\begin{bmatrix}\Phi^{M<}_{\substack{e\\o}mn}\\ \Phi^{M,1<}_{\substack{e\\o}mn}\\ \Phi^{M,2<}_{\substack{e\\o}mn}\\ 0\end{bmatrix}\right\}, \quad (16b)
$$

$$
\begin{bmatrix}H_\phi^0\\ H_\phi^1\\ H_\phi^2\\ H_\phi^3\end{bmatrix}=\frac{ik_0^2}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}-D_{mn}\frac{dP_n^m(\cos\theta)}{d\theta}\ \begin{matrix}\cos\\ \sin\end{matrix}\ (m\phi)
$$

$$
\cdot\Bigg\{j_n(k_0r)\begin{bmatrix}0\\ \left(\Phi^{N,r,1>}_{\substack{e\\o}mn}+\Phi^{N,\phi,1>}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,2>}_{\substack{e\\o}mn}+\Phi^{N,\phi,2>}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r>}_{\substack{e\\o}mn}+\Phi^{N,\phi>}_{\substack{e\\o}mn}\right)\end{bmatrix}
$$

$$
+h_n^{(1)}(k_0 r)\begin{bmatrix}\left(\Phi^{N,r<}_{\substack{e\\o}mn}+\Phi^{N,\phi<}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,1<}_{\substack{e\\o}mn}+\Phi^{N,\phi,1<}_{\substack{e\\o}mn}\right)\\ \left(\Phi^{N,r,2<}_{\substack{e\\o}mn}+\Phi^{N,\phi,2<}_{\substack{e\\o}mn}\right)\\ 0\end{bmatrix}\Bigg\}
$$

$$
\mp D_{mn}\frac{m}{\sin\theta}P_n^m(\cos\theta)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (m\phi)
$$

$$\cdot\left\{\frac{d[rj_n(k_0r)]}{k_0rdr}\begin{bmatrix}0\\ \Phi^{M,1>}_{\substack{e\\o}mn}\\ \Phi^{M,2>}_{\substack{e\\o}mn}\\ \Phi^{M>}_{\substack{e\\o}mn}\end{bmatrix}+\frac{d[rh_n^{(1)}(k_0r)]}{k_0rdr}\begin{bmatrix}\Phi^{M<}_{\substack{e\\o}mn}\\ \Phi^{M,1<}_{\substack{e\\o}mn}\\ \Phi^{M,2<}_{\substack{e\\o}mn}\\ 0\end{bmatrix}\right\}; \quad (16c)$$

where $j_n(k_0r)$ and $h_n^{(1)}(k_0r)$ are the spherical Bessel and Hankel functions of the first kind respectively, $P_n^m(\cos\theta)$ is the associated Legendre function, and the normalization coefficient D_{mn} is defined by

$$D_{mn}=\frac{(2n+1)}{n(n+1)}\frac{(n-m)!}{(n+m)!}.$$

The coefficients of the EM fields are expressed by

$$\begin{bmatrix}\Phi^{M<}_{\substack{e\\o}mn}\\ \Phi^{M,1<}_{\substack{e\\o}mn}\\ \Phi^{M,1>}_{\substack{e\\o}mn}\\ \Phi^{M,2<}_{\substack{e\\o}mn}\\ \Phi^{M,2>}_{\substack{e\\o}mn}\\ \Phi^{M>}_{\substack{e\\o}mn}\end{bmatrix}=\left\{\begin{bmatrix}\Psi^{M<}_{\substack{e\\o}13}+\Psi^{M<}_{\substack{e\\o}24}\\ \Psi^{M,1<}_{\substack{e\\o}13}+\Psi^{M,1<}_{\substack{e\\o}24}\\ \Psi^{M,1>}_{\substack{e\\o}13}+\Psi^{M,1>}_{\substack{e\\o}24}\\ \Psi^{M,2<}_{\substack{e\\o}13}+\Psi^{M,2<}_{\substack{e\\o}24}\\ \Psi^{M,2>}_{\substack{e\\o}13}+\Psi^{M,2>}_{\substack{e\\o}24}\\ \Psi^{M>}_{\substack{e\\o}13}+\Psi^{M>}_{\substack{e\\o}24}\end{bmatrix}\right\}, \quad (17a)$$

$$\begin{bmatrix}\Phi^{N,r<}_{\substack{e\\o}mn}\\ \Phi^{N,r,1<}_{\substack{e\\o}mn}\\ \Phi^{N,r,1>}_{\substack{e\\o}mn}\\ \Phi^{N,r,2<}_{\substack{e\\o}mn}\\ \Phi^{N,r,2>}_{\substack{e\\o}mn}\\ \Phi^{N,r>}_{\substack{e\\o}mn}\end{bmatrix}=\left\{\begin{bmatrix}\Psi^{N,r<}_{\substack{e\\o}13}+\Psi^{N,r<}_{\substack{e\\o}24}\\ \Psi^{N,r,1<}_{\substack{e\\o}13}+\Psi^{N,r,1<}_{\substack{e\\o}24}\\ \Psi^{N,r,1>}_{\substack{e\\o}13}+\Psi^{N,r,1>}_{\substack{e\\o}24}\\ \Psi^{N,r,2<}_{\substack{e\\o}13}+\Psi^{N,r,2<}_{\substack{e\\o}24}\\ \Psi^{N,r,2>}_{\substack{e\\o}13}+\Psi^{N,r,2>}_{\substack{e\\o}24}\\ \Psi^{N,r>}_{\substack{e\\o}13}+\Psi^{N,r>}_{\substack{e\\o}24}\end{bmatrix}\right\}, \quad (17b)$$

