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A VARIABLE METRIC ELECTRODYNAMICS. PLANE WAVES

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Abstract—Classical electrodynamics can be divided into two parts. In the first one, a need of introducing a plenty of directed quantities occurs, namely multivectors and differential forms but no scalar product is necessary. We call it premetric electrodynamics. In this part, principal equations of the theory can be tackled. The second part concerns solutions of the equations and requires establishing of a scalar product and, consequently, a metric. For anisotropic media two scalar products can be introduced depending on the electric permittivity and magnetic permeability tensors. In the case of plane electromagnetic waves both of them are needed because two constitutive equations are needed: one for the electric fields, the other for the magnetic field. We show which part of the description of plane electromagnetic waves is independent of scalar products, and where they become necessary.

1. INTRODUCTION

In the last decades, a way of presenting electromagnetism has been proposed based on a broad use of differential forms; see Refs [1–12]. Most of authors are content with algebraic definitions of the exterior forms: nice exceptions are Refs. [2, 8, 9] where visualizations by geometric images are shown. Not all presentations, also, put enough care to the use of odd forms. In Refs. [1–4], **D** is claimed to be a two-form. Only few authors applied odd forms in electrodynamics, under various names: *covariant W-p-vectors* [5], *twisted forms* [6, 9] or *odd forms* [10, 7].

When asked what directed quantity in three-dimensional space are the electric field strength **E** and electric induction **D** we usually answer: they are vectors. Similarly, when asked about the directed nature of the magnetic field strength **H** and the magnetic induction **B** we answer: they are pseudovectors or axial vectors. We do so because we do not realize that to exterior forms (called also differential forms if they depend on position) also attributes of magnitude and direction can be assigned. There are arguments showing that **E**, **D**, **H** and **B** are differential forms, hence they also can be considered as directed quantities. This characterization is basis-independent, hence it allows to proceed within the theory without tedious burden of indices.

The even and odd forms are necessary to formulate electrodynamics in a scalar-product-independent way. We call it *premetric electrodynamics*. It turns out that only the principal equations of this theory can be tackled in this manner: Maxwell's equations, the potentials, the Lorentz force, the continuity equation for charge, the boundary conditions. When one seeks their solutions, that is, specific electromagnetic fields as functions of position, a scalar product is needed for writing, among others, the constitutive equations involving electric permeability and magnetic permittivity. A special scalar product can be introduced in the case of anisotropic dielectric, for which the vectors $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ become parallel. In this manner the medium can be treated analogously to the isotropic one. Then the counterpart of the Coulomb field and the fields for many electrostatic problems can be found in a very natural way [13]. Analogously in the case of anisotropic magnetic medium another scalar product can be introduced, for which pseudovectors $\vec{\mathbf{H}}$ and $\vec{\mathbf{B}}$ are parallel. Then the medium appears to be isotropic, so the counterpart of the Biot-Savart law can be found and solutions of many magnetostatic problems [13] are possible. When considering plane electromagnetic waves electric and magnetic fields are present simultaneously, hence both scalar products have to be taken into account.

The present work depends strongly on previous one [13] where the geometric images of the multivectors and exterior forms (called directed quantities) have been introduced. A terminological change is now introduced: the ordinary quantities are now called even (even bivector, even one-form etc.) and pseudo-quantities are called odd. Also the premetric electrodynamics in the three-dimensional space has been presented in [13]. The electromagnetic field quantities have turned out to be of different nature: **E** is even one-form, **H** is odd one-form, **D** is odd two-form, and **B** is even two-form. This paper is devoted to the discussion of plane electromagnetic waves as long as possible without scalar products, and to show the place where the scalar products enter into considerations.

Principal types of directed quantities are recalled in Section 2. Useful products, namely the exterior product, contraction and scalar product, are recollected in Section 3. The first two products serve to formulate electrodynamics in a scalar-product-independent way what is done in the beginning of Section 4. We show also in it that two special scalar products can be introduced, related to electric permeability and magnetic permittivity, which enter the constitutive relations along with the contraction with the basic three-form.They form two kinds of the Hodge star operator.

In Section 5, we describe plane waves first in a metric-independent way and then with the use of two above mentioned scalar products. We convince that a one-form quantity, slowness, is more useful than the phase velocity which should be a vector quantity. For anisotropic medium, we find that two kinds of plane waves are possible which are called eigenwaves. The energy fluxes for these waves are not parallel to each other and serve to determine the velocity of the energy transport by the wave.

The present paper describes an approach to constitutive relations that is becoming popular in finite-element/finite-difference formulations of numerical electromagnetics. In the 90's computational electromagnetics people were trying to reformulate it with the aid of differential forms, but some problems remained, among them the way of introducing constitutive relations into the formalism. In the statics, the use of two suitably modified Hodge operators, electric and magnetic, for anisotropic media worked well, see e.g., [13, 14]. This idea caught on, and some research groups adopted the approach. The idea of applying metrics in constitutive relations has been used to treat absorbing boundary conditions [15]. The most recent development in this approach to constitutive relations is that of Obukhov and Hehl [16].

2. DIRECTED QUANTITIES IN THREE-DIMENSIONAL SPACE

The list of directed quantities in three-dimensional space consists of even and odd multivectors, even and odd forms.† For their more systematic introduction see Ref. [13]. Each *directed quantity* has a separate *direction* which consists of *attitude* and *orientation*. For the well known *vector*, depicted as a directed segment, the *direction* consists of a straight line (on which the vector lies), after Lounesto [17] called an *attitude*, and an arrow on that line which is called the *orientation*. The parallel vectors are said to have the same attitude.

Geometric images of the directed quantities are shown in Figures 1–9. These pictures should be treated as a recapitulation of more thorough definitions presented in [13].

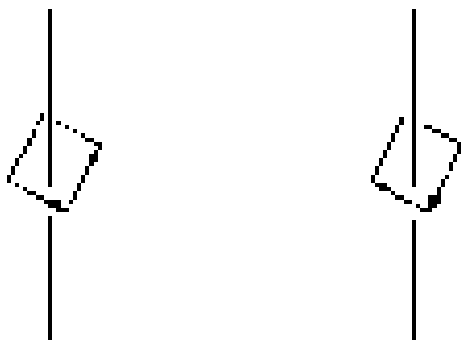


Figure 1. Two odd vectors of the same attitude and opposite orientations depicted by oriented parallelograms.

Table 1 collects relevant features of eight quantities which, in the presence of a metric, can be replaced by vectors (or pseudovectors). The upper part contains even and odd multivectors, the lower part contains their duals. Typical examples of geometric and electromagnetic quantities are added for all of them.

† Even vector is the ordinary or polar vector, odd vector is the pseudovector.

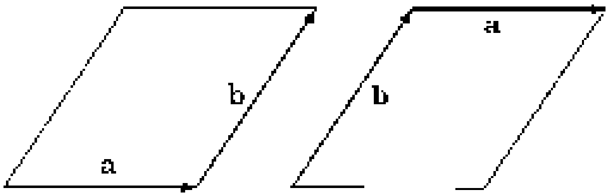


Figure 2. Even bivectors represented as exterior products of two vectors. Orientation is depicted as circulation on the boundary. Two bivectors $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$, opposite to each other.

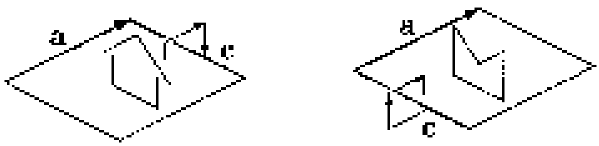


Figure 3. Odd bivectors represented as exterior products of even vector $\mathbf{a}$ and odd vector $\mathbf{c}$. Orientation is depicted as the arrow piercing the plane. Two odd bivectors $\mathbf{a} \wedge \mathbf{c}$ and $\mathbf{c} \wedge \mathbf{a}$, opposite to each other.

Table 1. Directed quantities replacing vectors.

features	even vector	odd vector	even bivector	odd bivector
attitude	straight line	straight line	plane	plane
orientation	arrow on	curved arrow around	curved arrow on	arrow piercing
magnitude	length	length	area	area
	l	l*	S	S*
	even one-form	odd one-form	even two-form	odd two-form
attitude	plane	plane	straight line	straight line
orientation	arrow piercing	curved arrow on	curved arrow around	arrow on
magnitude	inverse length	inverse length	inverse area	inverse area
	E	H	B	D

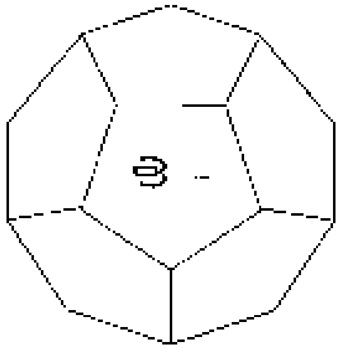


Figure 4. Solid body represents trivector. Orientation of the even trivector is marked as the helix, that of the odd trivector — as the sign.

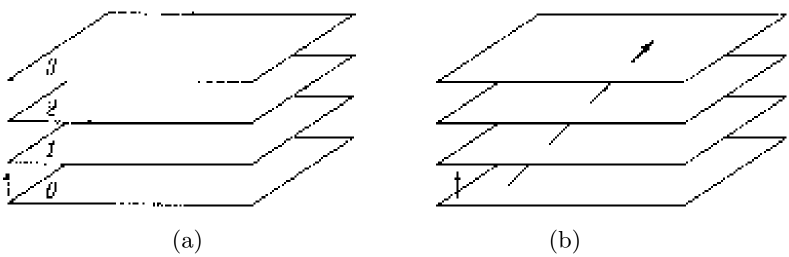


Figure 5. a) Family of parallel planes representing a one-form. Orientation depicted as the vertical arrow. b) By counting pierced layers between the planes we ascribe a number to a vector.

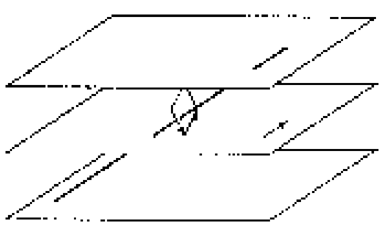
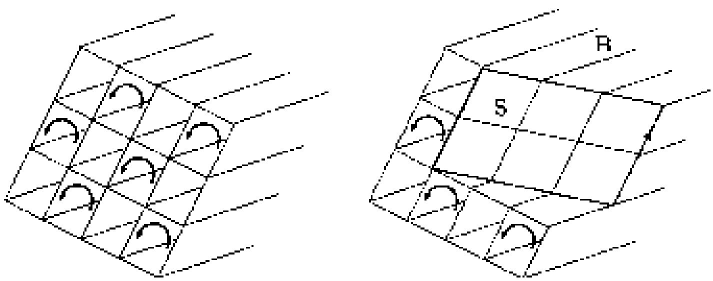


Figure 6. Family of parallel planes representing an odd one-form. Orientation depicted as a curved arrow on one of planes.



(a) (b)

Figure 7. a) Family of parallel pipes representing even two-form. Orientation depicted as a curved arrow around axis of the pipe. b) Value of two form **B** on bivector **S**.

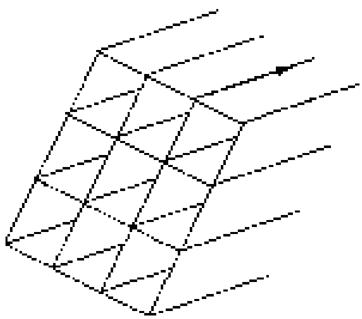


Figure 8. Family of parallel pipes representing odd two-form. Orientation depicted as rectilinear arrow along a pipe.

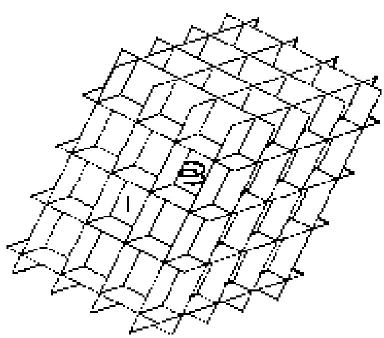


Figure 9. Family of cells representing a three-form. Orientation of the even three-form is marked by the helix and of the odd three-form — by the sign.

3. VARIOUS PRODUCTS

Let us denote by $\mathcal{M}$ the set of all even and odd multivectors including scalars. Each quantity has its grade, namely k-vectors have grade k. The exterior product can be defined for each pair of quantities from $\mathcal{M}$. Figures 2 and 3 show how bivectors are obtained as products of two vectors. The grades of factors add in such a product; but, when their sum is greater than three, the product is zero. Multiplication by two unit odd scalars[‡] r, l gives two natural isomorphisms between the subspaces of even k-vectors and odd k-vectors.

Let $\mathcal{F}$ denote the set of all even and odd forms. We can similarly define the exterior product for all pairs of elements from $\mathcal{F}$. As an example we show in Fig. 10 how one can build a two-form $\mathbf{B}$ as the product of two linear forms $\mathbf{f} \wedge \mathbf{g}$. We can say that the pipe of $\mathbf{B}$ is the intersection of the layers of the factors as the one-forms. The attitude of $\mathbf{B}$ is a straight line — the intersection of the attitudes (planes) of $\mathbf{f}$ and $\mathbf{g}$, the orientation is obtained by juxtaposing the arrows of both forms, second after the first.

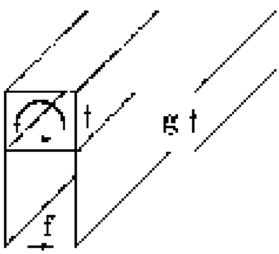


Figure 10. The exterior product $\mathbf{f} \wedge \mathbf{g}$ of two one-forms $\mathbf{f}$ and $\mathbf{g}$.

