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THEORY OF OPTICAL BULLETS

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Abstract—This paper is a theoretical study of solitons in multidimensions, also known as optical bullets, that is governed by the nonlinear Schrodinger's equation in 1 + 3 dimensions. The parameter dynamics of such multidimensional solitons has been obtained. The study is extended to obtain the adiabatic evolution of soliton parameters in presence of the perturbation terms. Furthermore, the parameter dynamics for the vector multidimensional solitons and including the presence of the perturbation terms has been obtained.

1. INTRODUCTION

The nonlinear Schrodinger's equation (NLSE) plays a vital role in various areas physical, biological and engineering sciences. It appears in many applied areas like Fluid Dynamics, Nonlinear Optics, Plasma Physics and Protein Chemistry just to name a few. In this paper we are going to study an important generalization of the NLSE known as the generalized nonlinear Schrodinger's equation (GNLSE) in $1+3$ dimensions that is given by:

$$iq_t + \frac{1}{2}\nabla^2 q + F\left(|q|^2\right) q = 0 \tag{1}$$

where, we have

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and F is a real valued algebraic function. This is the elliptic form of the GNLSE. The other form of the GNLSE is non-elliptic [27, 48]. Equation (1) is not integrable, in general. The nonintegrability is not necessarily related to the nonlinear term in (1). Higher order dispersion or birefringence, for example, can also make the system nonintegrable while it still remains Hamiltonian. The special case, $F(s) = s$, also known as the Kerr law of nonlinearity, is integrable in $1+1$ dimensions by the method of Inverse Scattering Transform (IST) first discovered by Zakharov and Shabat [1, 2]. The IST is the nonlinear analog of Fourier Transform that is used for solving linear partial differential equations. Schematically, the technique of IST and the Fourier transform are similar [2]. This special case falls in the category of S-integrable equations [24]. In this case, (1) is known as the cubic NLSE. The solutions are known as solitons. It arises in Fluid Dynamics, Nonlinear Optics and α-helix proteins in Protein Chemistry and many other areas [50].

In the context of optics, with a special functional form of F in (1) the pulse beam propagates in three dimensions and these are known as the '3-D envelope solitons' or more commonly *optical bullets* [7]. The term optical bullets was first coined by Silberberg in 1989 [7]. The three dimensional generalization of the NLSE admits spherical solitary waves. The GNLSE given by (1) contains both dispersion and the diffraction terms that will attempt to spread the pulse in both the longitudinal and transverse directions. Conversely, self-focussing effects will attempt to squeeze the pulse. It can thus be conceived that a stable self-trapped three-dimensional optical soliton could be formed when these opposing "forces" exactly balance. It is necessary to point out that the mere existence of solitary waves and this hand-waving "balance of forces" argument does not mean that these pulses *will* propagate without change. The standard nonlinear materials where the refractive index is strictly proportional to the intensity of the light do not allow stable light bullets. There are other refractive index models [7] that do permit the propagation of stable optical bullets. While true solitons, that are governed by the NLSE in $1+1$ dimensions with

Kerr law, survive collisions with no loss of energy, optical bullets are found to possess less energy than the input pulses. Thus optical bullets are not true solitons in a strict Mathematical sense. However, in the Physics literature and despite the protestations of a few pursuits, such robust pulses have pragmatically become to be known as solitons.

The stability problems for the Kerr law medium and for more general nonlinear medium has already been addressed [7]. Equation (1) has several symmetries in $1+2$ dimensions including the rotation, dilatation, Galelian transformations and Talanov lens transformations. On using the symmetry reductions, the exact solutions to (1) in $1+2$ dimensions was found for Kerr law case, in terms of Jacobi's elliptic functions. In addition, there also exists exact solution for the self-focussing and the self-defocussing cases that are multivalued at each point of real space. Moreover, non-stationary singular solutions were also obtained to (1) [7].

The general case where $F(s) \neq s$ takes us away from the IST picture as it is not of Painleve type [2]. Unlike the cubic NLSE which has an infinite number of conserved quantities, the GNLSE given by (1) has only a few. There are various types of non-Kerr law nonlinearity that are studied. A detailed account of the various types of non-Kerr law nonlinearity can be found elsewhere [14].

1.1. Integrals of Motion

As mentioned that equation (1), as it appears, is not integrable and neither it has infinitely many conserved quantities. In fact, there exists at least four integrals of motion. They are

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |q|^2 dxdydz \tag{2}$$

$$\boldsymbol{P} = \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(q\nabla q^* - q^*\nabla q\right) dxdydz \tag{3}$$

$$\boldsymbol{M} = \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{r}\times\left(q\nabla q^* - q^*\nabla q\right) dxdydz \tag{4}$$

$$H = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{1}{2}(\nabla q)\cdot(\nabla q^*) - f\left(|q|^2\right)\right] dxdydz \tag{5}$$

where

$$f(s) = \int_0^s F(s)ds$$

Here, in (4), $\boldsymbol{r}$ represents the position vector namely $\boldsymbol{r} = (x, y, z)$. The integrals of motion are respectively known as the *energy* (E), *linear momentum* $(\boldsymbol{P})$, *angular momentum* $(\boldsymbol{M})$ and the *Hamiltonian* (H).

We note that the first integral of motion, (2), is also known as the *mass, wave action, plasmon number* while in optics it is known as the *wave power*. Also, the conserved quantity related to the Talanov lens transformation was found in 1985 [2, 48]. It is important to remark here that in the special case of the power law of nonlinearity we have, an additional conserved quantity [48] that is given by

$$C = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|\boldsymbol{r}q + 2it\nabla q|^2 - \frac{4t^2}{p+1}|q|^{2p+2}\right)dxdydz \tag{6}$$

In fact, it was first pointed out by Kuznetsov and Turitsyn in 1985 that the invariant (C) in (6) is a consequence of the Noether's theorem [48].

2. SOLITON PARAMETER DYNAMICS

Since there is no inverse scattering solution to (1), we shall derive, in this section, the dynamics of soliton parameters from their corresponding definitions. For this, we shall assume that the soliton solution of (1), although not integrable, is given in the form [12–14]:

$$\begin{aligned} q(x,y,z;t) = A(t)g\left[B_1(t)\{x-\overline{x}(t)\}, B_2(t)\{y-\overline{y}(t)\}, B_3(t)\{z-\overline{z}(t)\}\right] \\ \exp[-i\kappa_1(t)\{x-\overline{x}(t)\}-i\kappa_2(t)\{y-\overline{y}(t)\}-i\kappa_3(t)\{z-\overline{z}(t)\} \\ +i\theta(t)] \end{aligned} \tag{7}$$

where g represents the shape of the soliton described by the GNLSE and it depends on the type of nonlinearity in (1). Also here, in (7), $A(t)$ represents the amplitude of the soliton, while $B_j(t)$ for $j = 1, 2, 3$ represents the width in the x, y and z direction respectively. We then have $\kappa_j(t)$ ($j = 1, 2, 3$) as the frequency of the soliton in the x, y and z direction respectively. Finally, $(\overline{x}(t), \overline{y}(t), \overline{z}(t))$ is the coordinate of the center of mass of the soliton. For convenience, we define the following integral:

$$\begin{aligned} I_{a,b,c,p}^{\alpha,\beta,\gamma} = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\tau_1)^a(\tau_2)^b(\tau_3)^c g^p(\tau_1,\tau_2,\tau_3) \\ & \cdot\left(\frac{\partial g}{\partial\tau_1}\right)^{\alpha}\left(\frac{\partial g}{\partial\tau_2}\right)^{\beta}\left(\frac{\partial g}{\partial\tau_3}\right)^{\gamma} d\tau_1 d\tau_2 d\tau_3 \end{aligned} \tag{8}$$

with non-negative integers a, b, c, p, α, β and γ where $\tau_1 = B_1(t)(x - \overline{x}(t))$, $\tau_2 = B_2(t)(y-\overline{y}(t))$ and $\tau_3 = B_3(t)(z-\overline{z}(t))$. For such a general form of the soliton given by (7), we have the integrals of motion, from

(2), (3), (4) and (5), respectively given by:

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |q|^2 dxdydz = \frac{A^2}{B_1B_2B_3} I_{0,0,0,2}^{0,0,0} \quad (9)$$

$$\begin{aligned} \boldsymbol{P} &= \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (q\nabla q^* - q^*\nabla q)\, dxdydz \\ &= -\frac{A^2}{B_1B_2B_3} I_{0,0,0,2}^{0,0,0}(\kappa_1,\kappa_2,\kappa_3) \end{aligned} \quad (10)$$

$$\begin{aligned} \boldsymbol{M} &= \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{r}\times(q\nabla q^* - q^*\nabla q)\, dxdydz \\ &= \frac{A^2}{B_1B_2B_3}\left(\frac{\kappa_2}{B_3} I_{0,0,1,2}^{0,0,0} - \frac{\kappa_3}{B_2} I_{0,1,0,2}^{0,0,0}, \right. \\ &\left. \frac{\kappa_3}{B_1} I_{1,0,0,2}^{0,0,0} - \frac{\kappa_1}{B_3} I_{0,0,1,2}^{0,0,0}, \frac{\kappa_1}{B_2} I_{0,1,0,2}^{0,0,0} - \frac{\kappa_2}{B_1} I_{1,0,0,2}^{0,0,0}\right) \end{aligned} \quad (11)$$

$$\begin{aligned} H &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{1}{2}(\nabla q)\cdot(\nabla q^*) - f\left(|q|^2\right)\right] dxdydz \\ &= \frac{A^2}{2B_1B_2B_3}\left[B_1^2 I_{0,0,0,0}^{2,0,0} + B_3^2 I_{0,0,0,0}^{0,2,0} + B_3^2 I_{0,0,0,0}^{0,0,2} \right. \\ &\left. + \left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right) I_{0,0,0,2}^{0,0,0}\right] - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^I f(s) dsdxdydz \end{aligned} \quad (12)$$

where the intensity I is given by $|q|^2$. The soliton parameters are defined as follows:

$$A(t) = \left[\frac{I_{0,0,0,2}^{0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^4 dxdydz}{I_{0,0,0,4}^{0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz}\right]^{\frac{1}{2}} \quad (13)$$

$$B_1(t) = \left[\frac{I_{2,0,0,2}^{0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz}{I_{0,0,0,2}^{0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau_1^2|q|^2 dxdydz}\right]^{\frac{1}{2}} \quad (14)$$

$$B_2(t) = \left[\frac{I_{0,2,0,2}^{0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz}{I_{0,0,0,2}^{0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau_2^2|q|^2 dxdydz}\right]^{\frac{1}{2}} \quad (15)$$

$$B_3(t) = \left[\frac{I^{0,0,0}_{0,0,2,2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz}{I^{0,0,0}_{0,0,0,2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau_3^2|q|^2 dxdydz}\right]^{\frac{1}{2}} \tag{16}$$

$$\kappa_1(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(qq_x^* - q^*q_x)\,dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz} \tag{17}$$

$$\kappa_2(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(qq_y^* - q^*q_y\right)dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz} \tag{18}$$

$$\kappa_3(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(qq_z^* - q^*q_z)\,dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz} \tag{19}$$

$$\overline{x}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x|q|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz} \tag{20}$$

$$\overline{y}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y|q|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz} \tag{21}$$

$$\overline{z}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}z|q|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2 dxdydz} \tag{22}$$