$$\begin{bmatrix} \Phi^{N,\phi<}_{\substack{e\\o}mn} \\ \Phi^{N,\phi,1<}_{\substack{e\\o}mn} \\ \Phi^{N,\phi,1>}_{\substack{e\\o}mn} \\ \Phi^{N,\phi,2<}_{\substack{e\\o}mn} \\ \Phi^{N,\phi,2>}_{\substack{e\\o}mn} \\ \Phi^{N,\phi>}_{\substack{e\\o}mn} \end{bmatrix} = \left\{ \begin{bmatrix} \Psi^{N,\phi<}_{\substack{e\\o}13} + \Psi^{N,\phi<}_{\substack{e\\o}24} \\ \Psi^{N,\phi,1<}_{\substack{e\\o}13} + \Psi^{N,\phi,1<}_{\substack{e\\o}24} \\ \Psi^{N,\phi,1>}_{\substack{e\\o}13} + \Psi^{N,\phi,1>}_{\substack{e\\o}24} \\ \Psi^{N,\phi,2<}_{\substack{e\\o}13} + \Psi^{N,\phi,2<}_{\substack{e\\o}24} \\ \Psi^{N,\phi,2>}_{\substack{e\\o}13} + \Psi^{N,\phi,2>}_{\substack{e\\o}24} \\ \Psi^{N,\phi>}_{\substack{e\\o}13} + \Psi^{N,\phi>}_{\substack{e\\o}24} \end{bmatrix} \right\}. \tag{17c}$$

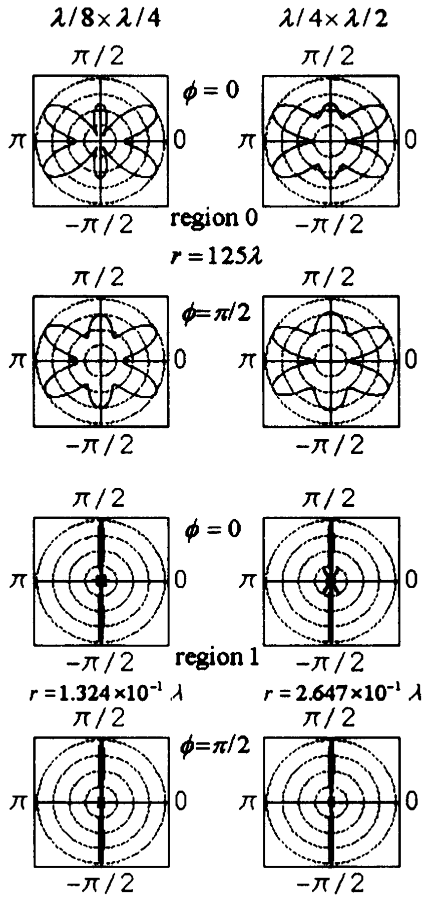

Figure 3. 2-D normalized EM radiation patterns in regions 0 and 1 of rectangular loop antennas.

4. NUMERICAL RESULTS

The current distributions are obtained using the MoM where the relationship between the loop and wire cross-section radii is $2\ln\frac{2\pi d}{c} \geq 10$. The analytical expressions of the current distribution are provided in the Appendix A and the convergence of the analysis is checked in details. It is realized from the computation that only four terms of the Fourier series are necessarily taken in the summation to ensure the accuracy while the fifth and higher order terms can be neglected.

Forty terms are considered for the numerical computation of the summation of the spherical Bessel and Hankel functions (*i.e.*, with respect to the index n). The 2-D normalized EM radiation patterns in the intermediate zones of the rectangular and square loop antennas are depicted in Figs. 3, 4 and 5, respectively. It should be pointed out that all numerical results are normalized by respectively their maxima.

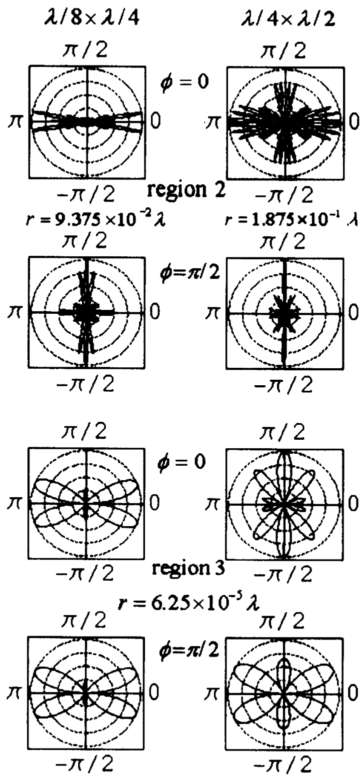

Figure 4. 2-D normalized EM radiation patterns in regions 2 and 3 of rectangular loop antennas.

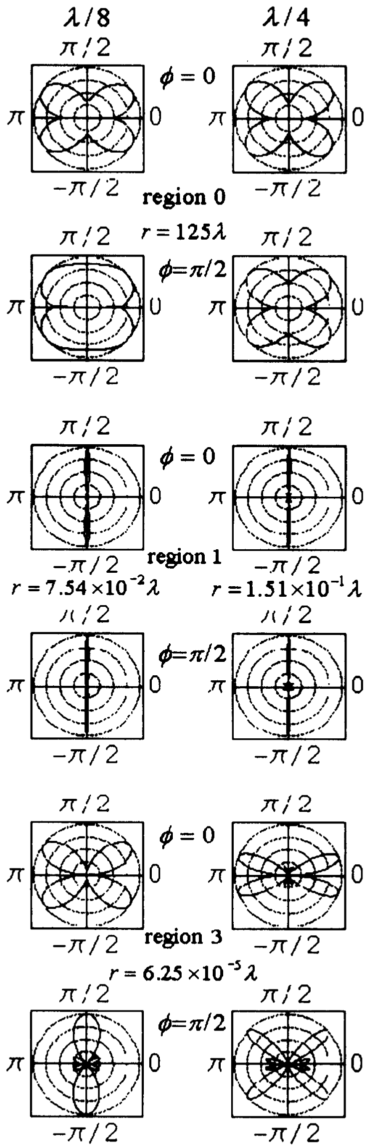

Figure 5. 2-D normalized EM radiation patterns in regions 0, 1 and 3 of square loop antennas.