Since each odd quantity N can be represented as $N = rM$ for some even quantity M, we may introduce the value of an even one-form $\mathbf{f}$ on an odd vector $\mathbf{b} = r\mathbf{a}$ as

$$\mathbf{f}[\mathbf{b}] = \mathbf{f}[r\mathbf{a}] = r\mathbf{f}[\mathbf{a}].$$

In this way, even one-form becomes a linear form on $\mathcal{M}_1$; that is, on the set of even and odd vectors. Analogous claims can be formulated about the two-forms as linear forms on $\mathcal{M}_2$ and three-forms as linear forms on $\mathcal{M}_3$.

Similarly, the value of an odd one-form $\mathbf{g} = r\mathbf{f}$ on an even vector $\mathbf{a}$ is defined as

$$\mathbf{g}[\mathbf{a}] = (r\mathbf{f})[\mathbf{a}] = r(\mathbf{f}[\mathbf{a}]).$$

‡ They are called *right* and *left*, and are opposite to each other.

Another kind of product can be introduced, namely, contraction. The *contraction* of the one-form $\mathbf{f}$ by the vector $\mathbf{a}$ is $\mathbf{a}\rfloor\mathbf{f} = \mathbf{f}[\mathbf{a}]$ where $\mathbf{f}[\mathbf{a}]$ is the value of $\mathbf{f}$ on $\mathbf{a}$. We define, similarly, the contraction from the other side: $\mathbf{f}\lfloor\mathbf{a} = \mathbf{f}[\mathbf{a}]$.

The *contraction* of the two-form $\mathbf{B} = \mathbf{f} \wedge \mathbf{g}$ (where $\mathbf{f},\ \mathbf{g}$ are one-forms) by the vector $\mathbf{a}$ is

$$\mathbf{a}\rfloor\mathbf{B} = \mathbf{a}\rfloor(\mathbf{f} \wedge \mathbf{g}) = \mathbf{f}[\mathbf{a}]\,\mathbf{g} - \mathbf{g}[\mathbf{a}]\,\mathbf{f}. \tag{1}$$

We can also define the contraction of $\mathbf{B}$ by $\mathbf{a}$ from the other side:

$$\mathbf{B}\lfloor\mathbf{a} = (\mathbf{f} \wedge \mathbf{g})\lfloor\mathbf{a} = \mathbf{f}\,\mathbf{g}[\mathbf{a}] - \mathbf{g}\,\mathbf{f}[\mathbf{a}] = \mathbf{g}[\mathbf{a}]\,\mathbf{f} - \mathbf{f}[\mathbf{a}]\,\mathbf{g}. \tag{2}$$

After comparing this with (1), we notice that this contraction is *anticommutive*:

$$\mathbf{B}\lfloor\mathbf{a} = -\mathbf{a}\rfloor\mathbf{B}. \tag{3}$$

It is possible to check that the contraction of a two-form by a vector $\mathbf{a}$ is parallel to both factors.

The *contraction* of the three-form $w = \mathbf{g} \wedge \mathbf{B}$ by the vector $\mathbf{a}$ is

$$\mathbf{a}\rfloor w = \mathbf{a}\rfloor(\mathbf{g} \wedge \mathbf{B}) = \mathbf{g}[\mathbf{a}]\mathbf{B} - \mathbf{g} \wedge (\mathbf{a}\rfloor\mathbf{B}). \tag{4}$$

Since we can write $w = \mathbf{B} \wedge \mathbf{g}$ as well, we define also the contraction from the other side:

$$w\lfloor\mathbf{a} = (\mathbf{B} \wedge \mathbf{g})\lfloor\mathbf{a} = \mathbf{B}\,\mathbf{g}[\mathbf{a}] - (\mathbf{B}\lfloor\mathbf{a}) \wedge \mathbf{g}.$$

The last term is the exterior product of two one-forms; hence, it is anticommutive and we obtain

$$w\lfloor\mathbf{a} = \mathbf{g}[\mathbf{a}]\,\mathbf{B} + \mathbf{g} \wedge (\mathbf{B}\lfloor\mathbf{a}).$$

Now we use (3):

$$w\lfloor\mathbf{a} = \mathbf{g}[\mathbf{a}]\,\mathbf{B} - \mathbf{g} \wedge (\mathbf{a}\rfloor\mathbf{B}).$$

Comparing this with (4), we notice the commutivity:

$$w\lfloor\mathbf{a} = \mathbf{a}\rfloor w.$$

The result of this contraction is obviously a two-form. One can check that the contraction of a three-form by a vector is parallel to the vector.

Similar inductive procedure serves to define contractions of multivectors with one-forms. For instance, the counterpart of (1) for a one-form $\mathbf{k}$ is the definition:

$$\mathbf{k}\rfloor(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{k}\rfloor\mathbf{a})\,\mathbf{b} - \mathbf{a}(\mathbf{k}\rfloor\mathbf{b}). \tag{5}$$

Then one defines the contraction of the trivector T with the two-form $\mathbf{B} = \mathbf{f} \wedge \mathbf{g}$ as follows:

$$T\lfloor\mathbf{B} = T\lfloor(\mathbf{f} \wedge \mathbf{g}) = (T\lfloor\mathbf{f})\lfloor\mathbf{g}, \qquad \mathbf{B}\rfloor T = (\mathbf{f} \wedge \mathbf{g})\rfloor T = \mathbf{f}\rfloor(\mathbf{g}\rfloor T).$$

One can check that this product is commutive and parallel to the two-form.

We omit tedious calculations leading to the following identity including a one- form $\mathbf{k}$, a three-form I and a bivector $\mathbf{s}$:

$$\mathbf{k} \wedge (I\lfloor\mathbf{s}) = I\lfloor(\mathbf{k}\rfloor\mathbf{s}). \tag{6}$$

The both sides of this equality are two-forms. For vector $\mathbf{b}$ in place of $\mathbf{s}$, another identity is fulfilled:

$$\mathbf{k} \wedge (I\lfloor\mathbf{b}) = \mathbf{k}(\mathbf{b})\, I. \tag{7}$$

A *scalar product* is the bilinear form g which is symmetric, positive definite and nondegenerate. Let $g(\mathbf{v}, \mathbf{u})$ denote the value of g on two vectors $\mathbf{v}$ and $\mathbf{u}$. When the vectors are expressed in a basis $\{\mathbf{e}_j\}$, that is, $\mathbf{v} = v^i\mathbf{e}_i$, $\mathbf{u} = u^j\mathbf{e}_j$,[§] then the bilinearity allows to write down

$$g(\mathbf{v}, \mathbf{u}) = v^i g_{ij} u^j, \quad \text{where} \quad g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j).$$

The matrix $\mathcal{G} = \{g_{ij}\}$ is known as the metric tensor, we call it the *scalar product matrix.*

For a given vector basis $\{\mathbf{e}_j\}$ a unique one-form basis $\{\mathbf{f}^k\}$ exists such that $\mathbf{f}^k(\mathbf{e}_j) = \delta^k_j$ – it is called *dual basis.* The scalar product of vectors induces the scalar product $\tilde{g}$ of one-forms by the formula

$$\tilde{g}(\mathbf{h}, \mathbf{k}) = h_i \tilde{g}^{ij} k_j, \quad \text{where} \quad \mathbf{h} = h_i\mathbf{f}^i, \;\; \mathbf{k} = k_j\mathbf{f}^j \tag{8}$$

and the matrix $\tilde{\mathcal{G}} = \{\tilde{g}^{ij}\}$ is inverse to $\mathcal{G}$:

$$\tilde{\mathcal{G}} = \mathcal{G}^{-1} \tag{9}$$

Let us consider two scalar products g_1 and g_2 in the same linear space such that g_1 is not proportional to g_2. As explained in [13], each of them has its own concentric spheres[‖] which for us are ellipsoids. We visualize them in Fig. 10 which for the sake of simplicity is made in two dimensions, hence the ellipsoids are shown as ellipses. One can find pairs of ellipses, one being g_1-sphere the other g_2-sphere such that they are tangent to each other. We show two such pairs in Fig. 11. Now we draw two vectors $\mathbf{c}_1$, $\mathbf{c}_2$ from the origin to the point of tangency of the ellipses and two vectors $\mathbf{b}_1$, $\mathbf{b}_2$ tangent to ellipses of two families simultaneously, see Fig. 12.

[§] The Einstein summation convention is assumed for repeated indices.

[‖] Sphere means a set of points with equal distances from the origin.

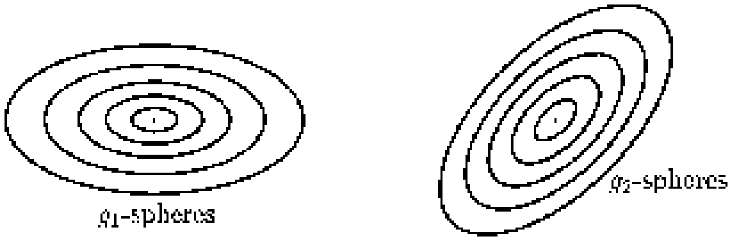


Figure 11. Each scalar product defines its spheres which look like ellipsoids.

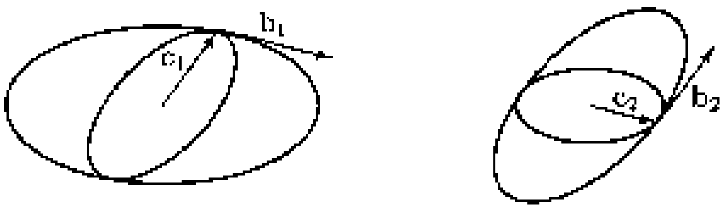


Figure 12. Two pairs of vectors orthogonal with respect to both scalar products.

From the properties of the scalar product $\mathbf{b}_1 \perp \mathbf{c}_1$ and $\mathbf{b}_2 \perp \mathbf{c}_2$ with respect to both scalar products g_i. It follows from this that $\mathbf{c}_1 \perp \mathbf{c}_2$ with respect to both g_i's. In the plane one can find only two attitudes for which vectors (like $\mathbf{c}_1$ and $\mathbf{c}_2$) have tips at points of tangency of ellipses from the two families. This means that in two dimensions when g_1 and g_2 are not proportional, there exist only two attitudes perpendicular to each other with respect to g_1 and g_2 simultaneously. It may happen that in three dimensions, only three such attitudes exist, we shall call them *principal attitudes* or *principal axes* of one scalar product with respect to the other. In such a case, the scalar product g_1 will be called *biaxial* with respect to g_2 and *vice versa.*

If one of the scalar products – let it be g_2 – is the ordinary scalar product of our space, its spheres are ordinary spheres, see Fig. 13 where the ellipsoids are represented by ellipses and the spheres by circles. Now the counterpart of Fig. 12 is as shown in Fig. 14, where we have an ellipsoid inscribed in a circle and a circle inscribed in an ellipsoid. In three dimensions, three vectors $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ exist which are perpendicular to each other, we say that they determine *principal*

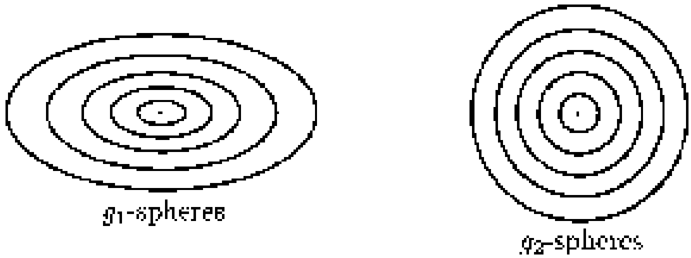


Figure 13. Ellipsoids become true spheres for the natural scalar product g_2.

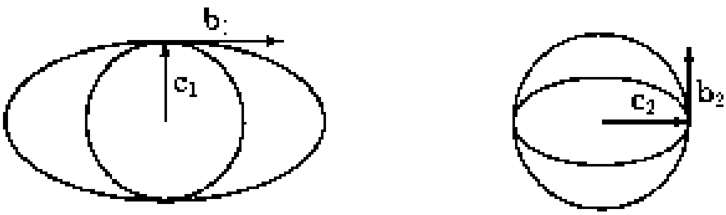


Figure 14. Ellipsoids become true spheres for the natural scalar product g_2.

attitudes or *principal axes* of g_1 with respect to the ordinary scalar product. At the tip point of $\mathbf{c}_1$, the two vectors $\mathbf{c}_2$ and $\mathbf{c}_3$ span a plane tangent to the ellipsoid and to the sphere simultaneously. The same is valid for the circular permutation of indices. This is the situation when the scalar product matrix $\mathcal{G}_1$, after putting into diagonal form, has three different values on the diagonal.

In the special case, when two eigenvalues of $\mathcal{G}_1$ are equal, the situation is different. In this case, there exists one vector $\mathbf{c}_1$ and the infinite family of vectors $\mathbf{c}_\alpha$ orthogonal to $\mathbf{c}_1$ with respect to both scalar products. They scan a plane perpendicular to $\mathbf{c}_1$. Then the sphere of g_1 is the ellipsoid of revolution. It is not strange then that there exists a ring of points of tangency of this ellipsoid to a sphere of g_2. The plane of this ring will be called *principal plane* and the attitude of $\mathbf{c}_1$ — *principal axis* of g_1. In this case the scalar product g_1 is called *uniaxial*.