Now, differentiating these parameters with respect to t and using (7) we arrive at the following evolution equations for the soliton parameters:

$$\frac{dE}{dt} = 0 \tag{23}$$

$$\frac{dA}{dt} = 0 \tag{24}$$

$$\frac{dB_1}{dt} = \frac{dB_2}{dt} = \frac{dB_3}{dt} = 0 \tag{25}$$

$$\frac{d\kappa_1}{dt} = \frac{d\kappa_2}{dt} = \frac{d\kappa_3}{dt} = 0 \tag{26}$$

$$\frac{d\overline{x}}{dt} = -\kappa_1 \quad (27)$$

$$\frac{d\overline{y}}{dt} = -\kappa_2 \quad (28)$$

$$\frac{d\overline{z}}{dt} = -\kappa_3 \quad (29)$$

$$\begin{aligned}\frac{d\theta}{dt} =& -\frac{\kappa_1^2+\kappa_2^2+\kappa_3^2}{2} \\ &+\frac{1}{2I_{0,0,0,2}^{0,0,0}}\left(B_1^2 I_{0,0,0,0}^{2,0,0} + B_2^2 I_{0,0,0,0}^{0,2,0} + B_3^2 I_{0,0,0,0}^{0,0,2}\right) \\ &-\frac{1}{I_{0,0,0,2}^{0,0,0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2(\tau_1,\tau_2,\tau_3)F\left(A^2 g^2(\tau_1,\tau_2,\tau_3)\right)d\tau_1 d\tau_2 d\tau_3\end{aligned} \quad (30)$$

Thus, from (23)–(26), we notice that the energy, amplitude, width and the frequency of the solitons remains constant. However, the center of mass and the phase of the soliton undergo a change as governed by (27)–(30). Here, (30) is obtained by differentiating (7) with respect to t and subtracting from its conjugate while utilizing (1). Thus, we have obtained the parameter dynamics for the solitons due to non-Kerr law nonlinearity. In particular, the case of Kerr-law nonlinearity where $F(s) = s$, we have, specially in $1+1$ dimension, the *sech* profile of the soliton that has already been studied elsewhere using the variational principle [28].

2.1. Perturbation Terms

We shall now consider the GNLSE along with its perturbation terms that is given by

$$iq_t + \frac{1}{2}\nabla^2 q + F\left(|q|^2\right)q = i\epsilon R[q,q^*] \quad (31)$$

Here, R is a spatio-differential operator and ϵ is a perturbation parameter with $0 < \epsilon \ll 1$. The perturbation parameter depends on the type of nonlinearity. For example, in the context of fiber optics where $F(s) = s^p,\ (0 < p < 2)$ in general, this perturbation parameter is called the relative width of the spectrum that arises due to quasi-monochromaticity [28]. In presence of the perturbation terms, we have the adiabatic dynamics of the soliton parameters as

$$\frac{dE}{dt} = \epsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q^*R + qR^*)dxdydz \quad (32)$$

$$\frac{dA}{dt} = \frac{\epsilon}{I^{0,0,0}_{0,0,0,4}}\frac{B_1B_2B_3}{2A^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q|^2\left(q^*R+qR^*\right)dxdydz \tag{33}$$

$$\begin{aligned}\frac{dB_1}{dt} &= \frac{\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{B_2B_3}{2A^2B_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q^*R+qR^*)dxdydz \\ &\quad -\frac{\epsilon}{I^{0,0,0}_{2,0,0,2}}\frac{B_2B_3}{2A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau_1^2(q^*R+qR^*)dxdydz\end{aligned} \tag{34}$$

$$\begin{aligned}\frac{dB_2}{dt} &= \frac{\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{B_1B_3}{2A^2B_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q^*R+qR^*)dxdydz \\ &\quad -\frac{\epsilon}{I^{0,0,0}_{0,2,0,2}}\frac{B_1B_3}{2A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau_2^2(q^*R+qR^*)dxdydz\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{dB_3}{dt} &= \frac{\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{B_1B_2}{2A^2B_3^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q^*R+qR^*)dxdydz \\ &\quad -\frac{\epsilon}{I^{0,0,0}_{0,0,2,2}}\frac{B_1B_2}{2A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau_3^2(q^*R+qR^*)dxdydz\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{d\kappa_1}{dt} &= \frac{i\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{B_1B_2B_3}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q_x^*R-q_xR^*)dxdydz \\ &\quad -\frac{\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{\kappa_1B_1B_2B_3}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q^*R+qR^*)dxdydz\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{d\kappa_2}{dt} &= \frac{i\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{B_1B_2B_3}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q_y^*R-q_yR^*)dxdydz \\ &\quad -\frac{\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{\kappa_2B_1B_2B_3}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q^*R+qR^*)dxdydz\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{d\kappa_3}{dt} &= \frac{i\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{B_1B_2B_3}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q_z^*R-q_zR^*)dxdydz \\ &\quad -\frac{\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{\kappa_3B_1B_2B_3}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q^*R+qR^*)dxdydz\end{aligned} \tag{39}$$

$$\frac{d\overline{x}}{dt} = -\kappa_1+\frac{\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{B_1B_2B_3}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(q^*R+qR^*)dxdydz \tag{40}$$

$$\frac{d\overline{y}}{dt} = -\kappa_2+\frac{\epsilon}{I^{0,0,0}_{0,0,0,2}}\frac{B_1B_2B_3}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y(q^*R+qR^*)dxdydz \tag{41}$$

$$\frac{d\overline{z}}{dt} = -\kappa_3 + \frac{\epsilon}{I^{0,0,0}_{0,0,0,2}} \frac{B_1 B_2 B_3}{A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z(q^* R + qR^*) dx dy dz \tag{42}$$

$$\begin{aligned}\frac{d\theta}{dt} &= -\frac{\kappa_1^2 + \kappa_2^2 + \kappa_3^2}{2} \\ &+ \frac{1}{2 I^{0,0,0}_{0,0,0,2}} \left(B_1^2 I^{2,0,0}_{0,0,0,0} + B_2^2 I^{0,2,0}_{0,0,0,0} + B_3^2 I^{0,0,2}_{0,0,0,0} \right) \\ &- \frac{1}{I^{0,0,0}_{0,0,0,2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^2(\tau_1, \tau_2, \tau_3) F\left(A^2 g^2(\tau_1, \tau_2, \tau_3)\right) d\tau_1 d\tau_2 d\tau_3 \\ &+ \frac{i\epsilon}{I^{0,0,0}_{0,0,0,2}} \frac{B_1 B_2 B_3}{2A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (q^* R - qR^*) dx dy dz \end{aligned} \tag{43}$$

Here, equations (32)–(42) are obtained by differentiating (9) and (13)–(22) respectively while using (31). However, (43) was obtained by differentiating (7) with respect to t and subtracting from its conjugate and using (31). We shall now rewrite (32)–(43) in the following form using the form of the soliton given by (7):

$$\frac{dE}{dt} = 2\epsilon \frac{A}{B_1 B_2 B_3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau_1, \tau_2, \tau_3) \Re \left[R e^{-i\phi} \right] d\tau_1 d\tau_2 d\tau_3 \tag{44}$$

$$\frac{dA}{dt} = \frac{\epsilon}{I^{0,0,0}_{0,0,0,4}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^3(\tau_1, \tau_2, \tau_3) \Re \left[R e^{-i\phi} \right] d\tau_1 d\tau_2 d\tau_3 \tag{45}$$

$$\begin{aligned}\frac{dB_1}{dt} &= \frac{\epsilon}{I^{0,0,0}_{0,0,0,2}} \frac{1}{AB_1^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau_1, \tau_2, \tau_3) \Re \left[R e^{-i\phi} \right] d\tau_1 d\tau_2 d\tau_3 \\ &- \frac{\epsilon}{I^{0,0,0}_{2,0,0,2}} \frac{1}{AB_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau_1^2 g(\tau_1, \tau_2, \tau_3) \Re \left[R e^{-i\phi} \right] d\tau_1 d\tau_2 d\tau_3 \end{aligned} \tag{46}$$

$$\begin{aligned}\frac{dB_2}{dt} &= \frac{\epsilon}{I^{0,0,0}_{0,0,0,2}} \frac{1}{AB_2^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau_1, \tau_2, \tau_3) \Re \left[R e^{-i\phi} \right] d\tau_1 d\tau_2 d\tau_3 \\ &- \frac{\epsilon}{I^{0,0,0}_{0,2,0,2}} \frac{1}{AB_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau_2^2 g(\tau_1, \tau_2, \tau_3) \Re \left[R e^{-i\phi} \right] d\tau_1 d\tau_2 d\tau_3 \end{aligned} \tag{47}$$

$$\begin{aligned}\frac{dB_3}{dt} &= \frac{\epsilon}{I^{0,0,0}_{0,0,0,2}} \frac{1}{AB_3^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau_1, \tau_2, \tau_3) \Re \left[R e^{-i\phi} \right] d\tau_1 d\tau_2 d\tau_3 \\ &- \frac{\epsilon}{I^{0,0,0}_{0,0,2,2}} \frac{1}{AB_3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau_3^2 g(\tau_1, \tau_2, \tau_3) \Re \left[R e^{-i\phi} \right] d\tau_1 d\tau_2 d\tau_3 \end{aligned} \tag{48}$$

$$\frac{d\kappa_1}{dt} = -\frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{\kappa_1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right] d\tau_1 d\tau_2 d\tau_3$$
$$-\frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\kappa_1 g(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right]\right.$$
$$\left. + B_1\frac{\partial g}{\partial\tau_1}\Im\left[Re^{-i\phi}\right]\right\} d\tau_1 d\tau_2 d\tau_3 \qquad (49)$$