5. CONCLUSIONS

This paper presents a MoM analysis of electrically large rectangular and square loop antennas. From the analysis, the radiated EM fields in the inner, outer and intermediate regions, of thin square and rectangular loop antennas are obtained. The cosine functions are chosen as the full-domain basis and weighting functions in the Galerkin's approach. Also, the dyadic Green's function in spherical coordinates is utilized in the derivation of the analytical expressions of the EM fields in the near, far and intermediate zones. The radiation patterns due to the derived current distributions are plotted in polar coordinates. The numerical results for radiation patterns in the far-zone are compared with results obtained in [20, Fig. 5.52] and an excellent agreement between the two results is obtained. This confirms partially the correctness of our theoretical derivation and numerical algorithm. Some new results are also presented in the paper for the antenna patterns due to large antennas.

APPENDIX A. ANALYTICAL EXPRESSIONS OF CURRENT DISTRIBUTIONS

In this appendix, the current distributions along the rectangular loop antenna wires of various dimensions are provided in closed form. They are expressed in terms of the spherical azimuth angle ϕ for different loop dimensions. These formulas are extremely useful for scientists and engineers who would like to obtain full-wave characteristics of the loop antennas since they can straightforwardly use these formulas without repeating the MoM analysis.

A.1. Case I: $\frac{\lambda}{8} \times \frac{\lambda}{4}$ rectangular loop

$$\begin{aligned} I_1(\phi) = \Big[&(0.08579+0.1316j)e^{j6\phi} + (-0.9956+2.220j)e^{j5\phi} \\ &+ (9.66+8.23j)e^{j4\phi} + (164.2-108.6j)e^{j3\phi} \\ &+ (1040+291.4j)e^{j2\phi} + (678.7-1221j)e^{j\phi} \\ &+ (509.3+304.3j) + (678.7-1221j)e^{-j\phi} \\ &+ (1040+291.4j)e^{-j2\phi} + (164.2-108.6j)e^{-j3\phi} \\ &+ (9.66+8.23j)e^{-j4\phi} + (-0.9956+2.22j)e^{-j5\phi} \\ &+ (0.08579+0.1316j)e^{-j6\phi}\Big]\, 10^{-7}, \end{aligned} \tag{A1}$$

$$I_2(\phi) = \Big[(-2.405-0.02027j)e^{j6\phi} + (-4.591-11.59j)e^{j5\phi}$$

$$
\begin{aligned}
&+(-62.28-13.49j)e^{j4\phi}+(311.6+232.2j)e^{j3\phi}\\
&+(-921.3-1196j)e^{j2\phi}+(2030-1955j)e^{j\phi}\\
&+(2187+551.2j)+(-1835-2148j)e^{-j\phi}\\
&+(-921.4-1196j)e^{-j2\phi}+(-220.8-324.7j)e^{-j3\phi}\\
&+(-36.78-64.76j)e^{-j4\phi}+(-4.591-11.59j)e^{-j5\phi}\\
&+\left.(0.01991+2.276j)e^{-j6\phi}\right]10^{-7},
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
I_3(\phi)=\ &\Big[(0.1244+0.1202j)e^{j6\phi}+(-0.2086+3.035j)e^{j5\phi}\\
&+(14.28+5.731j)e^{j4\phi}+(-138.8+145.2j)e^{j3\phi}\\
&+(1256+3464j)e^{j2\phi}+(-16.34+1868j)e^{j\phi}\\
&+(787.8+53.65j)+(-16.34+1868j)e^{-j\phi}\\
&+(1256+3464j)e^{-j2\phi}+(-138.8+145.2j)e^{-j3\phi}\\
&+(14.28+5.731j)e^{-j4\phi}+(-0.2086+3.035j)e^{-j5\phi}\\
&+(0.1244+0.1202j)e^{-j6\phi}\Big]10^{-7},
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
I_4(\phi)=\ &\Big[(-0.6271-2.391j)e^{j6\phi}+(-4.591-11.59j)e^{j5\phi}\\
&+(-12.40-62.51j)e^{j4\phi}+(-220.8-324.7j)e^{j3\phi}\\
&+(-237.6-125j)e^{j2\phi}+(-1835-2148j)e^{j\phi}\\
&+(416.9+2215j)+(2030+1955j)e^{-j\phi}\\
&+(-1138-979.3j)e^{-j2\phi}+(311.6+232.2j)e^{-j3\phi}\\
&+(-62.61-37.49j)e^{-j4\phi}+(11.11+4.192j)e^{-j5\phi}\\
&+(-1.901+0.1621j)e^{-j6\phi}\Big]10^{-7}.
\end{aligned}
\tag{A4}
$$