The scalar product of one-forms serves to define a *scalar product of two-forms*. Namely, for $\mathbf{B} = \mathbf{h} \wedge \mathbf{k}$, $\mathbf{D} = \mathbf{p} \wedge \mathbf{s}$ one defines it by the determinant

$$\tilde{g}(\mathbf{B}, \mathbf{D}) = \tilde{g}(\mathbf{h} \wedge \mathbf{k},\, \mathbf{p} \wedge \mathbf{s}) = \begin{vmatrix} \tilde{g}(\mathbf{h}, \mathbf{p}) & \tilde{g}(\mathbf{h}, \mathbf{s}) \\ \tilde{g}(\mathbf{k}, \mathbf{p}) & \tilde{g}(\mathbf{k}, \mathbf{s}) \end{vmatrix}. \qquad (10)$$

One may also introduce *inner products* of a two-form $\mathbf{D} = \mathbf{h} \wedge \mathbf{k}$ with a one-form $\mathbf{E}$ by the expressions

$$\tilde{g}(\mathbf{D}, \mathbf{E}) = \tilde{g}(\mathbf{h} \wedge \mathbf{k}, \mathbf{E}) = \mathbf{h}\, \tilde{g}(\mathbf{k}, \mathbf{E}) - \mathbf{k}\, \tilde{g}(\mathbf{h}, \mathbf{E})$$

and

$$\tilde{g}(\mathbf{E}, \mathbf{D}) = \tilde{g}(\mathbf{E}, \mathbf{h} \wedge \mathbf{k}) = \tilde{g}(\mathbf{E}, \mathbf{h})\, \mathbf{k} - \tilde{g}(\mathbf{E}, \mathbf{k})\, \mathbf{h}.$$

It is easy to notice that $\tilde{g}(\mathbf{D}, \mathbf{E}) = -\tilde{g}(\mathbf{E}, \mathbf{D})$. The inner product can be written in coordinates:

$$[\tilde{g}(\mathbf{D}, \mathbf{E})]_k = D_{kj} \tilde{g}^{ji} E_i, \qquad [\tilde{g}(\mathbf{E}, \mathbf{D})]_k = E_i \tilde{g}^{ij} D_{jk}. \tag{11}$$

The scalar product matrix $\mathcal{G}$ serves also to transform vectors into one-forms by the following formula for the basis vectors:

$$G(\mathbf{e}_i) = \mathbf{f}^k g_{ki},$$

and then, by linearity, for the arbitrary vectors. Similarly the inverse matrix $\tilde{\mathcal{G}}$ serves to transform one-forms into vectors:

$$\tilde{G}(\mathbf{f}^k) = \mathbf{e}_j \tilde{g}^{jk}.$$

Here $\mathbf{f}^i$'s are elements of a basis *dual* to the vector basis $\mathbf{e}_j$, which means $\mathbf{f}^i[\mathbf{e}_j] = \delta^i_j$. These linear operators are inverse of each other: $G\tilde{G} = \tilde{G}G = 1$. The linearity allows to find the image of arbitrary one-form $\mathbf{E} = E_k \mathbf{f}^k$:

$$\vec{\mathbf{E}} = \tilde{G}(\mathbf{E}) = \tilde{G}(E_k \mathbf{f}^k) = E_k \tilde{G}(\mathbf{f}^k) = \tilde{g}^{jk} E_k \mathbf{e}_j.$$

This means that vector coordinates E^j are related to form coordinates E_k by the relation

$$E^j = \tilde{g}^{jk} E_k. \tag{12}$$

After short calculations one may prove the following identities

$$(G(\mathbf{v}))(\mathbf{u}) = \mathbf{u} \rfloor G(\mathbf{v}) = g(\mathbf{u}, \mathbf{v}), \qquad \mathbf{f}(\tilde{G}(\mathbf{h})) = \mathbf{f} \rfloor \tilde{G}(\mathbf{h}) = \tilde{g}(\mathbf{f}, \mathbf{h}). \tag{13}$$

We define also operator $\tilde{G}^{\wedge 2}$, called *compound* or 2-*compound*, acting on two-forms by the prescription:

$$\tilde{G}^{\wedge 2}(\mathbf{h} \wedge \mathbf{k}) = \tilde{G}(\mathbf{h}) \wedge \tilde{G}(\mathbf{k}).$$

This object is obviously a bivector. Let us calculate its contraction with a one-form $\mathbf{f}$ with the use of (5):

$$\mathbf{f} \rfloor \tilde{G}^{\wedge 2}(\mathbf{h} \wedge \mathbf{k}) = \mathbf{f} \rfloor [\tilde{G}(\mathbf{h}) \wedge \tilde{G}(\mathbf{k})] = \left(\mathbf{f} \rfloor \tilde{G}(\mathbf{h})\right) \tilde{G}(\mathbf{k}) - \left(\mathbf{f} \rfloor \tilde{G}(\mathbf{k})\right) \tilde{G}(\mathbf{h}).$$

We apply now (13):

$$\mathbf{f}\rfloor\tilde{G}^{\wedge 2}(\mathbf{h}\wedge\mathbf{k}) = \tilde{g}(\mathbf{f},\mathbf{h})\tilde{G}(\mathbf{k}) - \tilde{g}(\mathbf{f},\mathbf{k})\tilde{G}(\mathbf{h}).$$

Both sides of this equation are vectors. We contract this with a three-form I and use the identity (6) to the left-hand side:

$$\mathbf{f}\wedge\left[I\lfloor\tilde{G}^{\wedge 2}(\mathbf{h}\wedge\mathbf{k})\right] = I\lfloor\left[\tilde{g}(\mathbf{f},\mathbf{h})\,\tilde{G}(\mathbf{k}) - \tilde{g}(\mathbf{f},\mathbf{k})\,\tilde{G}(\mathbf{h})\right]. \tag{14}$$

Now both sides are two-forms.

It is useful to introduce a kind of Hodge operator N_g which acts on one-forms. It is the composition of the mapping $\tilde{G}$ and the contraction with the basic odd three-form $\mathbf{f}_*^{123} = \mathbf{f}^1\wedge\mathbf{f}^2\wedge\mathbf{f}_*^3$:

$$N_g(\mathbf{E}) = \mathbf{f}_*^{123}\lfloor\tilde{G}(\mathbf{E}) =: \mathbf{D} \tag{15}$$

It maps even one-forms into odd two-forms or odd one-forms into even two-forms. Its inverse is the composition of inverses of the two mappings in the opposite order:

$$N_g^{-1}(\mathbf{D}) = -G(\mathbf{e}_{123}^*\lfloor\mathbf{D}) = \mathbf{E}$$

where $\mathbf{e}_{123}^*$ is the basic odd trivector.

Another kind of Hodge operator P_g can be introduced, acting on two-forms. It is the composition of the mapping $\tilde{G}^{\wedge 2}$ and the contraction with $\mathbf{f}_*^{123}$:

$$P_g(\mathbf{D}) = \mathbf{f}_*^{123}\lfloor\tilde{G}^{\wedge 2}(\mathbf{D}) \tag{16}$$

Lemma 3.1. The composition of the operators N_g and P_g has the form

$$P_g\left(N_g(\mathbf{E})\right) = -\left(\det\tilde{\mathcal{G}}\right)\,\mathbf{E}.$$

P r o o f. For $\mathbf{D} = \frac{1}{2}D_{ij}\,\mathbf{f}^i\wedge\mathbf{f}_*^j$ a calculation yields

$$\tilde{G}^{\wedge 2}(\mathbf{D}) = \frac{1}{2}\,\tilde{g}^{li}\tilde{g}^{kj}D_{ij}\,\mathbf{e}_l\wedge\mathbf{e}_k^*.$$

Then

$$P_g(\mathbf{D}) = \mathbf{f}_*^{123}\lfloor\left(\frac{1}{2}\,\tilde{g}^{li}\tilde{g}^{kj}D_{ij}\,\mathbf{e}_l\wedge\mathbf{e}_k^*\right) = \frac{1}{2}\,\tilde{g}^{li}\tilde{g}^{kj}D_{ij}\,\mathbf{f}_*^{123}\lfloor(\mathbf{e}_l\wedge\mathbf{e}_k^*)\,,$$

$$P_g(\mathbf{D}) = \frac{1}{2}D_{ij}\left[\left(\tilde{g}^{3i}\tilde{g}^{2j} - \tilde{g}^{2i}\tilde{g}^{3j}\right)\mathbf{f}^1 + \left(\tilde{g}^{1i}\tilde{g}^{3j} - \tilde{g}^{3i}\tilde{g}^{2j}\right)\mathbf{f}^2 + \left(\tilde{g}^{2i}\tilde{g}^{1j} - \tilde{g}^{1i}\tilde{g}^{2j}\right)\mathbf{f}^3\right]. \tag{17}$$

For $\mathbf{E}=E_k\mathbf{f}^k$ we obtain

$$\mathbf{D}=N_g(\mathbf{E})=\mathbf{f}_*^{123}\lfloor\tilde{G}\left(E_k\mathbf{f}^k\right)=\tilde{g}^{jk}E_k\,\mathbf{f}_*^{123}\lfloor\mathbf{e}_j=E_k\left(\tilde{g}^{3k}\mathbf{f}_*^{12}+\tilde{g}^{2k}\mathbf{f}_*^{31}+\tilde{g}^{1k}\mathbf{f}_*^{23}\right).$$

We recognize from this the following coordinates of $\mathbf{D}$:

$$D_{12}=\tilde{g}^{3k}E_k,\quad D_{31}=\tilde{g}^{2k}E_k,\quad D_{23}=\tilde{g}^{1k}E_k,$$

which can be written as $D_{ij}=\epsilon_{ijl}\,\tilde{g}^{lk}E_k$ with the Levi-Civita symbol ϵ_{ijl}. We substitute this into (17)

$$P_g\left(N_g(\mathbf{E})\right)=$$
$$\frac{1}{2}\epsilon_{ijl}\tilde{g}^{lk}E_k\left[\left(\tilde{g}^{3i}\tilde{g}^{2j}-\tilde{g}^{2i}\tilde{g}^{3j}\right)\mathbf{f}^1+\left(\tilde{g}^{1i}\tilde{g}^{3j}-\tilde{g}^{3i}\tilde{g}^{1j}\right)\mathbf{f}^2+\left(\tilde{g}^{2i}\tilde{g}^{1j}-\tilde{g}^{1i}\tilde{g}^{2j}\right)\mathbf{f}^3\right].$$

The antisymmetry of ϵ_{ijl} with respect to i,j allows to write

$$P_g(N_g(\mathbf{E}))=-\epsilon_{ijl}\tilde{g}^{lk}E_k\left(\tilde{g}^{2i}\tilde{g}^{3j}\,\mathbf{f}^1+\tilde{g}^{3i}\tilde{g}^{1j}\,\mathbf{f}^2+\tilde{g}^{1i}\tilde{g}^{2j}\,\mathbf{f}^3\right).$$

First coordinate of this one-form is

$$\left[P_g\left(N_g(\mathbf{E})\right)\right]_1=-\epsilon_{ijl}\tilde{g}^{2i}\tilde{g}^{3j}\tilde{g}^{lk}E_k=-\epsilon_{ijl}\tilde{g}^{2i}\tilde{g}^{3j}\tilde{g}^{kl}E_k.$$

where the symmetry of the matrix $\mathcal{G}$ was used. We write the summation over k explicitly

$$\left[P_g\left(N_g(\mathbf{E})\right)\right]_1=-\epsilon_{ijl}\tilde{g}^{2i}\tilde{g}^{3j}\tilde{g}^{1l}E_1-\epsilon_{ijl}\tilde{g}^{2i}\tilde{g}^{3j}\tilde{g}^{2l}E_2-\epsilon_{ijl}\tilde{g}^{2i}\tilde{g}^{3j}\tilde{g}^{3l}E_3.$$

The second term vanishes because it contains summation over i,l of symmetric symbol $\tilde{g}^{2i}\tilde{g}^{2l}$ with antisymmetric one ϵ_{ijl}. Similarly the last term vanishes, hence we are left with

$$\left[P_g\left(N_g(\mathbf{E})\right)\right]_1=-\epsilon_{ijl}\tilde{g}^{2i}\tilde{g}^{3j}\tilde{g}^{1l}E_1=-(\det\tilde{\mathcal{G}})\,E_1.$$

Similar calculations lead to the results

$$[P_g(N_g(\mathbf{E}))]_2=-(\det\tilde{\mathcal{G}})\,E_2,\quad [P_g(N_g(\mathbf{E}))]_3=-(\det\tilde{\mathcal{G}})\,E_3,$$

which can be summarized as

$$P_g(N_g(\mathbf{E}))=-(\det\tilde{\mathcal{G}})\,\mathbf{E}.\ \blacksquare$$

This lemma allows us to write

$$\mathbf{E}=-(\det\tilde{\mathcal{G}})^{-1}P_g(\mathbf{D})=-(\det\mathcal{G})\,P_g(\mathbf{D})\tag{18}$$

for $\mathbf{D}=N_g(\mathbf{E})$. In this manner $-(\det\mathcal{G})\,P_g$ is the inverse mapping to N_g.

Lemma 3.2. If a two-form $\mathbf{D}$ is the exterior product of two one-forms: $\mathbf{D} = \mathbf{h} \wedge \mathbf{H}$ then $\mathbf{E} = N_g^{-1}(\mathbf{D})$ is perpendicular to $\mathbf{h}$ and $\mathbf{H}$ with respect to scalar product $\tilde{g}$.