$$\frac{d\kappa_2}{dt} = -\frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{\kappa_2}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right] d\tau_1 d\tau_2 d\tau_3$$
$$-\frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\kappa_2 g(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right]\right.$$
$$\left. + B_2\frac{\partial g}{\partial\tau_2}\Im\left[Re^{-i\phi}\right]\right\} d\tau_1 d\tau_2 d\tau_3 \qquad (50)$$

$$\frac{d\kappa_3}{dt} = -\frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{\kappa_3}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right] d\tau_1 d\tau_2 d\tau_3$$
$$-\frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\kappa_3 g(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right]\right.$$
$$\left. + B_3\frac{\partial g}{\partial\tau_3}\Im\left[Re^{-i\phi}\right]\right\} d\tau_1 d\tau_2 d\tau_3 \qquad (51)$$

$$\frac{d\overline{x}}{dt} = -\kappa_1 + \frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xg(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right] d\tau_1 d\tau_2 d\tau_3 \qquad (52)$$

$$\frac{d\overline{y}}{dt} = -\kappa_2 + \frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yg(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right] d\tau_1 d\tau_2 d\tau_3 \qquad (53)$$

$$\frac{d\overline{z}}{dt} = -\kappa_3 + \frac{2\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} zg(\tau_1,\tau_2,\tau_3)\Re\left[Re^{-i\phi}\right] d\tau_1 d\tau_2 d\tau_3 \qquad (54)$$

$$\frac{d\theta}{dt} = -\frac{\kappa_1^2+\kappa_2^2+\kappa_3^2}{2}$$
$$+\frac{1}{2I_{0,0,0,2}^{0,0,0}}\left(B_1^2 I_{0,0,0,0}^{2,0,0} + B_2^2 I_{0,0,0,0}^{0,2,0} + B_3^2 I_{0,0,0,0}^{0,0,2}\right)$$
$$-\frac{1}{I_{0,0,0,2}^{0,0,0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2(\tau_1,\tau_2,\tau_3)F\left(A^2 g^2(\tau_1,\tau_2,\tau_3)\right) d\tau_1 d\tau_2 d\tau_3$$

$$+\frac{\epsilon}{I_{0,0,0,2}^{0,0,0}}\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau_1,\tau_2,\tau_3)\Im\left[Re^{-i\phi}\right]d\tau_1 d\tau_2 d\tau_3 \quad (55)$$

where $\Re$ and $\Im$ represent the real and imaginary parts respectively and

$$\phi = \kappa_1(t)\left\{x-\overline{x}(t)\right\}+\kappa_2(t)\left\{y-\overline{y}(t)\right\}+\kappa_3(t)\left\{z-\overline{z}(t)\right\}-\theta(t)$$

Equations (44)–(55) represent the adiabatic dynamics of perturbed multidimensional solitons or optical bullets that are governed by the NLSE with non-Kerr law nonlinearity.

3. VECTOR SOLITONS

The multidimensional vector solitons or vector optical bullets that are governed by the generalized NLSE that are going to be studied in this paper are of the following form:

$$iu_t+\frac{1}{2}\nabla^2 u+\left\{F\left(|u|^2\right)+\alpha F\left(|v|^2\right)\right\}u = 0 \quad (56)$$

$$iv_t+\frac{1}{2}\nabla^2 v+\left\{F\left(|v|^2\right)+\alpha F\left(|u|^2\right)\right\}v = 0 \quad (57)$$

Here, in equations (56) and (57) we have α is a constant. These equations also arise in other areas of Physics and Mathematical Physics. The phenomenon of soliton fusion and soliton fission can be explained by means of these mathematical models.

Equations (56) and (57) do not have infinitely many conserved quantities. In fact, they have at least three integrals of motion that are given by

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|u|^2+|v|^2\right)dxdydz \quad (58)$$

$$\boldsymbol{P} = \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(u\nabla u^*-u^*\nabla u)+(v\nabla v^*-v^*\nabla v)\}dxdydz \quad (59)$$

$$\boldsymbol{M} = \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\{\boldsymbol{r}\times(u\nabla u^*-u^*\nabla u)\} +\{\boldsymbol{r}\times(v\nabla v^*-v^*\nabla v)\}]dxdydz \quad (60)$$

In this section we shall derive the dynamics of vector soliton parameters from their corresponding definitions. For this, we shall assume that the solitons of (56) and (57), although not integrable, are given in the form [12, 13]:

$$u(x,y,z;t) = A^{(1)}(t)g\left[B_1^{(1)}(t)\left\{x-\overline{x}^{(1)}(t)\right\},B_2^{(1)}(t)\left\{y-\overline{y}^{(1)}(t)\right\},\right.$$

$$B_3^{(1)}(t)\left\{z-\overline{z}^{(1)}(t)\right\}\Big]\exp\Big[-i\kappa_1^{(1)}(t)\left\{x-\overline{x}^{(1)}(t)\right\} - i\kappa_2^{(1)}(t)\left\{y-\overline{y}^{(1)}(t)\right\} - i\kappa_3^{(1)}(t)\left\{z-\overline{z}^{(1)}(t)\right\} + i\theta^{(1)}(t)\Big] \tag{61}$$

$$v(x,y,z;t) = A^{(2)}(t)g\left[B_1^{(2)}(t)\left\{x-\overline{x}^{(2)}(t)\right\}, B_2^{(2)}(t)\left\{y-\overline{y}^{(2)}(t)\right\}, B_3^{(2)}(t)\left\{z-\overline{z}^{(2)}(t)\right\}\right]\exp\Big[-i\kappa_1^{(2)}(t)\left\{x-\overline{x}^{(2)}(t)\right\} - i\kappa_2^{(2)}(t)\left\{y-\overline{y}^{(2)}(t)\right\} - i\kappa_3^{(2)}(t)\left\{z-\overline{z}^{(2)}(t)\right\} + i\theta^{(2)}(t)\Big] \tag{62}$$

where g represents the shape of the soliton described by the GNLSE and it depends on the type of nonlinearity in (56) and (57). Also here, in (61) and (62), $A^{(l)}(t)$ with $l = 1, 2$ represents the amplitude of the solitons, while $B_j^{(l)}(t)$ for $j = 1, 2, 3$ represents the width in the x, y and z directions respectively. We then have $\kappa_j^{(l)}(t)$ as the frequency of the soliton in the x, y and z directions respectively. Finally, $(\overline{x}^{(l)}(t), \overline{y}^{(l)}(t), \overline{z}^{(l)}(t))$ represent the coordinates of the center of masses of the solitons. For convenience, we define the following integral:

$$I_{a,b,c,p}^{l,\alpha,\beta,\gamma} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_1^{(l)}\right)^a\left(\tau_2^{(l)}\right)^b\left(\tau_3^{(l)}\right)^c g^p\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right)\left(\frac{\partial g}{\partial \tau_1^{(l)}}\right)^\alpha\left(\frac{\partial g}{\partial \tau_2^{(l)}}\right)^\beta\left(\frac{\partial g}{\partial \tau_3^{(l)}}\right)^\gamma d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \tag{63}$$

with non-negative integers a, b, c, p, α, β, γ and $l = 1, 2$ where $\tau_1^{(l)} = B_1^{(l)}(t)(x - \overline{x}^{(l)}(t))$, $\tau_2^{(l)} = B_2^{(l)}(t)(y - \overline{y}^{(l)}(t))$ and $\tau_3^{(l)} = B_3^{(l)}(t)(z - \overline{z}^{(l)}(t))$. For such a general form of the soliton given by (61) and (62), we have the integrals of motion, (58), (59) and (60), respectively reduce to:

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|u|^2 + |v|^2\right)dxdydz = \frac{\left(A^{(1)}\right)^2}{B_1^{(1)}B_2^{(1)}B_3^{(1)}} I_{0,0,0,2}^{1,0,0,0} + \frac{\left(A^{(2)}\right)^2}{B_1^{(2)}B_2^{(2)}B_3^{(2)}} I_{0,0,0,2}^{2,0,0,0} \tag{64}$$

$$\boldsymbol{P} = \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(u\nabla u^* - u^*\nabla u) + (v\nabla v^* - v^*\nabla v)\}dxdydz$$

$$
\begin{aligned}
&= -I_{0,0,0,2}^{1,0,0,0}\frac{\left(A^{(1)}\right)^2}{B_1^{(1)}B_2^{(1)}B_3^{(1)}}\left(\kappa_1^{(1)},\kappa_2^{(1)},\kappa_3^{(1)}\right) \\
&\quad -I_{0,0,0,2}^{2,0,0,0}\frac{\left(A^{(2)}\right)^2}{B_1^{(2)}B_2^{(2)}B_3^{(2)}}\left(\kappa_1^{(2)},\kappa_2^{(2)},\kappa_3^{(2)}\right) \qquad (65)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{M} &= \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\{\boldsymbol{r}\times(u\nabla u^* - u^*\nabla u)\} \\
&\quad +\{\boldsymbol{r}\times(v\nabla v^* - v^*\nabla v)\}]dxdydz \\
&= \frac{\left(A^{(1)}\right)^2}{B_1^{(1)}B_2^{(1)}B_3^{(1)}}\left(\frac{\kappa_2^{(1)}}{B_3^{(1)}}I_{0,0,1,2}^{1,0,0,0} - \frac{\kappa_3^{(1)}}{B_2^{(1)}}I_{0,1,0,2}^{1,0,0,0},\right. \\
&\quad \left.\frac{\kappa_3^{(1)}}{B_1^{(1)}}I_{1,0,0,2}^{1,0,0,0} - \frac{\kappa_1^{(1)}}{B_3^{(1)}}I_{0,0,1,2}^{1,0,0,0}, \frac{\kappa_1^{(1)}}{B_2^{(1)}}I_{0,1,0,2}^{1,0,0,0} - \frac{\kappa_2^{(1)}}{B_1^{(1)}}I_{1,0,0,2}^{1,0,0,0}\right) \\
&\quad + \frac{\left(A^{(2)}\right)^2}{B_1^{(2)}B_2^{(2)}B_3^{(2)}}\left(\frac{\kappa_2^{(2)}}{B_3^{(2)}}I_{0,0,1,2}^{2,0,0,0} - \frac{\kappa_3^{(2)}}{B_2^{(2)}}I_{0,1,0,2}^{2,0,0,0},\right. \\
&\quad \left.\frac{\kappa_3^{(2)}}{B_1^{(2)}}I_{1,0,0,2}^{2,0,0,0} - \frac{\kappa_1^{(2)}}{B_3^{(2)}}I_{0,0,1,2}^{2,0,0,0}, \frac{\kappa_1^{(2)}}{B_2^{(2)}}I_{0,1,0,2}^{2,0,0,0} - \frac{\kappa_2^{(2)}}{B_1^{(2)}}I_{1,0,0,2}^{2,0,0,0}\right) \qquad (66)
\end{aligned}
$$