A.2. Case II: $\frac{\lambda}{4}\times\frac{\lambda}{2}$ rectangular loop

$$
\begin{aligned}
I_1(\phi)=\ &\Big[(-2.617+5.892j)e^{j6\phi}+(-35.29+27j)e^{j5\phi}\\
&+(19.37+131.8j)e^{j4\phi}+(831.8+361.8j)e^{j3\phi}\\
&+(1665+1213j)e^{j2\phi}+(1106-713.4j)e^{j\phi}\\
&+(717.6+1272j)+(1106-713.4j)e^{-j\phi}\\
&+(1665+1213j)e^{-j2\phi}+(831.8+361.8j)e^{-j3\phi}\\
&+(19.37+131.8j)e^{-j4\phi}+(-35.29+27j)e^{-j5\phi}\\
&+(-2.617+5.892j)e^{-j6\phi}\Big]10^{-7},
\end{aligned}
\tag{A5}
$$

$$\begin{aligned}
I_2(\phi) = \Big[&(-52.61 - 5.669j)e^{j6\phi} + (147.5 + 95.8j)e^{j5\phi} \\
&+ (-344.7 - 138.3j)e^{j4\phi} + (1037 + 1766j)e^{j3\phi} \\
&+ (-1366 - 615.2j)e^{j2\phi} + (163.5 + 1242j)e^{j\phi} \\
&+ (2708 - 806.1j) + (320 - 123.6j)e^{-j\phi} \\
&+ (1691 - 3510j)e^{-j2\phi} + (1320 - 1725j)e^{-j3\phi} \\
&+ (574.7 - 482.6j)e^{-j4\phi} + (177.2 - 85.95j)e^{-j5\phi} \\
&+ (42.41 - 6.625j)e^{-j6\phi}\Big] 10^{-7},
\end{aligned} \tag{A6}$$

$$\begin{aligned}
I_3(\phi) = \Big[&(3.282 + 5.935j)e^{j6\phi} + (-17.64 - 39.13j)e^{j5\phi} \\
&+ (146.3 + 38.51j)e^{j4\phi} + (-692.7 + 423.5j)e^{j3\phi} \\
&+ (1080 - 1357j)e^{j2\phi} + (653.1 + 1218j)e^{j\phi} \\
&+ (1458 - 631.5j) + (653.1 + 1218j)e^{-j\phi} \\
&+ (1080 - 1357j)e^{-j2\phi} + (-692.7 + 423.5j)e^{-j3\phi} \\
&+ (146.3 + 38.51j)e^{-j4\phi} + (-17.64 - 39.13j)e^{-j5\phi} \\
&+ (3.282 + 5.935j)e^{-j6\phi}\Big] 10^{-7},
\end{aligned} \tag{A7}$$

$$\begin{aligned}
I_4(\phi) = \Big[&(47.09 - 27.66j)e^{j6\phi} + (177.2 - 85.95j)e^{j5\phi} \\
&+ (349.4 - 126.1j)e^{j4\phi} + (1320 - 1725j)e^{j3\phi} \\
&+ (1421 - 469.3j)e^{j2\phi} + (320 - 1236j)e^{j\phi} \\
&+ (-2834 + 916.9j) + (163.5 + 1242j)e^{-j\phi} \\
&+ (-1204 - 3554j)e^{-j2\phi} + (1037 + 1766j)e^{-j3\phi} \\
&+ (-470.8 - 505.4j)e^{-j4\phi} + (147.5 + 95.8j)e^{-j5\phi} \\
&+ (-35.34 - 10.46j)e^{-j6\phi}\Big] 10^{-7}.
\end{aligned} \tag{A8}$$

Case III: $\frac{\lambda}{8}$ square loop

$$\begin{aligned}
I_1(\phi) = \Big[&(-0.0152 - 0.0085j)e^{j6\phi} + (-0.21 + 0.33j)e^{j5\phi} \\
&+ (14.23 + 19.44j)e^{j4\phi} + (-28.32 + 79.75j)e^{j3\phi} \\
&+ (284.9 + 127.3j)e^{j2\phi} + (395.2 - 839.4j)e^{j\phi} \\
&+ (346.4 + 165.1j) + (395.2 - 839.4j)e^{-j\phi} \\
&+ (284.9 + 127.3j)e^{-j2\phi} + (-28.32 + 79.75j)e^{-j3\phi} \\
&+ (14.23 + 19.44j)e^{-j4\phi} + (-0.2106 + 0.3319j)e^{-j5\phi}
\end{aligned}$$

$$+ (-0.01519 - 0.008525j)e^{-j6\phi}\Big] 10^{-7}, \tag{A9}$$

$$\begin{aligned} I_2(\phi) = \Big[&(0.0184 + 0.0071j)e^{j6\phi} + (-0.352 - 0.257j)e^{j5\phi} \\ &+ (15.39 + 17.56j)e^{j4\phi} + (95.57 + 8.611j)e^{j3\phi} \\ &+ (-352 - 86.94j)e^{j2\phi} + (1000 + 420.8j)e^{j\phi} \\ &+ (426.6 + 120.7j) + (-964.6 + 506.3j)e^{-j\phi} \\ &+ (-319.2 - 184.3j)e^{-j2\phi} + (-84.4 - 58.1j)e^{-j3\phi} \\ &+ (25.93 + 6.244j)e^{-j4\phi} + (0.3955 + 0.2113j)e^{-j5\phi} \\ &+ (0.0171 + 0.01053j)e^{-j6\phi}\Big] 10^{-7}, \end{aligned} \tag{A10}$$