P r o o f. Let $\mathbf{h}$ be an even one-form and $\mathbf{H}$ an odd one-form, then

$$-\mathbf{e}^*_{123} \lfloor (\mathbf{h} \wedge \mathbf{H}) = (h_3 H_2 - h_2 H_3)\,\mathbf{e}_1 + (h_1 H_3 - h_3 H_1)\mathbf{e}_2 + (h_2 H_1 - h_1 H_2)\mathbf{e}_3$$

and

$$\begin{aligned}\mathbf{E} &= G\,[-\mathbf{e}^*_{123} \lfloor (\mathbf{h} \wedge \mathbf{H})] \\ &= (h_3 H_2 - h_2 H_3) G(\mathbf{e}_1) + (h_1 H_3 - h_3 H_1) G(\mathbf{e}_2) + (h_2 H_1 - h_1 H_2) G(\mathbf{e}_3) \\ &= [g_{1i}(h_3 H_2 - h_2 H_3) + g_{2i}(h_1 H_3 - h_3 H_1) + g_{3i}(h_2 H_1 - h_1 H_2)]\,\mathbf{f}^i.\end{aligned}$$

Now calculate the scalar product of $\mathbf{E}$ and $\mathbf{h}$:

$$\begin{aligned}\tilde{g}(\mathbf{E}, \mathbf{h}) &= \tilde{g}^{ij} E_i h_j \\ &= \Big[\tilde{g}^{ij} g_{1i}(h_3 H_2 - h_2 H_3) + \tilde{g}^{ij} g_{2i}(h_1 H_3 - h_3 H_1) + \tilde{g}^{ij} g_{3i}(h_2 H_1 - h_1 H_2)\Big]\, h_j.\end{aligned}$$

Matrix $\mathcal{G}$ is symmetric and $\tilde{\mathcal{G}}\mathcal{G} = 1$, hence

$$\begin{aligned}\tilde{g}(\mathbf{E}, \mathbf{h}) &= \Big[\delta_1^j (h_3 H_2 - h_2 H_3) + \delta_2^j (h_1 H_3 - h_3 H_1) + \delta_3^j (h_2 H_1 - h_1 H_2)\Big]\, h_j \\ &= (h_3 H_2 - h_2 H_3) h_1 + (h_1 H_3 - h_3 H_1) h_2 + (h_2 H_1 - h_1 H_2) h_3 = 0.\end{aligned}$$

Similar calculations lead to the result

$$\tilde{g}(\mathbf{E}, \mathbf{H}) = (h_3 H_2 - h_2 H_3) H_1 + (h_1 H_3 - h_3 H_1) H_2 + (h_2 H_1 - h_1 H_2) H_3 = 0. \quad \blacksquare$$

4. DESCRIPTION OF ELECTRODYNAMICS

4.1. Premetric Description

We now present a list of physical quantities with their designation as directed quantities along with short justifications.

The most natural vectorial quantity is the *displacement vector* $\mathbf{l}$ which is of the same nature as the *radius vector* $\mathbf{r}$ of a point in space relative to a reference point. Of course, the *velocity* $\mathbf{v} = d\mathbf{r}/dt$, the derivative of $\mathbf{r}$ with respect to a scalar variable t, is also a vector. The same is true of the *acceleration* $\mathbf{a} = d\mathbf{v}/dt$ and the electric dipole moment $\mathbf{d} = q\mathbf{l}$.

The best physical model of the even bivector is a flat electric circuit. Its magnitude is just the area encompassed by the circuit; its attitude is the plane of the circuit and orientation is given by the sense of the current. This bivector could be called a *directed area* $\mathbf{S}$ of the circuit. A connected bivectorial quantity is then the *magnetic moment* $\mathbf{m} = I\mathbf{S}$ of the circuit, where I is the electric current.

An even one-form occurs naturally in the description of the waves. The locus of points is space with the same phase of a plane wave is just a plane. The family of planes with phases differing by a natural number can be viewed as the geometric image of the physical quantity known as the *wave vector* $\mathbf{k}$ with magnitude $2\pi/\lambda$, where λ is the wavelength. This physical quantity in its true directed nature is the one-form, not a vector, thus, in my opinion, it deserves another name. One possibility is *wave density*. If I am allowed to create an English word I would propose shorter name *wavity*.

Another even one-form quantity is the *electric field strength* $\mathbf{E}$, since we consider it to be a linear map of the infinitesimal vector $d\mathbf{r}$ into the infinitesimal potential difference: $-dV = \mathbf{E}[d\mathbf{r}]$. The *magnetic induction* $\mathbf{B}$ is an example of a two-form quantity, since it can be treated as a linear map of the directed area bivector $d\mathbf{s}$ into the magnetic flux: $d\Phi = \mathbf{B}[d\mathbf{s}]$.

Now for some examples of odd quantities; the area $\mathbf{s}^*$ of a surface, through which a flow is measured, is the first one. The side of the surface from which a substance (mass, energy, electric charge, etc.) passes is important. Hence, the orientation of $\mathbf{s}^*$ can be marked as an arrow piercing the surface. We claim that the *area of a flow* is an odd bivector quantity. Accordingly, the *current density* $\mathbf{j}$ has to be an odd two-form quantity. It corresponds to the linear map $dI = \mathbf{j}[d\mathbf{s}^*]$ of the infinitesimal area $d\mathbf{s}^*$ into the infinitesimal electric current dI.

The nature of the *electric induction* $\mathbf{D}$ can be deduced from the following prescription of its measurement. Take two identical metal discs, place one disc on top of the other, electrically discharge them and then place them in the presence of a field. As you separate the discs, the opposite sign charges induced on them are also separated. Now measure one of them with the aid of a Faraday cage. It turns out that for a small enough disc the charge is proportional to its area. One will agree that the *disc area* $d\mathbf{s}^*$ is an odd bivector since its magnitude is the area, its attitude is the plane and its orientation is given by an arrow showing which disc is to be connected with the Faraday cage. Because of the proportionality relation $dQ = \mathbf{D}[d\mathbf{s}^*]$, we ascertain that the *electric induction* is a linear map of the odd bivectors into scalars, i.e. it is an odd two-form.

The operational definition of the *magnetic field strength* $\mathbf{H}$ is as follows: Take a very small wireless solenoid prepared from a superconducting material. Close the circuit in a region of space where the magnetic field vanishes. Afterwards, introduce the circuit into an arbitrary region in the field. A superconductor has the property that the magnetic flux enclosed by it is always the same; a current will be induced to compensate for this external field flux. Now measure the current dI flowing through the superconductor. It turns out to be proportional to the solenoid length: $dI = \mathbf{H}[d\mathbf{l}^*]$. The *solenoid length* $d\mathbf{l}^*$ in this experiment is apparently an odd vector, hence the *magnetic field strength* $\mathbf{H}$ is an odd one-form.

As shown above, the four electromagnetic field quantities are of different directed nature: the electric field strength $\mathbf{E}$ is the even one-form, the electric induction $\mathbf{D}$ is the odd two-form, the magnetic field strength $\mathbf{H}$ is the odd one-form, and the magnetic induction $\mathbf{B}$ is the even two-form. With the aid of the exterior derivative

$$\mathbf{d} = \mathbf{f}^1 \frac{\partial}{\partial x^1} + \mathbf{f}^2 \frac{\partial}{\partial x^2} + \mathbf{f}^3 \frac{\partial}{\partial x^3},$$

where $\mathbf{f}^i$ are basic one-forms, one can express the differential Maxwell's equations in terms of differential forms:

$$\mathbf{d} \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{19}$$

$$\mathbf{d} \wedge \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}, \tag{20}$$

$$\mathbf{d} \wedge \mathbf{B} = 0, \tag{21}$$

$$\mathbf{d} \wedge \mathbf{D} = \rho. \tag{22}$$

where ρ is the density of charge odd three-form and $\mathbf{j}$ is the electric current density odd two-form.

The continuity equation for the electric charge, $\mathbf{d} \wedge \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$, can be derived from (20) and (22).

By virtue of the Poincaré lemma, one obtains from eq. (21) the existence of the directed potential $\mathbf{A}$ such that $\mathbf{B} = \mathbf{d} \wedge \mathbf{A}$. After substituting this into eq. (19) one obtains again due to the Poincaré lemma the scalar potential Φ such that $\mathbf{E} = -\mathbf{d}\Phi - \frac{\partial \mathbf{A}}{\partial t}$.

The energy density of the electromagnetic field can be expressed by the formula:

$$w = \frac{1}{2}(\mathbf{E} \wedge \mathbf{D} + \mathbf{B} \wedge \mathbf{H}).$$

The energy flux density of the electromagnetic field, as all flux densities, should be an odd two-form. Only the exterior product $\mathbf{H} \wedge \mathbf{E}$ (or $\mathbf{E} \wedge \mathbf{H}$) gives such a quantity which replaces the traditional Poynting vector:

$$\mathbf{S} = \mathbf{E} \wedge \mathbf{H}.$$

It may be called the *Poynting two-form.*

The *Lorentz force* $\mathbf{F}$ acting on the electric charge q moving with velocity $\mathbf{v}$ can be expressed with the aid of contraction between $\mathbf{B}$ and $\mathbf{v}$:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{B}\lfloor\mathbf{v}.$$

4.2. Variable Metric

In an isotropic medium, the electric field vector $\vec{\mathbf{E}}$ is parallel to the electric induction vector $\vec{\mathbf{D}}$. In terms of the forms this sentence has to be formulated as follows: In an isotropic medium planes of the electric field one-form $\mathbf{E}$ are perpendicular to the lines of the electric induction two-form $\mathbf{D}$. Since we know that in an anisotropic medium the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ are no longer parallel, we should express this so: the forms $\mathbf{E}$ and $\mathbf{D}$ are no longer perpendicular. But perpendicularity depends on a scalar product. In this manner a question arises: can we find another scalar product, appropriate for a given medium, such that the same forms become perpendicular to each other?

We tackle this question now. We all live in the three-dimensional space in which the *natural scalar product* exists, because we know which vectors are orthogonal and we ascribe a length to each of them independently of attitude. Knowing this natural scalar product, we may, in agreement with § 3, change even and odd one- and two-forms into even vectors $\vec{\mathbf{E}}$, $\vec{\mathbf{D}}$ or odd vectors $\vec{\mathbf{B}}$, $\vec{\mathbf{H}}$, which are present in the physics textbooks. With the natural orthonormal basis, we relate vector coordinates (with the upper indices) to one-form coordinates (with lower indices) according to formulas:

$$E^i = E_i, \qquad H^i = H_i. \tag{23}$$

On the other hand, relations concerning the electric and magnetic induction can be written with the aid of Levi-Civita symbol:

$$D_{ij} = \epsilon_{ijk} D^k, \qquad B_{ij} = \epsilon_{ijk} B^k. \tag{24}$$

This corresponds to the formulas

$$\mathbf{D} = \mathbf{f}_*^{123}\lfloor\vec{\mathbf{D}}, \qquad \mathbf{B} = \mathbf{f}_*^{123}\lfloor\vec{\mathbf{B}}. \tag{25}$$

where $\mathbf{f}_*^{123} = \mathbf{f}^1 \wedge \mathbf{f}^2 \wedge \mathbf{f}_*^3$ is the basic odd three-form, obtained from the one-form basis.

It is the vector notation in which the *constitutive relations* usually are written down: $\vec{\mathbf{D}} = \varepsilon(\vec{\mathbf{E}})$, $\vec{\mathbf{B}} = \mu(\vec{\mathbf{H}})$ in which, for so called *linear media*, symbols ε and μ denote linear mappings.

From now on we consider the electric field only. For the electric quantities, the constitutive relation has the form:

$$D^1 = \varepsilon_0(\varepsilon^{11}E^1 + \varepsilon^{12}E^2 + \varepsilon^{13}E^3),$$

$$D^2 = \varepsilon_0(\varepsilon^{21}E^1 + \varepsilon^{22}E^2 + \varepsilon^{23}E^3),$$

$$D^3 = \varepsilon_0(\varepsilon^{31}E^1 + \varepsilon^{32}E^2 + \varepsilon^{33}E^3),$$

where ε^{ij} are (dimensionless) elements of the *relative electric permittivity* matrix $\mathcal{E}$ of the medium, whereas ε_0 is the *electric permittivity of vacuum.* After using (23) we write this as

$$D^1 = \varepsilon_0(\varepsilon^{11}E_1 + \varepsilon^{12}E_2 + \varepsilon^{13}E_3),$$

$$D^2 = \varepsilon_0(\varepsilon^{21}E_1 + \varepsilon^{22}E_2 + \varepsilon^{23}E_3),$$

$$D^3 = \varepsilon_0(\varepsilon^{31}E_1 + \varepsilon^{32}E_2 + \varepsilon^{33}E_3),$$

or, according to the summation convention

$$D^i = \varepsilon_0\varepsilon^{ij}E_j. \tag{26}$$

By virtue of (24) we write this in terms of form coordinates:

$$D_{ij} = \varepsilon_0\epsilon_{ijk}\varepsilon^{kl}E_l.$$

Calculate now the exterior product $\mathbf{E} \wedge \mathbf{D}$:

$$\mathbf{E} \wedge \mathbf{D} = \left(E_1\mathbf{f}^1 + E_2\mathbf{f}^2 + E_3\mathbf{f}^3\right) \wedge \left(D_{12}\mathbf{f}_*^{12} + D_{23}\mathbf{f}_*^{23} + D_{31}\mathbf{f}_*^{31}\right)$$

$$= (E_1D_{23} + E_2D_{31} + E_3D_{12})\ \mathbf{f}_*^{123} = \left(E_1D^1 + E_2D^2 + E_3D^3\right)\ \mathbf{f}_*^{123}$$

$$= E_iD^i\ \mathbf{f}_*^{123} = \varepsilon_0\ \left(E_i\varepsilon^{ij}E_j\right)\ \mathbf{f}_*^{123}.$$

Hence the energy density of the electric field is

$$w_e = \frac{1}{2}\mathbf{E} \wedge \mathbf{D} = \frac{1}{2}\varepsilon_0\left(E_i\varepsilon^{ij}E_j\right)\ \mathbf{f}_*^{123}.$$

It is an odd three-form, but its single scalar coordinate,

$$|w_e| = \frac{1}{2}\varepsilon_0E_i\varepsilon^{ij}E_j, \tag{27}$$

is an image of the bilinear mapping of one-forms into scalars. We see that a scalar product of one-forms should be introduced for which the permittivity matrix is the scalar product matrix. Comparing (27) with (8) allows us to introduce $\tilde{\mathcal{G}}_\varepsilon = \mathcal{E}$, and due to (9)

$$\mathcal{G}_\varepsilon = \mathcal{E}^{-1}.$$

We formulate our observation as follows: matrix $\mathcal{G}_\varepsilon$, inverse to the matrix of relative electric permittivity, determines a scalar product appropriate for a given anisotropic dielectric.