The soliton parameters for (61) are defined as follows [12, 13]:

$$
A^{(1)}(t) = \left[\frac{I_{0,0,0,2}^{1,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^4dxdydz}{I_{0,0,0,4}^{1,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2dxdydz}\right]^{\frac{1}{2}} \qquad (67)
$$

$$
B_1^{(1)}(t) = \left[\frac{I_{2,0,0,2}^{1,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2dxdydz}{I_{0,0,0,2}^{1,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_1^{(1)}\right)^2|u|^2dxdydz}\right]^{\frac{1}{2}} \qquad (68)
$$

$$
B_2^{(1)}(t) = \left[\frac{I_{0,2,0,2}^{1,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2dxdydz}{I_{0,0,0,2}^{1,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_2^{(1)}\right)^2|u|^2dxdydz}\right]^{\frac{1}{2}} \qquad (69)
$$

$$B_3^{(1)}(t) = \left[\frac{I_{0,0,2,2}^{1,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2 dxdydz}{I_{0,0,0,2}^{1,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_3^{(1)}\right)^2|u|^2 dxdydz}\right]^{\frac{1}{2}} \tag{70}$$

$$\kappa_1^{(1)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(uu_x^* - u^*u_x\right)dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2 dxdydz} \tag{71}$$

$$\kappa_2^{(1)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(uu_y^* - u^*u_y\right)dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2 dxdydz} \tag{72}$$

$$\kappa_3^{(1)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(uu_z^* - u^*u_z\right)dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2 dxdydz} \tag{73}$$

$$\overline{x}^{(1)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x|u|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2 dxdydz} \tag{74}$$

$$\overline{y}^{(1)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y|u|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2 dxdydz} \tag{75}$$

$$\overline{z}^{(1)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}z|u|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^2 dxdydz} \tag{76}$$

Now, differentiating these parameters with respect to t and using (61) and (62) we arrive at the following evolution equations for the soliton parameters:

$$\frac{dE}{dt} = 0 \tag{77}$$

$$\frac{dA^{(1)}}{dt} = 0 \tag{78}$$

$$\frac{dB_1^{(1)}}{dt} = \frac{dB_2^{(1)}}{dt} = \frac{dB_3^{(1)}}{dt} = 0 \tag{79}$$

$$\frac{d\kappa_1^{(1)}}{dt} = \frac{d\kappa_2^{(1)}}{dt} = \frac{d\kappa_3^{(1)}}{dt} = 0 \tag{80}$$

$$\frac{d\overline{x}^{(1)}}{dt} = -\kappa_1^{(1)} \tag{81}$$

$$\frac{d\overline{y}^{(1)}}{dt} = -\kappa_2^{(1)} \tag{82}$$

$$\frac{d\overline{z}^{(1)}}{dt} = -\kappa_3^{(1)} \tag{83}$$

$$\begin{aligned}\frac{d\theta^{(1)}}{dt} = & -\frac{\left(\kappa_1^{(1)}\right)^2+\left(\kappa_2^{(1)}\right)^2+\left(\kappa_3^{(1)}\right)^2}{2} - \alpha F\left(\left(A^{(2)}\right)^2 g^2\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)\right) \\ & + \frac{1}{2I_{0,0,0,2}^{1,0,0,0}}\left\{\left(B_1^{(1)}\right)^2 I_{0,0,0,0}^{1,2,0,0} + \left(B_2^{(1)}\right)^2 I_{0,0,0,0}^{1,0,2,0} + \left(B_3^{(1)}\right)^2 I_{0,0,0,0}^{1,0,0,2}\right\} \\ & - \frac{1}{I_{0,0,0,2}^{1,0,0,0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right) \\ & \cdot F\left(\left(A^{(1)}\right)^2 g^2\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right)\right) d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)}\end{aligned} \tag{84}$$

For the soliton in (57) we have the corresponding parameters defined

$$A^{(2)}(t) = \left[\frac{I_{0,0,0,2}^{2,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^4 dxdydz}{I_{0,0,0,4}^{2,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2 dxdydz}\right]^{\frac{1}{2}} \tag{85}$$

$$B_1^{(2)}(t) = \left[\frac{I_{2,0,0,2}^{2,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2 dxdydz}{I_{0,0,0,2}^{2,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_1^{(2)}\right)^2|v|^2 dxdydz}\right]^{\frac{1}{2}} \tag{86}$$

$$B_2^{(2)}(t) = \left[\frac{I_{0,2,0,2}^{2,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2 dxdydz}{I_{0,0,0,2}^{2,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_2^{(2)}\right)^2|v|^2 dxdydz}\right]^{\frac{1}{2}} \tag{87}$$

$$B_3^{(2)}(t) = \left[\frac{I_{0,0,2,2}^{2,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2 dxdydz}{I_{0,0,0,2}^{2,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_3^{(2)}\right)^2|v|^2 dxdydz}\right]^{\frac{1}{2}} \tag{88}$$

$$\kappa_1^{(2)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(vv_x^* - v^*v_x)\,dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2dxdydz} \tag{89}$$

$$\kappa_2^{(2)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(vv_y^* - v^*v_y\right)dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2dxdydz} \tag{90}$$

$$\kappa_3^{(2)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(vv_z^* - v^*v_z)\,dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2dxdydz} \tag{91}$$

$$\overline{x}^{(2)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x|v|^2dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2dxdydz} \tag{92}$$

$$\overline{y}^{(2)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y|v|^2dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2dxdydz} \tag{93}$$

$$\overline{z}^{(2)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}z|v|^2dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v|^2dxdydz} \tag{94}$$

Again, differentiating these parameters with respect to t we arrive at the following evolution equations for the soliton parameters:

$$\frac{dA^{(2)}}{dt} = 0 \tag{95}$$

$$\frac{dB_1^{(2)}}{dt} = \frac{dB_2^{(2)}}{dt} = \frac{dB_3^{(2)}}{dt} = 0 \tag{96}$$

$$\frac{d\kappa_1^{(2)}}{dt} = \frac{d\kappa_2^{(2)}}{dt} = \frac{d\kappa_3^{(2)}}{dt} = 0 \tag{97}$$

$$\frac{d\overline{x}^{(2)}}{dt} = -\kappa_1^{(2)} \tag{98}$$

$$\frac{d\overline{y}^{(2)}}{dt} = -\kappa_2^{(2)} \tag{99}$$

$$\frac{d\overline{z}^{(2)}}{dt} = -\kappa_3^{(2)} \tag{100}$$

$$\begin{aligned}\frac{d\theta^{(2)}}{dt} = & -\frac{\left(\kappa_1^{(2)}\right)^2+\left(\kappa_2^{(2)}\right)^2+\left(\kappa_3^{(2)}\right)^2}{2} - \alpha F\left(\left(A^{(1)}\right)^2 g^2\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right)\right)\\ & +\frac{1}{2I_{0,0,0,2}^{2,0,0,0}}\left\{\left(B_1^{(2)}\right)^2 I_{0,0,0,0}^{2,2,0,0}+\left(B_2^{(2)}\right)^2 I_{0,0,0,0}^{2,0,2,0}+\left(B_3^{(2)}\right)^2 I_{0,0,0,0}^{2,0,0,2}\right\}\\ & -\frac{1}{I_{0,0,0,2}^{2,0,0,0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)\\ & \cdot F\left(\left(A^{(2)}\right)^2 g^2\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)\right) d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}\end{aligned} \tag{101}$$

Thus, from (77)–(80), we notice that the energy, amplitude, width and the frequency of the solitons remains constant. However, the center of mass and the phase of the soliton undergo a change as governed by (81)–(84). Here, (84) is obtained by differentiating (61) with respect to t and subtracting from its conjugate while utilizing (56). We have a similar situation for the soliton parameters of (57). Thus, we have obtained the parameter dynamics for the solitons due to non-Kerr law nonlinearity.

3.1. Perturbation Terms

We shall now consider the GNLSE along with its perturbation terms that are given by

$$iu_t + \frac{1}{2}\nabla^2 q + \left\{F\left(|u|^2\right) + \alpha F\left(|v|^2\right)\right\} u = i\epsilon R_1[u,u^*;v,v^*] \tag{102}$$

$$iv_t + \frac{1}{2}\nabla^2 v + \left\{F\left(|v|^2\right) + \alpha F\left(|u|^2\right)\right\} v = i\epsilon R_2[v,v^*;u,u^*] \tag{103}$$

Here, R_1 and R_2 are, once again, spatio-differential operators and ϵ is a perturbation parameter with $0 < \epsilon \ll 1$. In equations (102) and (103), the second nonlinear term on the left side is known as the cross-phase modulation (XPM) term in the context of optics. So, treating these XPM terms as the perturbation terms in addition to the regular perturbation terms that is already on the right side of (102), we arrive at the adiabatic parameter dynamics of the solitons as:

$$\frac{dE}{dt} = \epsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{(u^*R_1 + uR_1^*) + (v^*R_2 + vR_2^*)\right\} dxdydz \tag{104}$$

$$\frac{dA^{(1)}}{dt} = \frac{\epsilon}{I_{0,0,0,4}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)} B_3^{(1)}}{2\left(A^{(1)}\right)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^2 \left(u^* R_1 + u R_1^*\right) dxdydz \tag{105}$$

$$\begin{aligned}\frac{dB_1^{(1)}}{dt} = &\frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_2^{(1)} B_3^{(1)}}{2\left(A^{(1)}\right)^2 \left(B_1^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^* R_1 + u R_1^*\right) dxdydz \\ &- \frac{\epsilon}{I_{2,0,0,2}^{1,0,0,0}} \frac{B_2^{(1)} B_3^{(1)}}{2\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_1^{(1)}\right)^2 \left(u^* R_1 + u R_1^*\right) dxdydz\end{aligned} \tag{106}$$

$$\begin{aligned}\frac{dB_2^{(1)}}{dt} = &\frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_3^{(1)}}{2\left(A^{(1)}\right)^2 \left(B_2^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^* R_1 + u R_1^*\right) dxdydz \\ &- \frac{\epsilon}{I_{0,2,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_3^{(1)}}{2\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_2^{(1)}\right)^2 \left(u^* R_1 + u R_1^*\right) dxdydz\end{aligned} \tag{107}$$