$$\begin{aligned} I_3(\phi) = \Big[&(-0.0198 + 0.00062j)e^{j6\phi} + (0.038 - 0.044j)e^{j5\phi} \\ &+ (15.17 + 13.9j)e^{j4\phi} + (-37.51 - 84.96j)e^{j3\phi} \\ &+ (343.4 - 85.89j)e^{j2\phi} + (211.4 + 1036j)e^{j\phi} \\ &+ (429.9 - 82.91j) + (211.4 + 1036j)e^{-j\phi} \\ &+ (343.4 - 85.89j)e^{-j2\phi} + (-37.51 - 84.96j)e^{-j3\phi} \\ &+ (15.17 + 13.9j)e^{-j4\phi} + (0.03801 - 0.04419j)e^{-j5\phi} \\ &+ (-0.01983 + 0.000625j)e^{-j6\phi}\Big] 10^{-7}, \end{aligned} \tag{A11}$$

$$\begin{aligned} I_4(\phi) = \Big[&(0.0157 + 0.012j)e^{j6\phi} + (0.4 + 0.21j)e^{j5\phi} \\ &+ (16.28 + 20.87j)e^{j4\phi} + (-84.4 - 58.1j)e^{j3\phi} \\ &+ (-299.2 - 204.3j)e^{j2\phi} + (-964.6 - 506.3j)e^{j\phi} \\ &+ (361.5 + 255.4j) + (1000 + 420.8j)e^{-j\phi} \\ &+ (-345.8 - 114.2j)e^{-j2\phi} + (95.57 + 8.611j)e^{-j3\phi} \\ &+ (4.909 + 17.41j)e^{-j4\phi} + (-0.3519 - 0.2570j)e^{-j5\phi} \\ &+ (0.01763 + 0.008925j)e^{-j6\phi}\Big] 10^{-7}. \end{aligned} \tag{A12}$$

Case IV: $\frac{\lambda}{4}$ square loop

$$\begin{aligned} I_1(\phi) = \Big[&(-0.4018 - 0.6655j)e^{j6\phi} + (-8.068 + 3.686j)e^{j5\phi} \\ &+ (-72.88 + 218.3j)e^{j4\phi} + (-263.5 + 315.3j)e^{j3\phi} \\ &+ (515.8 + 574.8j)e^{j2\phi} + (772.5 - 641.8j)e^{j\phi} \\ &+ (626.2 + 768.4j) + (772.5 - 641.8j)e^{-j\phi} \\ &+ (515.8 + 574.8j)e^{-j2\phi} + (-263.5 + 315.3j)e^{-j3\phi} \end{aligned}$$

$$+\,(-72.88+218.3j)e^{-j4\phi}+(-8.068+3.686j)e^{-j5\phi}$$
$$+\,(-0.4018-0.6655j)e^{-j6\phi}\Big]\,10^{-7}, \tag{A13}$$

$$\begin{aligned}I_2(\phi) = \Big[&(0.9244+0.5033j)e^{j6\phi}+(-2.529-11.49j)e^{j5\phi}\\&+(33.98+223.9j)e^{j4\phi}+(466.8+5.128j)e^{j3\phi}\\&+(-1056-230.2j)e^{j2\phi}+(1247+1083j)e^{j\phi}\\&+(1328+391.9j)+(-842.6-1476j)e^{-j\phi}\\&+(-409.2-1217j)e^{-j2\phi}+(29.05-608.4j)e^{-j3\phi}\\&+(201.9-33.21j)e^{-j4\phi}+(9.769+10.72j)e^{-j5\phi}\\&+(0.469+1.203j)e^{-j6\phi}\Big]\,10^{-7},\end{aligned} \tag{A14}$$

$$\begin{aligned}I_3(\phi) = \Big[&(-0.5442+0.1875j)e^{j6\phi}+(-0.9462-6.786j)e^{j5\phi}\\&+(71.04+110.7j)e^{j4\phi}+(-178.9-173.7j)e^{j3\phi}\\&+(417.2-301.6j)e^{j2\phi}+(372.5+575.9j)e^{j\phi}\\&+(576.6-362.2j)+(372.5+575.9j)e^{-j\phi}\\&+(417.2-301.6j)e^{-j2\phi}+(-178.9-173.7j)e^{-j3\phi}\\&+(71.04+110.7j)e^{-j4\phi}+(-0.9462-6.786j)e^{-j5\phi}\\&+(-0.5442+0.1874j)e^{-j6\phi}\Big]\,10^{-7},\end{aligned} \tag{A15}$$

$$\begin{aligned}I_4(\phi) = \Big[&(0.0208+1.073j)e^{j6\phi}+(9.769+10.72j)e^{j5\phi}\\&+(-201.3+370.8j)e^{j4\phi}+(29.05-608.4j)e^{j3\phi}\\&+(-113.7-1070j)e^{j2\phi}+(-842.6-1476j)e^{j\phi}\\&+(112.5+1367j)+(1247+1083j)e^{-j\phi}\\&+(-1054-535.2j)e^{-j2\phi}+(466.8+5.128j)e^{-j3\phi}\\&+(-65.77+90.05j)e^{-j4\phi}+(-2.529-11.49j)e^{-j5\phi}\\&+(0.7442+0.8655j)e^{-j6\phi}\Big]\,10^{-7}.\end{aligned} \tag{A16}$$