It has been stated in § 3 that the inverse $G_\varepsilon^{-1} = \tilde{G}_\varepsilon$ of the metric tensor serves to change one-forms into vectors. Thus we could write $\vec{\mathbf{E}} = \tilde{G}_\varepsilon(\mathbf{E})$. This is not the usual vector of the electric field intensity, hence it is more safe to denote it differently, for instance by prime:

$$\vec{\mathbf{E}}' = \tilde{G}_\varepsilon(\mathbf{E}). \tag{28}$$

By dint of (12) this mapping assumes the following form on the coordinates

$$E'^i = \tilde{g}^{ij} E_j = \varepsilon^{ij} E_j.$$

Only the coefficient ε_0 is lacking to obtain (26) which can be written in the vector notation:

$$\vec{\mathbf{D}} = \varepsilon_0 \tilde{G}_\varepsilon(\mathbf{E}). \tag{29}$$

This relation can be expressed physically: electric permittivity changes the electric field intensity into the electric induction, and mathematically: as the inverse of the metric tensor (with the additional coefficient ε_0) it changes the one-form $\mathbf{E}$ into the vector $\vec{\mathbf{D}}$. After using notation (28) we write down

$$\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}}'. \tag{30}$$

Notice that the vectors $\vec{\mathbf{D}}$ and $\vec{\mathbf{E}}'$ are parallel, and the relation between them looks like in the vacuum. Thus, when one introduces a basis orthonormal with respect to the metric g_ε, the coordinates of $\vec{\mathbf{D}}$ and $\vec{\mathbf{E}}'$ are related to each other like in isotropic medium, hence all electrostatic problems can be solved for this medium just by rewriting the well know results in these new coordinates. This has been done in [13] for the Coulomb law and the plane capacitor. In this sense the metric can be variable, accommodated to a given anisotropic medium.

We may combine mapping (29) with (25) and obtain the equality

$$\mathbf{D} = \mathbf{f}_*^{123} \lfloor (\varepsilon_0 \tilde{G}_\varepsilon(\mathbf{E})) = \varepsilon_0 \mathbf{f}_*^{123} \lfloor \tilde{G}_\varepsilon(\mathbf{E}) \tag{31}$$

as a relation between the even one-form $\mathbf{E}$ and odd two-form $\mathbf{D}$. This is a composition of two linear mappings: first the inverse of the metric tensor connected with a given dielectric (that is, such that $\tilde{\mathcal{G}}_\varepsilon = \mathcal{E}$), and then the contraction with the odd three-form $\mathbf{f}_*^{123}$ made from the one-form basis, orthonormal in ordinary metric, not connected with the dielectric. At the end the multiplication by the scalar electric permittivity ε_0 of the vacuum is performed. The mapping $\mathbf{E} \to \mathbf{D}$ given by (31) is a kind of Hodge star operator.

Let us calculate the inner product of the forms $\mathbf{D}$ and $\mathbf{E}$ according to the scalar product $\tilde{g}_\varepsilon$. We first rewrite formula (11) accommodated to our situation:

$$[\tilde{g}_\varepsilon(\mathbf{D}, \mathbf{E})]_k = D_{kj}\tilde{g}_\varepsilon^{ji}E_i = D_{kj}\varepsilon^{ji}E_i.$$

Use (24) and (26)

$$[\tilde{g}_\varepsilon(\mathbf{D}, \mathbf{E})]_k = \varepsilon_0^{-1}\epsilon_{klj}D^l D^j.$$

The symbol ϵ_{klj} is antisymmetric, the product $D^l D^j$ is symmetric in the indices l, j, thus the sum is zero. Hence

$$\tilde{g}_\varepsilon(\mathbf{D}, \mathbf{E}) = 0. \tag{32}$$

We have shown that the odd two-form $\mathbf{D}$ is perpendicular to the even one-form $\mathbf{E}$ in the metric given by the permittivity. This is a formal proof of the expectation expressed in the first paragraph if this subsection.

Typically, the matrices of permittivity $\mathcal{E} = \{\varepsilon^{ij}\}$ and permeability $\mathcal{M} = \{\mu^{ij}\}$ are not proportional, hence the application of two scalar products is necessary: the electric one g_ε and the magnetic one g_μ. We omit the analogous reasoning leading to the result that the counterparts of relations (29) and (31) for the magnetic field assume the form

$$\vec{\mathbf{B}} = \mu_0\tilde{G}_\mu(\mathbf{H}), \tag{33}$$

$$\mathbf{B} = \mu_0\mathbf{f}_*^{123}\lfloor\tilde{G}_\mu(\mathbf{H}), \tag{34}$$

where metric tensor $\tilde{G}_\mu$ is related to the magnetic permeability via the matrix $\mathcal{M}$. The mapping $\mathbf{H} \to \mathbf{B}$ given by (34) is another kind of Hodge star operator. Moreover, the identity

$$\tilde{g}_\mu(\mathbf{B}, \mathbf{H}) = 0$$

holds, meaning that the even two-form $\mathbf{B}$ is perpendicular to the odd one-form $\mathbf{H}$ in the metric given by the permeability.

Here, again, the vectors $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}' = \tilde{G}_\mu(\mathbf{H})$ are parallel, so the relation between them looks like in the isotropic medium. Thus, with the use of a variable metric, accommodated to a given anisotropic medium, all magnetostatic problems can be solved for this medium just by rewriting the well know results in new coordinates orthogonal for the new metric. This has been done in [13] for the Biot-Savart law, the plane magnetic capacitor and the solenoid.

By virtue of Lemma 3.1 we write the relation inverse to (34):

$$\mathbf{H} = -\mu_0^{-1}(\det \mathcal{M})^{-1}\mathbf{f}_*^{123}\lfloor \tilde{G}^{\wedge 2}(\mathbf{B}). \tag{35}$$

5. PLANE WAVES

5.1. Premetric Description

We are searching for solutions of the free Maxwell equations in shape of the plane wave:

$$\mathbf{E}(\mathbf{r}, t) = \psi(\mathbf{k}[\mathbf{r}] - \omega t)\mathbf{E}_0, \qquad \mathbf{B}(\mathbf{r}, t) = \psi(\mathbf{k}[\mathbf{r}] - \omega t)\mathbf{B}_0,$$

$$\mathbf{H}(\mathbf{r}, t) = \psi(\mathbf{k}[\mathbf{r}] - \omega t)\mathbf{H}_0, \qquad \mathbf{D}(\mathbf{r}, t) = \psi(\mathbf{k}[\mathbf{r}] - \omega t)\mathbf{D}_0, \tag{36}$$

where $\psi(\cdot)$ is a scalar function of a scalar argument, ω is a scalar constant, and $\mathbf{k}[\mathbf{r}]$ is the value of the linear form $\mathbf{k}$ of the wavity on the radius vector $\mathbf{r}$. In the expected solution (36) all fields maintain their attitudes for all times and positions; only magnitudes and orientations may change. This means that the wave is linearly polarized. Why these solutions are called plane? Because they are constant where the function $k(\mathbf{r}) = \mathbf{k}[\mathbf{r}]$ is constant, i.e. on planes. The arbitrariness of ψ means that the plane wave (36) needs not be time-harmonic. When the one-form $\mathbf{k}$ is used, the problem of propagation direction is open. This direction, understood as one-dimensional direction of a vector, can not be chosen as perpendicular to the planes of the constant fields, because no scalar product is discriminated.

The argument $\phi = \mathbf{k}[\mathbf{r}] - \omega t$ of ψ, called *phase*, depends linearly both on position and time. The loci of points of constant phase are still planes, but these planes move when time flows. Can a phase velocity be introduced? If position $\mathbf{r}$ becomes a function of time this means that we introduce the motion $\mathbf{r}(t)$ of a fictitious particle. Introduce thus a motion (of course uniform) such that, the fictitious particle is always on a plane of fixed phase:

$$\phi = \mathbf{k}[\mathbf{r}(t)] - \omega t = \text{ const.}$$

Differentiate this equality with respect to time:

$$\frac{d\mathbf{k}[\mathbf{r}(t)]}{dt} - \omega = 0.$$

Since the mapping $\mathbf{r} \to \mathbf{k}[\mathbf{r}]$ is linear and continuous, the derivative may be put under the argument of $\mathbf{k}$:

$$\mathbf{k}\left[\frac{d\mathbf{r}(t)}{dt}\right] = \omega,$$

that is

$$\mathbf{k}[\mathbf{v}] = \omega,$$

where $\mathbf{v} = d\mathbf{r}/dt$ denotes velocity of the fictitious particle. We obtain:

$$\omega^{-1}\mathbf{k}[\mathbf{v}] = 1. \tag{37}$$

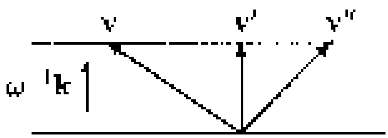


Figure 15. There are many velocities reciprocal to $\omega^{-1}\mathbf{k}$.

There is a lot of velocities $\mathbf{v}$ for which the one-form $\omega^{-1}\mathbf{k}$ gives value one, see Fig. 15. All of them may be called *reciprocal* to $\omega^{-1}\mathbf{k}$. Which one can we admit as the phase velocity? As long as scalar product is not present, none of them. Once a scalar product is introduced, we choose velocity perpendicular to the planes of constant phase. For such velocity $\mathbf{v}$, relation (37) assumes the form

$$\omega^{-1}\vec{\mathbf{k}} \cdot \mathbf{v} = \omega^{-1}kv = 1,$$

where $\vec{\mathbf{k}}$ is the vector also perpendicular to the planes of constant phase, $k = |\mathbf{k}|$, and hence $v = \frac{\omega}{k}$, which is the well known formula for the phase velocity. The equivalent formula $k/\omega = v^{-1}$ says that the quotient $\omega^{-1}k$ is the inverse of the phase velocity. If we want to designate $\omega^{-1}k$ by a separate word, the *slowness* is most suitable. (Less the velocity, the greater slowness – its physical dimension is s/m.) Returning to (37), we may say that the linear form $\mathbf{h} = \omega^{-1}\mathbf{k}$ is *phase slowness*. This notion can be introduced when no scalar product is present, which is not the case for the phase velocity.

We proceed to consider the Maxwell equations. The equation $\mathbf{d} \wedge \mathbf{E} + \partial\mathbf{B}/\partial t = 0$ gives the condition

$$\psi' \ \mathbf{k} \wedge \mathbf{E}_0 - \psi'\omega\mathbf{B}_0 = 0$$

(where prime denotes the derivative with respect to the whole argument) and allows to write

$$\mathbf{B}_0 = \omega^{-1}\mathbf{k} \wedge \mathbf{E}_0 \quad \text{and} \quad \mathbf{B} = \omega^{-1}\mathbf{k} \wedge \mathbf{E} = \mathbf{h} \wedge \mathbf{E}. \tag{38}$$

This equality expressed in the traditional language has the shape $\vec{\mathbf{B}} = \omega^{-1}\vec{\mathbf{k}} \times \vec{\mathbf{E}}$. We see also from (38) that the two forms are parallel: $\mathbf{B} \parallel \mathbf{E}$, which in terms of vectors is written as $\vec{\mathbf{B}} \cdot \vec{\mathbf{E}} = 0$. Moreover, $\mathbf{B} \parallel \mathbf{k}$ which corresponds to $\vec{\mathbf{B}} \cdot \vec{\mathbf{k}} = 0$. We illustrate all this observations on Fig. 16.

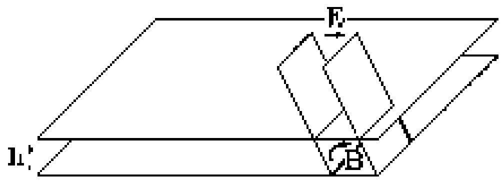


Figure 16. Spatial relations between **h**, **E** and **B**.

Next Maxwell equation, $\mathbf{d} \wedge \mathbf{B} = 0$, gives the condition

$$\psi' \ \mathbf{k} \wedge \mathbf{B}_0 = 0.$$

This condition and this Maxwell equation are automatically satisfied by virtue of (38).

The third free Maxwell equation $\mathbf{d} \wedge \mathbf{H} - \partial \mathbf{D}/\partial t = 0$ reduces to the condition

$$\psi' \ \mathbf{k} \wedge \mathbf{H}_0 + \psi' \omega \mathbf{D}_0 = 0$$

and allows us to write down

$$\mathbf{D}_0 = -\omega^{-1}\mathbf{k} \wedge \mathbf{H}_0 \quad \text{and} \quad \mathbf{D} = -\omega^{-1}\mathbf{k} \wedge \mathbf{H} = -\mathbf{h} \wedge \mathbf{H}. \tag{39}$$

Therefore, the fourth free Maxwell equation $\mathbf{d} \wedge \mathbf{D} = 0$ is automatically satisfied. The constant forms $\mathbf{E}_0$, $\mathbf{H}_0$, present in relations (38) and (39), for the time being, are arbitrary. It is visible from (39) that $\mathbf{D} \parallel \mathbf{H}$ and $\mathbf{D} \parallel \mathbf{k}$; we display this on Fig. 17. Condition (39) can be translated on the traditional language as $\vec{\mathbf{D}} = -\omega^{-1}\vec{\mathbf{k}} \times \vec{\mathbf{H}}$, and the parallelity conditions as $\vec{\mathbf{D}} \cdot \vec{\mathbf{H}} = 0$ and $\vec{\mathbf{D}} \cdot \vec{\mathbf{k}} = 0$.