$$\begin{aligned}\frac{dB_3^{(1)}}{dt} = &\frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)}}{2\left(A^{(1)}\right)^2 \left(B_3^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^* R_1 + u R_1^*\right) dxdydz \\ &- \frac{\epsilon}{I_{0,0,2,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)}}{2\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_3^{(1)}\right)^2 \left(u^* R_1 + u R_1^*\right) dxdydz\end{aligned} \tag{108}$$

$$\begin{aligned}\frac{d\kappa_1^{(1)}}{dt} = &\frac{i\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u_x^* R_1 - u_x R_1^*\right) dxdydz \\ &- \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{\kappa_1^{(1)} B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^* R_1 + u R_1^*\right) dxdydz\end{aligned} \tag{109}$$

$$\begin{aligned}\frac{d\kappa_2^{(1)}}{dt} = &\frac{i\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u_y^* R_1 - u_y R_1^*\right) dxdydz \\ &- \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{\kappa_1^{(1)} B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^* R_1 + u R_1^*\right) dxdydz\end{aligned} \tag{110}$$

$$\frac{d\kappa_3^{(1)}}{dt} = \frac{i\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(u_z^* R_1 - u_z R_1^*\right) dxdydz$$
$$- \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{\kappa_1^{(1)} B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(u^* R_1 + u R_1^*\right) dxdydz \quad (111)$$

$$\frac{d\overline{x}^{(1)}}{dt} = -\kappa_1^{(1)} + \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \left(u^* R_1 + u R_1^*\right) dxdydz \quad (112)$$

$$\frac{d\overline{y}^{(1)}}{dt} = -\kappa_2^{(1)} + \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y \left(u^* R_1 + u R_1^*\right) dxdydz \quad (113)$$

$$\frac{d\overline{z}^{(1)}}{dt} = -\kappa_3^{(1)} + \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)} B_3^{(1)}}{\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z \left(u^* R_1 + u R_1^*\right) dxdydz \quad (114)$$

$$\frac{d\theta^{(1)}}{dt} = -\frac{\left(\kappa_1^{(1)}\right)^2 + \left(\kappa_2^{(1)}\right)^2 + \left(\kappa_3^{(1)}\right)^2}{2} - \alpha F\left(\left(A^{(2)}\right)^2 g^2\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)\right)$$
$$+ \frac{1}{2 I_{0,0,0,2}^{1,0,0,0}} \left\{\left(B_1^{(1)}\right)^2 I_{0,0,0,0}^{1,2,0,0} + \left(B_2^{(1)}\right)^2 I_{0,0,0,0}^{1,0,2,0} + \left(B_3^{(1)}\right)^2 I_{0,0,0,0}^{1,0,0,2}\right\}$$
$$- \frac{1}{I_{0,0,0,2}^{1,0,0,0}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^2\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)$$
$$\cdot F\left(\left(A^{(1)}\right)^2 g^2\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)\right) d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)}$$
$$+ \frac{i\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{B_1^{(1)} B_2^{(1)} B_3^{(1)}}{2\left(A^{(1)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(u^* R_1 - u R_1^*\right) d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \quad (115)$$

Similarly, for the soliton in (103) we have the adiabatic parameter dynamics given by:

$$\frac{dA^{(2)}}{dt} = \frac{\epsilon}{I^{2,0,0,0}_{0,0,0,4}} \frac{B_1^{(2)} B_2^{(2)} B_3^{(2)}}{2\left(A^{(2)}\right)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |v|^2 \left(v^* R_2 + v R_2^*\right) dxdydz \tag{116}$$

$$\begin{aligned}\frac{dB_1^{(2)}}{dt} = {} & \frac{\epsilon}{I^{2,0,0,0}_{0,0,0,2}} \frac{B_2^{(2)} B_3^{(2)}}{2\left(A^{(2)}\right)^2 \left(B_1^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v^* R_2 + v R_2^*\right) dxdydz \\ & - \frac{\epsilon}{I^{2,0,0,0}_{2,0,0,2}} \frac{B_2^{(2)} B_3^{(2)}}{2\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_1^{(2)}\right)^2 \left(v^* R_2 + v R_2^*\right) dxdydz\end{aligned} \tag{117}$$

$$\begin{aligned}\frac{dB_2^{(2)}}{dt} = {} & \frac{\epsilon}{I^{2,0,0,0}_{0,0,0,2}} \frac{B_1^{(2)} B_3^{(2)}}{2\left(A^{(2)}\right)^2 \left(B_2^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v^* R_2 + v R_2^*\right) dxdydz \\ & - \frac{\epsilon}{I^{2,0,0,0}_{0,2,0,2}} \frac{B_1^{(2)} B_3^{(2)}}{2\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_2^{(2)}\right)^2 \left(v^* R_2 + v R_2^*\right) dxdydz\end{aligned} \tag{118}$$

$$\begin{aligned}\frac{dB_3^{(2)}}{dt} = {} & \frac{\epsilon}{I^{2,0,0,0}_{0,0,0,2}} \frac{B_1^{(2)} B_2^{(2)}}{2\left(A^{(2)}\right)^2 \left(B_3^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v^* R_2 + v R_2^*\right) dxdydz \\ & - \frac{\epsilon}{I^{2,0,0,0}_{0,0,2,2}} \frac{B_1^{(2)} B_2^{(2)}}{2\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_3^{(2)}\right)^2 \left(u^* R_2 + u R_2^*\right) dxdydz\end{aligned} \tag{119}$$

$$\begin{aligned}\frac{d\kappa_1^{(2)}}{dt} = {} & \frac{i\epsilon}{I^{2,0,0,0}_{0,0,0,2}} \frac{B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v_x^* R_2 - v_x R_2^*\right) dxdydz \\ & - \frac{\epsilon}{I^{2,0,0,0}_{0,0,0,2}} \frac{\kappa_1^{(2)} B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v^* R_2 + v R_2^*\right) dxdydz\end{aligned} \tag{120}$$

$$\begin{aligned}\frac{d\kappa_2^{(2)}}{dt} = {} & \frac{i\epsilon}{I^{2,0,0,0}_{0,0,0,2}} \frac{B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v_y^* R_2 - v_y R_2^*\right) dxdydz \\ & - \frac{\epsilon}{I^{2,0,0,0}_{0,0,0,2}} \frac{\kappa_1^{(2)} B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v^* R_2 + v R_2^*\right) dxdydz\end{aligned} \tag{121}$$

$$\frac{d\kappa_3^{(2)}}{dt} = \frac{i\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(v_z^* R_2 - v_z R_2^*\right) dxdydz$$
$$- \frac{\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{\kappa_1^{(2)} B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(v^* R_2 + v R_2^*\right) dxdydz \tag{122}$$

$$\frac{d\overline{x}^{(2)}}{dt} = -\kappa_1^{(2)} + \frac{\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \left(v^* R_2 + v R_2^*\right) dxdydz \tag{123}$$

$$\frac{d\overline{y}^{(2)}}{dt} = -\kappa_2^{(2)} + \frac{\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y \left(v^* R_2 + v R_2^*\right) dxdydz \tag{124}$$

$$\frac{d\overline{z}^{(2)}}{dt} = -\kappa_3^{(2)} + \frac{\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{B_1^{(2)} B_2^{(2)} B_3^{(2)}}{\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z \left(v^* R_2 + v R_2^*\right) dxdydz \tag{125}$$

$$\frac{d\theta^{(2)}}{dt} = -\frac{\left(\kappa_1^{(2)}\right)^2 + \left(\kappa_2^{(2)}\right)^2 + \left(\kappa_3^{(2)}\right)^2}{2} - \alpha F\left(\left(A^{(1)}\right)^2 g^2\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)\right)$$
$$+ \frac{1}{2I_{0,0,0,2}^{2,0,0,0}} \left\{\left(B_1^{(2)}\right)^2 I_{0,0,0,0}^{2,2,0,0} + \left(B_2^{(2)}\right)^2 I_{0,0,0,0}^{2,0,2,0} + \left(B_3^{(2)}\right)^2 I_{0,0,0,0}^{2,0,0,2}\right\}$$
$$- \frac{1}{I_{0,0,0,2}^{2,0,0,0}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^2\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$
$$\cdot F\left(\left(A^{(2)}\right)^2 g^2\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)\right) d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}$$
$$+ \frac{i\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{B_1^{(2)} B_2^{(2)} B_3^{(2)}}{2\left(A^{(2)}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(v^* R_2 - v R_2^*\right) d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \tag{126}$$

Here, equations (104)–(114) are obtained by differentiating (58) and (67)–(76) respectively while using (102). However, (115) was obtained by differentiating (61) with respect to t and subtracting from its conjugate and using (102). Similarly, the set (116)–(126) was obtained. We can, subsequently, rewrite these set of equations in the following convenient form:

$$\begin{aligned}\frac{dE}{dt} &= 2\epsilon\frac{A^{(1)}}{B_1^{(1)}B_2^{(1)}B_3^{(1)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right) \\ &\quad \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)}d\tau_2^{(1)}d\tau_3^{(1)} \\ &\quad +2\epsilon\frac{A^{(2)}}{B_1^{(2)}B_2^{(2)}B_3^{(2)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right) \\ &\quad \Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)}d\tau_2^{(2)}d\tau_3^{(2)} \end{aligned} \tag{127}$$

$$\begin{aligned}\frac{dA^{(1)}}{dt} &= \frac{\epsilon}{I_{0,0,0,4}^{1,0,0,0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^3\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right) \\ &\quad \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)}d\tau_2^{(1)}d\tau_3^{(1)} \end{aligned} \tag{128}$$

$$\begin{aligned}\frac{dB_1^{(1)}}{dt} &= \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}}\frac{1}{A^{(1)}\left(B_1^{(1)}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right) \\ &\quad \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)}d\tau_2^{(1)}d\tau_3^{(1)} \\ &\quad -\frac{\epsilon}{I_{2,0,0,2}^{1,0,0,0}}\frac{1}{A^{(1)}B_1^{(1)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_1^{(1)}\right)^2 g\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right) \\ &\quad \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)}d\tau_2^{(1)}d\tau_3^{(1)} \end{aligned} \tag{129}$$

$$\begin{aligned}\frac{dB_2^{(1)}}{dt} &= \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}}\frac{1}{A^{(1)}\left(B_2^{(1)}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right) \\ &\quad \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)}d\tau_2^{(1)}d\tau_3^{(1)} \\ &\quad -\frac{\epsilon}{I_{0,2,0,2}^{1,0,0,0}}\frac{1}{A^{(1)}B_2^{(1)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_2^{(1)}\right)^2 g\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right) \\ &\quad \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)}d\tau_2^{(1)}d\tau_3^{(1)} \end{aligned} \tag{130}$$