APPENDIX B. EXPLICIT EXPRESSIONS OF COEFFICIENTS OF THE SERIES

The intermediate expressions of the coefficients defined in (17a)–(17b) are determined as follows:

$$\Psi^{M<}_{\substack{e\\o}13} = \left[\int_{-\phi_0}^{\phi_0} I_1 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_3\right] j_n\!\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}ca}\,d\phi' \tag{B1}$$

$$\Psi^{M,1<}_{\substack{e\\o}13} = \left[\int_{-\frac{\pi}{2}+\phi_1}^{\frac{\pi}{2}-\phi_1} I_1 + \int_{\frac{\pi}{2}+\phi_1}^{\frac{3\pi}{2}-\phi_1} I_3\right] j_n\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}ca}\,d\phi' \tag{B2}$$

$$\begin{aligned}\Psi^{M,1>}_{\substack{e\\o}13} = &\left[\int_{2\pi-\phi_0}^{-\frac{\pi}{2}+\phi_1} + \int_{\frac{\pi}{2}-\phi_1}^{\phi_0}\right] I_1 h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}ca}\,d\phi' \\ &+\left[\int_{\pi-\phi_0}^{\frac{\pi}{2}+\phi_1} + \int_{\frac{3\pi}{2}-\phi_1}^{\pi+\phi_0}\right] I_3 h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}ca}\,d\phi'\end{aligned} \tag{B3}$$

$$\Psi^{M,2<}_{\substack{e\\o}13} = \left[\int_{-\frac{\pi}{2}+\phi_1}^{\frac{\pi}{2}-\phi_1} I_1 + \int_{\frac{\pi}{2}+\phi_1}^{\frac{3\pi}{2}-\phi_1} I_3\right] j_n\left(\frac{k_0 a}{|\cos\phi'|}\right) \tag{B4}$$

$$\begin{aligned}\Psi^{M,2>}_{\substack{e\\o}13} = &\left[\int_{2\pi-\phi_0}^{-\frac{\pi}{2}+\phi_1} + \int_{\frac{\pi}{2}-\phi_1}^{\phi_0}\right] I_1 h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}ca}\,d\phi' \\ &+\left[\int_{\pi-\phi_0}^{\frac{\pi}{2}+\phi_1} + \int_{\frac{3\pi}{2}-\phi_1}^{\pi+\phi_0}\right] I_3 h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}ca}\,d\phi'\end{aligned} \tag{B5}$$

$$\Psi^{M>}_{\substack{e\\o}13} = \left[\int_{-\phi_0}^{\phi_0} I_1 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_3\right] h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}ca}\,d\phi'. \tag{B6}$$

$$\Psi^{M<}_{\substack{e\\o}24} = \left[\int_{-\phi_0}^{\phi_0} I_2 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_4\right] j_n\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}sb}\,d\phi' \tag{B7}$$

$$\Psi^{M,1<}_{\substack{e\\o}24} = \left[\int_{\frac{\pi}{2}-\phi_2}^{\frac{\pi}{2}+\phi_2} I_2 + \int_{\frac{3\pi}{2}-\phi_2}^{\frac{3\pi}{2}+\phi_2} I_4\right] j_n\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}sb}\,d\phi' \tag{B8}$$

$$\Psi^{M,1>}_{\substack{e\\o}24} = \left[\int_{\phi_0}^{\frac{\pi}{2}-\phi_2} + \int_{\frac{\pi}{2}+\phi_2}^{\pi-\phi_0}\right] I_2 h_n^{(1)}\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}sb}\,d\phi' \tag{B9}$$

$$+\left[\int_{\frac{3\pi}{2}+\phi_2}^{2\pi-\phi_0} + \int_{\pi+\phi_0}^{\frac{3\pi}{2}-\phi_2}\right] I_4 h_n^{(1)}\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}sb}\,d\phi' \tag{B10}$$

$$\Psi^{M,2<}_{\substack{e\\o}24} = 0 \tag{B11}$$

$$\Psi^{M,2>}_{\substack{e\\o}24} = \left[\int_{\phi_0}^{\pi-\phi_0} I_2 + \int_{\pi+\phi_0}^{2\pi-\phi_0} I_4\right] h_n^{(1)}\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}sb}\,d\phi' \tag{B12}$$

$$\Psi^{M>}_{\substack{e\\o}24} = \left[\int_{-\phi_0}^{\phi_0} I_2 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_4\right] h_n^{(1)}\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}sb}\,d\phi' \tag{B13}$$

$$\Psi^{N,r<}_{\substack{e\\o}13} = \left[\int_{-\phi_0}^{\phi_0} I_1 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_3\right] j_n\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}sa}\,d\phi' \tag{B14}$$

$$\Psi_{\substack{e\\o}13}^{N,r,1<} = \left[\int_{-\frac{\pi}{2}+\phi_1}^{\frac{\pi}{2}-\phi_1} I_1 + \int_{\frac{\pi}{2}+\phi_1}^{\frac{3\pi}{2}-\phi_1} I_3\right] j_n\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}sa}\,d\phi' \tag{B15}$$

$$\begin{aligned}\Psi_{\substack{e\\o}13}^{N,r,1>} = &\left[\int_{2\pi-\phi_0}^{-\frac{\pi}{2}+\phi_1} + \int_{\frac{\pi}{2}-\phi_1}^{\phi_0}\right] I_1 h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}sa}\,d\phi' \\ &+ \left[\int_{\pi-\phi_0}^{\frac{\pi}{2}+\phi_1} + \int_{\frac{3\pi}{2}-\phi_1}^{\pi+\phi_0}\right] I_3 h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}sa}\,d\phi'\end{aligned} \tag{B16}$$

$$\Psi_{\substack{e\\o}13}^{N,r,2<} = \left[\int_{-\frac{\pi}{2}+\phi_1}^{\frac{\pi}{2}-\phi_1} I_1 + \int_{\frac{\pi}{2}+\phi_1}^{\frac{3\pi}{2}-\phi_1} I_3\right] j_n\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}sa}\,d\phi' \tag{B17}$$