Summarizing, we write now the plane-wave solutions of the Maxwell's equations:

$$\mathbf{E}(\mathbf{r}, t) = \psi(\mathbf{k}[\mathbf{r}] - \omega t)\mathbf{E}_0, \qquad \mathbf{B}(\mathbf{r}, t) = \psi(\mathbf{k}[\mathbf{r}] - \omega t)\mathbf{u} \wedge \mathbf{E}_0,$$

$$\mathbf{H}(\mathbf{r}, t) = \psi(\mathbf{k}[\mathbf{r}] - \omega t)\mathbf{H}_0, \qquad \mathbf{D}(\mathbf{r}, t) = -\psi(\mathbf{k}[\mathbf{r}] - \omega t)\mathbf{u} \wedge \mathbf{H}_0, \tag{40}$$

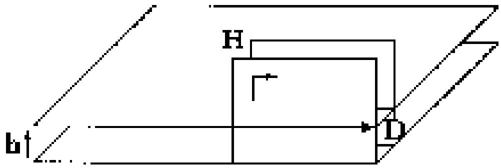


Figure 17. Spatial relations between **h**, **H** and **D**.

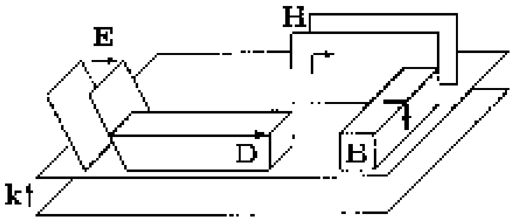


Figure 18. Field quantities and **h** in their spatial relations.

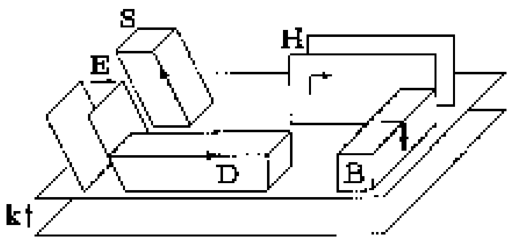


Figure 19. Spatial relation of **S** to other quantities.

where $\mathbf{E}_0$, $\mathbf{H}_0$ are arbitrary constant one-forms. This could be considered as a premetric form of the plane waves. The configuration of four field quantities is depicted on Fig. 18.

As we know from Fig. 10, the one-dimensional attitude of the Poynting odd two-form as the exterior product $\mathbf{S} = \mathbf{E} \wedge \mathbf{H}$ of one-forms is the intersection of the planes of the two factors. If **E** and **H** are as on Fig. 18, the attitude of **S** is oblique with respect to the planes of **k**. We repeat it here as Fig. 19 with the Poynting form added. It is visible from it that its direction is inclined in the vertical plane parallel to **D**. The direction of **S** must be conceded as the direction of wave propagation. We have now settlement of the dilemma considered immediately after Fig. 15: phase velocity is not given uniquely, hence it should be abandoned in the anisotropic medium. On the other hand, the propagation direction of the plane wave is determined by the energy flux density i.e., by the Poynting odd two-form.

The energy densities of the electric and magnetic field,

$$w_e = \frac{1}{2}\mathbf{E} \wedge \mathbf{D}, \qquad w_m = \frac{1}{2}\mathbf{H} \wedge \mathbf{B},$$

after a use of (38) and (39) are

$$w_e = \frac{1}{2}\mathbf{E} \wedge (\mathbf{H} \wedge \mathbf{h}), \qquad w_m = \frac{1}{2}\mathbf{H} \wedge (\mathbf{h} \wedge \mathbf{E}),$$

that is the contributions of two fields are exactly the same

$$w_e = \frac{1}{2}(\mathbf{E} \wedge \mathbf{H}) \wedge \mathbf{h}, \qquad w_m = \frac{1}{2}(\mathbf{E} \wedge \mathbf{H}) \wedge \mathbf{h}.$$

This means that in the plane electromagnetic wave, the same amount of energy is contained in the electric as in the magnetic field, compare Eq. (4.4) in Ref. [18].

The energy of the whole electromagnetic field for the considered plane waves is thus

$$w = w_e + w_m = (\mathbf{E} \wedge \mathbf{H}) \wedge \mathbf{h} = \mathbf{S} \wedge \mathbf{h}. \tag{41}$$

We obtained interesting identity relating the energy density with the energy flux density.

One may also introduce the velocity of the energy transport by the electromagnetic wave. In the traditional approach (solely in terms of vectors and scalars) the following equality is written

$$\vec{\mathbf{S}} = w\vec{\mathbf{v}}, \tag{42}$$

which defines $\vec{\mathbf{v}}$ as the energy transport velocity. This relation is analogous to $\vec{\mathbf{j}} = \rho\vec{\mathbf{v}}$ linking the current density $\vec{\mathbf{j}}$ with the density ρ of charges and their velocity $\vec{\mathbf{v}}$.

Because of another type of the directed quantities we should write (42) rather as the contraction

$$\mathbf{S} = w\lfloor\mathbf{v}. \tag{43}$$

It is an inverse relation to (41). Can one calculate $\mathbf{v}$ from this formula? In this purpose it is worth to introduce and odd trivector T inverse to the odd three-form w in the sense that the condition

$$w[T] = 1$$

is satisfied. It is possible to calculate that $T = |w|^{-1}\mathbf{e}^*_{123}$. If w is the energy density, T may be called *volume of unit energy*, since it has the physical dimension $\frac{\text{volume}}{\text{energy}}$. We may look at (43) as a mapping of a vector into an odd two-form The inverse mapping is the contraction of the odd two-form with the basic odd trivector, hence we have

$$\mathbf{v} = -T\lfloor\mathbf{S} = -|w|^{-1}\,\mathbf{e}^*_{123}\lfloor\mathbf{S}. \tag{44}$$

Since the odd trivector T is positive, the direction of $\mathbf{v}$ is the same as that of $\mathbf{S}$. In this way (44) is the sought formula for the *velocity of the energy transport by the electromagnetic field.*

In this sense we may claim that the phase slowness and the velocity of the energy transport by the plane electromagnetic wave are mutually inverse quantities. The energy transport velocity is one of vectors depicted on Fig. 15.

5.2. The Use of Scalar Products

When considering plane waves we admit simultaneous presence of the electric and magnetic fields. If the matrices of electric permittivity $\mathcal{E}$ and magnetic permeability $\mathcal{M}$ are not proportional, the application of two scalar products is necessary: the electric one $g_\varepsilon(\cdot,\cdot)$ and the magnetic one $g_\mu(\cdot,\cdot)$.

After comparing relations (31) and (34) with equation (15) we notice that the former define two Hodge-like operators N_ε and N_μ by

$$\varepsilon_0 N_\varepsilon(\mathbf{E}) = \varepsilon_0\mathbf{f}^{123}_*\lfloor\tilde{G}_\varepsilon(\mathbf{E}) = \mathbf{D}, \tag{45}$$

and

$$\mu_0 N_\mu(\mathbf{H}) = \mu_0\mathbf{f}^{123}_*\lfloor\tilde{G}_\mu(\mathbf{H}) = \mathbf{B}. \tag{46}$$

By dint of (38) and (39) we have $\mathbf{B} = \mathbf{h}\wedge\mathbf{E}$, $\mathbf{D} = -\mathbf{h}\wedge\mathbf{H}$, hence we apply Lemma 3.2 and obtain the following orthogonality conditions

$$\tilde{g}_\varepsilon(\mathbf{E},\mathbf{h}) = \tilde{g}_\varepsilon(\mathbf{E},\mathbf{H}) = 0, \tag{47}$$

$$\tilde{g}_\mu(\mathbf{H},\mathbf{h}) = \tilde{g}_\mu(\mathbf{H},\mathbf{E}) = 0. \tag{48}$$

The perpendicularity $\mathbf{E}\perp\mathbf{h}$ implies $\mathbf{E}\perp\mathbf{k}$ and similarly for $\mathbf{H}$:

$$\tilde{g}_\varepsilon(\mathbf{E},\mathbf{k}) = 0, \qquad \tilde{g}_\mu(\mathbf{H},\mathbf{k}) = 0. \tag{49}$$

In order to calculate the dielectric scalar products of two-forms we use eq. (10):

$$\tilde{g}_\varepsilon(\mathbf{B},\mathbf{D}) = \tilde{g}_\varepsilon(\mathbf{h}\wedge\mathbf{E},\,-\mathbf{h}\wedge\mathbf{H}) = -\begin{vmatrix} \tilde{g}_\varepsilon(\mathbf{h},\mathbf{h}) & \tilde{g}_\varepsilon(\mathbf{h},\mathbf{H}) \\ \tilde{g}_\varepsilon(\mathbf{E},\mathbf{h}) & \tilde{g}_\varepsilon(\mathbf{E},\mathbf{H}) \end{vmatrix}.$$

$$= -\tilde{g}_\varepsilon(\mathbf{h},\mathbf{h})\tilde{g}_\varepsilon(\mathbf{E},\mathbf{H}) + \tilde{g}_\varepsilon(\mathbf{h},\mathbf{H})\tilde{g}_\varepsilon(\mathbf{E},\mathbf{h}).$$

Due to (47), we obtain

$$\tilde{g}_\varepsilon(\mathbf{B},\mathbf{D}) = -\tilde{g}_\varepsilon(\mathbf{h},\mathbf{h})\cdot 0 + \tilde{g}_\varepsilon(\mathbf{h},\mathbf{H})\cdot 0 = 0. \tag{50}$$

Similarly, with the use of (48), we get for the magnetic scalar product

$$\tilde{g}_\mu(\mathbf{B},\mathbf{D}) = -\tilde{g}_\mu(\mathbf{h},\mathbf{h})\cdot 0 + 0\cdot\tilde{g}_\mu(\mathbf{E},\mathbf{h}) = 0.$$

All the obtained perpendicularity conditions can be summarized in the following Table 2:

Table 2. Orthogonality relations between quantities of one wave.

	$\tilde{g}_\mu(\mathbf{E},\mathbf{H}) = 0$	$\tilde{g}_\mu(\mathbf{B},\mathbf{D}) = 0$	$\tilde{g}_\mu(\mathbf{k},\mathbf{H}) = 0$
$\tilde{g}_\varepsilon(\mathbf{k},\mathbf{E}) = 0$	$\tilde{g}_\varepsilon(\mathbf{E},\mathbf{H}) = 0$	$\tilde{g}_\varepsilon(\mathbf{B},\mathbf{D}) = 0$	

Four of these relations were given in vectorial language as Equations (4.2) and (4.3) inf Ref. [18].

5.3. Eigenwaves

The orthogonality of $\mathbf{B}$ and $\mathbf{D}$ according to two scalar products $\tilde{g}_\varepsilon$ and $\tilde{g}_\mu$ means that the one-dimensional attitudes of $\mathbf{B}$ and $\mathbf{D}$ are perpendicular to each other in two metrics. This is the situation described in Sec. 3 when discussing Fig. 12. As it was mentioned there, in two dimensions (this is the case now because $\mathbf{D}$ and $\mathbf{B}$ are parallel to the planes of $\mathbf{k}$) when g_ε and g_μ are not proportional, there exists only two attitudes, represented by two vectors $\mathbf{c}_1$ and $\mathbf{c}_2$, perpendicular to each other in both scalar products. This implies that two kinds of plane waves of the form (36) exist such that:

$$\mathbf{D}^{(1)} \,||\, \mathbf{c}_1 \quad \text{and} \quad \mathbf{B}^{(1)} \,||\, \mathbf{c}_2, \tag{51}$$

or

$$\mathbf{D}^{(2)} \,||\, \mathbf{c}_2 \quad \text{and} \quad \mathbf{B}^{(2)} \,||\, \mathbf{c}_1. \tag{52}$$

They are called *eigenwaves*. The relations (51, 52) imply

$$\tilde{g}_\varepsilon\left(\mathbf{D}^{(1)},\mathbf{D}^{(2)}\right) = 0, \qquad \tilde{g}_\mu\left(\mathbf{D}^{(1)},\mathbf{D}^{(2)}\right) = 0, \tag{53}$$

$$\tilde{g}_\varepsilon\left(\mathbf{B}^{(1)},\mathbf{B}^{(2)}\right) = 0, \qquad \tilde{g}_\mu\left(\mathbf{B}^{(1)},\mathbf{B}^{(2)}\right) = 0. \tag{54}$$

The problem will have the same degree of complication, if we assume that the medium is magnetically isotropic, and only electrically anisotropic. Therefore, we assume that the magnetic scalar product g_μ coincides with the natural scalar product g of the physical space. We know that $\mathbf{B}$, $\mathbf{D}$ are parallel to the planes of $\mathbf{k}$. All the magnetic orthogonalities in first raw of Table 2 mean the natural orthogonalities of respective quantities. This has been taken into account in Figs. 18 and 19 .

Since the relative permittivity matrix $\mathcal{E}$ defines the dielectric scalar product g_ε, its properties determine properties the dielectric metric. Namely, the principal axes of $\mathcal{E}$ coincide with the principal attitudes of g_ε with respect to g, mentioned in Sec. 3. If three eigenvalues of $\mathcal{E}$ are distinct, the scalar product g_ε is bi-axial and the medium should be called *bi-axial*. If two eigenvalues of $\mathcal{E}$ are equal, g_ε is uniaxial and the medium is called *uniaxial*.