$$\begin{aligned}\frac{dB_3^{(1)}}{dt} &= \frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}}\frac{1}{A^{(1)}\left(B_3^{(1)}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(1)},\tau_2^{(1)},\tau_3^{(1)}\right) \\ &\quad \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)}d\tau_2^{(1)}d\tau_3^{(1)} \end{aligned}$$

$$
\begin{aligned}
& -\frac{\epsilon}{I_{0,0,2,2}^{1,0,0,0}} \frac{1}{A^{(1)} B_3^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\tau_3^{(1)}\right)^2 g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right) \\
& \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \qquad (131)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\kappa_1^{(1)}}{dt} = & -\frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{\kappa_1^{(1)}}{A^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right) \\
& \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \\
& -\frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{1}{A^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{\kappa_1^{(1)} g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)\right. \\
& \left.\Re\left[R_1 e^{-i\phi_1}\right] + B_1^{(1)} \frac{\partial g}{\partial \tau_1^{(1)}} \Im\left[R_1 e^{-i\phi_1}\right]\right\} d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \qquad (132)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\kappa_2^{(1)}}{dt} = & -\frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{\kappa_2^{(1)}}{A^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right) \\
& \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \\
& -\frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{1}{A^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{\kappa_2^{(1)} g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)\right. \\
& \left.\Re\left[R_1 e^{-i\phi_1}\right] + B_2^{(1)} \frac{\partial g}{\partial \tau_2^{(1)}} \Im\left[R_1 e^{-i\phi_1}\right]\right\} d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \qquad (133)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\kappa_3^{(1)}}{dt} = & -\frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{\kappa_3^{(1)}}{A^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right) \\
& \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \\
& -\frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{1}{A^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{\kappa_3^{(1)} g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)\right. \\
& \left.\Re\left[R_1 e^{-i\phi_1}\right] + B_3^{(1)} \frac{\partial g}{\partial \tau_3^{(1)}} \Im\left[R_1 e^{-i\phi_1}\right]\right\} d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \qquad (134)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\overline{x}^{(1)}}{dt} = & -\kappa_1^{(1)} + \frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{1}{A^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right) \\
& \Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \qquad (135)
\end{aligned}
$$

$$
\frac{d\overline{y}^{(1)}}{dt} = -\kappa_2^{(1)} + \frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{1}{A^{(1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)
$$

$$\Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \tag{136}$$

$$\frac{d\overline{z}^{(1)}}{dt} = -\kappa_3^{(1)} + \frac{2\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{1}{A^{(1)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} zg\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)$$
$$\Re\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \tag{137}$$

$$\frac{d\theta^{(1)}}{dt} = -\frac{\left(\kappa_1^{(1)}\right)^2 + \left(\kappa_2^{(1)}\right)^2 + \left(\kappa_3^{(1)}\right)^2}{2}$$
$$-\alpha F\left(\left(A^{(2)}\right)^2 g^2\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)\right)$$
$$+\frac{1}{2I_{0,0,0,2}^{1,0,0,0}}\left\{\left(B_1^{(1)}\right)^2 I_{0,0,0,0}^{1,2,0,0} + \left(B_2^{(1)}\right)^2 I_{0,0,0,0}^{1,0,2,0} + \left(B_3^{(1)}\right)^2 I_{0,0,0,0}^{1,0,0,2}\right\}$$
$$-\frac{1}{I_{0,0,0,2}^{1,0,0,0}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)$$
$$\cdot F\left(\left(A^{(1)}\right)^2 g^2\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)\right) d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)}$$
$$+\frac{\epsilon}{I_{0,0,0,2}^{1,0,0,0}} \frac{1}{A^{(1)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)$$
$$\Im\left[R_1 e^{-i\phi_1}\right] d\tau_1^{(1)} d\tau_2^{(1)} d\tau_3^{(1)} \tag{138}$$

Similarly, for the soliton in (103) we have the adiabatic parameter dynamics, in the convenient form given by:

$$\frac{dA^{(2)}}{dt} = \frac{\epsilon}{I_{0,0,0,4}^{2,0,0,0}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^3\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_2^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \tag{139}$$

$$\frac{dB_1^{(2)}}{dt} = \frac{\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}\left(B_1^{(2)}\right)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}$$
$$-\frac{\epsilon}{I_{2,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)} B_1^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_1^{(2)}\right)^2 g\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \tag{140}$$

$$\frac{dB_2^{(2)}}{dt} = \frac{\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}\left(B_2^{(2)}\right)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}$$
$$-\frac{\epsilon}{I_{0,2,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}B_2^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_2^{(2)}\right)^2 g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \tag{141}$$

$$\frac{dB_3^{(2)}}{dt} = \frac{\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}\left(B_3^{(2)}\right)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}$$
$$-\frac{\epsilon}{I_{0,0,2,2}^{2,0,0,0}} \frac{1}{A^{(2)}B_3^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_3^{(2)}\right)^2 g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \tag{142}$$

$$\frac{d\kappa_1^{(2)}}{dt} = -\frac{2\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{\kappa_1^{(2)}}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}$$
$$-\frac{2\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\kappa_1^{(2)} g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)\right.$$
$$\left.\Re\left[R_2 e^{-i\phi_2}\right] + B_1^{(2)} \frac{\partial g}{\partial \tau_1^{(2)}} \Im\left[R_2 e^{-i\phi_2}\right]\right\} d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \tag{143}$$

$$\frac{d\kappa_2^{(2)}}{dt} = -\frac{2\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{\kappa_2^{(2)}}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)$$
$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}$$
$$-\frac{2\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\kappa_2^{(2)} g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)\right.$$
$$\left.\Re\left[R_2 e^{-i\phi_2}\right] + B_2^{(2)} \frac{\partial g}{\partial \tau_2^{(2)}} \Im\left[R_2 e^{-i\phi_2}\right]\right\} d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \tag{144}$$

$$\frac{d\kappa_3^{(2)}}{dt} = \frac{i\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{\kappa_3^{(2)}}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(2)},\tau_2^{(2)},\tau_3^{(2)}\right)$$

$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}$$

$$-\frac{2\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \kappa_3^{(2)} g\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)\right.$$

$$\left.\Re\left[R_2 e^{-i\phi_2}\right] + B_3^{(2)} \frac{\partial g}{\partial \tau_2^{(2)}} \Im\left[R_2 e^{-i\phi_2}\right]\right\} d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \quad (145)$$

$$\frac{d\overline{x}^{(2)}}{dt} = -\kappa_1^{(2)} + \frac{2\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x g\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$

$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \quad (146)$$

$$\frac{d\overline{y}^{(2)}}{dt} = -\kappa_2^{(2)} + \frac{2\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y g\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$

$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \quad (147)$$

$$\frac{d\overline{z}^{(2)}}{dt} = -\kappa_3^{(2)} + \frac{2\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z g\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$

$$\Re\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \quad (148)$$

$$\frac{d\theta^{(2)}}{dt} = -\frac{\left(\kappa_1^{(2)}\right)^2 + \left(\kappa_2^{(2)}\right)^2 + \left(\kappa_3^{(2)}\right)^2}{2}$$

$$-\alpha F\left(\left(A^{(1)}\right)^2 g^2\left(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}\right)\right)$$

$$+\frac{1}{2I_{0,0,0,2}^{2,0,0,0}} \left\{\left(B_1^{(2)}\right)^2 I_{0,0,0,0}^{2,2,0,0} + \left(B_2^{(2)}\right)^2 I_{0,0,0,0}^{2,0,2,0} + \left(B_3^{(2)}\right)^2 I_{0,0,0,0}^{2,0,0,2}\right\}$$

$$-\frac{1}{I_{0,0,0,2}^{2,0,0,0}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$

$$\cdot F\left(\left(A^{(2)}\right)^2 g^2\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)\right) d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)}$$

$$+\frac{\epsilon}{I_{0,0,0,2}^{2,0,0,0}} \frac{1}{A^{(2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(2)}, \tau_2^{(2)}, \tau_3^{(2)}\right)$$

$$\Im\left[R_2 e^{-i\phi_2}\right] d\tau_1^{(2)} d\tau_2^{(2)} d\tau_3^{(2)} \quad (149)$$

Here we have used the notation

$$\phi_l = \kappa_1^{(l)}(t)\left\{x - \overline{x}^{(l)}(t)\right\} + \kappa_2^{(l)}(t)\left\{y - \overline{y}^{(l)}(t)\right\}$$

$$+\kappa_3^{(l)}(t)\left\{z - \overline{z}^{(l)}(t)\right\} - \theta^{(l)}(t)$$

Thus, we have the dynamical system or the parameter dynamics of the vector multidimensional solitons. These can be used to study various phenomenon in optical bullets like the soliton fusion, soliton fission, soliton tunnelling and soliton repulsion, and other aspects in presence of the perturbation terms.

4. GENERALIZED VECTOR SOLITONS

The multidimensional generalized vector solitons that are governed by the generalized NLSE that are going to be studied in this section are of the following form:

$$iq_t^{(l)} + \frac{1}{2}\nabla^2 q^{(l)} + \left\{ F\left(\left|q^{(l)}\right|^2\right) + \sum_{m\neq l}^{N} \alpha_m F\left(\left|q^{(m)}\right|^2\right)\right\} q^{(l)} = 0 \quad (150)$$

Here, in (150) we have α_m are constants. These equations also arise in various other areas of Physics and Mathematical Physics. In particular, it can be used in the study of the dynamics of vector optical bullets in Nonlinear Optics.