$$\begin{aligned}\Psi_{\substack{e\\o}13}^{N,r,2>} = &\left[\int_{2\pi-\phi_0}^{-\frac{\pi}{2}+\phi_1} + \int_{\frac{\pi}{2}-\phi_1}^{\phi_0}\right] I_1 h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}sa}\,d\phi' \\ &+ \left[\int_{\pi-\phi_0}^{\frac{\pi}{2}+\phi_1} + \int_{\frac{3\pi}{2}-\phi_1}^{\pi+\phi_0}\right] I_3 h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}sa}\,d\phi'\end{aligned} \tag{B18}$$

$$\Psi_{\substack{e\\o}13}^{N,r>} = \left[\int_{-\phi_0}^{\phi_0} I_1 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_3\right] h_n^{(1)}\left(\frac{k_0 a}{|\cos\phi'|}\right)\alpha_{\substack{e\\o}sa}\,d\phi' \tag{B19}$$

$$\Psi_{\substack{e\\o}24}^{N,r<} = \left[\int_{-\phi_0}^{\phi_0} I_2 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_4\right] j_n\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}cb}\,d\phi' \tag{B20}$$

$$\Psi_{\substack{e\\o}24}^{N,r,1<} = \left[\int_{\frac{\pi}{2}-\phi_2}^{\frac{\pi}{2}+\phi_2} I_2 + \int_{\frac{3\pi}{2}-\phi_2}^{\frac{3\pi}{2}+\phi_2} I_4\right] j_n\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}cb}\,d\phi' \tag{B21}$$

$$\begin{aligned}\Psi_{\substack{e\\o}24}^{N,r,1>} = &\left[\int_{\phi_0}^{\frac{\pi}{2}-\phi_2} + \int_{\frac{\pi}{2}+\phi_2}^{\pi-\phi_0}\right] I_2 h_n^{(1)}\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}cb}\,d\phi' \\ &+ \left[\int_{\frac{3\pi}{2}+\phi_2}^{2\pi-\phi_0} + \int_{\pi+\phi_0}^{\frac{3\pi}{2}-\phi_2}\right] I_4 h_n^{(1)}\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}cb}\,d\phi'\end{aligned} \tag{B22}$$

$$\Psi_{\substack{e\\o}24}^{N,r,2<} = 0 \tag{B23}$$

$$\Psi_{\substack{e\\o}24}^{N,r,2>} = \left[\int_{\phi_0}^{\pi-\phi_0} I_2 + \int_{\pi+\phi_0}^{2\pi-\phi_0} I_4\right] h_n^{(1)}\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}cb}\,d\phi' \tag{B24}$$

$$\Psi_{\substack{e\\o}24}^{N,r>} = \left[\int_{-\phi_0}^{\phi_0} I_2 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_4\right] h_n^{(1)}\left(\frac{k_0 b}{|\sin\phi'|}\right)\alpha_{\substack{e\\o}cb}\,d\phi' \tag{B25}$$

$$\Psi_{\substack{e\\o}13}^{N,\phi<} = \left[\int_{-\phi_0}^{\phi_0} I_1 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_3\right] \frac{d[x_0 j_n(x_0)]}{dx_0}\beta_{\substack{e\\o}ca}\,d\phi' \tag{B26}$$

$$\Psi^{N,\phi,1<}_{\substack{e\\o}13} = \left[\int_{-\frac{\pi}{2}+\phi_1}^{\pi-\phi_1} I_1 + \int_{\frac{\pi}{2}+\phi_1}^{\frac{3\pi}{2}-\phi_1} I_3\right] \frac{d[x_0 j_n(x_0)]}{dx_0} \beta_{\substack{e\\o}ca} d\phi' \tag{B27}$$

$$\Psi^{N,\phi,1>}_{\substack{e\\o}13} = \left[\int_{2\pi-\phi_0}^{-\frac{\pi}{2}+\phi_1} + \int_{\frac{\pi}{2}-\phi_1}^{\phi_0}\right] I_1 \frac{d\left[x_0 h_n^{(1)}(x_0)\right]}{dx_0} \beta_{\substack{e\\o}ca} d\phi'$$
$$+ \left[\int_{\pi-\phi_0}^{\frac{\pi}{2}+\phi_1} + \int_{\frac{3\pi}{2}-\phi_1}^{\pi+\phi_0}\right] I_3 \frac{d\left[x_0 h_n^{(1)}(x_0)\right]}{dx_0} \beta_{\substack{e\\o}ca} d\phi' \tag{B28}$$

$$\Psi^{N,\phi,2<}_{\substack{e\\o}13} = \left[\int_{-\frac{\pi}{2}+\phi_1}^{\frac{\pi}{2}-\phi_1} I_1 + \int_{\frac{\pi}{2}+\phi_1}^{\frac{3\pi}{2}-\phi_1} I_3\right] \frac{d[x_0 j_n(x_0)]}{dx_0} \beta_{\substack{e\\o}ca} d\phi' \tag{B29}$$

$$\Psi^{N,\phi,2>}_{\substack{e\\o}13} = \left[\int_{2\pi-\phi_0}^{-\frac{\pi}{2}+\phi_1} + \int_{\frac{\pi}{2}-\phi_1}^{\phi_0}\right] I_1 \frac{d\left[x_0 h_n^{(1)}(x_0)\right]}{dx_0} \beta_{\substack{e\\o}ca} d\phi'$$
$$+ \left[\int_{\pi-\phi_0}^{\frac{\pi}{2}+\phi_1} + \int_{\frac{3\pi}{2}-\phi_1}^{\pi+\phi_0}\right] I_3 \frac{d\left[x_0 h_n^{(1)}(x_0)\right]}{dx_0} \beta_{\substack{e\\o}ca} d\phi' \tag{B30}$$