Let us combine relation (39) with the constitutive equations. By virtue of (35) and (38) we write

$$\mathbf{H}=-\mu_0^{-1}\,(\det\mathcal{M})^{-1}\,\mathbf{f}_*^{123}\,\Big\lfloor\tilde{G}_\mu^{\wedge 2}(\mathbf{B})=-\mu_0^{-1}(\det\mathcal{M})^{-1}\mathbf{f}_*^{123}\Big\lfloor\,\tilde{G}_\mu^{\wedge 2}(\mathbf{h}\wedge\mathbf{E}) \tag{55}$$

and substitute this to (39):

$$\mathbf{D}=-\mathbf{h}\wedge\mathbf{H}=\mu_0^{-1}\left(\det\mathcal{M})^{-1}\mathbf{h}\wedge\left[\mathbf{f}_*^{123}\lfloor\tilde{G}^{\wedge 2}(\mathbf{h}\wedge\mathbf{E}\right)\right].$$

Identity (14) allows us to write

$$\mathbf{D}=\mu_0^{-1}\,(\det\mathcal{M})^{-1}\,\mathbf{f}_*^{123}\lfloor\,\left[\tilde{g}_\mu(\mathbf{h},\mathbf{h})\,\tilde{G}_\mu(\mathbf{E})-\tilde{g}_\mu(\mathbf{h},\mathbf{E})\,\tilde{G}_\mu(\mathbf{h})\right].$$

We insert expression (31) in the left hand side

$$\varepsilon_0\mathbf{f}_*^{123}\lfloor\tilde{G}_\varepsilon(\mathbf{E})=\mu_0^{-1}(\det\mathcal{M})^{-1}\,\mathbf{f}_*^{123}\lfloor\left[\tilde{g}_\mu(\mathbf{h},\mathbf{h})\,\tilde{G}_\mu(\mathbf{E})-\tilde{g}_\mu(\mathbf{h},\mathbf{E})\,\tilde{G}_\mu(\mathbf{h})\right].$$

The contraction with the three-form $\mathbf{f}_*^{123}$ is reversible, hence one may write down

$$\varepsilon_0\mu_0(\det\mathcal{M})\tilde{G}_\varepsilon(\mathbf{E})=\tilde{g}_\mu(\mathbf{h},\mathbf{h})\,\tilde{G}_\mu(\mathbf{E})-\tilde{g}_\mu(\mathbf{h},\mathbf{E})\,\tilde{G}_\mu(\mathbf{h}). \tag{56}$$

In this manner we arrived at a linear equation for the electric field $\mathbf{E}$. Its solutions determine the eigenwaves.

Since we assumed that the magnetic scalar product coincides with the natural scalar product, we may write

$$\det\mathcal{M}=1,\qquad\tilde{G}_\mu(\mathbf{E})=\vec{\mathbf{E}},\qquad\tilde{G}_\mu(\mathbf{h})=\vec{\mathbf{h}}, \tag{57}$$

hence

$$\varepsilon_0\mu_0\,\tilde{G}_\varepsilon(\mathbf{E}) = \tilde{g}_\mu(\mathbf{h},\mathbf{h})\,\vec{\mathbf{E}} - \tilde{g}_\mu(\mathbf{h},\mathbf{E})\,\vec{\mathbf{h}}.$$

We may also write $\tilde{g}_\mu(\mathbf{h},\mathbf{E}) = \vec{\mathbf{h}}\cdot\vec{\mathbf{E}}$ and $\tilde{g}_\mu(\mathbf{h},\mathbf{h}) = h^2$ where h is the magnitude of the slowness:

$$\varepsilon_0\mu_0\,\tilde{G}_\varepsilon(\mathbf{E}) = h^2\vec{\mathbf{E}} - (\vec{\mathbf{h}}\cdot\vec{\mathbf{E}})\,\vec{\mathbf{h}}.$$

This equation can be found in Section 97 of Ref. [19]. If the vector basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ is orthonormal according to the natural scalar product, and such that $\mathbf{u} = u\mathbf{f}^3$, this implies $\vec{\mathbf{h}} = h\,\mathbf{e}_3$ and we obtain

$$\varepsilon_0\mu_0\,\tilde{G}_\varepsilon(\mathbf{E}) = h^2\vec{\mathbf{E}} - h^2E_3\,\vec{\mathbf{e}}_3,$$

$$\tilde{G}_\varepsilon(\mathbf{E}) = \frac{h^2}{\varepsilon_0\mu_0}\left(\vec{\mathbf{E}} - E_3\,\vec{\mathbf{e}}_3\right). \tag{58}$$

This is a simplified version of equation (56). After comparing it with eq. (29) we observe that vector $\vec{\mathbf{D}}$ is proportional to the component of $\vec{\mathbf{E}}$, perpendicular to $\vec{\mathbf{h}}$.

When expressed by coordinates of $\mathbf{E}$, condition (58) assumes the form

$$\begin{aligned}
\varepsilon^{11}E_1 + \varepsilon^{12}E_2 + \varepsilon^{13}E_3 &= \frac{h^2}{\varepsilon_0\mu_0}\,E_1,\\
\varepsilon^{21}E_1 + \varepsilon^{22}E_2 + \varepsilon^{23}E_3 &= \frac{h^2}{\varepsilon_0\mu_0}\,E_2,\\
\varepsilon^{31}E_1 + \varepsilon^{32}E_2 + \varepsilon^{33}E_3 &= 0.
\end{aligned}$$

We conclude from this that the coordinates of $\mathbf{E}$ can not be arbitrary— they have to fulfill the above system of linear equations. We get rid of one coordinate and one equation by substituting the relation

$$E_3 = -\frac{\varepsilon^{31}}{\varepsilon^{33}}E_1 - \frac{\varepsilon^{32}}{\varepsilon^{33}}E_2, \tag{59}$$

obtained from the third equation, to first two ones:

$$\left(\varepsilon^{11} - \frac{\varepsilon^{13}\varepsilon^{31}}{\varepsilon^{33}}\right)E_1 + \left(\varepsilon^{12} - \frac{\varepsilon^{13}\varepsilon^{32}}{\epsilon_{33}}\right)E_2 = \frac{u^2}{\varepsilon_0\mu_0}E_1, \tag{60}$$

$$\left(\varepsilon^{21} - \frac{\varepsilon^{23}\varepsilon^{31}}{\varepsilon^{33}}\right)E_1 + \left(\varepsilon^{22} - \frac{\varepsilon^{23}\varepsilon^{32}}{\varepsilon^{33}}\right)E_2 = \frac{u^2}{\varepsilon_0\mu_0}E_2, \tag{61}$$

Notice that at the left-hand side matrix $A = \{a^{ij}\}$ is present with elements
$$a^{ij} = \varepsilon^{ij} - \frac{\varepsilon^{i3}\varepsilon^{3j}}{\varepsilon^{33}}$$
Since matrix $\mathcal{E}$ is symmetric, A has the same property. Equations (60, 61) constitute the following eigenvalue equation for A:
$$A\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \frac{h^2}{\varepsilon_0\mu_0}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \tag{62}$$
Matrix A is symmetric, hence its eigenvalues and eigenvectors exist. If it has two distinct eigenvalues $a_{(1)}$ and $a_{(2)}$, we obtain two distinct conditions for the phase slowness:
$$\frac{h^2}{\varepsilon_0\mu_0} = a_{(1)}, \quad \text{or} \quad \frac{h^2}{\varepsilon_0\mu_0} = a_{(2)},$$
hence
$$h = \sqrt{a_{(1)}\varepsilon_0\mu_0} \quad \text{or} \quad h = \sqrt{a_{(2)}\varepsilon_0\mu_0}.$$
This gives different phase slowness for two solutions of (62):
$$h_{(i)} = \sqrt{a_{(i)}\varepsilon_0\mu_0},$$
for $i \in \{1, 2\}$. According to this, for fixed circular frequency ω we have two values for the wavity coordinate $k_{(i)} = \omega h_{(i)}$. If the electric permittivity does not depend on the frequency of the electromagnetic field, the phase slowness coordinate is the same for various frequencies and is characteristic of the given medium and of a chosen two-dimensional direction in it. The eigenequation (62) can be also written as:
$$\sum_{m=1}^{2} a^{jm} E_m^{(i)} = a_{(i)} E_j^{(i)}. \tag{63}$$
In this manner we ascertained that only columns $\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$ satisfying the eigenequation (62) give solutions of the free Maxwell equations; these are the *eigenwaves*.

For our choice of basis such that $\mathbf{h} = h\,\mathbf{f}^3$, we obtain
$$\mathbf{h} \wedge \mathbf{E} = h\,\mathbf{f}^3 \wedge (E_1\mathbf{f}^1 + E_2\mathbf{f}^2 + E_3\mathbf{f}^3) = hE_1\mathbf{f}^{31} + hE_2\mathbf{f}^{32}.$$
Then $\tilde{G}^{\wedge 2}(\mathbf{h} \wedge \mathbf{E}) = h(E_1\mathbf{e}_{31} + E_2\mathbf{e}_{32})$. Inserting this into (55) yields
$$\mathbf{H} = -\mu_0^{-1}\mathbf{f}_*^{123} \lfloor h(E_1\mathbf{e}_{31} + E_2\mathbf{e}_{23}) = \frac{h}{\mu_0}\left(-E_2\mathbf{f}_*^1 + E_1\mathbf{f}_*^2\right). \tag{64}$$

From this, we derive

$$\mathbf{D} = -\mathbf{h} \wedge \mathbf{H} = -h\,\mathbf{f}^3 \wedge \frac{h}{\mu_0}\left(-E_2\mathbf{f}_*^1 + E_1\mathbf{f}_*^2\right) = \frac{h^2}{\mu_0}\left(E_1\mathbf{f}_*^{23} + E_2\mathbf{f}_*^{31}\right). \tag{65}$$

The electric field, after taking into account (59), is

$$\mathbf{E} = E_1\mathbf{f}^1 + E_2\mathbf{f}^2 - (\varepsilon^{33})^{-1}(\varepsilon^{31}E_1 + \varepsilon^{32}E_2)\,\mathbf{f}^3. \tag{66}$$

5.4. Relations Between the Eigenwaves

We shall now calculate some other scalar products between forms of the same kind, corresponding to different indices $i \in \{1,2\}$ enumerating the eigenwaves. To find the magnetic scalar product $\tilde{g}_\mu(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})$ we use (64):

$$\tilde{g}_\mu\left(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}\right) = \frac{h_{(1)}h_{(2)}}{\mu_0^2}\left(E_1^{(1)}E_1^{(2)} + E_2^{(1)}E_2^{(2)}\right).$$

Two eigenvectors $\begin{pmatrix} E_1^{(1)} \\ E_2^{(1)} \end{pmatrix}$ and $\begin{pmatrix} E_1^{(2)} \\ E_2^{(2)} \end{pmatrix}$ of one symmetric matrix A are perpendicular columns, hence we obtain

$$g_\mu\left(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}\right) = 0. \tag{67}$$

We see that the forms $\mathbf{H}^{(i)}$ are magnetically orthogonal.

Let us elaborate on the dielectric scalar product of the two one-forms $\mathbf{E}^{(i)}$

$$\tilde{g}_\varepsilon\left(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}\right) = \sum_{i,j=1}^{3} E_i^{(1)}\varepsilon^{ij}E_j^{(2)}.$$

We apply (59):

$$\begin{aligned}
\tilde{g}_\varepsilon\left(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}\right) &= E_1^{(1)}\varepsilon^{11}E_1^{(2)} + E_1^{(1)}\varepsilon^{12}E_2^{(2)} - E_1^{(1)}\frac{\varepsilon^{13}}{\varepsilon^{33}}\left(\varepsilon^{31}E_1^{(2)} + \varepsilon^{32}E_2^{(2)}\right) \\
&+ E_2^{(1)}\varepsilon^{21}E_1^{(2)} + E_2^{(1)}\varepsilon^{22}E_2^{(2)} - E^{(1)}\frac{\varepsilon^{23}}{\varepsilon^{33}}\left(\varepsilon^{31}E_1^{(2)} + \varepsilon^{32}E_2^{(2)}\right) \\
&- \left(\varepsilon^{33}\right)^{-1}\left(\varepsilon^{31}E_1^{(1)} + \varepsilon^{32}E_2^{(1)}\right)\varepsilon^{31}E_1^{(2)} - \left(\varepsilon^{33}\right)^{-1}\left(\varepsilon^{31}E_1^{(1)} + \varepsilon^{32}E_2^{(1)}\right)\varepsilon^{32}E_2^{(2)} \\
&+ \left(\varepsilon^{33}\right)^{-1}\left(\varepsilon^{31}E_1^{(1)} + \varepsilon^{32}E_2^{(1)}\right)\left(\varepsilon^{31}E_1^{(2)} + \varepsilon^{32}E_2^{(2)}\right).
\end{aligned}$$

After reduction of similar terms we are left with

$$\tilde{g}_\varepsilon\left(\mathbf{E}^{(1)},\mathbf{E}^{(2)}\right)=E_1^{(1)}\left(\varepsilon^{11}-\frac{\varepsilon^{13}\varepsilon^{31}}{\varepsilon^{33}}\right)E_1^{(2)}+E_1^{(1)}\left(\varepsilon^{12}-\frac{\varepsilon^{13}\varepsilon^{32}}{\varepsilon^{33}}\right)E_2^{(2)}$$
$$+E_2^{(1)}\left(\varepsilon^{21}-\frac{\varepsilon^{23}\varepsilon^{31}}{\varepsilon^{33}}\right)E_1^{(2)}+E_2^{(1)}\left(\varepsilon^{22}-\frac{\varepsilon^{23}\varepsilon^{32}}{\varepsilon^{33}}\right)E_2^{(2)}.$$

We recognize elements of A, hence

$$\tilde{g}_\varepsilon(\mathbf{E}^{(1)},\mathbf{E}^{(2)})=\sum_{i,j=1}^{2}E_i^{(1)}a^{ij}E_j^{(2)}.$$

Since $\begin{pmatrix}E_1^{(2)}\\ E_2^{(2)}\end{pmatrix}$ is the eigenvector to the eigenvalue $a_{(2)}$, we obtain

$$\tilde{g}_\varepsilon(\mathbf{E}^{(1)},\mathbf{E}^{(2)})=a_{(2)}(E_1^{(1)}E_1^{(2)}+E_2^{(1)}E_2^{(2)})=0. \tag{68}$$

In this manner we see that the forms $\mathbf{E}^{(i)}$ are dielectrically orthogonal.