Equation (150) does not have infinitely many conserved quantities. In fact, it has at least three integrals of motion that are given by

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{l=1}^{N} \left|q^{(l)}\right|^2 dxdydz \quad (151)$$

$$\boldsymbol{P} = \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{l=1}^{N} \left(q^{(l)}\nabla q^{(l)*} - q^{(l)*}\nabla q^{(l)}\right) dxdydz \quad (152)$$

$$\boldsymbol{M} = \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{l=1}^{N} \boldsymbol{r} \times \left(q^{(l)}\nabla q^{(l)*} - q^{(l)*}\nabla q^{(l)}\right) dxdydz \quad (153)$$

In this section we shall derive the dynamics of soliton parameters for (150) from their corresponding definitions. For this, we shall assume that the solitons of (150) although not integrable, are given in the form [9, 10]:

$$\begin{aligned} q^{(l)}(x,y,z;t) \;=\;& A^{(l)}(t) g\left[B_1^{(l)}(t)\left\{x - \overline{x}^{(l)}(t)\right\},\right. \\ & \left. B_2^{(l)}(t)\left\{y - \overline{y}^{(l)}(t)\right\}, B_3^{(l)}(t)\left\{z - \overline{z}^{(l)}(t)\right\}\right] \\ & \exp\left[-i\kappa_1^{(l)}(t)\left\{x - \overline{x}^{(l)}(t)\right\} - i\kappa_2^{(l)}(t)\left\{y - \overline{y}^{(l)}(t)\right\}\right. \\ & \left. - i\kappa_3^{(l)}(t)\left\{z - \overline{z}^{(l)}(t)\right\} + i\theta^{(l)}(t)\right] \end{aligned} \quad (154)$$

where g represents the shape of the soliton described by the GNLSE and it depends on the type of nonlinearity in (150). Also here, in (154), $A^{(l)}(t)$ with for $1 \leq l \leq N$ represents the amplitude of the solitons, while $B_j^{(l)}(t)$ for $j = 1, 2, 3$ represents the width in the x, y and z directions respectively. We then have $\kappa_j^{(l)}(t)$ as the frequency of the soliton in the x, y and z directions respectively. Finally, $(\overline{x}^{(l)}(t), \overline{y}^{(l)}(t), \overline{z}^{(l)}(t))$ represent the coordinates of the center of masses of the solitons. In this section we shall use, for convenience, the integral given by (63) with the exception that here we have $1 \leq l \leq N$. Thus, the integrals of motion are given by

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{l=1}^{N}\left|q^{(l)}\right|^2 dxdydz = \sum_{l=1}^{N}\frac{\left(A^{(l)}\right)^2}{B_1^{(l)}B_2^{(l)}B_3^{(l)}} I_{0,0,0,2}^{l,0,0,0} \quad (155)$$

$$\begin{aligned}
\boldsymbol{P} &= \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{l=1}^{N}\left(q^{(l)}\nabla q^{(l)*} - q^{(l)*}\nabla q^{(l)}\right) dxdydz \\
&= -\sum_{l=1}^{N} I_{0,0,0,2}^{l,0,0,0}\frac{\left(A^{(l)}\right)^2}{B_1^{(l)}B_2^{(l)}B_3^{(l)}}\left(\kappa_1^{(l)}, \kappa_2^{(l)}, \kappa_3^{(l)}\right) \qquad (156)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{M} &= \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{r}\times\sum_{l=1}^{N}\left(q^{(l)}\nabla q^{(l)*} - q^{(l)*}\nabla q^{(l)}\right) dxdydz \\
&= \sum_{l=1}^{N}\frac{\left(A^{(l)}\right)^2}{B_1^{(l)}B_2^{(l)}B_3^{(l)}}\left(\frac{\kappa_2^{(l)}}{B_3^{(l)}} I_{0,0,1,2}^{l,0,0,0} - \frac{\kappa_3^{(l)}}{B_2^{(l)}} I_{0,1,0,2}^{l,0,0,0},\right. \\
&\left.\frac{\kappa_3^{(l)}}{B_1^{(l)}} I_{1,0,0,2}^{l,0,0,0} - \frac{\kappa_1^{(l)}}{B_3^{(l)}} I_{0,0,1,2}^{l,0,0,0}, \frac{\kappa_1^{(l)}}{B_2^{(l)}} I_{0,1,0,2}^{l,0,0,0} - \frac{\kappa_2^{(l)}}{B_1^{(l)}} I_{1,0,0,2}^{l,0,0,0}\right) \qquad (157)
\end{aligned}$$

The soliton parameters for (150) are defined as follows:

$$A^{(l)}(t) = \left[\frac{I_{0,0,0,2}^{l,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^4 dxdydz}{I_{0,0,0,4}^{l,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz}\right]^{\frac{1}{2}} \qquad (158)$$

$$B_1^{(l)}(t) = \left[\frac{I_{2,0,0,2}^{l,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz}{I_{0,0,0,2}^{l,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_1^{(l)}\right)^2|q^{(l)}|^2 dxdydz}\right]^{\frac{1}{2}} \qquad (159)$$

$$B_2^{(l)}(t) = \left[\frac{I_{0,2,0,2}^{l,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz}{I_{0,0,0,2}^{l,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_2^{(l)}\right)^2|q^{(l)}|^2 dxdydz}\right]^{\frac{1}{2}} \quad (160)$$

$$B_3^{(l)}(t) = \left[\frac{I_{0,0,2,2}^{l,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz}{I_{0,0,0,2}^{l,0,0,0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_3^{(l)}\right)^2|q^{(l)}|^2 dxdydz}\right]^{\frac{1}{2}} \quad (161)$$

$$\kappa_1^{(l)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)}q_x^{(l)*}-q^{(l)*}q_x^{(l)}\right)dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz} \quad (162)$$

$$\kappa_2^{(l)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)}q_y^{(l)*}-q^{(l)*}q_y^{(l)}\right)dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz} \quad (163)$$

$$\kappa_3^{(l)}(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)}q_z^{(l)*}-q^{(l)*}q_z^{(l)}\right)dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz} \quad (164)$$

$$\overline{x}^{(l)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x|q^{(l)}|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz} \quad (165)$$

$$\overline{y}^{(l)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y|q^{(l)}|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz} \quad (166)$$

$$\overline{z}^{(l)}(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}z|q^{(l)}|^2 dxdydz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2 dxdydz} \quad (167)$$

We note that, these definitions of the soliton parameters are defined for $1 \leq l \leq N$. Now, differentiating these parameters with respect to t and using (154) we arrive at the following evolution equations for the soliton parameters:

$$\frac{dE}{dt} = 0 \quad (168)$$

$$\begin{aligned}
\frac{dA^{(l)}}{dt} &= 0 && (169)\\
\frac{dB_1^{(l)}}{dt} &= \frac{dB_2^{(l)}}{dt} = \frac{dB_3^{(l)}}{dt} = 0 && (170)\\
\frac{d\kappa_1^{(l)}}{dt} &= \frac{d\kappa_2^{(l)}}{dt} = \frac{d\kappa_3^{(l)}}{dt} = 0 && (171)\\
\frac{d\overline{x}^{(l)}}{dt} &= -\kappa_1^{(l)} && (172)\\
\frac{d\overline{y}^{(l)}}{dt} &= -\kappa_2^{(l)} && (173)\\
\frac{d\overline{z}^{(l)}}{dt} &= -\kappa_3^{(l)} && (174)
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta^{(l)}}{dt} = &-\frac{\left(\kappa_1^{(l)}\right)^2+\left(\kappa_2^{(l)}\right)^2+\left(\kappa_3^{(l)}\right)^2}{2}\\
&-\sum_{m\neq l}^{N} \alpha_m F\left(\left(A^{(m)}\right)^2 g^2\left(\tau_1^{(m)},\tau_2^{(m)},\tau_3^{(m)}\right)\right)\\
&+\frac{1}{2I_{0,0,0,2}^{l,0,0,0}}\left\{\left(B_1^{(l)}\right)^2 I_{0,0,0,0}^{l,2,0,0}+\left(B_2^{(l)}\right)^2 I_{0,0,0,0}^{l,0,2,0}+\left(B_3^{(l)}\right)^2 I_{0,0,0,0}^{l,0,0,2}\right\}\\
&-\frac{1}{I_{0,0,0,2}^{l,0,0,0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right)\\
&\cdot F\left(\left(A^{(l)}\right)^2 g^2\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right)\right) d\tau_1^{(l)}d\tau_2^{(l)}d\tau_3^{(l)} \qquad (175)
\end{aligned}$$

Thus, from (168)–(171), we notice that the energy, amplitude, width and the frequency of the solitons remains constant. However, the center of mass and the phase of the soliton undergo a change as governed by (172)–(175). Here, (175) is obtained by differentiating (154) with respect to t and subtracting from its conjugate while utilizing (150).

4.1. Perturbation Terms

In this section, we shall now consider the GNLSE along with its perturbation terms that are given by

$$iq_t^{(l)}+\frac{1}{2}\nabla^2 q^{(l)}+\left\{F\left(\left|q^{(l)}\right|^2\right)+\sum_{m\neq l}^{N}\alpha_m F\left(\left|q^{(m)}\right|^2\right)\right\}q^{(l)}=i\epsilon R_l\left[q^{(l)},q^{(l)*}\right] \tag{176}$$

Here, again, R_l $(1 \leq l \leq N)$ are spatio-differential operators and ϵ is a perturbation parameter with $0 < \epsilon \ll 1$. In equation (176), once again, the second nonlinear term on the left side is known as the cross-phase modulation (XPM) term in the context of optics. So, treating these XPM terms as the perturbation terms in addition to the regular perturbation terms that is already on the right side of (176), we arrive at the adiabatic parameter dynamics of the soliton as:

$$\frac{dE}{dt} = \epsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{l=1}^{N}\left(q^{(l)*}R_l+q^{(l)}R_l^*\right)dxdydz \tag{177}$$

$$\frac{dA^{(l)}}{dt} = \frac{\epsilon}{I_{0,0,0,4}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}B_3^{(l)}}{2\left(A^{(l)}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q^{(l)}|^2\left(q^{(l)*}R_l+q^{(l)}R_l^*\right)dxdydz \tag{178}$$

$$\begin{aligned}\frac{dB_1^{(l)}}{dt} =& \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_2^{(l)}B_3^{(l)}}{2\left(A^{(l)}\right)^2\left(B_1^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)*}R_l+q^{(l)}R_l^*\right)dxdydz\\ &-\frac{\epsilon}{I_{2,0,0,2}^{l,0,0,0}}\frac{B_2^{(l)}B_3^{(l)}}{2\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_1^{(l)}\right)^2\left(q^{(l)*}R_l+q^{(l)}R_l^*\right)dxdydz\end{aligned} \tag{179}$$

$$\begin{aligned}\frac{dB_2^{(l)}}{dt} =& \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_3^{(l)}}{2\left(A^{(l)}\right)^2\left(B_2^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)*}R_l+q^{(l)}R_l^*\right)dxdydz\\ &-\frac{\epsilon}{I_{0,2,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_3^{(l)}}{2\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_2^{(l)}\right)^2\left(q^{(l)*}R_l+q^{(l)}R_l^*\right)dxdydz\end{aligned} \tag{180}$$

$$\begin{aligned}\frac{dB_3^{(l)}}{dt} =& \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}}{2\left(A^{(l)}\right)^2\left(B_3^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)*}R_l+q^{(l)}R_l^*\right)dxdydz\\ &-\frac{\epsilon}{I_{0,0,2,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}}{2\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_3^{(l)}\right)^2\left(q^{(l)*}R_l+q^{(l)}R_l^*\right)dxdydz\end{aligned} \tag{181}$$