$$\Psi^{N,\phi>}_{\substack{e\\o}13} = \left[\int_{-\phi_0}^{\phi_0} I_1 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_3\right] \frac{d\left[x_0 h_n^{(1)}(x_0)\right]}{dx_0} \beta_{\substack{e\\o}ca} d\phi', \tag{B31}$$

$$\Psi^{N,\phi<}_{\substack{e\\o}24} = \left[\int_{-\phi_0}^{\phi_0} I_2 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_4\right] \frac{d[y_0 j_n(y_0)]}{dy_0} \beta_{\substack{e\\o}sb} d\phi' \tag{B32}$$

$$\Psi^{N,\phi,1<}_{\substack{e\\o}24} = \left[\int_{\frac{\pi}{2}-\phi_2}^{\frac{\pi}{2}+\phi_2} I_2 + \int_{\frac{3\pi}{2}-\phi_2}^{\frac{3\pi}{2}+\phi_2} I_4\right] \frac{d[y_0 j_n(y_0)]}{dy_0} \beta_{\substack{e\\o}sb} d\phi' \tag{B33}$$

$$\Psi^{N,\phi,1>}_{\substack{e\\o}24} = \left[\int_{\phi_0}^{\frac{\pi}{2}-\phi_2} + \int_{\frac{\pi}{2}+\phi_2}^{\pi-\phi_0}\right] I_2 \frac{d\left[y_0 h_n^{(1)}(y_0)\right]}{dy_0} \beta_{\substack{e\\o}sb} d\phi'$$
$$+ \left[\int_{\frac{3\pi}{2}+\phi_2}^{2\pi-\phi_0} + \int_{\pi+\phi_0}^{\frac{3\pi}{2}-\phi_2}\right] I_4 \frac{d\left[y_0 h_n^{(1)}(y_0)\right]}{dy_0} \beta_{\substack{e\\o}sb} d\phi' \tag{B34}$$

$$\Psi^{N,\phi,2<}_{\substack{e\\o}24} = 0 \tag{B35}$$

$$\Psi^{N,\phi,2>}_{\substack{e\\o}24} = \left[\int_{\phi_0}^{\pi-\phi_0} I_2 + \int_{\pi+\phi_0}^{2\pi-\phi_0} I_4\right] \frac{d\left[y_0 h_n^{(1)}(y_0)\right]}{dy_0} \beta_{\substack{e\\o}sb} d\phi' \tag{B36}$$

$$\Psi^{N,\phi>}_{\substack{e\\o}24} = \left[\int_{-\phi_0}^{\phi_0} I_2 + \int_{\pi-\phi_0}^{\pi+\phi_0} I_4\right] \frac{d\left[y_0 h_n^{(1)}(y_0)\right]}{dy_0} \beta_{\substack{e\\o}sb} d\phi' \tag{B37}$$

$$\alpha_{\substack{e\\o}ca} = -(2-\delta_{m0})\frac{dP_n^m(0)}{d\theta}\begin{array}{c}\cos\\ \sin\end{array}(m\phi') \times \cos(\phi')\left(\frac{a}{|\cos\phi'|}\right)^2 \tag{B38}$$

$$\alpha_{\substack{e\\o}sb} = -(2-\delta_{m0})\frac{dP_n^m(0)}{d\theta}\begin{array}{c}\cos\\ \sin\end{array}(m\phi') \times \sin(\phi')\left(\frac{b}{|\sin\phi'|}\right)^2 \tag{B39}$$

$$\alpha_{\substack{e\\o}sa} = (2-\delta_{m0})\frac{n(n+1)P_n^m(0)}{k_0}\begin{array}{c}\cos\\ \sin\end{array}(m\phi') \times \sin(\phi')\left(\frac{a}{|\cos\phi'|}\right) \tag{B40}$$

$$\alpha_{\substack{e\\o}cb} = (2-\delta_{m0})\frac{n(n+1)P_n^m(0)}{k_0}\begin{array}{c}\cos\\ \sin\end{array}(m\phi') \times \cos(\phi')\left(\frac{b}{|\sin\phi'|}\right) \tag{B41}$$

$$\beta_{\substack{e\\o}ca} = \mp k_0(2-\delta_{m0})mP_n^m(0)\begin{array}{c}\sin\\ \cos\end{array}(m\phi') \times \cos(\phi')\left(\frac{a}{|\cos\phi'|}\right) \tag{B42}$$

$$\beta_{\substack{e\\o}sb} = \mp(2-\delta_{m0})mP_n^m(0)\begin{array}{c}\sin\\ \cos\end{array}(m\phi') \times \sin(\phi')\left(\frac{b}{|\sin\phi'|}\right) \tag{B43}$$

$$x_0 = \frac{k_0 a}{|\cos\phi'|} \tag{B44}$$

$$y_0 = \frac{k_0 b}{|\sin\phi'|} \tag{B45}$$

where $\phi_1 = \sin^{-1}(a/r)$ and $\phi_2 = \cos^{-1}(b/r)$. The associated Legendre function $P_n^m(0)$ and its first-order derivative $dP_n^m(0)/d\theta$ are given by

$$\frac{dP_n^m(0)}{d\theta} = -\frac{2^{m+1}\sin\left[\frac{1}{2}(n+m)\pi\right]\Gamma\left(\frac{n+m}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-m+1}{2}\right)}, \tag{B46}$$

$$P_n^m(0) = \frac{2^m \cos\left[\frac{1}{2}(n+m)\pi\right] \Gamma\left(\frac{n+m+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-m}{2}+1\right)}. \tag{B47}$$

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