The perpendicularity relations (53), (54), (67) and (68) are summarized in next Table 3:

Table 3. Orthogonality relations between quantities of two eigenwaves.

	$\tilde{g}_\mu(\mathbf{B}^{(1)},\mathbf{B}^{(2)})=0$	$\tilde{g}_\mu(\mathbf{D}^{(1)},\mathbf{D}^{(2)})=0$	$\tilde{g}_\mu(\mathbf{H}^{(1)},\mathbf{H}^{(2)})=0$
$\tilde{g}_\varepsilon(\mathbf{E}^{(1)},\mathbf{E}^{(2)})=0$	$\tilde{g}_\varepsilon(\mathbf{B}^{(1)},\mathbf{B}^{(2)})=0$	$\tilde{g}_\varepsilon(\mathbf{D}^{(1)},\mathbf{D}^{(2)})=0$	

Only four of these relations can be found in Ref. [18] as Eqs. (4.15)–(4.18).

5.5. Energy Flux and Energy Density

One-dimensional attitude of the Poynting odd two-form as the exterior product $\mathbf{S}=\mathbf{E}\wedge\mathbf{H}$ of one-forms is the intersection of the planes of the two factors. If $\mathbf{E}$ and $\mathbf{H}$ are as on Fig. 18, the attitude of $\mathbf{S}$ is oblique with respect to the planes of $\mathbf{k}$. The energy flux density for the eigenwaves, according to (55), is

$$\mathbf{S}=\mathbf{E}\wedge\mathbf{H}=-{\mu_0}^{-1}\left(\det\mathcal{M}\right)^{-1}\mathbf{E}\wedge\mathbf{f}_*^{123}\lfloor\tilde{G}_\mu^{\wedge 2}\left(\mathbf{h}\wedge\mathbf{E}\right).$$

We apply (14)

$$\mathbf{S}={\mu_0}^{-1}\left(\det\mathcal{M}\right)^{-1}\mathbf{f}_*^{123}\lfloor\left[\tilde{g}_\mu(\mathbf{E},\mathbf{E})\,\tilde{G}_\mu(\mathbf{h})-\tilde{g}_\mu(\mathbf{E},\mathbf{h})\,\tilde{G}_\mu(\mathbf{E})\right]. \tag{69}$$

By substituting (57) we obtain

$$\mathbf{S} = \mu_0^{-1} \mathbf{f}_*^{123} \lfloor \left[E^2 \vec{\mathbf{h}} - (\vec{\mathbf{h}} \cdot \vec{\mathbf{E}}) \, \vec{\mathbf{E}} \right].$$

After replacing the two-form $\mathbf{S}$ by the vector $\vec{\mathbf{S}}$ this can be written as

$$\vec{\mathbf{S}} = \mu_0^{-1} \left[E^2 \vec{\mathbf{h}} - \left(\vec{\mathbf{h}} \cdot \vec{\mathbf{E}} \right) \, \vec{\mathbf{E}} \right] = \frac{E^2}{\mu_0} \vec{\mathbf{h}}_\perp$$

where $\vec{\mathbf{h}}_\perp$ is the component of $\vec{\mathbf{h}}$ perpendicular to $\vec{\mathbf{E}}$.

Let us calculate the energy density

$$w = \mathbf{h} \wedge \mathbf{S} = \mu_0^{-1} (\det \mathcal{M})^{-1} \mathbf{h} \wedge \left\{ \mathbf{f}_*^{123} \lfloor \left[\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \tilde{G}_\mu(\mathbf{h}) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \tilde{G}_\mu(\mathbf{E}) \right] \right\}$$

with the use of identity (7)

$$w = \mu_0^{-1} \left(\det \mathcal{M} \right)^{-1} \left[\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \; \mathbf{h}(\tilde{G}_\mu(\mathbf{h})) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \, \mathbf{h}(\tilde{G}_\mu(\mathbf{E})) \right] \mathbf{f}_*^{123}$$

and of identity (13)

$$w = \mu_0^{-1} (\det \mathcal{M})^{-1} \left[\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \, \tilde{g}_\mu(\mathbf{h}, \mathbf{h}) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \, \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \right] \mathbf{f}_*^{123}. \quad (70)$$

After substitution of (69) and (70) to (44) we obtain

$$\mathbf{v} = \frac{\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \, \tilde{G}_\mu(\mathbf{h}) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \, \tilde{G}_\mu(\mathbf{E})}{\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \, \tilde{g}_\mu(\mathbf{h}, \mathbf{h}) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \, \tilde{g}_\mu(\mathbf{E}, \mathbf{h})} \quad (71)$$

We may calculate the value of one-form $\mathbf{h}$ on vector $\mathbf{v}$:

$$\mathbf{h}[\mathbf{v}] = \frac{\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \, \mathbf{h}(\tilde{G}_\mu(\mathbf{h})) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \, \mathbf{h}(\tilde{G}_\mu(\mathbf{E}))}{\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \, \tilde{g}_\mu(\mathbf{h}, \mathbf{h}) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \, \tilde{g}_\mu(\mathbf{E}, \mathbf{h})}$$

$$= \frac{\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \, \tilde{g}_\mu(\mathbf{h}, \mathbf{h}) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \, \tilde{g}_\mu(\mathbf{E}, \mathbf{h})}{\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \, \tilde{g}_\mu(\mathbf{h}, \mathbf{h}) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \, \tilde{g}_\mu(\mathbf{E}, \mathbf{h})} = 1.$$

In this sense we may claim that the phase slowness and the velocity of the energy transport by the plane electromagnetic wave are mutually reciprocal quantities. The energy transport velocity is one of vectors depicted in Fig. 15.

Having used the identity $g(\tilde{G}(\mathbf{h}), \tilde{G}(\mathbf{E})) = \tilde{g}(\mathbf{h}, \mathbf{E})$ one may calculate also the magnetic scalar product of the vector $\mathbf{v}$ with itself:

$$g_\mu(\mathbf{v}, \mathbf{v}) = \frac{\tilde{g}_\mu(\mathbf{E}, \mathbf{E})}{\tilde{g}_\mu(\mathbf{E}, \mathbf{E}) \tilde{g}_\mu(\mathbf{h}, \mathbf{h}) - \tilde{g}_\mu(\mathbf{E}, \mathbf{h}) \tilde{g}_\mu(\mathbf{E}, \mathbf{h})}. \quad (72)$$

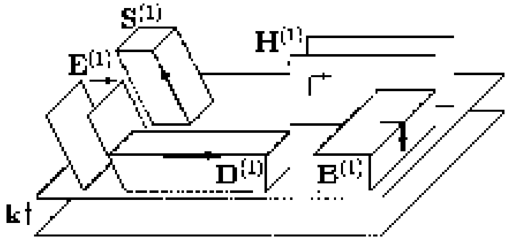


Figure 20. Configuration of relevant quantities for first eigenwave.

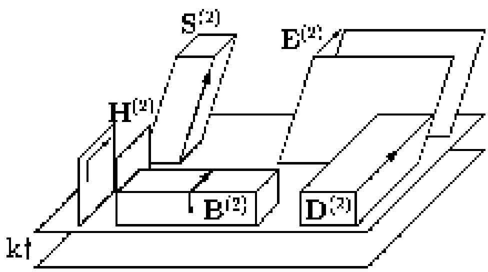


Figure 21. Configuration of relevant quantities for second eigenwave.

This shows that for the same magnitude of the electric field of two eigenwaves (i.e. for the same expression $\tilde{g}_\mu(\mathbf{E},\mathbf{E})$) the magnitudes of phase velocities of two eigenwaves are different because the phase slownesses are different.

We may assume that Fig. 19 shows the first eigenwave. We repeat it here as Fig. 20 with the indices of the eigenwave added. In such a case the second eigenwave should be as in Fig. 21. A careful look at the two pictures allows to notice that quantities describing the second eigenwave are obtained from the first one by rotation about 90 degrees around the vertical axis.

We notice that the Poynting odd two-forms, corresponding to the two eigenwaves may have different directions. In such a case two plane eigenwaves with the same frequency and the same planes of constant phase propagate differently, i.e., they transport the energy in distinct directions and have distinct magnitudes of phase slownesses and, therefore, different phase velocities.

It is worth to ponder why the eigenwaves are not needed in isotropic medium. Now, the matrix $\mathcal{E}$ is then diagonal, $\varepsilon^{i3} = \epsilon^{3j} = 0$ and the elements of A given by $a^{ij} = \varepsilon^{ij} - \frac{\varepsilon^{i3}\varepsilon^{3j}}{\varepsilon^{33}}$ should be written as $a^{ij} = \varepsilon^{ij} = \varepsilon_r \delta^{ij}$. Thus in the isotropic medium, matrix A becomes the multiple of the unit matrix, hence each column $\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$ is its eigenvector and no two of them are distinguished. Also one eigenvalue exists and in this connection only one phase slowness occurs in the given medium $h = \sqrt{\varepsilon_r \varepsilon_0 \mu_0}$. Then formula (59) for arbitrary E_1, E_2 gives $E_3 = 0$, which means that $\mathbf{E}$ is also magnetically perpendicular to the wavity one-form $\mathbf{k}$. In the traditional language, this means that vectors $\vec{\mathbf{E}}$ and $\vec{\mathbf{k}}$ are perpendicular.

6. CONCLUSION

For the description of plane waves in an anisotropic medium, the metric independent formalism is used, based on the differential forms. An argument is presented, showing that the phase velocity vector $\mathbf{v}$ is not unique and a *phase slowness* one-form $\mathbf{h}$ should be introduced such that $\mathbf{h}[\mathbf{v}] = 1$ for any phase velocity vector $\mathbf{v}$. It is uniquely related to wavity one-form $\mathbf{k}$ via the relation $\mathbf{h} = \omega^{-1}\mathbf{k}$.

The Maxwell equations imply two relations between the electromagnetic fields $\mathbf{B} = \mathbf{h} \wedge \mathbf{E}$ and $\mathbf{D} = -\mathbf{h} \wedge \mathbf{H}$. The both two-form fields $\mathbf{B}$ and $\mathbf{D}$ are parallel to the phase slowness. The Poynting two-form $\mathbf{S} = \mathbf{E} \wedge \mathbf{H}$ and the energy density w of the electromagnetic field are related via $w = \mathbf{S} \wedge \mathbf{h}$. These relations belong to the premetric description of the plane electromagnetic wave.

The constitutive equations for linear media allow to introduce two scalar products: $\tilde{g}_\varepsilon$ for the electric field and $\tilde{g}_\mu$ for the magnetic field. In this way the metric can vary from medium to medium, therefore the name: variable metric electrodynamics. It turns out for the plane waves that the perpendicularity conditions $\tilde{g}_{\varepsilon,\mu}(\mathbf{E}, \mathbf{H}) = 0$ and $\tilde{g}_{\varepsilon,\mu}(\mathbf{B}, \mathbf{D}) = 0$ are satisfied for both scalar products. The perpendicularity of the one-forms $\mathbf{E}$ and $\mathbf{H}$ to the slowness occurs only in one of the scalar products: $\tilde{g}_\varepsilon(\mathbf{h}, \mathbf{E}) = 0$, $\tilde{g}_\mu(\mathbf{h}, \mathbf{H}) = 0$.

When the scalar product matrices $\tilde{\mathcal{G}}_\varepsilon$ and $\tilde{\mathcal{G}}_\mu$ are not proportional, only two linearly independent plane waves exist, called *eigenwaves*, and for them the following perpendicularity conditions are satisfied: $\tilde{g}_{\varepsilon,\mu}(\mathbf{B}^{(1)}, \mathbf{B}^{(2)}) = 0$, $\tilde{g}_{\varepsilon,\mu}(\mathbf{D}^{(1)}, \mathbf{D}^{(2)}) = 0$, $\tilde{g}_\varepsilon(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}) = 0$ and $\tilde{g}_\mu(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}) = 0$. The two eigenwaves have phase slowness of distinct magnitude but the same direction.

For each eigenwave the phase velocity vector can be introduced, parallel to the Poynting two-form $\mathbf{S}$ through the relation $\mathbf{S} = w\lfloor\mathbf{v}$. The explicit formula (71) is given for $\mathbf{v}$ and the relation $\mathbf{h}[\mathbf{v}] = 1$ is checked. The phase velocities of two eigenwaves have distinct directions and magnitudes.

The presented formalism is suitable also for the magnetically anisotropic medium. In such a case it is sufficient to choose for g_μ not the natural scalar product, but the scalar product for which $\tilde{\mathcal{G}}_\mu = \mathcal{M}$, the matrix of relative magnetic permittivity.

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