$$\frac{d\kappa_1^{(l)}}{dt} = \frac{i\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q_x^{(l)*}R_l - q_x^{(l)}R_l^*\right)dxdydz$$
$$-\frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{\kappa_1^{(l)}B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)*}R_l + q^{(l)}R_l^*\right)dxdydz \tag{182}$$

$$\frac{d\kappa_2^{(l)}}{dt} = \frac{i\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q_y^{(l)*}R_l - q_y^{(l)}R_l^*\right)dxdydz$$
$$-\frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{\kappa_1^{(l)}B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)*}R_l + q^{(l)}R_l^*\right)dxdydz \tag{183}$$

$$\frac{d\kappa_3^{(l)}}{dt} = \frac{i\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q_z^{(l)*}R_l - q_z^{(l)}R_l^*\right)dxdydz$$
$$-\frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{\kappa_1^{(l)}B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)*}R_l + q^{(l)}R_l^*\right)dxdydz \tag{184}$$

$$\frac{d\overline{x}^{(l)}}{dt} = -\kappa_1^{(l)} + \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\left(q^{(l)*}R_l + q^{(l)}R_l^*\right)dxdydz \tag{185}$$

$$\frac{d\overline{y}^{(l)}}{dt} = -\kappa_2^{(l)} + \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\left(q^{(l)*}R_l + q^{(l)}R_l^*\right)dxdydz \tag{186}$$

$$\frac{d\overline{z}^{(l)}}{dt} = -\kappa_3^{(l)} + \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}B_3^{(l)}}{\left(A^{(l)}\right)^2}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\left(q^{(l)*}R_l + q^{(l)}R_l^*\right)dxdydz \tag{187}$$

$$\frac{d\theta^{(l)}}{dt} = -\frac{\left(\kappa_1^{(l)}\right)^2+\left(\kappa_2^{(l)}\right)^2+\left(\kappa_3^{(l)}\right)^2}{2}$$

$$
\begin{aligned}
&-\sum_{m\neq l}^{N}\alpha_m F\left(\left(A^{(m)}\right)^2 g^2\left(\tau_1^{(m)},\tau_2^{(m)},\tau_3^{(m)}\right)\right)\\
&+\frac{1}{2I_{0,0,0,2}^{l,0,0,0}}\left\{\left(B_1^{(l)}\right)^2 I_{0,0,0,0}^{l,2,0,0}+\left(B_2^{(l)}\right)^2 I_{0,0,0,0}^{l,0,2,0}+\left(B_3^{(l)}\right)^2 I_{0,0,0,0}^{l,0,0,2}\right\}\\
&-\frac{1}{I_{0,0,0,2}^{l,0,0,0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right)\\
&\cdot F\left(\left(A^{(l)}\right)^2 g^2\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right)\right) d\tau_1^{(l)}d\tau_2^{(l)}d\tau_3^{(l)}\\
&+\frac{i\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{B_1^{(l)}B_2^{(l)}B_3^{(l)}}{2\left(A^{(l)}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q^{(l)*}R_l-q^{(l)}R_l^*\right)d\tau_1^{(l)}d\tau_2^{(l)}d\tau_3^{(l)}
\end{aligned}
\tag{188}
$$

Here, equations (177)–(187) are obtained by differentiating (151) and (158)–(167) respectively while using (176). However, (188) was obtained by differentiating (164) with respect to t and subtracting from its conjugate and using (176). Introducing the notation,

$$
\begin{aligned}
\phi_l &= \kappa_1^{(l)}(t)\left\{x-\overline{x}^{(l)}(t)\right\}+\kappa_2^{(l)}(t)\left\{y-\overline{y}^{(l)}(t)\right\}\\
&\quad+\kappa_3^{(l)}(t)\left\{z-\overline{z}^{(l)}(t)\right\}-\theta^{(l)}(t)
\end{aligned}
$$

we can, subsequently, reduce these set of equations to the following form:

$$
\begin{aligned}
\frac{dE}{dt} &= 2\epsilon\sum_{l=1}^{N}\frac{A^{(l)}}{B_1^{(l)}B_2^{(l)}B_3^{(l)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right)\\
&\quad\Re\left[R_l e^{-i\phi_l}\right]d\tau_1^{(l)}d\tau_2^{(l)}d\tau_3^{(l)}
\end{aligned}
\tag{189}
$$

$$
\begin{aligned}
\frac{dA^{(l)}}{dt} &= \frac{\epsilon}{I_{0,0,0,4}^{l,0,0,0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^3\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right)\\
&\quad\Re\left[R_l e^{-i\phi_l}\right]d\tau_1^{(l)}d\tau_2^{(l)}d\tau_3^{(l)}
\end{aligned}
\tag{190}
$$

$$
\begin{aligned}
\frac{dB_1^{(l)}}{dt} &= \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}}\frac{1}{A^{(l)}\left(B_1^{(l)}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right)\\
&\quad\Re\left[R_l e^{-i\phi_l}\right]d\tau_1^{(l)}d\tau_2^{(l)}d\tau_3^{(l)}\\
&\quad-\frac{\epsilon}{I_{2,0,0,2}^{l,0,0,0}}\frac{1}{A^{(l)}B_1^{(l)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tau_1^{(l)}\right)^2 g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right)
\end{aligned}
$$

$$\Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \tag{191}$$

$$\begin{aligned}
\frac{dB_2^{(l)}}{dt} &= \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}\left(B_2^{(l)}\right)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right) \\
&\quad \Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \\
&\quad - \frac{\epsilon}{I_{0,2,0,2}^{l,0,0,0}} \frac{1}{A^{(l)} B_2^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_2^{(l)}\right)^2 g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right) \\
&\quad \Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)}
\end{aligned} \tag{192}$$

$$\begin{aligned}
\frac{dB_3^{(l)}}{dt} &= \frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}\left(B_3^{(l)}\right)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right) \\
&\quad \Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \\
&\quad - \frac{\epsilon}{I_{0,0,2,2}^{l,0,0,0}} \frac{1}{A^{(l)} B_3^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tau_3^{(l)}\right)^2 g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right) \\
&\quad \Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)}
\end{aligned} \tag{193}$$

$$\begin{aligned}
\frac{d\kappa_1^{(l)}}{dt} &= -\frac{2\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{\kappa_1^{(l)}}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right) \\
&\quad \Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \\
&\quad - \frac{2\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \kappa_1^{(l)} g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right) \right. \\
&\quad \left. \Re\left[R_l e^{-i\phi_l}\right] + B_1^{(l)} \frac{\partial g}{\partial \tau_1^{(l)}} \Im\left[R_l e^{-i\phi_l}\right] \right\} d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)}
\end{aligned} \tag{194}$$

$$\begin{aligned}
\frac{d\kappa_2^{(l)}}{dt} &= -\frac{2\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{\kappa_2^{(l)}}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right) \\
&\quad \Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \\
&\quad - \frac{2\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \kappa_2^{(l)} g\left(\tau_1^{(l)},\tau_2^{(l)},\tau_3^{(l)}\right) \right. \\
&\quad \left. \Re\left[R_l e^{-i\phi_l}\right] + B_2^{(l)} \frac{\partial g}{\partial \tau_2^{(l)}} \Im\left[R_l e^{-i\phi_l}\right] \right\} d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)}
\end{aligned} \tag{195}$$

$$
\begin{aligned}
\frac{d\kappa_3^{(l)}}{dt} &= \frac{i\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{\kappa_3^{(l)}}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right) \\
&\Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \\
&-\frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\kappa_3^{(l)} g\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right)\right. \\
&\left.\Re\left[R_l e^{-i\phi_l}\right] + B_3^{(l)} \frac{\partial g}{\partial \tau_3^{(l)}} \Im\left[R_l e^{-i\phi_l}\right]\right\} d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \qquad (196)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\overline{x}^{(l)}}{dt} &= -\kappa_1^{(l)} + \frac{2\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xg\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right) \\
&\Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \qquad (197)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\overline{y}^{(l)}}{dt} &= -\kappa_2^{(l)} + \frac{2\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yg\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right) \\
&\Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \qquad (198)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\overline{z}^{(l)}}{dt} &= -\kappa_3^{(l)} + \frac{2\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} zg\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right) \\
&\Re\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \qquad (199)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\theta^{(l)}}{dt} &= -\frac{\left(\kappa_1^{(l)}\right)^2 + \left(\kappa_2^{(l)}\right)^2 + \left(\kappa_3^{(l)}\right)^2}{2} \\
&-\sum_{m\neq l}^{N} \alpha_m F\left(\left(A^{(m)}\right)^2 g^2\left(\tau_1^{(m)}, \tau_2^{(m)}, \tau_3^{(m)}\right)\right) \\
&+\frac{1}{2I_{0,0,0,2}^{l,0,0,0}} \left\{\left(B_1^{(l)}\right)^2 I_{0,0,0,0}^{l,2,0,0} + \left(B_2^{(l)}\right)^2 I_{0,0,0,0}^{l,0,2,0} + \left(B_3^{(l)}\right)^2 I_{0,0,0,0}^{l,0,0,2}\right\} \\
&-\frac{1}{I_{0,0,0,2}^{l,0,0,0}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right) \\
&\cdot F\left(\left(A^{(l)}\right)^2 g^2\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right)\right) d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \\
&+\frac{\epsilon}{I_{0,0,0,2}^{l,0,0,0}} \frac{1}{A^{(l)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\tau_1^{(l)}, \tau_2^{(l)}, \tau_3^{(l)}\right) \\
&\Im\left[R_l e^{-i\phi_l}\right] d\tau_1^{(l)} d\tau_2^{(l)} d\tau_3^{(l)} \qquad (200)
\end{aligned}
$$

5. CONCLUSION

In this paper we have obtained, for the first time, the parameter dynamics of the multidimensional solitons in 1 + 3 dimensions that are governed by the generalized NLSE. In addition, we have obtained the adiabatic parameter dynamics of the solitons in presence of the perturbation terms. These parameter dynamics of the multidimensional solitons can be also derived using the variational principle. The Dynamical System of the soliton parameters, thus obtained, is very useful in various areas of applied nonlinear science like Fluid Dynamics, Non-linear Optics, Plasma Physics. For example, in the case of Nonlinear Optics, one can use these parameter dynamics to study *dromions* [3] and *optical bullets* analytically.

We note that, besides the deterministic perturbation terms that are considered here, one encounters, in reality, the stochastic type perturbation. The adiabatic soliton parameter dynamics due to such type of perturbations will be reported in a future publication.

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